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Intermediary Leverage Cycles and Financial Stability

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Abstract

We present a theory of financial intermediary leverage cycles within a dynamic model of the macroeconomy. Intermediaries face risk-based funding constraints that give rise to procyclical leverage and a procyclical share of intermediated credit. The pricing of risk varies as a function of intermediary leverage, and asset return exposure to intermediary leverage shocks earns a positive risk premium. Relative to an economy with constant leverage, financial intermediaries generate higher consumption growth and lower consumption volatility in normal times, at the cost of endogenous systemic financial risk. The severity of systemic crisis depends on intermediaries' leverage and net worth. Regulations that tighten funding constraints affect the systemic risk-return trade-off by lowering the likelihood of systemic crises at the cost of higher pricing of risk.

Key words: systemic risk, macroprudential policy, DSGE, amplification, capital regulation, financial intermediation

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I. Introduction

The financial crisis of 2007-09 highlighted the central role that financial intermediaries play in the propagation of fundamental shocks. While this has renewed the debate around financial stability policies, a widely accepted conceptual framework for the study of prudential policies within a macroeconomic setting has not yet emerged. In this paper, we develop a dynamic stochastic general equilibrium model in which the leverage cycle of financial intermediaries propagates fundamental shocks. The model features endogenous solvency risk of the financial sector, allowing us to study the impact of prudential policies on the trade-off between systemwide distress and the pricing of risk during normal times.

We depart from the emerging literature[1] on dynamic macroeconomic models with financial intermediaries by assuming that intermediaries have to hold equity in proportion to the riskiness of their total assets. Our approach gives rise to predictions about the intermediary leverage cycle and the pricing of risk that alternative theories do not capture. The model gives rise to the procyclical leverage behavior emphasized by Adrian and Shin [2010a], and the procyclicality of intermediated credit documented by Adrian, Colla, and Shin [2012]. Furthermore, prices of risk fluctuate as a function of intermediary leverage, and the price of risk of leverage is positive, features that have been documented by Adrian, Moench, and Shin [2010] and Adrian, Etula, and Muir [2013b].

In our theory, financial intermediaries have two roles. While both households and intermediaries can own existing firms' capital, intermediaries have access to a better capital creation technology, capturing financial institutions' ability to allocate capital and monitor borrowers. The second role of intermediaries is to provide risk bearing capacity by having inside equity. Intermediaries' ability to bear risk fluctuates over time due to the risk sensitive nature of their funding constraint.

Costly adjustments to the real capital stock lead to the intermediary leverage cycle, which translates into an endogenous amplification of shocks. While fundamental shocks are assumed to be homoskedastic, equilibrium asset prices and equilibrium consumption growth

[1] Brunnermeier and Sannikov [2011, 2012], He and Krishnamurthy [2012a, 2013], Gertler and Kiyotaki [2012], and Gertler, Kiyotaki, and Queralto [2011] all have recently proposed equilibrium theories with a financial sector.

exhibit stochastic volatility. When adverse shocks to intermediary balance sheets are sufficiently large, intermediaries experience systemic solvency risk and need to restructure. We assume that such systemic risk occurs when intermediaries' net worth falls below a threshold. Intermediaries deleverage by writing down debt, imposing losses on households. Whether systemic financial crisis are benign or generate large consumption losses depends on the severity of the shocks, the leverage of intermediaries, and their relative net worth.

Our model gives rise to the "volatility paradox" of Brunnermeier and Sannikov [2012] and Adrian and Brunnermeier [2011]: Times of low volatility tend to be associated with a buildup of leverage, which in turn increases forward-looking systemic risk. We also study the systemic risk-return trade-off: Low prices of risk today tend to be associated with larger forward-looking systemic risk measures, suggesting that measures of asset price valuations are useful indicators for systemic risk assessments.

In a benchmark model with a constant intermediary leverage constraint, the resulting equilibrium growth of investment, price of capital, and the risk-free rate are constant. Fluctuations in output of the benchmark economy are entirely due to productivity shocks, and output is fully insulated from liquidity shocks. In contrast, in the model with a risk based funding constraint, liquidity shocks spill over to real activity, and productivity shocks are amplified. When funding constraints are risk based, intermediaries provide consumption smoothing services to households during normal times that result in higher growth rates, at the cost of occasional realizations of systemic risk states with large consumption drops. The tightness of risk based intermediary constraints thus regulate a systemic risk return trade-off.

Since our theory captures important empirical regularities about the dynamic interactions between the financial sector and the macroeconomy, it provides a conceptual framework for financial stability policies. In this paper, we focus primarily on one form of prudential policy, which concerns the tightness of intermediaries' funding constraint and can be interpreted as capital regulation. Our paper is among the few that consider the role of (macro)prudential policies in dynamic equilibrium models explicitly (see Goodhart, Kashyap, Tsomocos, and Vardoulakis [2012], Angelini, Neri, and Panetta [2011], Bianchi and Mendoza [2011], and Nuño and Thomas [2012] for alternative settings). Our main findings are intuitive. We show that households' welfare dependence on the tightness of the intermediaries' capital constraint is inversely U-shaped: very loose constraints generate excessive risk taking of intermediaries relative to household preferences, while very tight funding constraints inhibit intermediaries' risk taking and investment. This trade-off maps closely into the debate on optimal regulation. It should be noted that these results rely on our assumption that intermediaries finance themselves only in the public debt market, thus violating the necessary assumptions for the Modigliani and Miller [1958] capital structure irrelevance result. While the impact of prudential regulation would be less pronounced if intermediaries were able to issue equity, any positive cost of equity issuance would preserve the systemic risk-return trade-off.

The rest of the paper is organized as follows. We describe the model in Section II. The equilibrium interactions and outcomes are outlined in Section III. We investigate the creation of systemic risk and its welfare implications in Section IV. Conclusions are presented in Section V. Technical details are relegated to the appendix.

I.A. Related Literature

This paper is related to several strands of the literature. Geanakoplos [2003] and Fostel and Geanakoplos [2008] show that leverage cycles can cause contagion and issuance rationing in a general equilibrium model with heterogeneous agents, incomplete markets, and endogenous collateral. Brunnermeier and Pedersen [2009] further show that market liquidity and traders' access to funding are co-dependent, leading to liquidity spirals. Our model differs from that of Fostel and Geanakoplos [2008] as our asset markets are dynamically complete and debt contracts are not collateralized. The leverage cycle in our model comes from the risk-based leverage constraint of the financial intermediaries and is intimately related to the funding liquidity of Brunnermeier and Pedersen [2009]. Unlike their model, however, the funding liquidity that matters in our setup is that of the financial intermediaries, not that of speculative traders.

This paper is also related to studies of amplification in models of the macroeconomy. The seminal paper in this literature is Bernanke and Gertler [1989], which shows that the condition of borrowers' balance sheets is a source of output dynamics. Net worth increases during economic upturns, increasing investment and amplifying the upturn, while the opposite dynamics hold in a downturn. Kiyotaki and Moore [1997] show that small shocks can be amplified by credit restrictions, giving rise to large output fluctuations. Instead of focusing on financial frictions in the demand for credit as Bernanke–Gertler and Kiyotaki–Moore do, our theory focuses on frictions in the supply of credit. Another important distinction is that the intermediaries in our economy face leverage constraints that depend on current volatility, which give rise to procyclical leverage. In contrast, the leverage constraints of Kiyotaki–Moore are state independent.

Gertler and Kiyotaki [2012] and Gertler et al. [2011] extend the accelerator mechanism of Bernanke and Gertler [1989] and Kiyotaki and Moore [1997] to financial intermediaries. Gertler et al. [2011] consider a model in which financial intermediaries can issue outside equity and short-term debt, making intermediary risk exposure an endogenous choice. Gertler and Kiyotaki [2012] further extend the model to allow for household liquidity shocks as in Diamond and Dybvig [1983]. While these models are similar in spirit to that presented in this paper, our model is more parsimonious in nature and allows for endogenous defaultable debt. We can thus investigate the creation of systemic default and the effectiveness of macroprudential policy in mitigating these risks.

Our theory is closely related to the work of He and Krishnamurthy [2012a, 2013] and Brunnermeier and Sannikov [2011, 2012], who explicitly introduce a financial sector into dynamic models of the macroeconomy. While our setup shares many conceptual and technical features of this work, our points of departure are empirically motivated. We allow households to invest via financial intermediaries as well as directly in the capital stock, a feature strongly supported by the data, which gives rise to important substitution effects between directly granted and intermediated credit. In the setup of He–Krishnamurthy, investment is always intermediated. Furthermore, our model features procyclical intermediary leverage, while theirs is countercyclical. Finally, systemic risk of the intermediary sector is at the heart of our analysis, while He–Krishnamurthy and Brunnermeier–Sannikov focus primarily on the amplification of shocks. In fact, in the set-up of He–Krishnamurthy, the financial sector is only constrained in times of crises. That means that the consumption-CAPM holds during normal times, and intermediary wealth enters the pricing kernel in times of crises only. In contrast, in our approach, intermediary state variables (wealth and leverage) always enter into the pricing kernel.

The interactions between the households, the financial intermediaries, and the productive sector lead to a highly nonlinear system. We consider the nonlinearity a desirable feature, as the model is able to capture strong amplification effects. Our theory features both endogenous risk amplification (where fundamental volatility is amplified as in Danielsson, Shin, and Zigrand [2011]), as well as the creation of endogenous systemic risk.

In our theory, equilibrium dynamics are functions of two intermediary state variables: their leverage and their net worth. In contrast, in other equilibrium models with heterogenous agents, the relevant state variables are typically wealth shares, not leverage. For example, in Rampini and Viswanathan [2012], the second variable is household wealth. In Dumas [1989] and Wang [1996], the state variables are aggregate output and the ratio of the marginal utilities of the two types of agents.

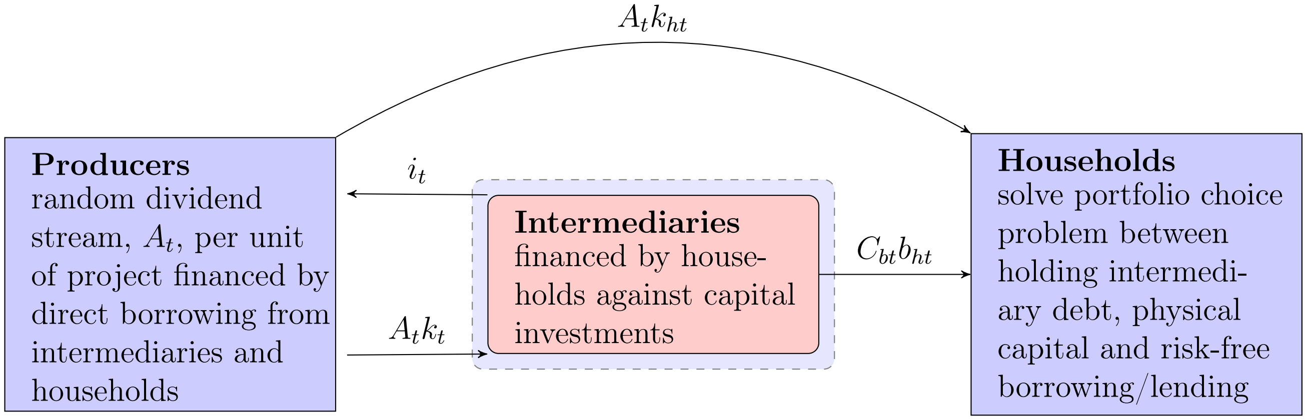
II. The Model

We begin with a single consumption good economy, where the unique good at time $t > 0$ is used as the numeraire. There are three types of agents in the economy: producers, financial intermediaries, and households. We abstract from modeling the decisions of the producers and focus instead on the interaction between the intermediary sector and the households. The basic structure of the economy is represented in Figure I.

II.A. Production

We consider an economy with two active types of agents: financially sophisticated intermediaries and unsophisticated households. While both types of agents can own capital, only financial intermediaries can create new capital through investment. We denote by K_t the aggregate amount of capital in the economy at time $t \geq 0$ and assume that each unit of capital produces A_t units of the consumption good. The total output in the economy at

Figure I: Economy Structure



time t is given by

$$Y_t = A_t K_t,$$

where the stochastic productivity of capital $\{A_t = e^{a_t}\}_{t \geq 0}$ follows a geometric diffusion process of the form

$$da_t = \bar{a} dt + \sigma_a dZ_{at},$$

with $(Z_{at})_{0 \leq t < +\infty}$ a standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Each unit of capital in the economy depreciates at a rate λ_k, so that the capital stock in the economy evolves as

$$dK_t = (I_t - \lambda_k) K_t dt,$$

where I_t is the reinvestment rate per unit of capital in place. Thus, output in the economy evolves according to

$$dY_t = \left(I_t - \lambda_k + \bar{a} + \frac{\sigma_a^2}{2} \right) Y_t dt + \sigma_a Y_t dZ_{at}.$$

Notice that the quantity $A_t K_t$ corresponds to the "efficiency" capital of Brunnermeier and Sannikov [2012], with a constant productivity rate of 1.

There is a fully liquid market for physical capital in the economy, in which both the financial intermediaries and the households are allowed to participate. To keep the economy scale-invariant, we denote by $p_{kt} A_t$ the price of one unit of capital at time t in terms of the consumption good.

II.B. Households

There is a unit mass of risk-averse, infinitely lived households in the economy. We assume that the households in the economy are identical, such that the equilibrium outcomes are determined by the decisions of the representative household. The households, however, are exposed to a preference shock, modeled as a change-of-measure variable in the household's utility function. This reduced-form approach allows us to remain agnostic as to the exact source of this second shock: With this specification, it can arise either from time-variation in the households' risk aversion or from time-variation in households' beliefs. In particular, we assume that the representative household evaluates different consumption paths $\{c_t\}_{t\geq 0}$ according to

$$\mathbb{E}\left[\int_0^{+\infty} e^{-(\xi_t+\rho_h t)} \log c_t dt\right],$$

where ρ_h is the subjective time discount of the representative household, and c_t is the consumption at time t. Here, $\exp(-\xi_t)$ is the Radon-Nikodym derivative of the measure induced by households' time-varying preferences or beliefs with respect to the physical measure. For simplicity, we assume that $\{\xi_t\}_{t\geq 0}$ evolves as a Brownian motion, correlated with the productivity shock, Z_{at}:

$$d\xi_t = \sigma_\xi \rho_{\xi,a} dZ_{at} + \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} dZ_{\xi t},$$

where $\{Z_{\xi t}\}$ is a standard Brownian motion of $(\Omega, \mathcal{F}_t, \mathbb{P})$, independent of Z_{at}. In the current setting, with households constrained in their portfolio allocation, $\exp(-\xi_t)$ can be interpreted as a time-varying liquidity preference shock, as in Allen and Gale [1994], Diamond and Dybvig [1983], and Holmström and Tirole [1998] or as a time-varying shock to the preference for early resolution of uncertainty, as in Bhamra, Kuehn, and Strebulaev [2010a,b]. In particular, when the households receive a positive $d\xi_t$ shock, their effective discount rate increases, leading to a higher demand for liquidity.

The households finance their consumption through holdings of physical capital, holdings of risky intermediary debt, and short-term risk-free borrowing and lending. Unlike the intermediary sector, the households do not have access to the investment technology. Thus, the physical capital k_{ht} held by households evolves according to

$$dk_{ht} = -\lambda_k k_{ht} dt.$$

When a household buys k_{ht} units of capital at price $p_{kt}A_t$, by Itô's lemma, the value of capital evolves according to

$$\frac{d\left(k_{ht}p_{kt}A_t\right)}{k_{ht}p_{kt}A_t} = \frac{dA_t}{A_t} + \frac{dp_{kt}}{p_{kt}} + \frac{dk_{ht}}{k_{ht}} + \left\langle \frac{dp_{kt}}{p_{kt}}, \frac{dA_t}{A_t} \right\rangle.$$

Each unit of capital owned by the household also produces A_t units of output, so the total return to one unit of household wealth invested in capital is

$$dR_{kt} = \underbrace{\frac{A_t k_{ht}}{k_{ht}p_{kt}A_t} dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(k_{ht}p_{kt}A_t\right)}{k_{ht}p_{kt}A_t}}_{\text{capital gains}} \equiv \mu_{Rk,t}dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t}.$$

In addition to direct capital investment, the households can invest in risky intermediary debt. To keep the balance sheet structure of the financial institutions time-invariant, we assume that the bonds mature at a constant rate λ_b, so that the time t probability of a bond maturing before time $t+dt$ is $\lambda_b dt$. Notice that this corresponds to an infinite-horizon version of the "stationary balance sheet" assumption of Leland and Toft [1996]. Thus, the risky debt holdings b_{ht} of households follow

$$db_{ht} = \left(\beta_t - \lambda_b\right) b_{ht} dt,$$

where β_t is the issuance rate of new debt. The bonds pay a floating coupon $C_{bt}A_t$ until maturity, with the coupon payment determined in equilibrium to clear the risky bond market. Similarly to capital, risky bonds are liquidly traded, with the price of a unit of intermediary debt at time t in terms of the consumption good given by $p_{bt}A_t$. Hence, the total return from one unit of household wealth invested in risky debt is

$$\begin{aligned} dR_{bt} &= \underbrace{\frac{(C_{bt} + \lambda_b - \beta_t p_{bt}) A_t b_{ht}}{b_{ht} p_{bt} A_t} dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d (b_{ht} p_{bt} A_t)}{b_{ht} p_{bt} A_t}}_{\text{capital gains}} \\ &\equiv \mu_{Rb,t} dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi t}. \end{aligned}$$

When a household with total wealth w_{ht} buys k_{ht} units of capital and b_{ht} units of risky intermediary debt, it invests the remaining $w_{ht} - p_{kt}k_{ht} - p_{bt}b_{ht}$ at the risk-free rate r_{ft}, so that household wealth evolves as

$$dw_{ht} = r_{ft}w_{ht} + p_{kt}A_t k_{ht} (dR_{kt} - r_{ft}dt) + p_{bt}A_t b_{ht} (dR_{bt} - r_{ft}dt) - c_t dt. \tag{1}$$

We assume that the households face no-shorting constraints, such that

$$\begin{aligned} k_{ht} &\geq 0 \\ b_{ht} &\geq 0. \end{aligned}$$

Thus, the households solve

$$\max_{\{c_t, k_{ht}, b_{ht}\}} \mathbb{E}\left[\int_0^{+\infty} e^{-(\xi_t + \rho_h t)} \log c_t dt\right], \tag{2}$$

subject to the household wealth evolution 1 and the no-shorting constraints. We have the following result.

Lemma 1. *The household's optimal consumption choice satisfies*

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

In the unconstrained region, the household's optimal portfolio choice is given by

$$\begin{bmatrix} \pi_{kt} \\ \pi_{bt} \end{bmatrix} = \left(\begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix} \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix} \right)^{-1} \begin{bmatrix} \mu_{Rk,t} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{bmatrix}$$
$$- \sigma_{\xi} \begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}^{-1} \begin{bmatrix} \rho_{\xi,a} \\ \sqrt{1-\rho_{\xi,a}^2} \end{bmatrix}.$$

Proof. See Appendix A. □

Thus, the household with the time-varying beliefs chooses consumption as a myopic investor but with a lower rate of discount. The optimal portfolio choice of the household, on the other hand, also includes an intratemporal hedging component for variations in the Radon-Nikodym derivative, $\exp(-\xi_t)$. Since intermediary debt is locally risk-less, however, households do not self-insure against intermediary default. Appendix A provides also the optimal portfolio choice in the case when the household is constrained. In our simulations, the household never becomes constrained as the intermediary wealth never reaches zero.

II.C. Financial Intermediaries

There is a unit mass of risk-neutral, infinitely lived financial intermediaries in the economy. In the basic formulation, we assume that all the financial intermediaries are identical and thus the equilibrium outcomes are determined by the behavior of the representative intermediary. We abstract from modeling the dividend payment decision ("consumption") of the intermediary sector and assume that an intermediary invests maximally if the opportunity arises. In particular, financial intermediaries create new capital through capital investment. Denote by k_t the physical capital held by the representative intermediary at time t and by $i_t A_t$ the investment rate per unit of capital. Then the stock of capital held by the representative intermediary evolves according to

$$dk_t = (\Phi(i_t) - \lambda_k)\, k_t dt.$$

Here, $\Phi(\cdot)$ reflects the costs of (dis)investment. We assume that $\Phi(0) = 0$, so in the absence of new investment, capital depreciates at the economy-wide rate λ_k. Notice that the above formulation implies that costs of adjusting capital are higher in economies with a higher level of capital productivity, corresponding to the intuition that more developed economies are more specialized. We follow Brunnermeier and Sannikov [2012] in assuming that investment carries quadratic adjustment costs, so that Φ has the form

$$\Phi(i_t) = \phi_0 \left(\sqrt{1 + \phi_1 i_t} - 1 \right),$$

for positive constants ϕ_0 and ϕ_1.

Each unit of capital owned by the intermediary produces $A_t (1 - i_t)$ units of output net of investment. As a result, the total return from one unit of intermediary capital invested in physical capital is given by

$$dr_{kt} = \underbrace{\frac{(1 - i_t) A_t k_t}{k_t p_{kt} A_t} dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d(k_t p_{kt} A_t)}{k_t p_{kt} A_t}}_{\text{capital gains}} = dR_{kt} + \left(\Phi(i_t) - \frac{i_t}{p_{kt}} \right) dt.$$

Compared to the households, the financial intermediaries earn an extra return to holding firm capital to compensate them for the cost of investment. This extra return is partially passed on to the households as coupon payments on the intermediaries' debt.

It should be noted that financial intermediaries serve two functions in our economy. First, they generate new investment. Second, they provide capital that provides risk-bearing capacity to the households. Compare this with the notion of intermediation of He and Krishnamurthy [2012a,b, 2013]. In their model, intermediaries allow households to access the risky investment technology: Without the intermediary sector, the households can only invest in the risk-free rate. Instead, the households enter into a profit-sharing agreement with the intermediary, with the profits distributed according to the initial wealth contributions. Our model is going to give rise to procyclical share of financial intermediation, a fact very strongly supported by the data, which the setup of He and Krishnamurthy does not capture.

