Moral Hazard and Long-Run Incentives*

Preliminary and Incomplete

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Abstract

This paper considers dynamic moral hazard settings, in which the agent's actions have consequences over a long horizon. The agent's limited liability constraints makes it difficult to tie the agent's compensation to long-run outcomes. To maintain incentives, the optimal contract delays the agent's compensation and ties it to future performance. Some of the agent's compensation is deferred past termination, which is triggered when the value of deferred compensation drops to a specific, strictly positive threshold. The agent's pay-performance sensitivity, and therefore risk exposure, build up towards a target level during employment, and decrease after termination.

1 Introduction.

This paper studies dynamic agency problems, where the impact of the agent's actions is observed with delay. These situations are common in practice. CEO's actions do not have immediate impact on firm profitability, but rather they affect what happens to the firm over a long horizon. The success of private equity funds is not fully revealed until they sell their illiquid investments.

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The quality of mortgages given by a broker is not known until several years down the road. There has been a lot of informal discussion of issues that arise in these settings. Deferred compensation and clawback provisions come up frequently. However, it has been difficult to design a formal tractable framework to analyze these issues. This paper puts forth one such framework.

The main casual intuition about these situations is that the agent has to be exposed to the risk of the project's long-term performance. However, the agent's limited liability is one major challenge to designing incentives. Due to limited liability, natural incentive mechanisms look like options, which reward the agent for the upside but bound the agent's responsibility for downside. It's a tails I win, heads you lose situation. The stronger incentives on the margin, the sooner is the limited liability constraint hit. This raises many questions. What is the best way to design incentives near the constraint? When is it optimal to fire the agent? Should some of the agent's pay remain deferred after the agent is fired? Is there any danger that, due to limited liability, the agent tries to extract as many private benefits from the project as possible, before he is fired?

This paper develops a model to tackle these issues. We build a dynamic agency model, in which the agent continuously puts unobservable effort until he may be fired, and continuously consumes a compensation flow. Current effort affects observable output over the entire future, with gradually diminishing impact. We derive the optimal contract among all history-dependent contracts. That is, the agent's compensation and termination of employment can depend on the history of output in an arbitrary way, and some compensation may be paid out after termination.

The optimal contract has interesting features. First, the agent's pay-performance sensitivity generally increases with tenure. That is, as the horizon over which the agent could affect the project increases, the optimal contract exposes the agent to more project risk. Second, the agent's pay-performance sensitivity decreases after bad performance, due to limited liability. Eventually, the agent may be fired. Third, the agent has unvested compensation even when he is fired. This compensation is tied to performance after termination, and is paid out gradually.

The following example, for the application of CEO compensation, illustrates the qualitative features of the optimal contract. Consider a contract that

- during employment, per quarter, grants the CEO \$1 million of incentive compensation, consisting of a \$3 million stock and a \$2 million loan, into a special account until vesting
- terminates the CEO if the value of the stock in the account ever drops below 80% of the value of the loan
- both before and after termination, allows 10% of the shares in the incentive account to be sold per quarter; uses the proceeds to pay down up to 10% of the outstanding loan, with any remaining money paid out to the CEO

This contract uses deferred compensation, which consists of a levered stake in the company. The CEO achieves a required exposure to firm performance by investing his own compensation, as well as money borrowed from the company, in company stock. During employment, more unvested compensation is added to the incentive account, increasing the CEO's exposure to company performance. The agent is allowed to go underwater a bit, but large enough drops in the stock price lead to termination. Compensation vests gradually, even after termination. The vesting rate of 10% is set to reflect how quickly company performance becomes less informative about the CEO's effort impact in the past.[1] Even near the termination threshold, a significant probability that company stock recovers prevents the agents incentives from deteriorating completely. A call option 20% out of the money with expiration in 3 years still has Δ of about 0.5 when the stock price volatility is 30%.

Our formal analysis assumes that the impact of the agent's effort on future output is exponentially decaying, at rate κ, and allows for arbitrary history-dependent contracts. It is convenient to compare an arbitrary contract to an exotic option, with the underlying being the relevant performance measure, such as the stock price. In our analysis, we measure the payouts of the derivative in the units of the agent's utility rather than money, to account for the agents risk preferences and the cost of effort. The option pays at multiple dates in the future. The value of the of the option W_t is the agent's future expected utility from the contract, taking into account

[1]The 10% vesting rate in the example is roughly consistent with the rate at which firms abnormal earnings are thought to revert to the mean on average, see Fama and French (2000), and leads to an average sale time of about 3 years for the shares remaining in the account at the time of termination.

how payouts and termination time depend on the underlying. Option Δ, the sensitivity of value to the underlying, is related to the agent's incentives on the margin. However, unlike in standard models where the agent's effort affects only concurrent performance measures, in our model not only current but also future Δ's affect the agent's incentives on the margin. In the CEO compensation example above, $3 million worth of stock granted to the agent in two years (conditional on survival), vesting on average after three more years, add to the agent's incentives at time 0. The agent takes into account the probability that he is still employed and receives those shares in two years, as well as his ability to affect firm performance after two years with current effort. Formally, we show that on the margin the agent's incentives at time t are defined by

$$\Phi_t = E_t \left[\int_0^\infty e^{-(r+\kappa)(t+s)} \Delta_{t+s} \, ds \right], \tag{1}$$

where r is the agent's discount rate and Δ_{t+s} is the sensitivity "option value" to the underlying at time $t+s$. Quantity Φ_t plays an important role in optimal contract design. Pay-performance sensitivity Δ_t is set taking into account its impact on the agent's incentives over the time interval $[0, t]$, as well as the costs due to the agent's risk aversion and the possibility of inefficient termination.

While Φ_t affects the agent's incentives on the margin, we are also concerned about the agent's incentives to take fundamentally different effort strategies. For example, if the agent puts low effort for several periods, he would expect the firm to perform poorly in the future. If Δ of the agent's contract decreases with the price of the underlying, like Δ of a call option, the agent's incentives to put effort in the future deteriorate. Thus, low effort in the past reinforces the agent's incentives to lower effort, making it well possible that the agent's optimal effort strategy involves putting low or no effort throughout. After all, the agent's liability for losses is bounded, and he may be able to collect significant benefits before he is fired.

If the agent has right incentives on the margin, is there a way to guarantee that the agent cannot benefit by deviating significantly? We show that one does not have to worry about large deviations under the condition that Φ_t does not change too fast with changes in the underlying. That is, the agent's incentives to undertake significant deviations depend on the expected Γs of his contract, i.e. the rate at which Δ_t changes with the underlying. Intuitively, on one hand, even if the agent faces strong incentives *on the margin*, i.e. Δ_t are large, he will not have incentives to put effort if his exposure to losses is severely limited. In this situation, Δ would have to quickly drop to 0 after losses, i.e. Γ is high. On the other hand, large deviations are less troubling in the CEO compensation example above if shares granted to the CEO have a long vesting horizon. Similarly, compensation contracts based on longer-dated options have lower Γ.

This paper is organized as follows. In Section 2 we lay out a basic model, in which the impact of the agent's effort on future output has an exponential decay rate κ. As $\kappa \to \infty$, the model is reduced to a more standard model analyzed in Sannikov (2008). In Section 3 we analyze the agent's incentives on the margin, and provide a condition when first-order incentive constraints guarantee the optimality of the agent's effort. With these results, we are also able to transform the principal's relaxed problem, maximizing profit subject to just the first-order incentive constraints - to an optimal stochastic control problem. In Section 4, we characterize the solution of the relaxed problem using the Lagrangian approach to stochastic control, which is related to the work of Yong and Zhou (1999). The optimal contract is characterized by the multipliers ν_t on the agent's utility and λ_t on the agent's incentives. While λ_t explicitly determines the evolution of ν_t and the agent's compensation, as well as the sensitivity of compensation to output, the evolution of λ_t depends on the agent's implicit effort.

In Sections 5 and 6, we introduce an additional assumption, which significantly simplifies the form of the optimal contract and guarantees that the validity of our first-order approach. Specifically, we assume that the signal about the agent's effort contains significant noise. Even when the signal about the agent's effort is noisy, pay for performance is beneficial if the agent's effort can have a significant effect on value. We argue that these assumptions fit particularly well the application of CEO compensation, as it is difficult to filter out the impact of CEO effort on firm value due to volatility. In this environment, in Section 5 we provide explicit formulas that determine how the agent's pay depend on output, given a termination rule. The problem of determining optimal termination time itself boils down to an optimal exercise time of a real option. Specifically, in the optimal contract the agent's pay c_t maximizes

$$\nu_t u(c) - c,$$

where ν_t is a martingale with sensitivity to performance given by $const \cdot \hat{\lambda}_t$, $\hat{\lambda}_t = 1 - e^{-(r+\kappa)t}$ until the termination time τ, and $\hat{\lambda}_t = e^{-(r+\kappa)(\tau-t)}\hat{\lambda}_\tau$ after time τ. We see that the contract is extremely simple. In Section 6, we adapt our characterization of the optimal contract to the application of CEO compensation, in which firm scale follows a geometric Brownian motion.