The intermediaries finance their investment in new capital projects by issuing risky floating coupon bonds to the households. Denoting by β_t the issuance rate of bonds at time t, the stock of bonds b_t on a representative intermediary's balance sheet evolves as

$$db_t = (\beta_t - \lambda_b)\, b_t dt.$$

Each unit of debt issued by the intermediary pays $C_{bt} A_t$ units of output until maturity and A_t units of output at maturity. The total net cost of one unit of intermediary debt is therefore given by

$$dr_{bt} = \underbrace{\frac{(C_{bt} + \lambda_b - \beta_t p_{bt})\, A_t b_t}{b_t p_{bt} A_t} dt}_{\text{dividend−price ratio}} + \underbrace{\frac{d\,(b_t p_{bt} A_t)}{b_t p_{bt} A_t}}_{\text{capital gains}} = dR_{bt}.$$

Thus, the cost of debt to the intermediary equals the return on holding bank debt for the households.

Consider now the budget constraint of an intermediary in this economy. An intermediary in this economy holds capital investment projects (k_t) on the assets side of its balance sheet and has bonds (b_t) on the liability side. In mathematical terms, we can express the corresponding budget constraint as

$$p_{kt} A_t k_t = p_{bt} A_t b_t + w_t, \tag{3}$$

where w_t is the implicit value of equity in the intermediary. Thus, in terms of flows, the intermediary's equity value evolves according to

$$dw_t = k_t p_{kt} A_t dr_{kt} - b_t p_{bt} A_t dr_{bt}. \tag{4}$$

The key assumption of this paper concerns the funding of the intermediary. We assume that intermediary borrowing is restricted by a risk-based capital constraint, similar to the value at risk (VaR) constraint of Danielsson et al. [2011]. In particular, we assume that

$$\alpha \sqrt{\frac{1}{dt} \langle k_t d\,(p_{kt} A_t) \rangle^2} = w_t, \tag{5}$$

where $\langle\cdot\rangle^2$ is the quadratic variation operator. That is, we assume that the intermediaries are restricted to retain enough equity to cover a certain fraction of losses on their assets. Unlike a traditional *VaR* constraint, this does not keep the volatility of intermediary equity constant, leaving the intermediary sector exposed to solvency risk. The risk-based capital constraint implies a time-varying leverage constraint θ_t, defined by

$$\theta_t = \frac{p_{kt} A_t k_t}{w_t} = \frac{1}{\alpha \sqrt{\frac{1}{dt} \left\langle \frac{d(p_{kt} A_t)}{p_{kt} A_t} \right\rangle^2}}.$$

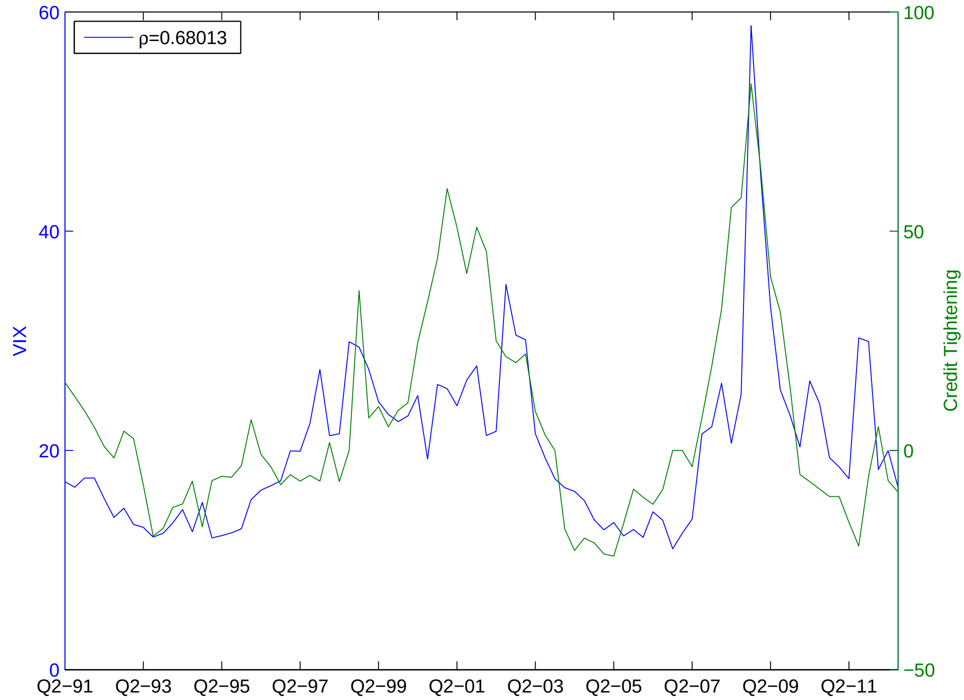
Thus, the per-dollar total *VaR* of assets is negatively related to intermediary leverage, a feature documented by Adrian and Shin [2010a], who provide a micro foundation for the risk based capital constraint.[2]

The risk based capital constraint of intermediaries is directly related to the way in which financial intermediaries manage market risk. Trading operations of major banks – most of which are undertaken in the security broker-dealer subsidiaries – are managed by allocating equity in relation to the *VaR* of trading assets. Constraint 5 directly captures such behavior. Banking books, on the other hand, are managed either according to credit risk models, or using historical cost accounting rules with loss provisioning. Although the constraint 5 does not directly capture these features of banks' risk management, empirical evidence suggests that the risk based funding constraint is a good behavioral assumption for bank lending. In particular, the proxy for the tightness of credit supply conditions reported by the Senior Loan Officer Survey of the Federal Reserve is highly correlated with measures of aggregate volatility such as the VIX (see Figure II). A higher level of asset volatility is thus associated with tighter lending conditions of commercial banks, which constraint 5 captures.

We assume that the risk based capital constraint 5 always binds. We view this as a technological constraint. The assumption of a permanently binding funding constraint is in sharp contrast to the models of Brunnermeier and Sannikov [2011, 2012]. In that alternative approach, there is no risk based leverage constraint. Instead, intermediaries manage

[2]An alternative interpretation would be in terms of a counterbalancing force to a government subsidy (such as access to a better investment technology than other sectors of the economy) provide to the banking sector. As pointed out in Kareken and Wallace [1978], government subsidies distort the risk-taking decisions of banks, precipitating the need for government regulation of risk taking.

Figure II: Market Volatility and Credit Supply Conditions



NOTES: VIX refers to the Chicago Board Options Exchange (CBOE) market volatility index. The credit tightening indicator refers to the measure of lending standards for commercial and industrial loans to large and medium firms, as reported in the Board of Governors of the Federal Reserve System Senior Loan Officer Opinion Survey. ρ is the linear correlation between the two series. Source: Haver DLX.

their leverage so as to make sure that they have a big enough buffer to make their debt instantaneously risk free. The intertemporal risk management of the intermediary is then driving their effective risk aversion, pinning down their leverage and balance sheet growth. In contrast, in our approach, intermediaries always leverage to the maximum, and do not have to make intertemporal decisions about the tightness of their funding constraint. We choose our assumption for its power in generating empirical predictions that are closely aligned with the data. Furthermore, there is anecdotal evidence that intermediaries tend to leverage maximally. Finally, the binding constraint also captures the short termism of financial intermediaries.

The parameter α determines how much equity the intermediary has to hold for each dollar of asset volatility. We view this parameter α as a policy parameter that is pinned down by regulation. α can be interpreted as the tightness of risk based capital requirements, similar to the capital requirements of the Basel Committee on Banking Supervision.

The representative intermediary maximizes equity holder value to solve

$$\max_{\{k_t,\beta_t,i_t\}} \quad \mathbb{E}\left[\int_0^{\tau_D} e^{-\rho t} w_t dt\right], \tag{6}$$

subject to the dynamic intermediary budget constraint 4 and the risk-based capital constraint constraint 5. Here, ρ is the subjective discount rate of the intermediary and τ_D is the (random) time at which the representative intermediary becomes distressed and has to be restructured. We assume that distress occurs when the intermediary equity falls below an exogenously specified threshold, so that

$$\tau_D = \inf_{t\geq 0}\left\{w_t \leq \bar{\omega} p_{kt} A_t K_t\right\}.$$

Notice that, since the distress boundary grows with the scale of the economy, the intermediary can never outgrow the possibility of distress. When the intermediary is restructured, the management of the intermediary changes. The new management defaults of the debt of the previous intermediary, reducing leverage to $\underline{\theta}$, but maintains the same level of capital as before. The inside equity of the new intermediary is thus

$$w_{\tau_D^+} = \bar{\omega}\frac{\theta_{\tau_D}}{\underline{\theta}} p_{k\tau_D} A_{\tau_D} K_{\tau_D}.$$

Finally, we define the term structure of distress risk to be

$$\delta_t(T) = \mathbb{P}\left(\tau_D \leq T \,|\, (w_t,\theta_t)\right).$$

Here, $\delta_t(T)$ is the time t probability of default occurring before time T. Notice that, since the fundamental shocks in the economy are Brownian, and all the agents in the economy have perfect information, the local distress risk is zero. We refer to the default of the intermediary as systemic risk, as there is a single intermediary in the economy, so its distress is systemic.

In our simulations, we use parameter values for $\bar{\omega}$ that are positive (not zero), thus viewing intermediaries default state as a restructuring event.

II.D. Equilibrium

Definition 1. *An equilibrium in this economy is a set of price processes* $\{p_{kt}, p_{bt}, C_{bt}\}_{t\geq 0}$*, a set of household decisions* $\{k_{ht}, b_{ht}, c_t\}_{t\geq 0}$*, and a set of intermediary decisions* $\{k_t, \beta_t, i_t, \theta_t\}_{t\geq 0}$ *such that the following apply:*

1. *Taking the price processes and the intermediary decisions as given, the household's choices solve the household's optimization problem 2, subject to the household budget constraint 1.*

2. *Taking the price processes and the household decisions as given, the intermediary's choices solve the intermediary optimization problem 6, subject to the intermediary wealth evolution 3 and the risk-based capital constraint 5.*

3. *The capital market clears:*

$$K_t = k_t + k_{ht}.$$

4. *The risky bond market clears:*

$$b_t = b_{ht}.$$

5. *The risk-free debt market clears:*

$$w_{ht} = p_{kt} A_t k_{ht} + p_{bt} A_t b_{ht}.$$

6. *The goods market clears:*

$$c_t = A_t \left(K_t - i_t k_t \right).$$

Notice that the bond markets' clearing conditions imply

$$p_{kt}A_tK_t = w_{ht} + w_t.$$

Notice also that the aggregate capital in the economy evolves as

$$dK_t = -\lambda_k K_t dt + \Phi(i_t) k_t dt.$$

III. Solution

We solve for the equilibrium in terms of two state variables. The first state variable is θ_t, the leverage of the intermediary sector. As for the second state variable, we define the fraction of the total wealth in the economy held by the financial intermediaries as

$$\omega_t = \frac{w_t}{w_t + w_{ht}} = \frac{w_t}{p_{kt}A_tK_t}.$$

With this definition, the share of total wealth in the economy held by the households is $(1 - \omega_t)$. The vector of state variables in the economy is then

$$(\theta_t, \omega_t).$$

Notice that, by construction, the household belief shocks are expectation-neutral, and thus their level is not a state variable in the economy. Similarly, we have defined prices in the economy to scale with the level of productivity, A_t, so productivity itself is not a state variable in the scaled version of the economy. We will characterize the equilibrium outcomes in terms of these variables, with the equilibrium conditions determining the time series evolution of θ_t and ω_t in terms of the primitive shocks in the economy, $(Z_{at},\ Z_{\xi,t})$. In particular, we will make use of the following representations

$$\frac{d\omega_t}{\omega_t} = \mu_{\omega t}dt + \sigma_{\omega a,t}dZ_{at} + \sigma_{\omega\xi,t}dZ_{\xi t}$$
$$\frac{d\theta_t}{\theta_t} = \mu_{\theta t}dt + \sigma_{\theta a,t}dZ_{at} + \sigma_{\theta\xi,t}dZ_{\xi t}.$$

Notice that, by observing the evolution of A_t, as well as the two state variables in the economy, we can isolate the time series evolution of the shocks to household beliefs, $(Z_{\xi t})_{t\geq 0}$. Notice finally that the *VaR* constraint implies

$$\alpha^{-2}\theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2.$$

Thus, the riskiness of the return to holding capital increases as intermediary leverage decreases. We plot the theoretical and the empirical trade-off between leverage growth and volatility in Figure III. Clearly, higher levels of the VIX tend to precede declines in broker-dealer leverage (right panel). In the model, this translates into a negative relationship between lagged growth rate of asset return volatility and intermediary leverage growth (left panel). The negative relationship between broker-dealer leverage and the VIX is further investigated in Adrian and Shin [2010a,b].[3] While the evidence from Figure III is from broker dealers, it also has an empirical counterpart for the banking book. As discussed earlier, the lending standards of banks vary tightly with the VIX, indicating that new lending of commercial banks is highly correlated with measures of market volatility.

Table II reports the coefficients and the R^2 of the regression of broker-dealer leverage growth on lagged growth in implied volatility in the data (first column) and in the model. For the model, we report the mean, median, 5% and 95% realizations of the coefficients in a sample of 10000 paths. The paths are simulated at a monthly frequency for 70 years, using the parameters in Table I.

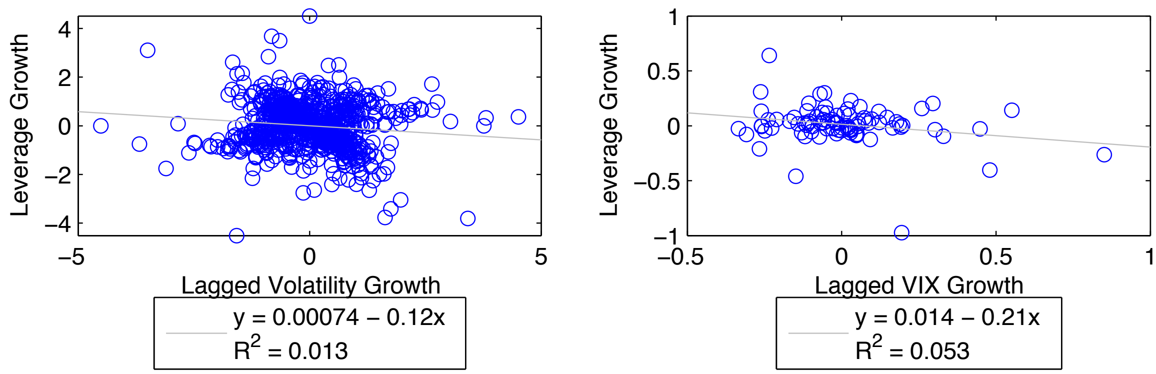
III.A. Capital evolution

Recall from the intermediary's leverage constraint that

$$\theta_t = \frac{p_{kt}A_t k_t}{w_t}.$$

[3]While Adrian and Shin [2010b] show that fluctuations in primary dealer repo—which is a proxy for fluctuations in broker-dealer leverage — tend to forecast movements in the VIX, Figure III shows that higher levels of the VIX precede declines in broker-dealer leverage. We use the lagged VIX as the VIX is implied volatility and hence a forward-looking measure (though the negative relationship also holds for contemporaneous VIX). Adrian and Shin [2010a] use the *VaR* data of major securities broker-dealers to show a negative association between broker-dealer leverage growth and the *VaRs* of the broker dealers.

Figure III: Intermediary Leverage and Lagged Volatility Growth



NOTES: The relationship between the growth rate of leverage of financial institutions and the lagged growth rate of implied volatility. Right panel: quarterly growth of broker-dealer leverage (y-axis) versus lagged quarterly growth of the Chicago Board Options Exchange (CBOE) market volatility index (VIX) (x-axis); left panel: quarterly growth of intermediary leverage, θ_t, (y-axis) versus lagged quarterly growth of capital return volatility, $\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$, ($x$-axis) for a representative path. Data on broker-dealer leverage comes from Flow of Funds Table L.129. Data from the model is simulated using parameters in Table I at a monthly frequency for 70 years.

Table I: Parameters

Parameter	Value
$\bar{a}$	0.0651
σ_a	0.388
ρ	0.06
$\rho_h - \sigma_\xi^2/2$	0.05
ϕ_0	0.1
ϕ_1	20
λ_k	0.03
$\rho_{\xi,a}$	0
σ_ξ	0.0388
α	2.5

NOTES: Parameters used in simulations. The parameters of the productivity growth process ($\bar{a}$, σ_a), the parameters of the investment technology (ϕ_0, ϕ_1), subjective discount rates (ρ_h, ρ), and depreciation (λ_k) are taken from Brunnermeier and Sannikov [2012].

Using our definition of ω_t, we can thus express the amount of capital held by the financial institutions as

$$k_t = \frac{\theta_t w_t}{p_{kt} A_t} = \theta_t \omega_t K_t.$$

Table II: Intermediary Leverage and Lagged Volatility Growth

	Data	Mean	5%	Median	95%
β_0	0.014	0.000	-0.003	0.000	0.003
β_1	-0.208	-0.105	-0.187	-0.104	-0.025
R^2	0.053	0.013	0.001	0.011	0.035

NOTES: The relationship between the growth rate of leverage of financial institutions and the lagged growth rate of implied volatility. The "Data" column reports the coefficients estimated using broker-dealer leverage growth as the dependent variable and the growth rate of the Chicago Board Options Exchange (CBOE) market volatility index (VIX) as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized growth rate of leverage, θ_t, of the intermediaries as the dependent variable, and growth rate of total volatility of the return on capital, $\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on broker-dealer leverage comes from Flow of Funds Table L.129.

Applying Itô's lemma, we obtain

$$dk_t = \omega_t K_t d\theta_t + \theta_t K_t d\omega_t + \theta_t \omega_t dK_t + K_t \langle d\theta_t, d\omega_t \rangle .$$

Recall, on the other hand, that the intermediary's capital evolves as

$$dk_t = \left(\Phi\left(i_t\right) - \lambda_k\right) k_t dt.$$

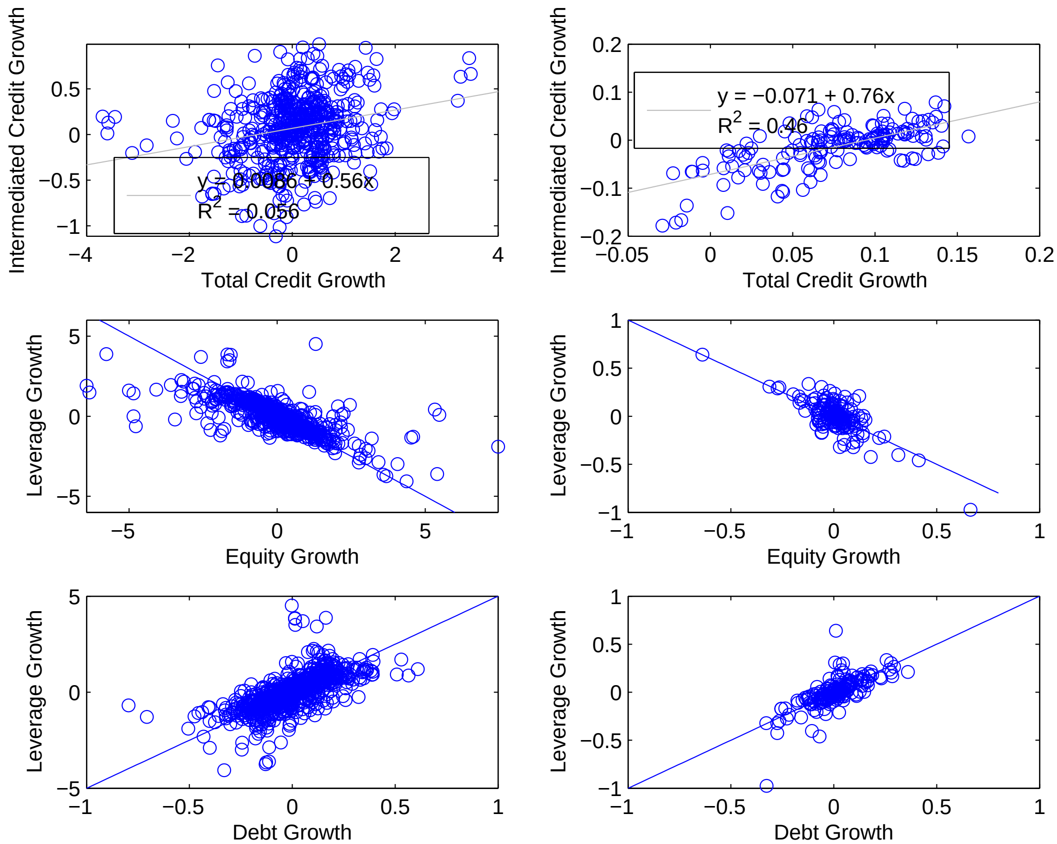
Equating coefficients, we obtain

$$\begin{aligned}
\sigma_{\theta a,t} &= -\sigma_{\omega a,t} \\
\sigma_{\theta \xi,t} &= -\sigma_{\omega \xi,t} \\
\mu_{\theta t} &= \underbrace{\Phi\left(i_t\right)\left(1 - \theta_t \omega_t\right)}_{\text{asset growth rate}} - \mu_{\omega t} + \underbrace{\sigma_{\theta a,t}^2 + \sigma_{\theta \xi,t}^2}_{\text{risk adjustment}}.
\end{aligned}$$

Thus, intermediary leverage is perfectly negatively correlated with the share of wealth held by the financial intermediaries. This reflects the fact that capital stock is not immediately adjustable, so changes in the value of intermediary assets translate one-for-one into changes in intermediary leverage. Notice further that the intermediary faces a trade-off in the growth rate of its leverage, $\mu_{\theta t}$, and the growth rate of its wealth share in the economy, $\mu_{\omega t}$.

Figure IV plots the growth of the share of intermediated credit as a function of total credit growth, showing the strong positive relationship in the model and the data. This positive relationship has been previously documented in Adrian et al. [2012] and shows the procyclical nature of intermediated finance. The coefficients of the corresponding regression are reported in Table III, with the linear coefficient remaining positive even for extreme paths. The middle panel of Figure IV shows the procyclical nature of the leverage of financial intermediaries. Leverage tends to expand when balance sheets grow, a fact that has been documented by Adrian and Shin [2010b] for the broker-dealer sector and by Adrian et al. [2012] for the commercial banking sector. The lower panel shows that the procyclical leverage translates into countercyclical equity growth, both in the data and in the model. We should note that the procyclical leverage of financial intermediaries is closely tied to the risk-based capital constraint. In contrast, previous literature has found it challenging to generate this feature and in fact exhibits countercyclical leverage (see Brunnermeier and Sannikov [2011, 2012], He and Krishnamurthy [2012a, 2013], Bernanke and Gertler [1989], Kiyotaki and Moore [1997], Gertler and Kiyotaki [2012], and Gertler et al. [2011]).