Literature Review. This paper is related to the strands on literature on agency models and executive compensation. Papers such as Radner (1985), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990) and Phelan and Townsend (1991) provide foundations for the analysis if repeated principal-agent interactions. In these settings, in each period the agent's effort affects the probability distribution of a signal observed by the principal, and the optimal contract can be presented in a recursive form. That is, in these settings the agent's continuation value completely summarizes the agent's incentives. Using the recursive structure, Sannikov (2008) provides a continuous-time model of repeated agency, in which it is possible to explicitly characterize the optimal contract using an ordinary differential equation.

The problem addressed in this paper is non-standard, as the agent's action can have future consequences. That is, the agent's current effort affects firm's unobservable fundamentals, which have impact on cash flows over the long run. Therefore, to summarize incentives in our setting, one also has to keep track of the derivative of the agent's payoff with respect to fundamentals. This leads to the so-called first-order approach, which has been used recently to analyze a number of environments. Kapicka (2011) and Williams (2011) use the first-order approach in environments where the agent has private information. DeMarzo and Sannikov (2011) and He, Wei and Yu (2012) study environments with learning, where the agent's actions can affect the principal's belief about fundamentals. While we also use the first-order approach, we also provide a sufficient conditions for its validity. That is, we characterize a class of contracts, for which first-order conditions guarantee full optimality.

There are not many papers where the impact of the agent's effort is observed with delay. Hopenhayn and Jarque (2010) consider a setting where the agent's one-time effort input affects output over a long horizon. See also Jarque (2011). Edmans, Gabaix, Sadzik and Sannikov (2012) consider a scale-invariant setting where the agent can manipulate performance over a limited time horizon and do not allow for termination. In contrast, this paper considers a fairly general framework.

One especially attractive feature of this paper is the closed-form characterization of the optimal contract in environments with large noise. Such a clean characterization is rare in contracting environments. Holmstrom and Milgrom (1987) derive a linear contract for a very particular model with exponential utility. Edmans, Gabaix, Sadzik and Sannikov (2012) obtain a tractable contract in a scale-invariant setting. In contrast, we consider a setting that allows for general utility function and for termination.

This paper is also related to literature on managerial compensation. The model predicts that the agent's pay-performance sensitivity under the optimal contract increases gradually during employment. This is consistent with empirical evidence documented by Gibbons and Murphy (1992). At the same time, the model also suggests that some of CEO's compensation should be deferred after termination, a feature observed rarely in practice. DeMarzo and Sannikov (2006) and Biais, Mariotti, Plantin and Rochet (2007) study managerial compensation in the optimal contracting framework with a risk-neutral agent, but allow the agent's actions to have only contemporaneous effect on cash flows. In these settings, it is also optimal to defer some of the agent's compensation, but only until the time of termination. Deferred compensation creates more room to punish the agent in the future in case of bad performance. Backloaded compensation also helps employee retention, a point first made by Lazear (1979). Edmans, Gabaix, Sadzik and Sannikov (2012) study executive compensation with risk aversion, and derive an optimal contract that features rebalancing. If the firm's stock price goes down, the optimal contract replaces some cash in the agent's deferred compensation package with stock. This feature solves a common problem of call options, which lose their incentive power when they fall out of money as the stock price drops. This feature is also present in our setting, if CEO effort is easier to observe in smaller firms. However, because of limited liability, the agent's absolute pay-performance sensitivity falls after bad performance, leading to termination.

The assumption that the agent's effort is difficult to detect leads to significant tractability in this paper. How much value does CEO effort create, relative to the volatility of firm value? Regarding not effort but skill, Gabaix and Landier (2008) estimate that the best CEO is able to add 0.16% more to firm value than the 250th best CEO. This is certainly not a small number in money terms, \$16 million for a \$10 billion firm, but it is virtually undetectable in the face of stock price volatility. This observation suggests that performance evaluation is extremely noisy in the context of CEO compensation.