Figure IV: Intermediary Balance Sheet Evolution



NOTES: Procyclicality of intermediary balance sheets. Top panels: The relationship between total credit in the economy and the amount of credit extended through the financial intermediary sector, with the left panel plotting the realized growth rate of capital held by intermediaries, k_t, (y-axis) versus the growth rate of total capital in the economy, K_t, (x-axis) for a representative path, and the right panel plotting the growth rate of credit extended by financial intermediaries to the non-financial corporate sector (y-axis) versus the growth rate of total credit to the non-financial corporate sector (x-axis). Middle panels: The relationship between intermediary leverage growth and intermediary equity growth, with the left panel plotting quarterly growth of intermediary leverage, θ_t, (y-axis) versus quarterly growth of intermediary wealth in the economy, ω_t, (x-axis) for a representative path, and the right panel plotting quarterly growth of broker-dealer leverage (y-axis) versus quarterly growth of scaled broker-dealer equity (x-axis). Lower panels: The relationship between intermediary leverage growth and debt growth, with the left panel plotting quarterly growth of intermediary leverage, θ_t, (y-axis) versus quarterly growth of household wealth in the economy, $1 - \omega_t$, (x-axis) for a representative path, and the right panel plotting quarterly growth of broker-dealer leverage (y-axis) versus quarterly growth of scaled broker-dealer debt (x-axis). In both the middle and the lower panels, the scaling factor is the total credit to the non-financial sector, from Flow of Funds Table L.102. Data on total credit to the nonfinancial corporate sector and the share of intermediated finance come from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets come from Flow of Funds Table L.129. Data from the model is simulated using parameters in Table I at a monthly frequency for 70 years.

Table III: Procyclicality of Intermediated Credit

	Data	Mean	5%	Median	95%
β_0	-0.071	-0.112	-0.203	-0.108	-0.040
β_1	0.756	0.434	0.190	0.433	0.680
R^2	0.460	0.048	0.009	0.045	0.101

NOTES: The relationship between total credit in the economy and the amount of credit extended through the financial intermediary sector. The "Data" column reports the coefficients estimated using the growth rate of credit extended by financial intermediaries to the non-financial corporate sector as the dependent variable, and the growth rate of total credit to the non-financial corporate sector as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized growth rate of capital held by intermediaries, k_t, as the dependent variable, and the growth rate of total capital in the economy, K_t, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on total credit to the nonfinancial corporate sector and the share of intermediated finance come from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets come from Flow of Funds Table L.129.

III.B. Household's problem

Notice that, with the notation introduced above, we also have

$$\pi_{kt} = \frac{p_{kt}A_t k_{ht}}{w_{ht}} = \frac{p_{kt}A_t\left(K_t - k_t\right)}{\left(1-\omega_t\right)p_{kt}A_t K_t} = \frac{1-\theta_t\omega_t}{1-\omega_t}$$
$$\pi_{bt} = 1 - \pi_{kt} = \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}.$$

Thus, the household holds a non-zero amount of intermediary debt while intermediary leverage exceeds one, and a non-zero amount of capital while the unlevered value of intermediary capital share is less than one. Using this result, we can express the excess return to holding capital as

$$\mu_{Rk,t} - r_{ft} = \underbrace{\left(\sigma^2_{ka,t} + \sigma^2_{k\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t}}_{\text{compensation for own risk}} + \underbrace{\left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\omega_t\left(\theta_t-1\right)}{1-\omega_t}}_{\text{compensation for risk of correlated asset}}$$
$$+ \underbrace{\sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho^2_{\xi,a}}\right)}_{\text{compensation for beliefs risk}}.$$

Thus, the excess return on holding capital directly has three components. The first compensates households for the direct risk of holding a claim to the volatile output stream, while the second compensates households for the riskiness of holding the correlated asset (risky intermediary debt). The remaining component is the hedging motive for holding capital and compensates households for the risk associated with fluctuations in the Radon-Nikodym derivative, $\exp(-\xi_t)$.

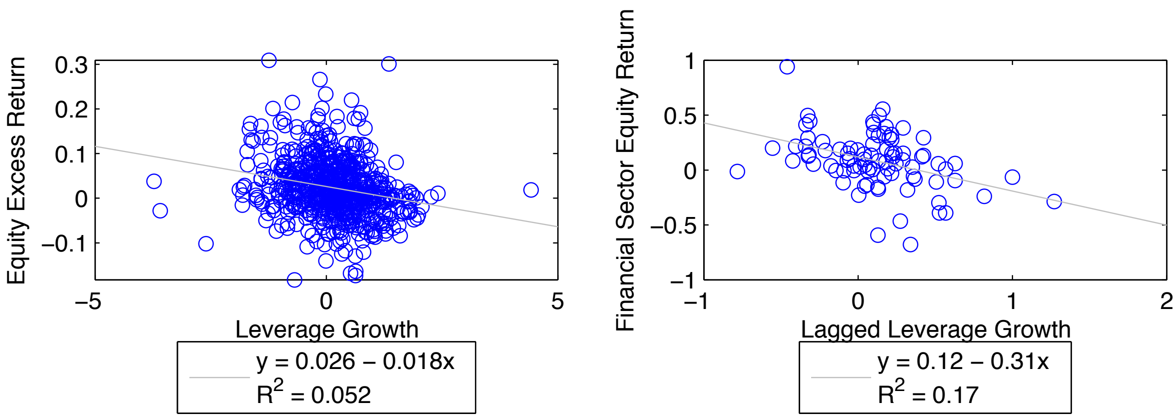
Similarly, the excess return to holding risky intermediary debt is given by

$$\mu_{Rb,t} - r_{ft} = \underbrace{\left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right) \frac{\omega_t (\theta_t - 1)}{1 - \omega_t}}_{\text{compensation for own risk}} + \underbrace{\left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{1 - \theta_t \omega_t}{1 - \omega_t}}_{\text{compensation for risk of correlated asset}}$$
$$+ \underbrace{\sigma_\xi \left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1 - \rho_{\xi,a}^2}\right)}_{\text{compensation for beliefs risk}}.$$

As with the excess return to direct capital investment, the excess return on risky intermediary debt has three components. The first compensates households for the direct risk of holding a claim to the volatile coupons, while the second compensates households for the riskiness of holding the correlated asset (direct capital investment). As with capital, the remaining component is the hedging motive for holding capital and compensates households for the risk associated with fluctuations in the Radon-Nikodym derivative, $\exp(-\xi_t)$.

In equilibrium, expected excess returns on both capital and risky debt are negatively related to the growth rate of intermediary leverage. The left panel of Figure V plots simulated excess returns as a function of intermediary leverage growth, while the right panel plots the same relationship in the data. In particular, we see that the excess return to capital increases as the growth rate of intermediary leverage decreases. In fact, Adrian et al. [2010] document that broker-dealer leverage growth is a good empirical proxy for the time variation of expected returns for a variety of stock and bond portfolios. This negative relationship within the model is further documented in Table IV, with the linear regression coefficient consistently negative across different path realizations.

Figure V: Excess Returns and Intermediary Leverage



NOTES: The relationship between the growth rate of leverage of financial institutions and the equity excess returns. Right panel: quarterly excess return to holding the S&P Financial Index (y-axis) versus lagged annual growth of broker-dealer leverage (x-axis) ; left panel: quarterly excess return to holding capital, dR_{kt}, (y-axis) versus lagged annual intermediary leverage growth, $d\theta_t$, (x-axis). Data on broker-dealer leverage comes from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics. Data from the model is simulated using parameters in Table I at a monthly frequency for 70 years.

Finally, notice that

$$\frac{dw_{ht}}{w_{ht}} = \frac{d\left(\left(1-\omega_t\right)p_{kt}A_tK_t\right)}{\left(1-\omega_t\right)p_{kt}A_tK_t}.$$

Thus, the expected rate of change in the financial intermediaries' wealth share in the economy is given by

$$\mu_{\omega t} = \underbrace{\left(\theta_t - 1\right)\left(\mu_{Rkt} - \mu_{Rb,t}\right)}_{\text{expected porfolio return}} - \underbrace{\left(\sigma_{ka,t}\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right)}_{\text{compensation for portfolio risk}}$$
$$+ \underbrace{\frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right]}_{\text{consumption provision to households}},$$

and the loadings of the financial intermediaries' wealth share in the economy on the two sources of fundamental risk are given by

$$\sigma_{\omega a,t} = \left(\theta_t - 1\right)\left(\sigma_{ka,t} - \sigma_{ba,t}\right)$$

Table IV: Excess Returns and Intermediary Leverage

	Data	Mean	5%	Median	95%
β_0	0.118	0.076	0.068	0.076	0.084
β_1	-0.310	-0.031	-0.038	-0.031	-0.024
R^2	0.167	0.100	0.064	0.100	0.143

NOTES: The relationship between excess returns and lagged broker-dealer leverage growth. The "Data" column reports the coefficients estimated using the quarterly return to holding the S&P Financial Index as the dependent variable, and lagged annual broker-dealer leverage growth as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized quarterly excess return to holding capital, dR_{kt} as the dependent variable, and lagged annual intermediary leverage growth, $d\theta_t$, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on broker-dealer leverage come from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics and Barclays.

$$\sigma_{\omega\xi,t} = (\theta_t - 1)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right).$$

That is, the risk loadings of the financial intermediaries' relative wealth reflect the ability of the financial intermediaries to absorb shocks to their balance sheets. The negative sign on the volatility of bond returns reflects the fact that losses in the value of the bonds benefit the intermediaries by reducing their debt burden.

III.C. Goods market clearing and price of capital

Recall that goods market clearing implies the households consume all output, except that used for investment

$$c_t = A_t\left(K_t - i_t k_t\right).$$

Recall further that the only real choice the intermediary has to make (since financing is restricted by the risk-based capital constraint) is in its optimal investment, given by

$$\frac{1}{p_{kt}} = \Phi'\left(i_t\right),$$

such that the equilibrium rate of investment is given by

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

As the price of capital increases, the book value of intermediary assets increases and the intermediaries are able to invest at a higher rate. Thus, in equilibrium, we must have

$$\underbrace{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)p_{kt}\left(1-\omega_t\right)}_{\text{household demand}} = \underbrace{1 - \frac{\theta_t\omega_t}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right)}_{\text{total supply}}.$$

The households' demand for the consumption good is driven by the households' wealth share in the economy, $1-\omega_t$, and the capital price p_{kt}. The supply of the consumption good, on the other hand, is determined by the financial intermediaries' wealth share in the economy, ω_t, financial intermediaries' leverage, θ_t, and the capital price. Denoting

$$\beta = \left(\frac{4}{\phi_0^2\phi_1}\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\right),$$

the price of capital solves

$$0 = p_{kt}^2\theta_t\omega_t + \beta p_{kt}\left(1-\omega_t\right) - \frac{4}{\phi_0^2\phi_1} - \frac{4\theta_t\omega_t}{\phi_0^2\phi_1^2},$$

or

$$p_{kt} = \frac{-\beta\left(1-\omega_t\right) + \sqrt{\beta^2\left(1-\omega_t\right)^2 + \frac{16}{\phi_0^2\phi_1^2}\theta_t\omega_t\left(\phi_1 + \theta_t\omega_t\right)}}{2\theta_t\omega_t}. \tag{7}$$

As an aside, notice that, for the intermediary to disinvest, we must have

$$\left(1-\omega_t\right) \geq \frac{\phi_0\phi_1}{2\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}.$$

Thus, the intermediary disinvests when the household is a large fraction of the economy—that is, when the intermediary has a relatively low value of equity. Applying Itô's lemma and equating coefficients, we obtain

$$
\begin{aligned}
[dZ_{at}]: \quad & \beta\omega_t\sigma_{\omega a,t} = \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\sigma_{ka,t} - \sigma_a\right) \\
[dZ_{\xi t}]: \quad & \beta\omega_t\sigma_{\omega\xi,t} = \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_{k\xi,t} \\
[dt]: \quad & 0 = \left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\left(1-\theta_t\omega_t\right)\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) \\
& + \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)p_{kt}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\sigma_{ka,t}\right) \\
& - \beta p_{kt}\omega_t\mu_{\omega t} + \theta_t\omega_t p_{kt}^2\left(\left(\sigma_{ka,t}-\sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\
& - \beta p_{kt}\omega_t\left(\left(\sigma_{ka,t}-\sigma_a\right)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right).
\end{aligned}
$$

Thus, in equilibrium, the financial intermediaries' wealth ratio in the economy reacts to shocks in the households' beliefs in the same direction as the return to capital.

III.D. Equilibrium

We summarize the resulting equilibrium outcomes in the following lemma.

Lemma 2. *In equilibrium, the expected excess return on capital and risky intermediary debt, as well as the expected return on intermediary equity, the risk-free rate, and the volatility of intermediary equity and intermediary debt, depends linearly on the volatility of the return to holding capital. In particular, we can express the endogenous variables as*

$$
\begin{aligned}
\mu_{Rk,t} &= \mathcal{K}_0\left(\omega_t,\theta_t\right) + \mathcal{K}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t} \\
\mu_{Rb,t} &= \mathcal{B}_0\left(\omega_t,\theta_t\right) + \mathcal{B}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{B}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t} \\
\mu_{\omega t} &= \mathcal{O}_0\left(\omega_t,\theta_t\right) + \mathcal{O}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} + \mathcal{O}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t} \\
\mu_{\theta t} &= \mathcal{S}_0\left(\omega_t,\theta_t\right) + \mathcal{S}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} - \mathcal{O}_\xi\left(\omega_t,\theta_t\right)\sigma_{k\xi,t} \\
r_{ft} &= \mathcal{R}_0\left(\omega_t,\theta_t\right) + \mathcal{R}_a\left(\omega_t,\theta_t\right)\sigma_{ka,t} \\
\sigma_{ba,t} &= \frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)}{\beta\omega_t\left(\theta_t-1\right)}\sigma_a - \frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\theta_t\omega_t\right)}{\beta\omega_t\left(\theta_t-1\right)}\sigma_{ka,t} \\
\sigma_{b\xi,t} &= -\frac{2\theta_t\omega_t p_{kt} + \beta\left(1-\theta_t\omega_t\right)}{\beta\omega_t\left(\theta_t-1\right)}\sigma_{k\xi,t}
\end{aligned}
$$

$$\sigma_{\theta a,t} = -\frac{2\theta_t\omega_t p_{kt} + \beta(1-\omega_t)}{\beta\omega_t}(\sigma_{ka,t} - \sigma_a)$$
$$\sigma_{\theta\xi,t} = -\frac{2\theta_t\omega_t p_{kt} + \beta(1-\omega_t)}{\beta\omega_t}\sigma_{k\xi,t},$$

where the coefficients $(\mathcal{K}_0, \mathcal{K}_a, \mathcal{B}_0, \mathcal{B}_a, \mathcal{B}_\xi, \mathcal{O}_0, \mathcal{O}_a, \mathcal{O}_\xi, \mathcal{S}_0, \mathcal{S}_a, \mathcal{R}_0, \mathcal{R}_a)$ *are non linear functions of the state variables* (ω_t, θ_t)*, given in the Appendix A. The loadings of the return to holding capital on the shock to household beliefs,* $\sigma_{k\xi,t}$*, and on the shock to productivity,* $\sigma_{ka,t}$*, are given, respectively, by*

$$\sigma_{k\xi,t} = -\sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \sigma_{ka,t}^2}$$
$$\sigma_{ka,t} = \frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\left(1 + \frac{1-\omega_t}{\omega_t(2\theta_t\omega_t p_{kt} + \beta(1-\omega_t))}\right).$$

Proof. See Appendix A. □

Notice that we pick the negative root in determining the exposure of capital to the household liquidity shocks, $\sigma_{k\xi,t}$. Intuitively, when the household experiences a negative liquidity shock, such that $dZ_{\xi t} < 0$, the household discount rate is increased, making households more impatient and decreasing the return to holding capital.

III.E. Equilibrium Asset Pricing

We can now characterize the equilibrium pricing kernel as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_{ft}dt - \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{ka,t} + \frac{\omega_t(\theta_t-1)}{1-\omega_t}\sigma_{ba,t} + \sigma_\xi\rho_{\xi,a}\right)dZ_{at}$$
$$- \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{k\xi,t} + \frac{\omega_t(\theta_t-1)}{1-\omega_t}\sigma_{b\xi,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\right)dZ_{\xi t}.$$

While it is natural to express the pricing kernel as a function of the fundamental shocks ξ and a, these are not readily observable. Instead, we follow the empirical literature and express the pricing kernel in terms of shocks to output and leverage. Define the standardized innovation to (log) output as

$$d\hat{y}_t = \sigma_a^{-1}\left(d\log Y_t - \mathbb{E}_t\left[d\log Y_t\right]\right) = dZ_{at},$$

and the standardized innovation to the growth rate of leverage of the intermediaries as

$$\begin{aligned} d\hat{\theta}_t &= \left(\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2\right)^{-\frac{1}{2}}\left(\frac{d\theta_t}{\theta_t} - \mathbb{E}_t\left[\frac{d\theta_t}{\theta_t}\right]\right) \\ &= \frac{\sigma_{\theta a,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}} dZ_{at} + \frac{\sigma_{\theta\xi,t}}{\sqrt{\sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2}} dZ_{\xi t}. \end{aligned}$$

Thus, we can express the pricing kernel as

$$\frac{d\Lambda_t}{\Lambda_t} = -r_{ft}dt - \eta_{\theta t}d\hat{\theta}_t - \eta_{yt}d\hat{y}_t,$$

where the price of risk associated with shocks to the growth rate of intermediary leverage is

$$\eta_{\theta t} = \sqrt{1 + \frac{\sigma_{\theta a,t}^2}{\sigma_{\theta\xi,t}^2}}\left(\frac{1-\theta_t\omega_t}{1-\omega_t}\sigma_{k\xi,t} + \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}\sigma_{b\xi,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\right),$$

and the price of risk associated with shocks to output is

$$\begin{aligned} \eta_{yt} = {} & \frac{1-\theta_t\omega_t}{1-\omega_t}\left(\sigma_{ka,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{k\xi,t}\right) + \frac{\omega_t\left(\theta_t - 1\right)}{1-\omega_t}\left(\sigma_{ba,t} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{b\xi,t}\right) \\ & + \sigma_\xi\left(\rho_{\xi,a} - \frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sqrt{1-\rho_{\xi,a}^2}\right). \end{aligned}$$

Thus, a two-factor Merton [1973] ICAPM holds, with shocks to intermediary leverage driving the uncertainty about future investment opportunities.

Substituting the equilibrium expressions for $\sigma_{\theta a,t}$, $\sigma_{\theta\xi,t}$ and $\sigma_{b\xi,t}$, we obtain

$$\eta_{\theta t} = \sqrt{1 + \frac{\left(\sigma_{ka,t} - \sigma_a\right)^2}{\sigma_{k\xi,t}^2}}\left(-\frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_{k\xi,t} + \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\right).$$

Since capital has a negative exposure to the households' preference shocks, the price of risk associated with shocks to intermediary leverage is positive, so leverage risk commands a positive risk premium. While the sign of the risk premium is always positive, the dependence of the price of leverage risk on the leverage growth rate is nonmonotonic. The empirical literature strongly favors the positive price of leverage risk for stock and bond returns (see Adrian et al. [2013b]) and a negative relationship between the price of risk and the growth rate of leverage (see Adrian et al. [2010]).

Similarly, the price of risk associated with shocks to output is given, in equilibrium, by

$$\begin{aligned}\eta_{yt} &= \frac{1-\theta_t\omega_t}{1-\omega_t}\left(\sigma_{ka,t}-\frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{k\xi,t}\right)+\frac{\omega_t\left(\theta_t-1\right)}{1-\omega_t}\left(\sigma_{ba,t}-\frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sigma_{b\xi,t}\right)\\ &\quad+\sigma_\xi\left(\rho_{\xi,a}-\frac{\sigma_{\theta a,t}}{\sigma_{\theta\xi,t}}\sqrt{1-\rho_{\xi,a}^2}\right)\\ &=\sigma_a+\sigma_\xi\left(\rho_{\xi,a}-\frac{\sigma_{ka,t}-\sigma_a}{\sigma_{k\xi,t}}\sqrt{1-\rho_{\xi,a}^2}\right).\end{aligned}$$

Unlike the price of leverage risk, the price of risk associated with shocks to output changes signs, depending on whether the equilibrium sensitivity of the return to holding capital to output shocks is lower or higher than the fundamental volatility. The time-varying nature of the direction of the risk premium for output shocks makes it difficult to detect in observed returns, suggesting an explanation for the poor performance of the production CAPM in the data.

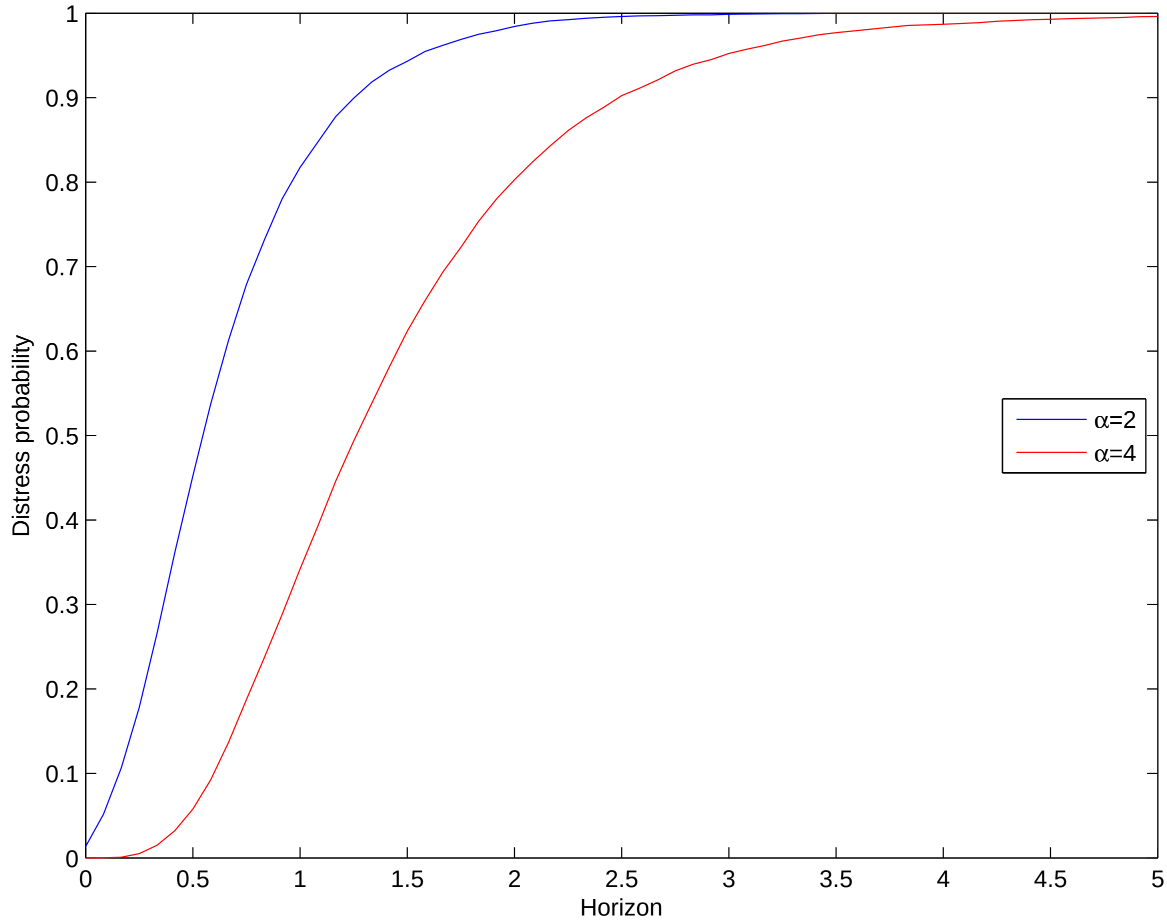
IV. Financial Stability and Household Welfare

In this section, we describe the term structure of the distress probability, $\delta_t(T)$, and, in particular, the effect of a tightening of the risk-based capital constraint. We then compare the equilibrium outcomes in our model to the equilibrium outcomes in one with constant leverage. Finally, we discuss some implications of the risk-based capital constraint for the welfare of the households in the economy.

IV.A. Intermediary distress

We begin by considering the term structure of the probability of distress. Figure VI plots the cumulative probability of the intermediary sector becoming distressed before a given

Figure VI: Term Structure of Default Probabilities



Notes: The term structure of intermediary cumulative default probabilities for two different levels of tightness of the risk-based capital constraint, α. The default probabilities are computed using 10000 simulations of the economy on a monthly frequency using the parameters in Table I.

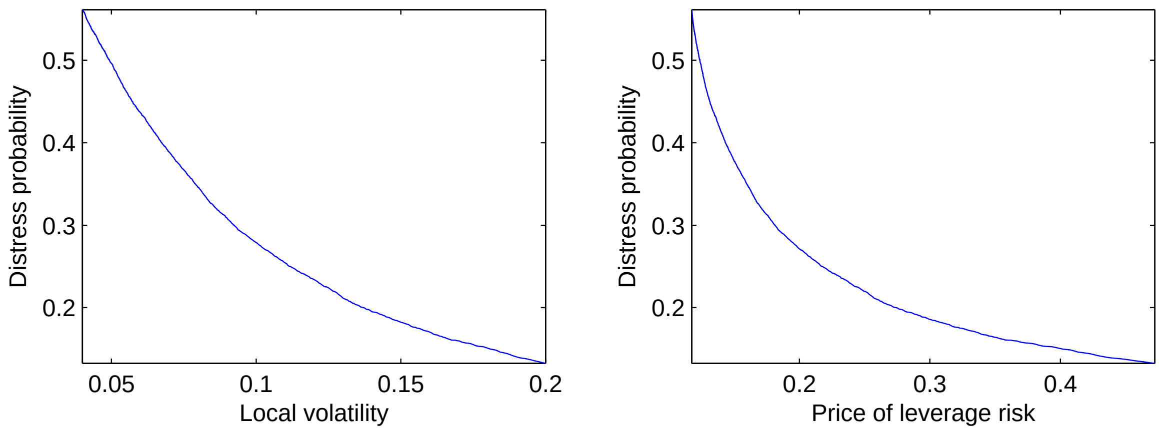
horizon.[4] As the capital constraint tightens, the whole term structure of cumulative distress probabilities shifts down indicating that the expected time to distress increases. Intuitively, as the funding constraint tightens, intermediaries' leverage becomes more constraint, making them borrow less from the households, and decreasing the probability to become distressed. Consider now the trade-off between the instantaneous riskiness of capital investment and the long-run fragilities in the economy. The left panel of Figure VII plots the six month

[4] Although this probability cannot be computed analytically, we can easily compute it using Monte Carlo simulations. For the computation of the probability of distress, as well as for the expected discounted present value of household utility, we simulated 10000 paths of the economy.

distress probability as a function of the current instantaneous volatility of the return to holding capital. We see that the model-implied quantities have the negative relationship observed in the run-up to the 2007-2009 financial crisis. This relation forms the crux of the volatility paradox: Periods of low volatility of the return to holding capital coincide with high intermediary leverage, which leads to high systemic solvency and liquidity risk. The volatility paradox was first described by Brunnermeier and Sannikov [2012], and empirically documented by Adrian and Brunnermeier [2011]. In the context of the model, local volatility is inversely proportional to leverage. As leverage increases, the intermediaries issue more risky debt, making distress more likely. This leads to the negative relationship between the probability of distress and current period return volatility. Notice that tightening the risk-based capital constraint does not change the shape of the trade-off but rather limits the admissible range of current period volatility. The right panel of Figure VII plots the trade-off between the six month distress probability and the price of risk associated with shocks to the growth rate of intermediary leverage. Since the price of leverage risk depends linearly on return volatility, an increase in contemporaneous risk increases the price of leverage risk while decreasing the long-term instability in the economy. This mechanism allows intermediaries to increase their risk exposure during periods of low volatility, which increases the risk of financial distress.

Intermediary distress is costly (in consumption terms) for the households. In Figure VIII, we plot a sample evolution of the economy, focusing on the evolution of output and consumption (upper panel), intermediary wealth share in the economy and intermediary leverage (middle panels), and of the realized return to intermediary debt (lower panel). Notice first that, while intermediaries' distress is usually preceded by high intermediary leverage, distress can occur even when intermediary leverage is relatively low. Moreover, intermediaries can maintain high levels of leverage without becoming distressed. Thus, high leverage is not a foolproof indicator of distress risk. The recapitalization of intermediaries comes at the cost of a consumption drop for the households, which can be quite significant. Since the restructuring of intermediaries is done through default on debt, household wealth (and, hence, consumption) exhibits sharp declines when intermediaries become distressed. It is worth emphasizing that the transmission mechanism from financial sector distress to real

Figure VII: Volatility Paradox



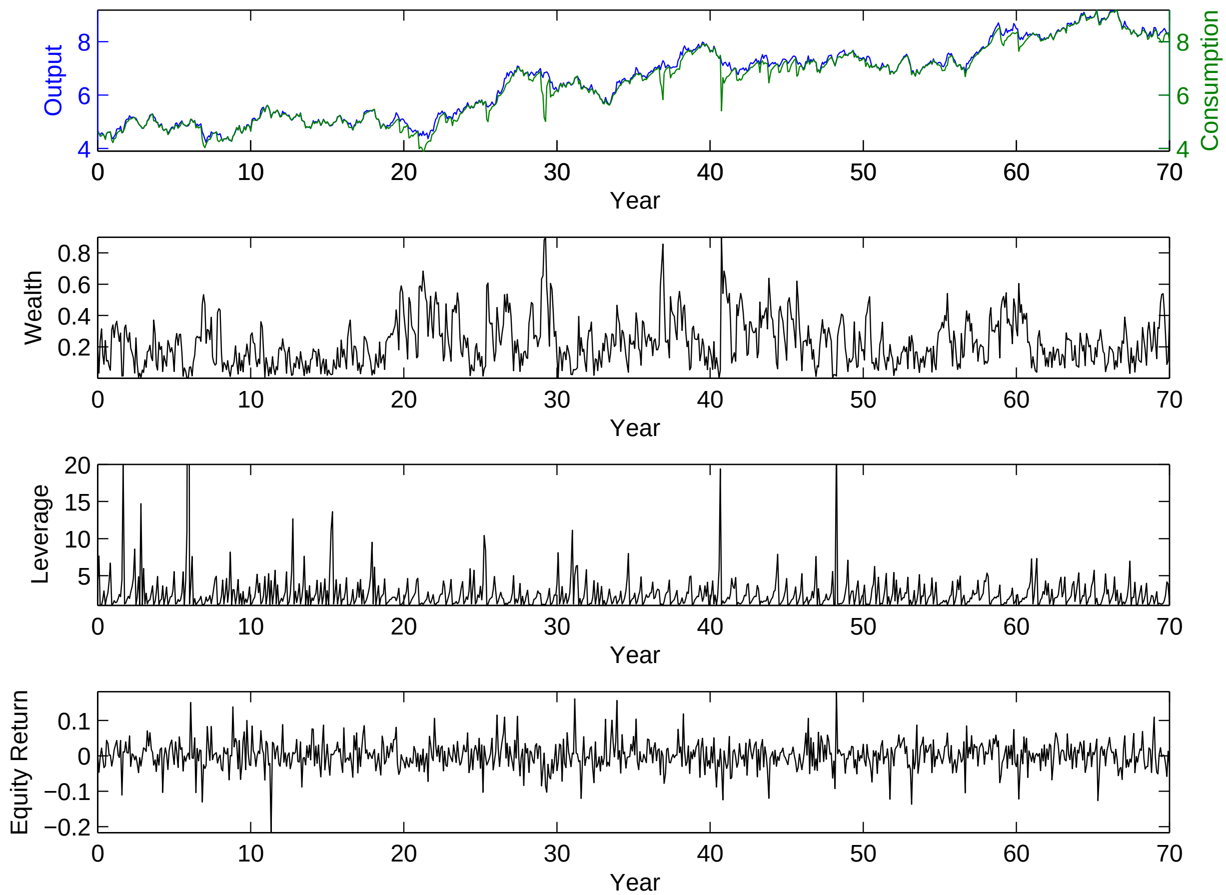
NOTES: Left panel: 6 month probability of intermediary default (y-axis) versus instantaneous volatility of equity returns, $\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}$, ($x$-axis); right panel: 6 month probability of intermediary default (y-axis) versus the risk price of standardized shocks to leverage, $\eta_{\theta t}$, (x-axis). The default probabilities are computed using 10000 simulations of the economy on a monthly frequency using the parameters in Table I.

economic activity is via two channels. The first is a wealth effect of households, which leads to an adjustment of the consumption path, and a reallocation of savings. The second channel is more direct, and consists in adjustments to the capital creation decision of intermediaries.

IV.B. Distortions and amplifications

The simulated path of the economy in Figure VIII illustrates the negative implications of intermediary distress for the households in the economy. The risk-based capital constraint faced by the intermediaries in our economy amplifies the fundamental shocks in the economy and distorts equilibrium outcomes. This amplification mechanism is illustrated in Figure IX: A shock to the relative wealth of the intermediaries reduces the equilibrium level of investment, reducing the price of capital, which makes the risk-based capital constraint bind more, reducing further the financial intermediaries' relative wealth. The amplification mechanism acts through the time-varying leverage constraint that is induced by the risk-sensitive capital constraint. To understand the mechanism better, we describe the equilibrium outcomes in an economy with constant leverage, and contrast the resulting dynamics with those in the full model.

Figure VIII: Sample Path of the Economy



In particular, consider an economy in which, instead of facing the risk-based capital constraint, the intermediaries face a constant leverage constraint, such that

$$\frac{p_{kt} A_t k_t}{w_t} = \bar{\theta},$$

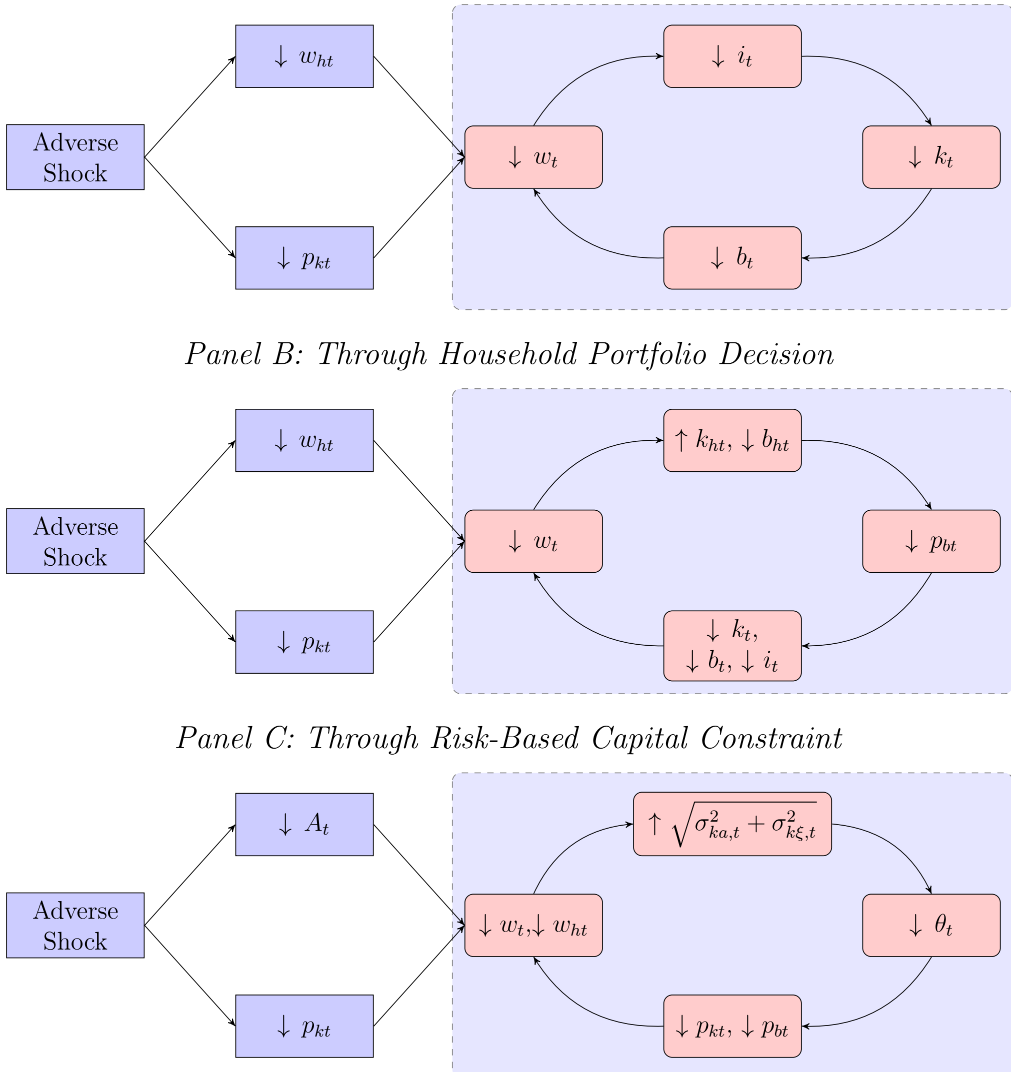
where $\bar{\theta}$ is a constant set by the prudential regulator. The equilibrium outcomes are summarized in the following lemma.

Lemma 3. *The economy with constant leverage converges to an economy with a constant wealth share of the intermediary sector in the economy*

$$\omega_t = \bar{\theta}^{-1}.$$

Figure IX: Shock Amplification

Panel A: Through Intermediary Investment Decision



In the steady state, the intermediary sector owns all the capital in the economy, with the expected excess return to holding capital given by

$$\mu_{Rk,t} - r_{ft} = \frac{1}{p_k} + \sigma_a^2 - \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \Phi\left(i_t\right),$$

and the expected excess return to holding bank debt given by

$$\mu_{Rb,t} - r_{ft} = \sigma_a^2,$$

with the riskiness of the returns equal to the riskiness of the productivity growth

$$\sigma_{ka,t} = \sigma_{ba,t} = \sigma_a$$
$$\sigma_{k\xi,t} = \sigma_{b\xi,t} = 0.$$

Proof. See Appendix A. □

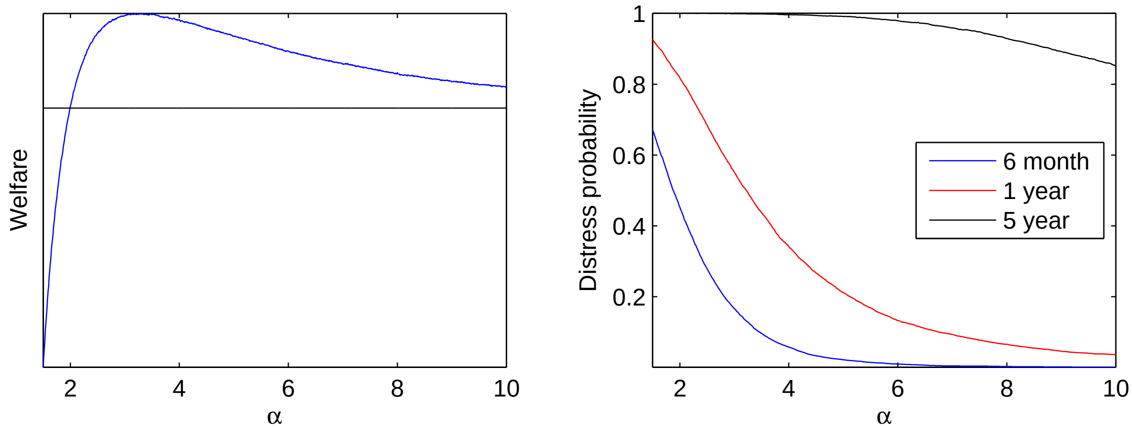
Thus, when the financial intermediaries face a constant leverage constraint, the intermediary sector does not amplify the fundamental shocks in the economy. Furthermore, since the intermediaries represent a constant fraction of the wealth of the economy with constant leverage, there is no risk of intermediary distress. Notice, however, that the excess return to holding capital compensates investors for the cost of capital adjustment. Thus, the financial system provides a channel through which market participants can share the cost of capital investment. It is important to point out that the household preference shocks ξ are not transmitted in this economy. Intuitively, since the households are no longer the marginal investors in the capital market, the price of capital only reflects shocks to intermediaries' pricing kernel which only varies with productivity shocks.

The benefit of having a financial system with a flexible leverage constraint is, then, increased output growth and more valuable capital, albeit at the cost of global stability. Since the rate of investment and the capital price are constant in this benchmark, the volatility of consumption growth equals the volatility of productivity growth, and the expected consumption growth rate equals the expected productivity growth rate. In our model, the financial intermediary sector allows households to smooth consumption, reducing the instantaneous volatility of consumption during good times, but at the cost of higher consumption growth volatility during times of financial distress. In particular, notice that, in the full model, volatility of consumption growth is given by

$$\left\langle \frac{dc_t}{c_t} \right\rangle^2 = \left(-\frac{2\theta_t \omega_t}{\beta \left(1-\omega_t\right)} p_{kt} \left(\sigma_{ka,t} - \sigma_a\right) + \sigma_a \right)^2 + \left(\frac{2\theta_t \omega_t}{\beta \left(1-\omega_t\right)} p_{kt} \sigma_{k\xi,t} \right)^2,$$

which is lower than the fundamental volatility σ_a^2 when $\sigma_{ka,t}$ is bigger than σ_a.

Figure X: Household Welfare



NOTES: Left panel: expected present value of household utility (y-axis) as a function of the tightness of risk-based capital constraint, α, (x-axis); right panel: 6 month, 1 year and 5 year cumulative default probabilities (y-axis) as a function of the tightness of risk-based capital constraint, α, (x-axis). Welfare and default probabilities are computed using 10000 simulations of the economy on a monthly frequency using the parameters in Table I, with household welfare computed over a 70 year horizon.

More formally, consider the trade-off in terms of the expected discounted present value of household utility. In Figure X, we plot the household welfare in the economy with pro-cyclical intermediary leverage as a function of the tightness of the risk-based capital constraint, as well as the the household welfare in the economy with constant leverage. Notice first that household welfare is not monotone in α: Initially, as the risk-based capital constraint becomes tighter, household welfare increases as distress risk decreases. For high enough levels of α, however, the household welfare decreases as the risk-based capital constraint becomes tighter. Intuitively, for low values of α, periods of financial distress (which are accompanied by sharp drops in consumption) are more frequent and the households become better off as the constraint becomes tighter. As α increases, the intermediaries become more stable, increasing household welfare. As α becomes too large, while probability of intermediary distress is still lower (see the right panel of Figure X),the risk-sharing function of the intermediaries is impeded, leading to lower household utility. Notice finally that household welfare in the economy with pro-cyclical leverage can be higher than that in the economy with constant leverage, even when a suboptimal α is chosen.

IV.C. *Stress tests*

By introducing preferences for the financial intermediaries, we can extend our model to study the impact of the use of stress tests as a macroprudential tool. By further introducing preferences for the prudential regulator, the model also provides implications for the optimal design of stress tests. We leave the formal treatment of these extensions for future work and provide here a sketch of how stress tests can be incorporated in the current setting.

Recall that, in our model, intermediary debt is subject to the risk-based capital constraint, which is a constraint on the local volatility of the asset side of the intermediary balance sheet

$$\theta_t^{-1} \geq \alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}.$$

Stress tests, on the other hand, can be interpreted as a constraint on the total volatility of the asset side of the balance sheet over a fixed time interval

$$\theta_t^{-1} \geq \vartheta\sqrt{\mathbb{E}_t\left[\int_t^T \left(\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2\right) ds\right]}.$$

Thus, in effect, stress tests can be thought of as a Stackelberg game between the policymaker and the financial intermediaries, with the policymaker moving first to choose the maximal allowable level of volatility over a time interval, and the intermediaries moving second to allocate the volatility allowance between different periods. Under the assumption that the prudential regulator designs stress tests to minimize total volatility, while the intermediaries maximize the expected discounted value of equity, the optimization problem for the intermediaries resembles the optimal robust control problem under model misspecification studied by Hansen and Sargent [2001, 2007]; Hansen, Sargent, and Tallarini [1999]; Hansen, Sargent, Turmuhambetova, and Williams [2006], among others

$$V_t(\vartheta) = \max_{\{i,\beta,k\}} \min_{q\in\mathcal{Q}(\vartheta)} \int \int_t^{\tau_D} e^{-\rho(s-t)} w_t(i,\beta,k)\, ds dq$$

subject to

$$\theta_t^{-1} \geq \vartheta \sqrt{\int_t^T \int \left(\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2\right) dq_s ds.}$$

Notice that, in the limit at $T \to t + dt$, this reduces to the risk-based capital constraint described above. In the language of Hansen et al. [2006], this is a *nonsequential* problem since the constraint is over a non-infinitesimal time horizon. The density function q is a density over the future realizations of the fundamental shocks $(dZ_{at}, dZ_{\xi t})$ in the economy, and $\mathcal{Q}$ is the set of densities that satisfies the stress-test constraint. Hansen et al. [2006] show how to move from the nonsequential robust controls problems to sequential problems. In particular, for the constraint formulation, they augment the state-space to include the continuation value of entropy and solve for the optimal value function that also depends on this continuation entropy.