Relative to existing literature on CEO compensation, this paper has two main advantages: *tractability*, as compensation in the optimal contract is determined explicitly and *generality*, as we can allow for an arbitrary utility function, and for termination.

2 The Model.

Consider a continuous-time environment, in which the agent is employed from time 0 until a contractually determined termination time $\tau \leq \infty$. Until time τ, the agent affects the firm's fundamentals with his effort $a_t \in [0, \infty)$. The principal observes neither the agent's effort nor fundamentals. Fundamentals evolve according to

$$d\delta_t = (a_t - \kappa \delta_t)\, dt, \quad \text{or equivalently} \quad \delta_t = e^{-\kappa t}\delta_0 + \int_0^t e^{-\kappa(t-s)} a_s\, ds, \tag{2}$$

where δ_0 is common knowledge. Fundamentals affect the output according to

$$dX_t = (r + \kappa)\delta_t\, dt + \sigma\, dZ_t, \tag{3}$$

where Z_t is a Brownian motion. From (2) and (3), the effect of effort on future output is exponentially decaying, and effort a_t creates value at time $s \geq t$ rate

$$e^{-\kappa(s-t)}(r + \kappa)a_t \tag{4}$$

for the risk-neutral principal, who discounts cash flows at rate r. Thus, effort a_t creates value at rate

$$\int_t^\infty e^{-r(s-t)} e^{-\kappa(s-t)}(r + \kappa)a_t\, ds = a_t.$$

Note that in this parameterization the value (4) that the agent creates from effort is independent of κ.

The agent's utility cost of effort is given by an increasing convex function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$. The agent also gets utility from payments he receives from the principal. His utility of consumption function $u : [0, \infty) \to [0, \infty)$ is increasing and concave, and satisfies $u(0) = 0$.

We consider fully history-dependent contracts, which specify a possible termination time τ as well as the agent's compensation c_t as functions of the entire past history of output $\{X_s, s \in [0, t]\}$. Importantly, the agent receives compensation even after time τ, since his effort before time τ affects output after time τ. The decision to fire the agent is irreversible. The principal can commit to any fully contingent contract. In the event that the agent is fired, the principal's outside option at time τ is given by

$$\delta_\tau + L.$$

This value could be determined endogenously as the payoff from hiring a new agent. Importantly, $L > 0$, so that the principal prefers to fire the agent if he cannot motivate him to put sufficient effort.

In response to a contract (c, τ), the agent will choose a strategy $a = \{a_t, 0 \leq t \leq \tau\}$ to maximize his payoff. The principal's payoff depends on the strategy that the agent chooses.

The optimal contract in this setting solves the following constrained optimization problem. The objective is to maximize the principal's expected profit

$$E^a \left[\int_0^\infty e^{-rt} \left(dX_t - c_t \, dt \right) \right] = \tag{5}$$

$$\delta_0 + E^a \left[\int_0^\tau e^{-rt} a_t \, dt + e^{-r\tau} L - \int_0^\infty e^{-rt} c_t \, dt \right],$$

where E^a denotes the expectation given the agent's strategy a. The constraints are

$$W_0 = E^a \left[\int_0^\infty e^{-rt} (u(c_t) - 1_{t \leq \tau} \, h(a_t)) \, dt \right], \tag{6}$$

where W_0 is the agent's required utility at time 0, and the *incentive constraints*

$$W_0 \geq E^{\hat{a}} \left[\int_0^\infty e^{-rt} (u(c_t) - 1_{t \leq \tau} \, h(\hat{a}_t)) \, dt \right] \tag{7}$$

for all alternative strategies $\hat{a}$.

Solving this problem involves finding not only the optimal contract (c, τ) but also the agent's optimal strategy a under this contract, since the strategy enters both the objective function and the constraints.

If we let $\kappa \to \infty$, then this model converges to a standard principal-agent model, in which the agent's effort adds only to concurrent output. Specifically, in this case the output is given by

$$dX_t = a_t \, dt + \sigma \, dZ_t, \tag{8}$$

instead of (3).

3 Incentive Constraints.

In this section, for an arbitrary contract we analyze the agent's incentives. Proposition 1 investigates the agent's incentives on the margin and obtains first-order conditions for when the agent's strategy is optimal. Proposition 3 identifies conditions when a strategy that satisfies the first-order incentive constraints ends up being optimal among all strategies.