In our setting, we can reformulate the optimization problem of the representative intermediary as

$$V_t(\vartheta) = \max_{\{i,\beta,k,\alpha_s\}} \mathbb{E}_t \left[\int_t^{\tau_D} e^{-\rho(s-t)} w_t(i, \beta, k)\, ds \right]$$

subject to

$$\frac{\theta_s^{-1}}{\alpha_s} \geq \sqrt{\sigma_{ka,s}^2 + \sigma_{k\xi,s}^2}$$
$$\theta_t^{-1} \geq \vartheta \sqrt{\mathbb{E}_t \left[\int_t^T \frac{\theta_s^{-2}}{\alpha_s^2} ds \right]}.$$

That is, the intermediaries choose an optimal capital plan at the time of the stress test to maximize the discounted present value of equity subject to satisfying the intertemporal volatility constraint imposed by the stress test. Locally, the portfolio allocation decision of the intermediaries satisfy a risk-based capital constraint, albeit with a time-varying α. However, along a given capital plan, the optimal decisions of both the households and the intermediaries are as described above. Stress tests are hence a natural but technically challenging extension of the current setup and are left for future exploration.

V. Conclusion

We present a dynamic, general equilibrium theory of financial intermediaries' leverage cycle as a conceptual basis for policies geared toward financial stability. In this setup, any change in prudential policies has general equilibrium effects that impact the pricing of financial and nonfinancial credit, the equilibrium volatilities of financial and real assets, and the allocation of consumption and investment goods. From a normative point of view, such effects are important to understand, as they ultimately determine the effectiveness of prudential policies.

The assumptions of our model are empirically motivated, and our theory captures many important stylized facts about financial intermediary dynamics that have been documented in the literature. There is both direct and intermediated credit by households, giving rise to substitution from intermediated credit to directly granted credit in times of tighter intermediary constraints. The risk-based funding constraint leads to procyclical intermediary leverage, matching empirical observations. Our theory generates the volatility paradox: times of low contemporaneous volatility allow high intermediary leverage, increases in forward-looking systemic risk. Finally, the time variation in pricing of risk is a function of leverage growth and the price of risk of asset exposure is positive, two additional features that are strongly borne out in the data.

The most important contribution of the paper is to directly study the impact of prudential policies on the likelihood of systemic liquidity and solvency risks. We uncover a systemic risk-return trade-off: Tighter intermediary capital requirements tend to shift the term structure of systemic risk downward, at the cost of increased risk pricing today. This trade-off forms the basis for the evaluation of costs and benefits associated with financial stability policies, as discussed in greater detail in Adrian, Covitz, and Liang [2013a].

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A Proofs

I.A. Household's optimization

Recall that the household solves the portfolio optimization problem:

$$\max_{\{c_t,\pi_{kt},\pi_{bt}\}} \quad \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

subject to the wealth evolution equation:

$$\begin{aligned} dw_{ht} &= r_{ft}w_{ht}dt + w_{ht}\pi_{kt}\left\{\left(\mu_{Rk,t}-r_{ft}\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t}\right\} \\ &\quad + w_{ht}\pi_{bt}\left\{\left(\mu_{Rb,t}-r_{ft}\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t}\right\} - c_t dt, \end{aligned}$$

and the no-shorting constraints:

$$\pi_{kt},\ \pi_{bt} \geq 0.$$

Instead of solving the dynamic optimization problem, we follow Cvitanić and Karatzas [1992] and rewrite the household problem in terms of a static optimization. Cvitanić and Karatzas [1992] extend the Cox and Huang [1989] martingale method approach to constrained optimization problems, such as the one that the households face in our economy.
Define $K = \mathbb{R}^2_+$ to be the convex set of admissible portfolio strategies and introduce the support function of the set $-K$ to be

$$\begin{aligned} \delta\left(x\right) = \delta\left(x|\, K\right) &\equiv \sup_{\vec{\pi}\in K}\left(-\vec{\pi}'x\right) \\ &= \begin{cases} 0, & x\in K \\ +\infty, & x\notin K \end{cases}. \end{aligned}$$

We can then define an auxiliary unconstrained optimization problem for the household, with the returns in the auxiliary asset market defined as

$$\begin{aligned} r^v_{ft} &= r_{ft} + \delta\left(\vec{v}_t\right) \\ dR^v_{kt} &= \left(\mu_{Rk,t} + v_{1t} + \delta\left(\vec{v}_t\right)\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi,t} \\ dR^v_{bt} &= \left(\mu_{Rb,t} + v_{2t} + \delta\left(\vec{v}_t\right)\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi,t}, \end{aligned}$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V\left(K\right)$ of square-integrable, progressively measurable processes taking values in K. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta^v_t}{\eta^v_t} = -\left(r_{ft} + \delta\left(\vec{v}_t\right)\right)dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma'_{Rt}\right)^{-1}d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative household then becomes

$$\max_{c_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t} \log c_t dt\right]$$

subject to the static budget constraint:

$$w_{h0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_t dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves the dual problem

$$\min_{v\in V(K)} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t} \tilde{u}\left(\lambda \eta_t^v\right) dt\right],$$

where $\tilde{u}\left(x\right)$ is the convex conjugate of $-u\left(-x\right)$

$$\tilde{u}\left(x\right) \equiv \sup_{z>0}\left[\log\left(zx\right) - zx\right] = -\left(1+\log x\right)$$

and λ is the Lagrange multiplier of the static budget constraint. Cvitanić and Karatzas [1992] show that, for the case of logarithmic utility, the optimal choice of v satisfies

$$\begin{aligned} v_t^* &= \arg\min_{x\in K}\left\{2\delta(x) + \left|\left|\left(\vec{\mu}_{Rt} - r_{ft} + x\right)' \sigma_{Rt}^{-1}\right|\right|^2\right\} \\ &= \arg\min_{x\in K}\left|\left|\left(\vec{\mu}_{Rt} - r_{ft} + x\right)' \sigma_{Rt}^{-1}\right|\right|^2. \end{aligned}$$

Thus,

$$\begin{aligned} v_{1t} &= \begin{cases} 0, & \mu_{Rk,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rk,t}, & \mu_{Rk,t} - r_{ft} < 0 \end{cases} \\ v_{2t} &= \begin{cases} 0, & \mu_{Rb,t} - r_{ft} \geq 0 \\ r_{ft} - \mu_{Rb,t}, & \mu_{Rb,t} - r_{ft} < 0 \end{cases}. \end{aligned}$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\xi_t-\rho_h t}}{c_t} - \lambda \eta_t^v,$$

or

$$c_t = \frac{e^{-\xi_t-\rho_h t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ht} = \mathbb{E}_t\left[\int_t^{+\infty} \eta_s^v c_s ds\right] = \mathbb{E}_t\left[\int_t^{+\infty} \frac{e^{-\xi_s-\rho_h s}}{\lambda} ds\right] = \frac{e^{-\xi_t-\rho_h t}}{\lambda\left(\rho_h - \sigma_\xi^2/2\right)}.$$

Thus

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

To solve for the household's optimal portfolio allocation, notice that:

$$\begin{aligned}
\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} &= -\rho_h dt - d\xi_t + \frac{1}{2} d\xi_t^2 \\
&= \left(-\rho_h + \frac{1}{2}\sigma_\xi^2\right) dt - \sigma_\xi \rho_{\xi,a} dZ_{at} - \sigma_\xi \sqrt{1-\rho_{\xi,a}^2} dZ_{\xi t}.
\end{aligned}$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ht}}{w_{ht}} + \frac{dw_{ht}}{w_{ht}}\frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\pi}_t' = \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)' \left(\sigma_{Rt}'\sigma_{Rt}\right)^{-1} - \sigma_\xi \begin{bmatrix} \rho_{\xi a} & \sqrt{1-\rho_{\xi a}^2} \end{bmatrix} \sigma_{Rt}^{-1}.$$

I.B. *Equilibrium outcomes*

To summarize, in equilibrium, we must have

$$\begin{aligned}
&\mu_{\theta t} = \Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) - \mu_{\omega t} + \sigma_{\theta a,t}^2 + \sigma_{\theta\xi,t}^2 \\
&\mu_{Rk,t} - r_{ft} = \left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right)\frac{1-\theta_t\omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \\
&+ \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&\mu_{Rb,t} - r_{ft} = \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)\frac{\theta_t\omega_t - \omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t} \\
&+ \sigma_\xi\left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&\mu_{\omega t} = \left(\theta_t - 1\right)\left(\mu_{Rkt} - \mu_{Rb,t}\right) + \left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) \\
&+ \frac{1-\omega_t}{\omega_t}\left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right)\theta_t\omega_t\right] \\
&\sigma_{\theta a,t} = -\left(\theta_t - 1\right)\left(\sigma_{ka,t} - \sigma_{ba,t}\right) \\
&\sigma_{\theta\xi,t} = -\left(\theta_t - 1\right)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right) \\
&\beta\left(\theta_t\omega_t - \omega_t\right)\sigma_{ba,t} = -\left(\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)\sigma_{ka,t}
\end{aligned}$$

$$
\begin{aligned}
&+ \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_a \\
&\beta\left(\theta_t\omega_t - \omega_t\right)\sigma_{b\xi,t} = -\left(\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}\right)\sigma_{k\xi,t} \\
&\alpha^{-2}\theta_t^{-2} = \sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 \\
&0 = \left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\left(1-\theta_t\omega_t\right)\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1-\theta_t\omega_t\right) \\
&+ \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)p_{kt}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a\sigma_{ka,t}\right) \\
&- \beta p_{kt}\omega_t\mu_{\omega t} + \theta_t\omega_t p_{kt}^2\left(\left(\sigma_{ka,t}-\sigma_a\right)^2 + \sigma_{k\xi,t}^2\right) \\
&- \beta p_{kt}\omega_t\left(\left(\sigma_{ka,t}-\sigma_a\right)\sigma_{\omega a,t} + \sigma_{k\xi,t}\sigma_{\omega\xi,t}\right).
\end{aligned}
$$

Notice that the first eight equations describe the evolutions of θ_t, ω_t, the return of risky intermediary debt R_{bt}, and the expected excess return to direct capital holding in terms of the two state variables, (θ_t, ω_t) and the loadings, $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$, of the return to direct capital holding on the two fundamental shocks in the economy.[5] The final two equations, then, express $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$ in terms of the state variables.

Before solving the final two equations, we simplify the equilibrium conditions. Notice first that

$$
\left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) = -\left(\theta_t - 1\right)\sigma_{ka,t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right) - \left(\theta_t - 1\right)\sigma_{k\xi,t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right),
$$

and

$$
\begin{aligned}
\mu_{Rkt} - \mu_{Rb,t} = &\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \sigma_{ka,t}\sigma_{ba,t} - \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{1-\theta_t\omega_t}{1-\omega_t} \\
&- \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2 - \sigma_{ka,t}\sigma_{ba,t} - \sigma_{k\xi,t}\sigma_{b\xi,t}\right)\frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \\
&+ \sigma_\xi\left(\left(\sigma_{ka,t} - \sigma_{ba,t}\right)\rho_{\xi,a} + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left(\mu_{Rkt} - \mu_{Rb,t}\right) + \frac{1}{\theta_t - 1}\left(\sigma_{ka,t}\sigma_{\theta a,t} + \sigma_{k\xi,t}\sigma_{\theta\xi,t}\right) = &-\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\left(\sigma_{ka,t} - \sigma_{ba,t}\right)^2 - \frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)^2 \\
&+ \sigma_\xi\left(\left(\sigma_{ka,t} - \sigma_{ba,t}\right)\rho_{\xi,a} + \left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right)\sqrt{1-\rho_{\xi,a}^2}\right).
\end{aligned}
$$

Using

$$
\begin{aligned}
&\beta\left(\theta_t\omega_t - \omega_t\right)\left(\sigma_{ka,t} - \sigma_{ba,t}\right) = \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\sigma_{ka,t} - \sigma_a\right) \\
&\beta\left(\theta_t\omega_t - \omega_t\right)\left(\sigma_{k\xi,t} - \sigma_{b\xi,t}\right) = \left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\sigma_{k\xi,t}
\end{aligned}
$$

[5] Recall that we have also expressed the price of capital in terms of the state variables.

we can thus express the drift of ω_t as

$$\begin{aligned}
\mu_{\omega t} = & -\frac{1}{\beta^2 \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right)^2 \left[\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2\right] \\
& + \frac{\sigma_\xi}{\beta \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right) \left(\left(\sigma_{ka,t} - \sigma_a\right) \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1-\rho_{\xi,a}^2}\right) \\
& + \frac{1-\omega_t}{\omega_t} \left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right) \theta_t \omega_t\right].
\end{aligned}$$

Substituting the risk-based capital constraint, this becomes

$$\begin{aligned}
\mu_{\omega t} = & -\frac{1}{\beta^2 \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right)^2 \left[\sigma_a^2 - 2\sigma_a \sigma_{ka,t} + \frac{\theta_t^{-2}}{\alpha^2}\right] \\
& + \frac{\sigma_\xi}{\beta \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right) \left(\left(\sigma_{ka,t} - \sigma_a\right) \rho_{\xi,a} + \sigma_{k\xi,t} \sqrt{1-\rho_{\xi,a}^2}\right) \\
& + \frac{1-\omega_t}{\omega_t} \left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right) \theta_t \omega_t\right] \\
& \equiv \mathcal{O}_0\left(\omega_t, \theta_t\right) + \mathcal{O}_a\left(\omega_t, \theta_t\right) \sigma_{ka,t} + \mathcal{O}_\xi\left(\omega_t, \theta_t\right) \sigma_{k\xi,t},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{O}_0\left(\omega_t, \theta_t\right) = & -\frac{1}{\beta^2 \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right)^2 \left[\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2}\right] \\
& - \frac{\sigma_\xi \sigma_a \rho_{\xi,a}}{\beta \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right) \\
& + \frac{1-\omega_t}{\omega_t} \left[\left(\rho_h - \frac{\sigma_\xi^2}{2}\right) - \frac{1}{p_{kt}} + \Phi\left(i_t\right) \theta_t \omega_t\right]
\end{aligned} \tag{8}$$

$$\mathcal{O}_a\left(\omega_t, \theta_t\right) = \frac{2\sigma_a}{\beta^2 \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right)^2 + \frac{\sigma_\xi \rho_{\xi,a}}{\beta \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right) \tag{9}$$

$$\mathcal{O}_\xi\left(\omega_t, \theta_t\right) = \frac{\sigma_\xi \sqrt{1-\rho_{\xi,a}^2}}{\beta \omega_t} \left(2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)\right). \tag{10}$$

Substituting into the drift rate of intermediary leverage

$$\begin{aligned}
\mu_{\theta t} & = \Phi\left(i_t\right)\left(1-\theta_t \omega_t\right) - \mu_{\omega t} + \sigma_{\theta a,t}^2 + \sigma_{\theta \xi,t}^2 \\
& = \mathcal{S}_0\left(\omega_t, \theta_t\right) + \mathcal{S}_a\left(\omega_t, \theta_t\right) \sigma_{ka,t} + \mathcal{S}_\xi\left(\omega_t, \kappa_t\right) \sigma_{k\xi,t},
\end{aligned}$$

where

$$\mathcal{S}_0\left(\omega_t, \theta_t\right) = \Phi\left(i_t\right)\left(1-\theta_t \omega_t\right) - \mathcal{O}_0\left(\omega_t, \theta_t\right) + \left(\frac{2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)}{\beta \omega_t}\right)^2 \frac{\theta_t^{-2}}{\alpha^2} \tag{11}$$

$$\mathcal{S}_a\left(\omega_t, \theta_t\right) = -2\sigma_a \left(\frac{2\theta_t \omega_t p_{kt} + \beta \left(1-\omega_t\right)}{\beta \omega_t}\right) - \mathcal{O}_a\left(\omega_t, \theta_t\right) \tag{12}$$

$$\mathcal{S}_\xi\left(\omega_t, \theta_t\right) = -\mathcal{O}_\xi\left(\omega_t, \theta_t\right). \tag{13}$$

Similarly, the excess return on capital is given by

$$
\begin{aligned}
\mu_{Rk,t} - r_{ft} &= \left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2\right) \frac{1-\theta_t\omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} \\
&+ \sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&= \frac{\theta_t^{-2}}{\alpha^2}\frac{1-\theta_t\omega_t}{1-\omega_t} - \frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} \\
&+ \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} + \sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&= \frac{-2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} \\
&+ \sigma_\xi \left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right).
\end{aligned}
$$

The excess return on intermediary debt is given by

$$
\begin{aligned}
&\mu_{Rb,t} - r_{ft} = \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right) \frac{\theta_t\omega_t - \omega_t}{1-\omega_t} + \left(\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}\right) \frac{1-\theta_t\omega_t}{1-\omega_t} \\
&+ \sigma_\xi \left(\sigma_{ba,t}\rho_{\xi,a} + \sigma_{b\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
&= \left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right)\left(\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\
&+ \left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right)\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right)^2 \sigma_a^2 \\
&- 2\left(\frac{\theta_t\omega_t - \omega_t}{1-\omega_t}\right)\left(\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right)\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
&- \left(\frac{1-\theta_t\omega_t}{\theta_t\omega_t - \omega_t}\right)\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} \\
&+ \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\right)\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\sigma_a\sigma_{ka,t} \\
&+ \sigma_\xi\rho_{\xi,a}\left[-\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\sigma_{ka,t} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\sigma_a\right] \\
&- \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\frac{\beta\left(1-\theta_t\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(\theta_t\omega_t - \omega_t\right)}\sigma_{k\xi,t}.
\end{aligned}
$$

Notice also that we can now derive the risk-free rate. Recall that, in the unconstrained region, the risk-free rate satisfies the household Euler equation

$$
r_{ft} = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \frac{1}{dt}\mathbb{E}\left[\frac{dc_t}{c_t}\right] - \frac{1}{dt}\mathbb{E}\left[\frac{\langle dc_t\rangle^2}{c_t^2} + \frac{\langle dc_t, d\xi_t\rangle^2}{c_t}\right].
$$

Applying Itô's lemma to the goods clearing condition, we obtain

$$\begin{aligned} dc_t &= d\left(K_t A_t - i_t k_t A_t\right) \\ &= A_t dK_t + \left(K_t - i_t k_t\right) dA_t - A_t k_t di_t - A_t i_t dk_t - k_t \left\langle di_t, dA_t \right\rangle. \end{aligned}$$

From the financial intermediaries' optimal investment choice, we have

$$\begin{aligned} di_t = &\frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a \left(\sigma_{ka,t} - \sigma_a\right) \right) dt \\ &+ \frac{\phi_0^2 \phi_1 p_{kt}^2}{4} \left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2 \right) dt \\ &+ \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \left(\sigma_{ka,t} - \sigma_a\right) dZ_{at} + \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} dZ_{\xi t}. \end{aligned}$$

Thus

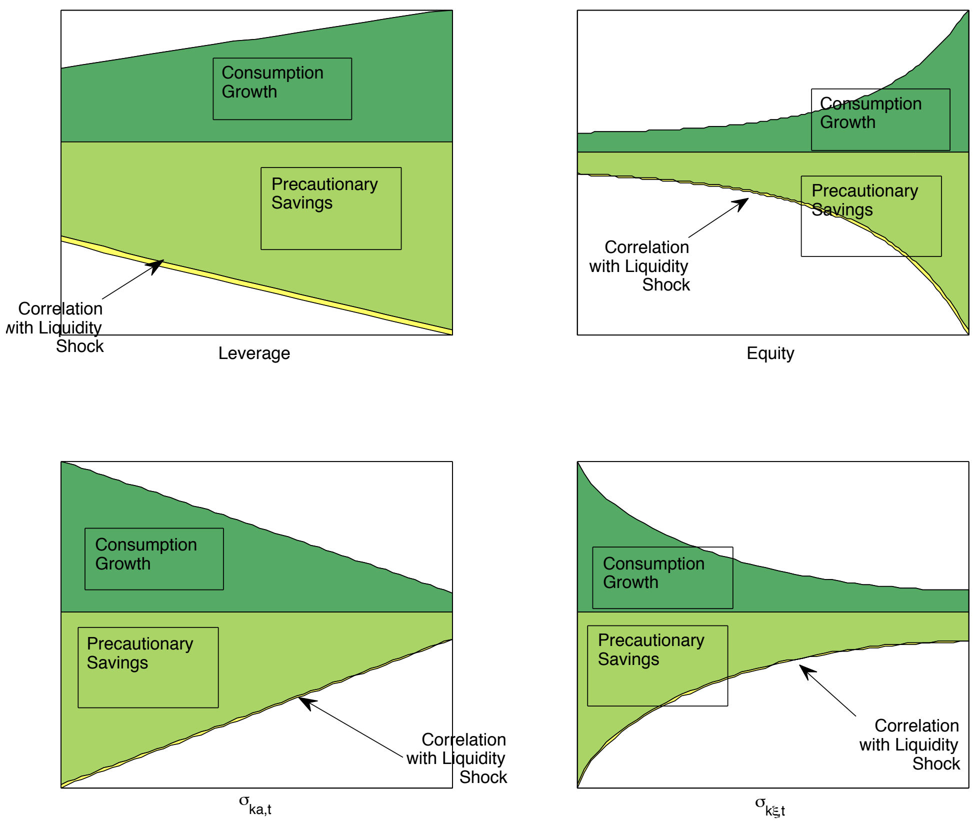
$$\begin{aligned} \frac{1}{dt}\mathbb{E}\left[\frac{dc_t}{c_t}\right] &= \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k + \frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t} \Phi\left(i_t\right)\left(1 - i_t\right) \\ &\quad - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k \right) \right) \\ &\quad - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{4} \left(\left(\sigma_{ka,t} - \sigma_a\right)^2 + \sigma_{k\xi,t}^2 \right) \\ \frac{1}{dt}\mathbb{E}\left[\frac{\left\langle dc_t \right\rangle^2}{c_t^2}\right] &= \left(\sigma_a - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \left(\sigma_{ka,t} - \sigma_a\right) \right)^2 \\ &\quad + \left(\left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} \right)^2 \\ \frac{1}{dt}\mathbb{E}\left[\frac{\left\langle dc_t d\xi_t \right\rangle^2}{c_t}\right] &= \sigma_\xi \left(\sigma_a - \left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{ka,t} \right) \rho_{\xi,a} \\ &\quad - \sigma_\xi \left(\left(\frac{\theta_t \omega_t}{1 - i_t \theta_t \omega_t}\right) \frac{\phi_0^2 \phi_1 p_{kt}^2}{2} \sigma_{k\xi,t} \right) \sqrt{1 - \rho_{\xi,a}^2}. \end{aligned}$$

Figure XI plots the three components of the risk-free rate as a function of intermediary leverage (upper left panel), the wealth share of the intermediary (upper right panel), and the volatility of the excess return to capital (lower panels). We see that the precautionary savings motive dominates the consumption growth component, and even more so as leverage increases.

Recall that, in equilibrium, we have

$$1 - i_t \theta_t \omega_t = \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) p_{kt} \left(1 - \omega_t\right),$$

Figure XI: Equilibrium Risk-Free Rate



NOTES: Equilibrium risk-free rate as a function of intermediary leverage (upper left panel), the wealth share of the intermediary (upper right panel), and the volatility of the excess return to capital (lower panels). "Consumption growth" refers to the component of the risk-free rate that's due to the expected consumption growth rate. "Precautionary saving" is measured as the contribution to the risk-free of the variance of the consumption growth rate. "Correlation with Liquidity Shock" is measured as the contribution to the risk-free of the correlation between the consumption growth rate and the liquidity shock $d\xi_t$. The equilibrium risk-free rate is then given as the lower boundary of the shaded region in each graph.

so that

$$\begin{aligned}\frac{1}{1-i_t\theta_t\omega_t}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 &= \left(\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)p_{kt}\left(1-\omega_t\right)\right)^{-1}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 \\ &= \frac{p_{kt}}{\beta\left(1-\omega_t\right)}\end{aligned}$$

and

$$1 + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\frac{\phi_0^2\phi_1}{4}p_{kt}^2 = \frac{\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}.$$

Substituting into the expression for the risk-free rate, we obtain

$$
\begin{aligned}
r_{ft} = & \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right) \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \\
& + \frac{\theta_t\omega_t}{1 - i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) - \frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}}\right) \\
& - \frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\sigma_a^2 + \frac{\theta_t^{-2}}{\alpha^2} - 2\sigma_{ka,t}\sigma_a\right) - \left(\frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)^2 \frac{\theta_t^{-2}}{\alpha^2} \\
& - \sigma_a^2\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)^2 + \frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_\xi\rho_{\xi,a}\sigma_{ka,t} \\
& + \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
& - \sigma_a\sigma_\xi\rho_{\xi,a}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right) + \frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t}.
\end{aligned}
$$

We can now solve for the return on capital. In particular, we have

$$
\begin{aligned}
\mu_{Rk,t} = & \, r_{ft} - \frac{2\omega_t\theta_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} \\
& + \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
= & \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \left(\bar{a} + \frac{\sigma_a^2}{2} - \lambda_k\right)\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} \\
& + \frac{\theta_t\omega_t}{1 - i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right) - \frac{2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}}\right) \\
& - \frac{3\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\frac{\theta_t^{-2}}{\alpha^2} + \frac{\beta\left(1-\omega_t\right) + 4\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_a\sigma_{ka,t} \\
& - \sigma_a^2\left(\frac{\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)} + \left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)^2\right) \\
& + \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right)\sigma_{ka,t}\sigma_a \\
& - \sigma_a\sigma_\xi\rho_{\xi,a}\left(\frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\right) - \left(\frac{2p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \\
& + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_\xi\rho_{\xi,a}\sigma_{ka,t} \\
& + \frac{\beta\left(1-\omega_t\right) + 2\theta_t\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\sigma_{k\xi,t}.
\end{aligned}
$$

Solving for $\mu_{Rk,t}$, we obtain

$$
\mu_{Rk,t} = \mathcal{K}_0\left(\omega_t, \theta_t\right) + \mathcal{K}_a\left(\omega_t, \theta_t\right)\sigma_{ka,t} + \mathcal{K}_\xi\left(\omega_t, \theta_t\right)\sigma_{k\xi,t},
$$

where

$$\begin{aligned}
\mathcal{K}_0(\omega_t,\theta_t) = & \frac{\beta(1-\omega_t)}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}\left(\rho_h - \frac{\sigma_\xi^2}{2} + \frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi(i_t)(1-i_t)\right) \\
& + \frac{2\theta_t\omega_t}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}} - \sigma_a\sigma_\xi\rho_{\xi,a} \\
& + \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k - \frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}\left(3 + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)}\right)\frac{\theta_t^{-2}}{\alpha^2} \\
& - \sigma_a^2\left(\frac{\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}} + \left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\right)\right) \qquad (14)\\
\mathcal{K}_a(\omega_t,\theta_t) = & \sigma_\xi\rho_{\xi,a} + \frac{\beta(1-\omega_t)+4\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}\sigma_a + \frac{4p_{kt}\theta_t\omega_t}{\beta(1-\omega_t)}\sigma_a \qquad (15)\\
\mathcal{K}_\xi(\omega_t,\theta_t) = & \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}. \qquad (16)
\end{aligned}$$

We can now express the risk-free rate in the economy as

$$\begin{aligned}
r_{ft} = & \mu_{Rk,t} + \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2} - \frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\sigma_a\sigma_{ka,t} \\
& - \sigma_\xi\left(\sigma_{ka,t}\rho_{\xi,a} + \sigma_{k\xi,t}\sqrt{1-\rho_{\xi,a}^2}\right) \\
& \equiv \mathcal{R}_0(\omega_t,\theta_t) + \mathcal{R}_a(\omega_t,\theta_t)\sigma_{ka,t},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_0(\omega_t,\theta_t) = & \mathcal{K}_0(\omega_t,\theta_t) + \frac{2\omega_t\theta_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2} \qquad (17)\\
\mathcal{R}_a(\omega_t,\theta_t) = & \mathcal{K}_a(\omega_t,\theta_t) - \frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\sigma_a - \sigma_\xi\rho_{\xi,a}. \qquad (18)
\end{aligned}$$

Substituting into the excess return on holding intermediary debt, we obtain

$$\mu_{Rb,t} = \mathcal{B}_0(\omega_t,\theta_t) + \mathcal{B}_a(\omega_t,\theta_t)\sigma_{ka,t} + \mathcal{B}_\xi(\omega_t,\theta_t)\sigma_{k\xi,t},$$

where

$$\begin{aligned}
\mathcal{B}_0(\omega_t,\theta_t) = & \mathcal{R}_0(\omega_t,\theta_t) + \left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)^2\frac{\theta_t^{-2}}{\alpha^2} \qquad (19)\\
& + \left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)^2\sigma_a^2 \\
& + \sigma_\xi\rho_{\xi,a}\sigma_a\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)} \\
& - \left(\frac{1-\theta_t\omega_t}{\theta_t\omega_t-\omega_t}\right)\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(1-\omega_t)}\frac{\theta_t^{-2}}{\alpha^2}
\end{aligned}$$

$$
\begin{aligned}
\mathcal{B}_a(\omega_t,\theta_t) &= \mathcal{R}_a(\omega_t,\theta_t) + \left(\frac{1-\theta_t\omega_t}{1-\omega_t}\right)\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\sigma_a \\
&\quad - 2\left(\frac{\theta_t\omega_t-\omega_t}{1-\omega_t}\right)\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\left(\frac{\beta(1-\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)\sigma_a \\
&\quad - \sigma_\xi\rho_{\xi,a}\left(\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}\right)
\end{aligned}
\tag{20}
$$

$$
\mathcal{B}_\xi(\omega_t,\theta_t) = \mathcal{R}_\xi(\omega_t,\theta_t) - \sigma_\xi\sqrt{1-\rho_{\xi,a}^2}\frac{\beta(1-\theta_t\omega_t)+2\theta_t\omega_t p_{kt}}{\beta(\theta_t\omega_t-\omega_t)}. \tag{21}
$$

Notice that

$$
p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\mathcal{K}_\xi(\omega_t,\theta_t) - \beta p_{kt}\omega_t\mathcal{O}_\xi(\omega_t,\theta_t) = 0.
$$

Using these results and the risk-based capital constraint, we can rewrite

$$
\begin{aligned}
0 &= \theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&\quad + p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a(\sigma_{ka,t}-\sigma_a)\right) \\
&\quad - \beta p_{kt}\omega_t\mu_{\omega_t} - p_{kt}(\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left((\sigma_{ka,t}-\sigma_a)^2 + \sigma_{k\xi,t}^2\right)
\end{aligned}
$$

as

$$
\begin{aligned}
0 &= \theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&\quad + p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left(\mu_{Rk,t} - \frac{1}{p_{kt}} - \bar{a} - \frac{\sigma_a^2}{2} + \lambda_k - \sigma_a(\sigma_{ka,t}-\sigma_a)\right) \\
&\quad - \beta p_{kt}\omega_t\mu_{\omega_t} - p_{kt}(\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2 - 2\sigma_{ka,t}\sigma_a\right) \\
&\equiv \mathcal{C}_0(\omega_t,\theta_t) + \mathcal{C}_a(\omega_t,\theta_t)\sigma_{ka,t},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{C}_0(\omega_t,\theta_t) &= \theta_t\omega_t(1-\theta_t\omega_t)\Phi(i_t)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&\quad - p_{kt}(\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right) \\
&\quad + p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\left(\mathcal{K}_0(\omega_t,\theta_t) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k\right) \\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_0(\omega_t,\theta_t)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\mathcal{C}_a(\omega_t,\theta_t) &= p_{kt}(2\theta_t\omega_t p_{kt}+\beta(1-\omega_t))\mathcal{K}_a(\omega_t,\theta_t) \\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_a(\omega_t,\theta_t) + p_{kt}\beta(1-\omega_t)\sigma_a.
\end{aligned}
\tag{23}
$$

Solving for $\sigma_{ka,t}$, we obtain

$$\sigma_{ka,t} = -\frac{\mathcal{C}_0\left(\omega_t,\theta_t\right)}{\mathcal{C}_a\left(\omega_t,\theta_t\right)}.$$

Substituting into the risk-based capital constraint, we obtain

$$\frac{\theta_t^{-2}}{\alpha^2} = \sigma_{k\xi,t}^2 + \left(\frac{\mathcal{C}_0\left(\omega_t,\theta_t\right)}{\mathcal{C}_a\left(\omega_t,\theta_t\right)}\right)^2,$$

so that

$$\sigma_{k\xi,t} = \sqrt{\frac{\theta_t^{-2}}{\alpha^2} - \left(\frac{\mathcal{C}_0\left(\omega_t,\theta_t\right)}{\mathcal{C}_a\left(\omega_t,\theta_t\right)}\right)^2}.$$

We can further simplify the above expressions by substituting for $\mathcal{O}_0$, $\mathcal{O}_a$, $\mathcal{K}_0$, and $\mathcal{K}_a$. Notice first that

$$\begin{aligned}
\mathcal{C}_a\left(\omega_t,\theta_t\right) &= p_{kt}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\mathcal{K}_a\left(\omega_t,\theta_t\right) \\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_a\left(\omega_t,\theta_t\right) + p_{kt}\beta\left(1-\omega_t\right)\sigma_a \\
&= p_{kt}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\mathcal{K}_a\left(\omega_t,\theta_t\right) - \sigma_\xi\rho_{\xi,a}\right) \\
&\quad + p_{kt}\left\{-\frac{2\sigma_a}{\beta}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2 + \beta\left(1-\omega_t\right)\sigma_a\right\} \\
&= p_{kt}\left(\beta\left(1-\omega_t\right) + 4\theta_t\omega_t p_{kt} + \frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\right)\sigma_a \\
&\quad + p_{kt}\left\{-\frac{2\sigma_a}{\beta}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2 + \beta\left(1-\omega_t\right)\sigma_a\right\} \\
&= \frac{2\sigma_a p_{kt}}{\beta}\left(\frac{\omega_t}{1-\omega_t}\right)\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{C}_0\left(\omega_t,\theta_t\right) &= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&\quad - p_{kt}\left(\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right) \\
&\quad + p_{kt}\left(2\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\mathcal{K}_0\left(\omega_t,\theta_t\right) - \frac{1}{p_{kt}} - \bar{a} + \frac{\sigma_a^2}{2} + \lambda_k\right) \\
&\quad - \beta p_{kt}\omega_t\mathcal{O}_0\left(\omega_t,\theta_t\right) \\
&= \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2 - \frac{4}{\phi_0^2\phi_1^2}\right) \\
&\quad - p_{kt}\left(\theta_t\omega_t p_{kt} + \beta\left(1-\omega_t\right)\right)\left(\frac{\theta_t^{-2}}{\alpha^2} + \sigma_a^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+\beta p_{kt}\left(1-\omega_t\right)\frac{\theta_t\omega_t}{1-i_t\theta_t\omega_t}\Phi\left(i_t\right)\left(1-i_t\right)\\
&-\theta_t\omega_t p_{kt}^2\left(3+\frac{4p_{kt}\theta_t\omega_t}{\beta\left(1-\omega_t\right)}\right)\frac{\theta_t^{-2}}{\alpha^2}\\
&-\sigma_a^2 p_{kt}\left(\theta_t\omega_t p_{kt}+\frac{\left(\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}\right)^2}{\beta\left(1-\omega_t\right)}\right)\\
&+\frac{p_{kt}}{\beta}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2\left(\sigma_a^2+\frac{\theta_t^{-2}}{\alpha^2}\right)-\beta p_{kt}\left(1-\omega_t\right)\Phi\left(i_t\right)\theta_t\omega_t
\end{aligned}
$$

Collecting like terms, we obtain

$$
\begin{aligned}
\mathcal{C}_0\left(\omega_t,\theta_t\right)=&\ \theta_t\omega_t\left(1-\theta_t\omega_t\right)\Phi\left(i_t\right)\left(p_{kt}^2-\frac{4}{\phi_0^2\phi_1^2}\right)\\
&-\beta p_{kt}\left(1-\omega_t\right)\theta_t\omega_t\Phi\left(i_t\right)\left(1-\frac{1-i_t}{1-i_t\theta_t\omega_t}\right)\\
&+\frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\left\{\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2-\beta\left(\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\right\}\\
&-\frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\theta_t\omega_t p_{kt}\left(3\beta+\frac{4p_{kt}\theta_t\omega_t}{1-\omega_t}\right)\\
&+\sigma_a^2\frac{p_{kt}}{\beta}\left\{\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2-\beta\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\right\}\\
&-\sigma_a^2\frac{p_{kt}}{\beta}\frac{\left(\beta\left(1-\omega_t\right)+2\theta_t\omega_t p_{kt}\right)^2}{\left(1-\omega_t\right)}\\
=&-\frac{\theta_t^{-2}}{\alpha^2}\frac{p_{kt}}{\beta}\frac{\omega_t}{1-\omega_t}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)^2\\
&-\sigma_a^2\frac{p_{kt}}{\beta}\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)\frac{\omega_t\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)+1-\omega_t}{1-\omega_t}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\sigma_{ka,t}&=-\frac{\mathcal{C}_0\left(\omega_t,\theta_t\right)}{\mathcal{C}_a\left(\omega_t,\theta_t\right)}\\
&=\frac{\theta_t^{-2}}{\alpha^2}+\sigma_a^2\left(1+\frac{1-\omega_t}{\omega_t\left(2\theta_t\omega_t p_{kt}+\beta\left(1-\omega_t\right)\right)}\right).
\end{aligned}
$$

I.C. Constant leverage benchmark

We begin by solving for the equilibrium dynamics of the intermediaries' wealth share in the economy. Recall that the capital held by the intermediaries is given by

$$k_t=\theta\omega_t K_t.$$

Applying Itô's lemma, we obtain

$$\begin{aligned}\frac{dk_t}{k_t} &= \frac{d\omega_t}{\omega_t} + \frac{dK_t}{K_t} \\ &= \left(\mu_{\omega t} + \Phi\left(i_t\right)\theta\omega_t - \lambda_k\right)dt + \sigma_{\omega a,t}dZ_{at} + \sigma_{\omega\xi,t}dZ_{\xi,t}.\end{aligned}$$

Recall, on the other hand, that intermediary capital evolves as

$$\frac{dk_t}{k_t} = \left(\Phi\left(i_t\right) - \lambda_k\right)dt.$$

Thus, equating coefficients, we obtain

$$\begin{aligned}\sigma_{\omega a,t} &= 0 \\ \sigma_{\omega\xi,t} &= 0 \\ \mu_{\omega t} &= \Phi\left(i_t\right)\left(1 - \theta\omega_t\right).\end{aligned}$$

Consider now the wealth evolution of the representative household. From the households' budget constraint, we have

$$\frac{dw_{ht}}{w_{ht}} = \left(r_{ft} - \rho_h + \frac{\sigma_\xi^2}{2}\right)dt + \frac{1-\bar{\theta}\omega_t}{1-\omega_t}\left(dR_{kt} - r_{ft}dt\right) + \frac{\omega_t\left(\bar{\theta}-1\right)}{1-\omega_t}\left(dR_{bt} - r_{ft}dt\right).$$

On the other hand, from the definition of ω_t, we obtain

$$\begin{aligned}\frac{dw_{ht}}{w_{ht}} &= \frac{d\left(\left(1-\omega_t\right)p_{kt}A_tK_t\right)}{\left(1-\omega_t\right)p_{kt}A_tK_t} \\ &= \frac{dp_{kt}}{p_{kt}} + \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\omega_t}{1-\omega_t}\frac{d\omega_t}{\omega_t} + \left\langle \frac{dp_{kt}}{p_{kt}}, \frac{dA_t}{A_t}\right\rangle.\end{aligned}$$

Equating coefficients once again and simplifying, we obtain

$$\begin{aligned}\sigma_{ba,t} &= \sigma_{ka,t} \\ \sigma_{b\xi,t} &= \sigma_{k\xi,t} \\ \mu_{Rb,t} &= \mu_{Rk,t} + \frac{1-\omega_t}{\omega_t\left(\bar{\theta}-1\right)}\left(\rho_h - \frac{\sigma_\xi^2}{2} - \frac{1}{p_{kt}}\right) + \Phi\left(i_t\right).\end{aligned}$$

We now turn to solving for the equilibrium price of capital. The goods clearing condition in this economy reduces to

$$\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)\left(1-\omega_t\right)p_{kt} = 1 - i_t\bar{\theta}\omega_t.$$

Substituting the optimal level of investment

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right),$$

we obtain that the price of capital satisfies

$$\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)(1-\omega_t)\,p_{kt} = 1 - \frac{\bar{\theta}\omega_t}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

Then the price of capital satisfies

$$0 = \bar{\theta}\omega_t p_{kt}^2 + \beta(1-\omega_t)\,p_{kt} - \frac{4}{\phi_0^2\phi_1^2}\left(\bar{\theta}\omega_t + \phi_1\right),$$

or:

$$p_{kt} = \frac{-\beta(1-\omega_t) + \sqrt{\beta^2(1-\omega_t)^2 + \frac{16\bar{\theta}\omega_t}{\phi_0^2\phi_1^2}\left(\bar{\theta}\omega_t + \phi_1\right)}}{2\bar{\theta}\omega_t}.$$

Applying Itô's lemma, we obtain

$$0 = \bar{\theta}\omega_t p_{kt}^2\left(2\frac{dp_{kt}}{p_{kt}} + \left\langle\frac{dp_{kt}}{p_{kt}}\right\rangle^2 + \frac{d\omega_t}{\omega_t}\right) + \beta(1-\omega_t)\,p_{kt}\frac{dp_{kt}}{p_{kt}} - \frac{\omega_t}{1-\omega_t}\beta p_{kt}\frac{d\omega_t}{\omega_t} + \frac{4}{\phi_0^2\phi_1^2}\bar{\theta}\frac{d\omega_t}{\omega_t}.$$

Equating coefficients and simplifying, we obtain

$$\begin{aligned}
\sigma_{ka,t} &= \sigma_a \\
\sigma_{k\xi,t} &= 0 \\
\mu_{Rk,t} &= \frac{1}{p_{kt}} + \bar{a} + \frac{\sigma_a^2}{2} - \lambda_k \\
&\quad - \frac{\Phi(i_t)\left(1-\bar{\theta}\omega_t\right)}{p_{kt}\left(2\bar{\theta}\omega_t p_{kt} + \beta(1-\omega_t)\right)}\left(\frac{4\bar{\theta}}{\phi_0^2\phi_1^2} - \frac{\omega_t}{1-\omega_t}\beta p_{kt} + \bar{\theta}\omega_t p_{kt}^2\right).
\end{aligned}$$

Finally, consider the equilibrium risk-free rate. Notice that

$$\begin{aligned}
\frac{dc_t}{c_t} &= \frac{d\left(\left(1-i_t\bar{\theta}\omega_t\right)A_tK_t\right)}{\left(1-i_t\bar{\theta}\omega_t\right)A_tK_t} \\
&= \frac{dA_t}{A_t} + \frac{dK_t}{K_t} - \frac{\bar{\theta}\omega_t}{1-i_t\bar{\theta}\omega_t}di_t - \frac{i_t\bar{\theta}\omega_t}{1-i_t\bar{\theta}\omega_t}\frac{d\omega_t}{\omega_t} - \frac{\bar{\theta}\omega_t}{1-i_t\bar{\theta}\omega_t}\left\langle di_t, \frac{dA_t}{A_t}\right\rangle,
\end{aligned}$$

and

$$di_t = d\left(\frac{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}{\beta}p_{kt}^2 - \frac{1}{\phi_1}\right) = \frac{\left(\rho_h - \frac{\sigma_\xi^2}{2}\right)}{\beta}p_{kt}^2\left(2\frac{dp_{kt}}{p_{kt}} + \left\langle\frac{dp_{kt}}{p_{kt}}\right\rangle^2\right).$$

Using

$$1 - i_t\bar{\theta}\omega_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right)(1-\omega_t)\,p_{kt},$$

the risk-free rate is thus given by

$$\begin{aligned} r_{ft} &= \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \frac{1}{dt}\mathbb{E}_t\left[\frac{dc_t}{c_t}\right] - \frac{1}{dt}\mathbb{E}_t\left[\left\langle\frac{dc_t}{c_t}\right\rangle\right] \\ &= \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) + \bar{a} - \frac{\sigma_a^2}{2} + \Phi\left(i_t\right)\bar{\theta}\omega_t - \lambda_k \\ &\quad - \frac{2\bar{\theta}\omega_t p_{kt}}{\beta\left(1-\omega_t\right)}\left(\mu_{Rk,t} - \frac{1}{p_{kt}} + \lambda_k - \bar{a} - \frac{\sigma_a^2}{2}\right) - \frac{i_t\bar{\theta}\omega_t}{1 - i_t\bar{\theta}\omega_t}\mu_{\omega t}. \end{aligned}$$

B Risk-Averse Intermediaries

In this Section, we outline the economy with risk-averse intermediaries and the numerical procedure for the equilibrium in the economy. We conclude by showing that empirical regularities are born-out in this economy as well, albeit in a weaker form than with risk-neutral intermediaries.

As before, we consider a continuous time, infinite horizon economy. Uncertainty is described by a two-dimensional, standard Brownian motion $Z_t = [Z_{at},\ Z_{\xi t}]'$ for $t \geq 0$, defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the augmented filtration generated by Z_t. There are three types of agents in the economy: (passive) producers, financially sophisticated intermediaries and unsophisticated households. We further differentiate the financial intermediaries into two types: bank (including dealers and investment banks) intermediaries and non-bank (such as pension funds, insurance companies and mutual funds) intermediaries. The structure of the economy is illustrated in Figure XII.

II.A. Production

There is a single consumption good in the economy, produced continuously. We assume that physical capital is the only input into the production of the consumption good, so that the total output in the economy at date $t \geq 0$ is

$$Y_t = A_t K_t,$$

where K_t is the aggregate amount of capital in the economy at time t, and the stochastic productivity of capital $\{A_t = e^{a_t}\}_{t\geq 0}$ follows a geometric diffusion process of the form

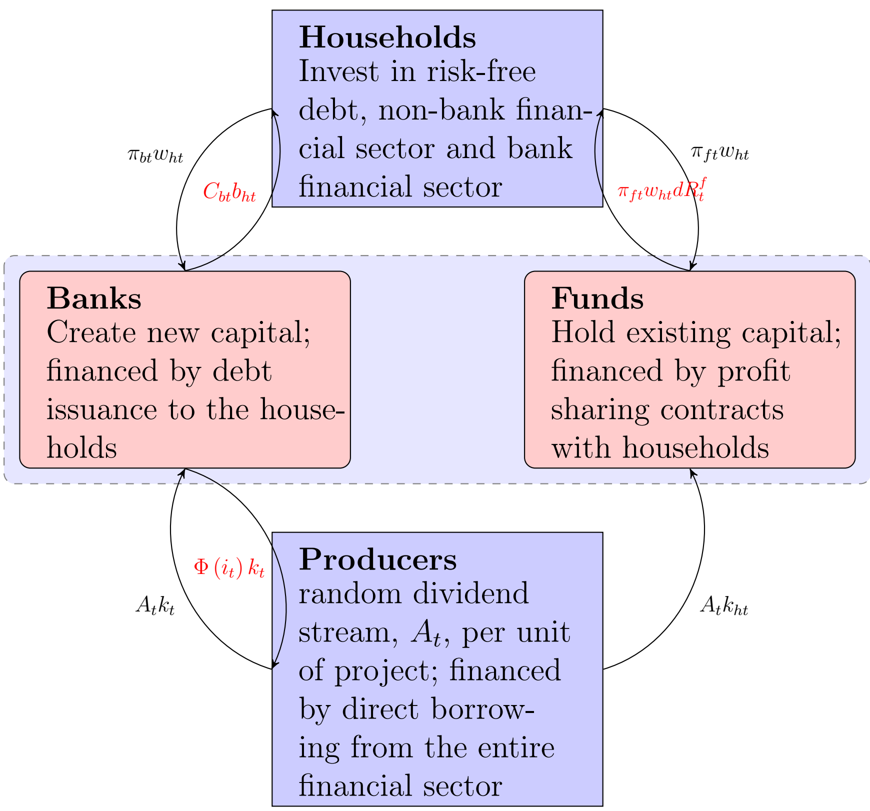
$$da_t = \bar{a}dt + \sigma_a dZ_{at}.$$

The stock of physical capital in the economy depreciates at a constant rate λ_k, so that the total physical capital in the economy evolves as

$$dK_t = \left(I_t - \lambda_k\right) K_t dt,$$

where I_t is the reinvestment rate per unit of capital in place. There is a fully liquid market for physical capital in the economy, in which both types of financial intermediaries are allowed

Figure XII: Structure of the economy



to participate. We denote by $p_{kt}A_t$ the price of one unit of capital at time $t \geq 0$ in terms of the consumption good.

II.B. Financial intermediary sector

There are two types of intermediaries in the economy: the bank intermediaries, subject to macroprudential policy constraints, and non-bank intermediaries, subject to a skin-in-the-game constraint. The bank intermediaries represent the levered financial institutions, such as commercial and investment banks and broker-dealers, in the economy, while the non-bank intermediaries represent institutions such as hedge and mutual funds, pension funds and insurance companies.

II.B.1. Non-bank financial sector

The non-bank financial sector in our model corresponds to the financial sector of He and Krishnamurthy [2012a,b, 2013]. In particular, there is a unit mass of risk-averse specialists that manage the non-bank intermediaries ("funds"), with each fund matched to a single agent. The fund managers are risk-averse, infinitely lived agents that evaluate consumption paths $\{c_{ft}\}_{t\geq 0}$ using

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \log c_{ft} dt\right],$$

where ρ is the subjective discount rate.

As in He and Krishnamurthy [2013], at every date t, each specialist is randomly matched with a household to form a fund, creating a continuum of identical bilateral relationships. The specialists execute trades on behalf of the fund in the capital and risk-free debt markets, while the household only decides on the allocation between risk-free debt and the two intermediary sectors. The match is broken at date $t+dt$, and the specialists and households are rematched. Denote the specialist wealth at time t by w_{ft}, and by H_t the household's wealth allocation to the non-bank intermediaries. As in He and Krishnamurthy [2012b, 2013], we assume that the non-bank intermediaries face a skin-in-the-game constraint when raising capital from the households. In particular, for every dollar of specialist wealth invested in the fund, the households only contribute up to $m > 0$ dollars, so that

$$H_t \leq m w_{ft}.$$

The constant m measures the tightness of the capital constraint faced by the specialists in the economy.

Finally, the specialist can allocate the total intermediary wealth freely between the risk-free debt and the existing capital stock in economy, with the total profits shared between the households and the specialists in accordance to their relative contributions to the fund. Assuming for simplicity that the specialist invests all of his post-consumption wealth in the fund, the representative specialist solves

$$\max_{\{c_{ft},\theta_{ft}\}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \log c_{ft} dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ft}}{w_{ft}} = \theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{ft}}{w_{ft}}dt,$$

where θ_{ft} is the fraction of the fund's equity allocated to the risky capital, r_{ft} is the equilibrium risk-free rate and dR_{kt} is the return on the existing capital in the economy

$$dR_{kt} = \underbrace{\frac{A_t k_{ht}}{k_{ht} p_{kt} A_t}dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(k_{ht} p_{kt} A_t\right)}{k_{ht} p_{kt} A_t}}_{\text{capital gains}} \equiv \mu_{Rk,t}dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t},$$

with k_{ht} the number of units of capital held by the funds. When $\theta_{ft} > 1$, the fund holds a levered position in risky capital. We have the following result.

Lemma 4. *The specialists consume a constant fraction of their wealth*

$$c_{ft} = \rho w_{ft},$$

and allocate the fund's capital as a mean-variance investor

$$\theta_{ft} = \frac{\mu_{Rk,t} - r_{ft}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}.$$

Proof. Introduce $\theta_{bf,t}$ to be the fraction of fund equity allocated to risky intermediary debt, and let $\vec{\theta}_{ft} \equiv [\theta_{ft}\ \theta_{bf,t}]'$ be the vector of portfolio choices of the fund at time t.
Define $\mathcal{K} = \mathbb{R} \times \{0\}$ to be the convex set of admissible portfolio strategies and introduce the support function of the set $-\mathcal{K}$ to be

$$\begin{aligned}\delta\left(x\right) = \delta\left(x|\,\mathcal{K}\right) &\equiv \sup_{\vec{\theta_f}\in\mathcal{K}}\left(-\vec{\theta}_f' x\right)\\ &= \begin{cases}0, & \text{if } x_1 = 0\\ +\infty, & \text{otherwise.}\end{cases}\end{aligned}$$

We can then define an auxiliary unconstrained optimization problem for the specialist, with the returns in the auxiliary asset market defined as

$$\begin{aligned} r_{ft}^v &= r_{ft} + \delta\left(\vec{v}_t\right)\\ dR_{kt}^v &= \left(\mu_{Rk,t} + v_{1t} + \delta\left(\vec{v}_t\right)\right)dt + \sigma_{ka,t}dZ_{at} + \sigma_{k\xi,t}dZ_{\xi t}\\ dR_{bt}^v &= \left(\mu_{Rb,t} + v_{2t} + \delta\left(\vec{v}_t\right)\right)dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi t},\end{aligned}$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V\left(\mathcal{K}\right)$ of square-integrable, progressively measurable processes taking values in $\mathcal{K}$. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta\left(\vec{v}_t\right)\right)dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma_{Rt}'\right)^{-1}d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative specialist then becomes

$$\max_{c_{ft}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t} \log c_{ft} dt\right],$$

subject to the static budget constraint

$$w_{f0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_{ft} dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves

$$\begin{aligned} \vec{v}_t^* &= \arg\min_{x_1=0} \left\{2\delta(x) + \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right)\right|\right|^2\right\} \\ &= \arg\min_{x_1=0} \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right)\right|\right|^2. \end{aligned}$$

Thus,

$$v_t^* = \begin{bmatrix} 0 \\ -\left(\mu_{Rb,t} - r_{ft}\right) \end{bmatrix}.$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\rho t}}{c_{ft}} - \lambda \eta_t^v,$$

or

$$c_{ft} = \frac{e^{-\rho t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ft} = \mathbb{E}_t\left[\int_t^{+\infty} \eta_s^v c_{fs} ds\right] = \mathbb{E}_t\left[\int_t^{+\infty} \frac{e^{-\rho s}}{\lambda} ds\right] = \frac{e^{-\rho t}}{\lambda \rho}.$$

Thus

$$c_{ft} = \rho w_{ft}.$$

To solve for the fund's optimal portfolio allocation, notice that:

$$\frac{d\left(\eta_t^v w_{ft}\right)}{\eta_t^v w_{ft}} = -\rho dt.$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_{ft}\right)}{\eta_t^v w_{ft}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ft}}{w_{ft}} + \frac{dw_{ft}}{w_{ft}}\frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\theta}_{ft} = \left(\sigma_{Rt}\sigma_{Rt}'\right)^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right).$$

□

II.B.2. Bank financial sector

In addition to the specialists that manage the institutions in the non-bank financial sector, there is also a unit mass of infinitely-lived, risk-averse bankers that manage the institutions ("banks") in the banking sector. The banking sector in our economy performs two functions in the economy. First, the bank intermediary is a financially constrained institution that has access to (a better) capital creation technology than the households and can thus intermediate between households and the productive sector, channeling funds from one to the other to the benefit of both parties. Since the intermediaries have wealth of their own, they serve another important function in the economy by providing risk-bearing capacity to the households.

In particular, bank financial intermediaries create new physical capital ("projects") through capital investment. Denote by $i_t A_t$ the investment rate and by $\Phi\left(i_t\right) A_t$ the new projects created per unit of capital held by the intermediaries. Here, $\Phi\left(\cdot\right)$ reflects the costs of (dis)investment. We assume that $\Phi\left(0\right) = 0$, so in the absence of new investment, capital depreciates at the economy-wide rate λ_k. Notice that the above formulation implies that costs of adjusting capital are higher in economies with a higher level of capital productivity, corresponding to the intuition that more developed economies are more specialized. We follow Brunnermeier and Sannikov [2012] in assuming that investment carries quadratic adjustment costs, so that Φ has the form

$$\Phi\left(i_t\right) = \phi_0\left(\sqrt{1+\phi_1 i_t} - 1\right),$$

for positive constants ϕ_0 and ϕ_1.

Unlike the non-bank intermediaries, the banks finance themselves by issuing long-term risky debt to the households. To keep the balance sheet structure of financial institutions time-invariant, we assume that the risky intermediary debt matures at a constant rate λ_b, so that the time t probability of a bond maturing before time $t+dt$ is $\lambda_b dt$. Notice that this corresponds to an infinite-horizon version of the "stationary balance sheet" assumption of Leland and Toft [1996]. The bonds pay a floating coupon $C_{dt}A_t$ until maturity, with the

coupon payment determined in equilibrium to clear the risky bond market. Similarly to capital, risky bonds are liquidly traded, with the price of a unit of intermediary debt at time t in terms of the consumption good given by $p_{bt}A_t$.
We assume that the representative banker evaluates consumption paths $\{c_{bt}\}_{t\geq 0}$ using

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{bt}dt\right].$$

Denote by w_t the (inside) equity of the representative intermediary at date $t \geq 0$, by k_t the intermediary's holdings of capital and by b_t the intermediary's stock of debt outstanding. Introduce

$$\theta_t = \frac{k_t p_{kt} A_t}{w_t}$$

to be the total leverage of the representative intermediary, and

$$\theta_{bt} = \frac{b_t p_{bt} A_t}{w_t}$$

to be the ratio of long term debt to equity. Then, the value of inside equity evolves as

$$\frac{dw_t}{w_t} = \theta_t\left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right)dt\right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt.$$

Here, dR_{bt} is the return to holding one unit of intermediary debt

$$dR_{bt} = \underbrace{\frac{\left(C_{dt} + \lambda_b - \text{б}_t p_{bt}\right)A_t b_t}{b_t p_{bt} A_t}dt}_{\text{dividend}-\text{price ratio}} + \underbrace{\frac{d\left(b_t p_{bt} A_t\right)}{b_t p_{bt} A_t}}_{\text{capital gains}} \equiv \mu_{Rb,t}dt + \sigma_{ba,t}dZ_{at} + \sigma_{b\xi,t}dZ_{\xi t},$$

where б_t is the new debt issuance at time t. We assume that the banking sector cannot participate in the instantaneous risk-free debt market, so that

$$\theta_t = 1 + \theta_{bt}.$$

Notice further that the excess return to holding capital from the viewpoint of the intermediaries also includes the net gain from investing in a new project, $\Phi\left(i_t\right)$, rather than buying the corresponding capital stock in the market, i_t/p_{kt}.
The key assumption of this paper concerns the funding of the banks. We assume that intermediary borrowing is restricted by a risk-based capital constraint, similar to the value at risk (VaR) constraint of Danielsson et al. [2011]. In particular, we assume that

$$w_t \geq \alpha\sqrt{\frac{1}{dt}\left\langle k_t d\left(p_{kt}A_t\right)\right\rangle^2}.$$

It should be noted that, since the banks are constrained, they create less new capital in equilibrium than they would without regulation. The risk-based constraint can be microfounded using a moral hazard problem, as in Adrian and Shin [2010a]; however, we abstract from the question of why the risk-based constraint exists in our economy.[6] Notice that this risk-based funding constraint boils down to a time varying leverage constraint

$$\theta_t \leq \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}}.$$

The value of inside equity of the representative banker evolves as

$$\frac{dw_t}{w_t} = \theta_t\left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right)dt\right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt.$$

Thus, the representative banker solves

$$\max_{\theta_t,\theta_{bt},i_t,c_{bt}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{bt}dt\right]$$

subject to the dynamic budget constraint

$$\frac{dw_t}{w_t} = \theta_t\left(dR_{kt} - r_{ft}dt + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right)dt\right) - \theta_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_{bt}}{w_t}dt,$$

the risk-based capital constraint

$$\theta_t \leq \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}},$$

and the adding-up constraint

$$\theta_{bt} = \theta_t - 1.$$

Lemma 5. *The representative banker optimally consumes at rate*

$$c_{bt} = \rho w_t$$

and invests in new projects at rate

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

While the capital constraint in not binding, the banking system leverage is

$$\theta_t = \frac{\sigma_{ba,t}^2 - \sigma_{ka,t}\sigma_{ba,t} + \sigma_{b\xi,t}^2 - \sigma_{k\xi,t}\sigma_{b\xi,t} - \left(\mu_{Rb,t} - r_{ft}\right) + \left(\mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)}.$$

[6] An alternative interpretation would be in terms of a counterbalancing force to a government subsidy (such as access to a better investment technology than other sectors of the economy) provide to the banking sector. As pointed out in Kareken and Wallace [1978], government subsidies distort the risk-taking decisions of banks, precipitating the need for government regulation of risk taking.

Proof. Notice first that the investment choice, i_t, only enters into the budget constraint directly. Thus, we can take the first order condition with respect to investment to obtain

$$\Phi\left(i_t\right)' = p_{kt}^{-1},$$

or

$$i_t = \frac{1}{\phi_1}\left(\frac{\phi_0^2\phi_1^2}{4}p_{kt}^2 - 1\right).$$

Consider now the optimal consumption and portfolio allocation choice of the representative banker. As with the specialists, we rewrite the dynamic optimization as a static problem. To make the derivation more concise, denote

$$\begin{aligned}\tilde{\mu}_{Rk,t} &= \mu_{Rk,t} + \left(\Phi\left(i_t\right) - \frac{i_t}{p_{kt}}\right)\\ \tilde{\mu}_{Rt} &= \left[\begin{array}{cc}\tilde{\mu}_{Rk,t} & \mu_{Rb,t}\end{array}\right]'.\end{aligned}$$

As in the previous section, let

$$\mathcal{K}_t = \left[x \in \mathbb{R}^2 \middle| \; x_1 \leq \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}},\; x_2 = 1 - x_1\right]$$

be the convex set of admissible portfolio strategies as date t, with the support function of $-\mathcal{K}_t$ given by

$$\delta_t\left(x\right) = \begin{cases} -x_2 + \frac{x_2 - x_1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} & \text{if} \quad x_1 \leq x_2 \\ +\infty & \text{otherwise}\end{cases}$$

and the barrier cone of $-\mathcal{K}_t$ by

$$\tilde{K} = \left[x \in \mathbb{R}^2 | x_1 \leq x_2\right].$$

The auxiliary state-price density in this case is

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta_t\left(\vec{v}_t\right)\right)dt - \left(\tilde{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)'\left(\sigma_{Rt}'\right)^{-1} d\vec{Z}_t,$$

where the optimal $\vec{v}_t$ satisfies

$$\vec{v}_t^* = \arg\min_{x \in \tilde{K}}\left[2\delta\left(x\right) + \left\|\sigma_{Rt}^{-1}\left(\tilde{\mu}_{Rt} - r_{ft}\right) + \sigma_{Rt}^{-1}x\right\|^2\right].$$

Denote the Lagrange multiplier on $x_1 \leq x_2$ by $2\phi_1$. Taking the first order conditions, we obtain

$$2\begin{bmatrix} -\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} \\ \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} - 1 \end{bmatrix} + 2\left(\sigma'_{Rt}\right)^{-1}\left(\sigma_{Rt}^{-1}\left(\tilde{\mu}_{Rt} - r_{ft}\right) + \sigma_{Rt}^{-1}x\right) + 2\phi_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0,$$

or

$$\vec{v}_t^* = \sigma_{Rt}\sigma'_{Rt}\begin{bmatrix} \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} \\ -\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} + 1 \end{bmatrix} + \phi_1\sigma_{Rt}\sigma'_{Rt}\begin{bmatrix} -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{bmatrix}.$$

Thus, if $\phi_1 = 0$, so that banks are at the risk-based capital constraint,

$$\begin{aligned}\vec{v}_t^* &= \sigma_{Rt}\sigma'_{Rt}\begin{bmatrix} \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} \\ -\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} + 1 \end{bmatrix} - \begin{bmatrix} \mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} \\ \mu_{Rb,t} - r_{ft} \end{bmatrix} \\ &= \begin{bmatrix} -\left(\widetilde{\mu}_{Rk,t} - r_{ft}\right) - \left(\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} - 1\right)\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) + \frac{1}{\alpha}\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2} \\ -\left(\mu_{Rb,t} - r_{ft}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)\left(\frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}} - 1\right) + \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2+\sigma_{k\xi,t}^2}}\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) \end{bmatrix}.\end{aligned}$$

Otherwise, $v_1^* = v_2^*$. Solving for ϕ_1, we obtain

$$\phi_1 = \frac{-\left(\widetilde{\mu}_{Rk,t} - r_{ft}\right) + \left(\mu_{Rb,t} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} + \frac{1}{\alpha\sqrt{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}} + \frac{\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)}.$$

Substituting into the equation for v_2^*, we obtain

$$\begin{aligned}v_1 = v_2 = &-\left(\widetilde{\mu}_{Rk,t} - r_{ft}\right)\frac{\left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right) - \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} \\ &-\left(\mu_{Rb,t} - r_{ft}\right)\frac{\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right)\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} + \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) \\ &-\frac{\left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right) - \left(\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)}\left(\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2 - \left(\sigma_{b\xi,t}\sigma_{k\xi,t} + \sigma_{ba,t}\sigma_{ka,t}\right)\right)\end{aligned}$$

Turn now to the solution of the augmented unconstrained problem. Analogously to the specialists, the representative banker solves

$$\max_{c_{bt}} \mathbb{E}\left[\int_0^{+\infty} e^{-\rho t}\log c_{bt}dt\right]$$

subject to the static budget constraint

$$w_0 = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_{bt} dt\right].$$

Taking the first order condition, we obtain

$$[c_{bt}]: \quad 0 = \frac{e^{-\rho t}}{c_{bt}} - \lambda \eta_t^v,$$

or

$$c_{bt} = \frac{e^{-\rho t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_t = \mathbb{E}_t\left[\int_t^{+\infty} \eta_s^v c_{bs} ds\right] = \mathbb{E}_t\left[\int_t^{+\infty} \frac{e^{-\rho s}}{\lambda} ds\right] = \frac{e^{-\rho t}}{\lambda \rho}.$$

Thus

$$c_{bt} = \rho w_t.$$

To solve for the bank's optimal portfolio allocation, notice that:

$$\frac{d\left(\eta_t^v w_t\right)}{\eta_t^v w_t} = -\rho dt.$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_t\right)}{\eta_t^v w_t} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_t}{w_t} + \frac{dw_t}{w_t}\frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\theta}_t = \left(\sigma_{Rt}\sigma_{Rt}'\right)^{-1}\left(\tilde{\mu}_{Rt} - r_{ft} + \vec{v}_t\right).$$

When the bank is unconstrained, $v_1^* = 0$ and the optimal portfolio allocation is

$$\begin{aligned}
\begin{bmatrix} \theta_t \\ -\theta_{bt} \end{bmatrix} &= \left(\begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}\begin{bmatrix} \sigma_{ka,t} & \sigma_{ba,t} \\ \sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}\right)^{-1}\begin{pmatrix} \mu_{Rk,t} + \Phi\left(i_t\right) - \frac{i_t}{p_{kt}} - r_{ft} + v_1 \\ \mu_{Rb,t} - r_{ft} + v_2 \end{pmatrix} \\
&= \begin{bmatrix} \frac{\sigma_{ba,t}^2 - \sigma_{ka,t}\sigma_{ba,t} + \sigma_{b\xi,t}^2 - \sigma_{k\xi,t}\sigma_{b\xi,t} - \left(\mu_{Rb,t} - r_{ft}\right) + \left(\mu_{Rk,t} + \Phi(i_t) - \frac{i_t}{p_{kt}} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} \\ \frac{\sigma_{ka,t}^2 - \sigma_{ba,t}\sigma_{ka,t} + \sigma_{k\xi,t}^2 - \sigma_{b\xi,t}\sigma_{k\xi,t} + \left(\mu_{Rb,t} - r_{ft}\right) - \left(\mu_{Rk,t} + \Phi(i_t) - \frac{i_t}{p_{kt}} - r_{ft}\right)}{\left(\left(\sigma_{ba,t} - \sigma_{ka,t}\right)^2 + \left(\sigma_{b\xi,t} - \sigma_{k\xi,t}\right)^2\right)} \end{bmatrix}.
\end{aligned}$$

□

Thus, while the capital constraint does not bind, the banks act as standard log-utility investors. In particular, since the optimal portfolio allocation does not include an intertemporal hedging demand, the banking intermediaries do not self-insure against the possibility of the capital constraint binding.

II.C. Households

We model the household sector as a continuum of mass one of homogeneous, risk-averse agents. The representative household is matched with the representative specialist to create the representative fund. The representative household is exposed to a preference shock, modeled as a change-of-measure variable in the household's utility function. In particular, we assume that the representative household evaluates possible consumption paths according to

$$\mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

where c_t is consumption at date $t \geq 0$ and ρ_h is the subjective discount rate. Here, $\exp\left(-\xi_t\right)$ is the Radon-Nikodym derivative of the measure induced by households' time-varying preferences or beliefs with respect to the physical measure. For simplicity, we normalize $\xi_0 = 0$ and assume that $\{\xi_t\}_{t\geq 0}$ evolves as a Brownian motion, uncorrelated with the productivity shock, Z_{at}

$$d\xi_t = \sigma_\xi dZ_{\xi t}.$$

The representative household finances its consumption through short-term risk-free borrowing and lending, an equity stake in the representative fund, and holdings of risky bank debt. Denote by π_{bt} the fraction of household wealth w_{ht} allocated to risky debt and by π_{kt} the fraction of household wealth allocated to the fund. The skin-in-the-game constraint for the specialists implies that the household allocation to the fund is constrained by

$$\pi_{kt}w_{ht} \leq m w_{ft}.$$

The households are also constrained from shorting either the fund or the risky bank debt. Thus, the representative household solves

$$\max_{\substack{\pi_{kt},\pi_{bt}\\ c_t}} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ht}}{w_{ht}} = \pi_{kt}\theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + \pi_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_t}{w_{ht}}dt,$$

the skin-in-the-game constraint

$$\pi_{kt}w_{ht} \leq m w_{ft},$$

and no shorting constraints

$$\pi_{kt} \geq 0$$
$$b_{ht} \geq 0.$$

The household takes the portfolio choice θ_{ft} and the wealth of the specialists w_{ft} as given when solving for the optimal consumption plan and portfolio allocation strategy. We have the following result.

Lemma 6. *The households' optimal consumption choice satisfies*

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2}\right) w_{ht}.$$

While the households are unconstrained in their wealth allocation, the households' optimal portfolio choice is given by

$$\begin{bmatrix} \pi_{kt} \\ \pi_{bt} \end{bmatrix} = \left(\begin{bmatrix} \theta_{ft}\sigma_{ka,t} & \theta_{ft}\sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix} \begin{bmatrix} \theta_{ft}\sigma_{ka,t} & \sigma_{ba,t} \\ \theta_{ft}\sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix} \right)^{-1} \begin{bmatrix} \theta_{ft}\left(\mu_{Rk,t} - r_{ft}\right) \\ \mu_{Rb,t} - r_{ft} \end{bmatrix}$$
$$- \begin{bmatrix} \theta_{ft}\sigma_{ka,t} & \sigma_{ba,t} \\ \theta_{ft}\sigma_{k\xi,t} & \sigma_{b\xi,t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \sigma_\xi \end{bmatrix}.$$

The household becomes constrained in its allocation to the fund sector when

$$\mu_{Rk,t} - r_{ft} \geq 2m \frac{\theta_{ft}\omega_{ft}}{1 - \omega_{ft} - \omega_t} \frac{\left(\sigma_{ka,t}\sigma_{b\xi,t} - \sigma_{k\xi,t}\sigma_{ba,t}\right)^2}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2} + \frac{\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2}\left(\mu_{Rb,t} - r_{ft}\right)$$
$$+ \frac{\left(\sigma_{ka,t}\sigma_{b\xi,t} - \sigma_{k\xi,t}\sigma_{ba,t}\right)\left(\sigma_{ba,t} - \sigma_{b\xi,t}\right)}{\sigma_{ba,t}^2 + \sigma_{b\xi,t}^2}\sigma_\xi.$$

Proof. Recall that the representative household solves

$$\max_{\substack{\pi_{kt},\pi_{bt} \\ c_t}} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t - \rho_h t} \log c_t dt\right],$$

subject to the dynamic budget constraint

$$\frac{dw_{ht}}{w_{ht}} = \pi_{kt}\theta_{ft}\left(dR_{kt} - r_{ft}dt\right) + \pi_{bt}\left(dR_{bt} - r_{ft}dt\right) + r_{ft}dt - \frac{c_t}{w_{ht}}dt,$$

the skin-in-the-game constraint

$$\pi_{kt} \leq m\frac{w_{ft}}{w_{ht}},$$

and no shorting constraints

$$\pi_{kt} \geq 0$$

$$\pi_{bt} \geq 0.$$

Notice that the representative household takes the skin-in-the-game constraint as given. In particular, denote the relative wealth of the specialist to be

$$\omega_{ft} = \frac{w_{ft}}{w_t + w_{ft} + w_{ht}}$$

and the relative wealth of the bankers to be

$$\omega_t = \frac{w_t}{w_t + w_{ft} + w_{ht}},$$

so that the constraint can be represented as

$$\pi_{kt} \leq m \frac{\omega_{ft}}{1 - \omega_{ft} - \omega_t}.$$

The representative household then takes $\omega_{ft}/(1-\omega_{ft}-\omega_t)$ as given in solving for the optimal consumption and portfolio allocation rules.
The time t convex set of admissible portfolio strategies is then $\mathcal{K}_t = \mathbb{R}_+ \times [0, m\omega_{ft}/(1 - \omega_{ft} - \omega_t)]$, with the support function

$$\begin{aligned}
\delta_t(x) = \delta_t(x|\, \mathcal{K}_t) &\equiv \sup_{\vec{\pi} \in \mathcal{K}_t} (-\vec{\pi}' x) \\
&= m \frac{\omega_{ft}}{1 - \omega_{ft} - \omega_t} x_1 \mathbf{1}_{x_1 \leq 0} + \infty x_2 \mathbf{1}_{x_2 \leq 0}.
\end{aligned}$$

For future use, introduce also the barrier cone $\tilde{\mathcal{K}}_t$ of $-\mathcal{K}_t$ to be

$$\tilde{\mathcal{K}}_t = \left\{ x \in \mathbb{R}^2 : \ x_2 \geq 0 \right\}.$$

We can then define an auxiliary unconstrained optimization problem for the household, with the returns in the auxiliary asset market defined as

$$\begin{aligned}
r_{ft}^v &= r_{ft} + \delta(\vec{v}_t) \\
dR_{kt}^v &= (\mu_{Rk,t} + v_{1t} + \delta(\vec{v}_t))\, dt + \sigma_{ka,t} dZ_{at} + \sigma_{k\xi,t} dZ_{\xi t} \\
dR_{bt}^v &= (\mu_{Rb,t} + v_{2t} + \delta(\vec{v}_t))\, dt + \sigma_{ba,t} dZ_{at} + \sigma_{b\xi,t} dZ_{\xi t},
\end{aligned}$$

for each $\vec{v}_t = [v_{1t}\ v_{2t}]'$ in the space $V(\mathcal{K})$ of square-integrable, progressively measurable processes taking values in $\mathcal{K}$. Corresponding to the auxiliary returns processes is an auxiliary state-price density

$$\frac{d\eta_t^v}{\eta_t^v} = -\left(r_{ft} + \delta(\vec{v}_t)\right) dt - \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t\right)' \left(\sigma_{Rt}'\right)^{-1} d\vec{Z}_t,$$

where

$$\vec{\mu}_{Rt} = \begin{bmatrix} \mu_{Rk,t} \\ \mu_{Rb,t} \end{bmatrix}; \quad \sigma_{Rt} = \begin{bmatrix} \sigma_{ka,t} & \sigma_{k\xi,t} \\ \sigma_{ba,t} & \sigma_{b\xi,t} \end{bmatrix}; \quad \vec{Z}_t = \begin{bmatrix} Z_{at} \\ Z_{\xi t} \end{bmatrix}.$$

The auxiliary unconstrained problem of the representative household then becomes

$$\max_{c_t} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\log c_t dt\right],$$

subject to the static budget constraint:

$$w_{h0} = \mathbb{E}\left[\int_0^{+\infty} \eta_t^v c_t dt\right].$$

The solution to the original constrained problem is then given by the solution to the unconstrained problem for the v that solves the dual problem

$$\min_{v\in V(\mathcal{K})} \mathbb{E}\left[\int_0^{+\infty} e^{-\xi_t-\rho_h t}\tilde{u}\left(\lambda\eta_t^v\right)dt\right],$$

where $\tilde{u}\left(x\right)$ is the convex conjugate of $-u\left(-x\right)$

$$\tilde{u}\left(x\right) \equiv \sup_{z>0}\left[\log\left(zx\right) - zx\right] = -\left(1+\log x\right)$$

and λ is the Lagrange multiplier of the static budget constraint. Extending the result of Cvitanić and Karatzas [1992] for logarithmic utility to the case of a random rate of time discounting, the optimal choice of v satisfies

$$\begin{aligned}\vec{v}_t^* &= \arg\min_{x\in\tilde{\mathcal{K}}_t}\left\{2\delta(x) + \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right) + \Sigma_\xi\right|\right|^2\right\} \\ &= \arg\min_{x\in\tilde{\mathcal{K}}_t}\left\{2m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t}x_1\mathbf{1}_{x_1\leq 0} + \left|\left|\sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft} + x\right) + \Sigma_\xi\right|\right|^2\right\},\end{aligned}$$

where

$$\Sigma_\xi = \begin{bmatrix} 0 \\ \sigma_\xi \end{bmatrix}.$$

Taking the first order conditions and solving for the variables of interest, we obtain

$$\begin{aligned} x_1 &= \begin{cases} 2m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t} - \sigma_{Rt}^{-1}\left(\vec{\mu}_{Rt} - r_{ft}\right), & \text{if } \mu_{Rk,t} - r_{ft} \geq 2m\frac{\omega_{ft}}{1-\omega_{ft}-\omega_t} \\ 0, & \text{otherwise} \end{cases}, \\ x_2 &= -\left(\mu_{Rb,t} - r_{ft}\right) + \frac{\sigma_{ka,t}\sigma_{ba,t} + \sigma_{k\xi,t}\sigma_{b\xi,t}}{\sigma_{ka,t}^2 + \sigma_{k\xi,t}^2}\left(\mu_{Rk,t} - r_{ft} + x_1\right).\end{aligned}$$

Consider now solving the auxiliary unconstrained problem. Taking the first order condition, we obtain

$$[c_t]: \quad 0 = \frac{e^{-\xi_t-\rho_h t}}{c_t} - \lambda\eta_t^v,$$

or

$$c_t = \frac{e^{-\xi_t - \rho_h t}}{\lambda \eta_t^v}.$$

Substituting into the static budget constraint, we obtain

$$\eta_t^v w_{ht} = \mathbb{E}_t \left[\int_t^{+\infty} \eta_s^v c_s ds \right] = \mathbb{E}_t \left[\int_t^{+\infty} \frac{e^{-\xi_s - \rho_h s}}{\lambda} ds \right] = \frac{e^{-\xi_t - \rho_h t}}{\lambda \left(\rho_h - \sigma_\xi^2 / 2 \right)}.$$

Thus

$$c_t = \left(\rho_h - \frac{\sigma_\xi^2}{2} \right) w_{ht}.$$

To solve for the household's optimal portfolio allocation, notice that:

$$\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = -\rho_h dt - d\xi_t + \frac{1}{2} d\xi_t^2 = \left(-\rho_h + \frac{1}{2}\sigma_\xi^2 \right) dt - \sigma_\xi dZ_{\xi t}.$$

On the other hand, applying Itô's lemma, we obtain

$$\frac{d\left(\eta_t^v w_{ht}\right)}{\eta_t^v w_{ht}} = \frac{d\eta_t^v}{\eta_t^v} + \frac{dw_{ht}}{w_{ht}} + \frac{dw_{ht}}{w_{ht}} \frac{d\eta_t^v}{\eta_t^v}.$$

Equating the coefficients on the stochastic terms, we obtain

$$\vec{\pi}_t' = \left(\vec{\mu}_{Rt} - r_{ft} + \vec{v}_t \right)' \left(\sigma_{Rt}' \sigma_{Rt} \right)^{-1} - \sigma_\xi \begin{bmatrix} 0 & 1 \end{bmatrix} \sigma_{Rt}^{-1}.$$

□

Thus, the households consume a constant fraction of their wealth, but at a slower rate than they would if they were not exposed to the liquidity preference shocks, ξ_t. The households' optimal allocation to capital has a intratemporal hedging component to compensate them for exposure to the liquidity shocks.

II.D. Equilibrium

Definition 2. *An equilibrium in the economy is a set of price processes* $\{p_{kt}, p_{bt}, r_{ft}\}_{t\geq 0}$*, a set of household decisions* $\{\pi_{kt}, b_{ht}, c_t\}_{t\geq 0}$*, a set of specialist decisions* $\{k_{ft}, c_{ft}\}_{t\geq 0}$*, and a set of intermediary decisions* $\{k_t, i_t, b_t, c_{bt}\}_{t\geq 0}$ *such that the following apply:*

1. *Taking the price processes, the specialist decisions and the intermediary decisions as given, the household's choices solve the household's optimization problem, subject to the household budget constraint, the no shorting constraints and the skin-in-the-game constraint for the funds.*

2. *Taking the price processes, the specialist decisions and the household decisions as given, the intermediary's choices solve the intermediary's optimization problem, subject to the intermediary budget constraint, and the regulatory constraint.*

3. *Taking the price processes, the household decisions and the intermediary decisions as given, the specialist's choices solve the specialist's optimization problem, subject to the specialist budget constraint.*

4. *The capital market clears at all dates*

$$k_t + k_{ft} = K_t.$$

5. *The risky bond market clears*

$$b_t = b_{ht}.$$

6. *The risk-free debt market clears*

$$w_t + w_{ft} + w_{ht} = p_{kt} A_t K_t.$$

7. *The goods market clears*

$$c_t + c_{bt} + c_{ft} + A_t k_t i_t = K_t A_t.$$

Notice that, in equilibrium, the stock of capital in the economy evolves as

$$dK_t = \left(\Phi\left(i_t\right) \frac{k_t}{K_t} - \lambda_k \right) K_t dt.$$

We solve for the equilibrium as a function of the three state variables, θ_t, ω_t and ω_{ft}. In particular, notice that we can express all the other equilibrium quantities in terms of the state variables and the sensitivities of the return to holding capital to output and liquidity shocks, $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$. We solve for these last two equilibrium quantities numerically as solutions to the system of equations that

1. Equates θ_t to the solution to the optimal portfolio allocation choice of the representative bank;

2. Equates the equilibrium risk-free rate to the risk-free rate given by the specialists' Euler equation.

The other equilibrium quantities can be expressed as follows.

1. Equilibrium price of capital, p_{kt}, (from goods market clearing) and optimal capital investment policy, i_t, (from bankers' optimization) as a function of the state variables only;

2. From capital market clearing, fund allocation to capital, θ_{ft}, as a function of the state variables only;

3. From debt market clearing, household allocation to debt, π_{bt}, as a function of state variables only;

4. From specialists' optimal capital allocation, expected excess return to holding capital, $\mu_{Rk,t} - r_{ft}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

5. From the equilibrium evolution of specialists' wealth, the sensitivities of the funds' wealth share in the economy, ω_{ft}, to output and liquidity shocks, $\sigma_{\omega_f a,t}$ and $\sigma_{\omega_f \xi,t}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

6. From the equilibrium evolution of the price of capital, the sensitivities of the bankers' leverage to output and liquidity shocks, $\sigma_{\theta a,t}$ and $\sigma_{\theta\xi,t}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

7. From the equilibrium evolution of bankers' wealth, the sensitivities of the return to holding risky debt to output and liquidity shocks, $\sigma_{ba,t}$ and $\sigma_{b\xi,t}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

8. From the households' optimal allocation to bank debt, expected excess return to holding capital, $\mu_{Rb,t} - r_{ft}$, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

9. From the households' optimal portfolio choice, household allocation to funds, π_{kt}, as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

10. From the equilibrium evolution of specialists' wealth, the expected growth rate of specialists' wealth share, $\mu_{\omega_f t}$ as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

11. From the equilibrium evolution of bankers' wealth, the expected growth rate of bankers' wealth share, $\mu_{\omega t}$ as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

12. From the equilibrium evolution of capital, the expected growth rate of banks' leverage, $\mu_{\theta t}$ as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$;

13. From the equilibrium drift of the price of capital, the risk-free rate r_{ft} as a function of the state variables and $\sigma_{ka,t}$ and $\sigma_{k\xi,t}$.

Consider now the equilibrium outcomes from the model. Table V reports the coefficients and the R^2 of the regression of broker-dealer leverage growth on lagged growth in implied volatility in the data (first column) and in the model. For the model, we report the mean, median, 5% and 95% realizations of the coefficients in a sample of 10000 paths. The paths are simulated at a monthly frequency for 70 years, using the parameters in Table I. As is the case for risk-neutral intermediaries, higher levels of return volatility (VIX in the data) tend to precede declines in broker-dealer leverage. Next, in Table VI, we report the results of regressing the growth rate of the share of intermediated credit on total credit growth, showing the strong positive relationship in the model and the data. Finally, we show the negative relationship between excess returns and lagged intermediary leverage growth in Table VII. Thus, the empirical regularities we demonstrate in the model with risk-neutral, constrained banks survive even when we introduce a second intermediary sector and allow banks to optimally choose their asset allocations subject to regulatory constraints.

Table V: Intermediary Leverage and Lagged Volatility Growth

	Data	Mean	5%	Median	95%
β_0	0.01	0.00	-0.01	0.00	0.00
β_1	-0.21	-1.88	-5.08	-1.54	-0.21
R^2	0.05	0.21	0.01	0.20	0.46

Notes: The relationship between the growth rate of leverage of financial institutions and the lagged growth rate of implied volatility with risk-averse intermediaries. The "Data" column reports the coefficients estimated using broker-dealer leverage growth as the dependent variable and the growth rate of the Chicago Board Options Exchange (CBOE) market volatility index (VIX) as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized growth rate of leverage, θ_t, of the intermediaries as the dependent variable, and growth rate of total volatility of the return on capital, $\sqrt{\sigma^2_{ka,t} + \sigma^2_{k\xi,t}}$, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on broker-dealer leverage comes from Flow of Funds Table L.129.

Table VI: Procyclicality of Intermediated Credit

	Data	Mean	5%	Median	95%
β_0	-0.071	-0.002	-0.017	0.000	0.012
β_1	0.756	0.013	0.01	0.022	0.097
R^2	0.460	0.014	0.000	0.003	0.063

Notes: The relationship between total credit in the economy and the amount of credit extended through the financial intermediary sector with risk-averse intermediaries. The "Data" column reports the coefficients estimated using the growth rate of credit extended by financial intermediaries to the non-financial corporate sector as the dependent variable, and the growth rate of total credit to the non-financial corporate sector as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized growth rate of capital held by intermediaries, k_t, as the dependent variable, and the growth rate of total capital in the economy, K_t, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on total credit to the nonfinancial corporate sector and the share of intermediated finance come from Flow of Funds Table L.102. Data on broker-dealer leverage, equity, and assets come from Flow of Funds Table L.129.

Table VII: Excess Returns and Intermediary Leverage

	Data	Mean	5%	Median	95%
β_0	0.118	0.044	0.025	0.045	0.062
β_1	-0.310	-0.058	-0.224	-0.030	-0.003
R^2	0.167	0.024	0.001	0.019	0.065

NOTES: The relationship between excess returns and lagged broker-dealer leverage growth with risk-averse intermediaries. The "Data" column reports the coefficients estimated using the quarterly return to holding the S&P Financial Index as the dependent variable, and lagged annual broker-dealer leverage growth as the explanatory variable. The "Mean", 5%, "Median" and 95% columns refer to moments of the distribution of coefficients estimated using 10000 simulated paths, with realized quarterly excess return to holding capital, dR_{kt} as the dependent variable, and lagged annual intermediary leverage growth, $d\theta_t$, as the explanatory variable. β_0 is the constant in the estimated regression, β_1 is the loading on the explanatory variable, and R^2 is the percent variance explained. Data on broker-dealer leverage come from Flow of Funds Table L.129 and that on the return to the S&P Financial Index from Haver Analytics and Barclays.