The results of this section allow us to conjecture the optimal contract via the first-order approach. The first-order approach replaces the full set of incentive constraints (7) with weaker first-order conditions for the agent's effort choice. This leads to a relaxation of the constrained optimization problem (5), which can be solved as an optimal stochastic control problem using Proposition 2. Then Proposition 3 allows us to verify when the conjectured contract satisfies all the incentive constraints.

Before presenting formal results, let us summarize the key findings of this section and draw parallels to the standard principal-agent problem, in which the agent's effort affects only concurrent output according to (8). For an arbitrary contract, a particularly important variable is the agent's continuation value defined according to

$$W_t = E_t^a \left[\int_t^\infty e^{-r(s-t)} (u(c_s) - 1_{s \le \tau}\, h(a_s))\, ds \right]. \tag{9}$$

The constraint (6), which is related to individual rationality, requires that the value of the contract to the agent at time 0 takes a specific value W_0. Furthermore, in the standard principal-agent setting, the marginal benefit to the agent of putting extra effort is determined by the sensitivity of W_t to output

$$dW_t/dX_t = \Delta_t.$$

The notation Δ is intentional to emphasize the analogy to option Delta: the sensitivity of option value to changes in the underlying X.

When effort a_t has an exponentially decaying impact on all future output, characterized by (2) and (3), then the marginal benefit of effort at time t depends on

$$\Phi_t = dW_t/d\delta_t = E_t^a \left[(r+\kappa) \int_t^\infty e^{-(r+\kappa)(s-t)} \Delta_s\, ds \right]. \tag{10}$$

Note that as $\kappa \to \infty$, $\Phi_t \to \Delta_t$ if Δ_t is continuous. Since the marginal cost of effort is given by $h'(a_t)$, the first-order condition for optimal effort choice is

$$h'(a_t) = \Phi_t.$$

There is an alternative expression for Φ_t, which is very useful for our analysis. Note that the value of fundamentals δ_t at time t affects the probability distribution over future paths of output. Girsanov Theorem implies that the rate at which fundamentals δ_t affect the probability distribution over future paths is given by

$$\zeta_t^{t+s} = \int_t^{t+s} e^{-\kappa(s'-t)} \frac{r+\kappa}{\sigma} \frac{dX_{s'} - (r+\kappa)\delta_{s'}\, ds'}{\sigma}. \tag{11}$$

Note that as δ_t goes up, the paths $\{X_{s'},\ s' \in [t,s]\}$, for which $dX_{s'}$ exceeds expectation $(r+\kappa)\delta_{s'}\, ds'$, become more likely.

It follows that

$$\Phi_t = dW_t/d\delta_t = E_t^a \left[\int_t^\infty e^{-r(s-t)} \zeta_t^s (u(c_s) - 1_{s \le \tau}\, h(a_s))\, ds \right]. \tag{12}$$

The First-order Incentive Constraints. The following proposition identifies the first-order conditions for a strategy a to be optimal under a contract (c, τ).

Proposition 1 *Fix a strategy a. A necessary condition for a to be optimal under the contract (c, τ) is that for all t,*

$$a_t \ \textit{maximizes} \ \Phi_t a - h(a), \tag{13}$$

where Φ_t is defined by (12).

From now on, denote by $a(\Phi_t)$ the effort that solves the problem (13). Under our assumptions, $a(h'(a)) = a$ for $a \geq 0$, and $a(\Phi) = 0$ for $\Phi \leq h'(0)$.

Proof. To identify the first-order incentive-compatibility constraint, consider a deviation away from the strategy a towards an alternative strategy $\hat{a}$. Formally, for $\phi \in [0,1]$, let the strategy $(1-\phi)a + \phi\hat{a}$ assign effort $(1-\phi)a_t + \phi\hat{a}_t$ to each history of output $\{X_s,\ s \in [0,t]\}$. Then, if the strategies a and $\hat{a}$ generate the paths of fundamentals δ and $\hat{\delta}$, the strategy $(1-\phi)a + \phi\hat{a}$ leads to the path of fundamentals $(1-\phi)\delta + \phi\hat{\delta}$.

If the agent follows the strategy $(1-\phi)a + \phi\hat{a}$ rather than a, then by Girsanov's Theorem, he changes the underlying probability measure by the relative density process $\xi_t(\phi)$ given by

$$\exp\left(-\frac{1}{2}\int_0^t \frac{\phi^2(r+\kappa)^2(\hat{\delta}_s-\delta_s)^2}{\sigma^2}\,ds + \int_0^t \frac{\phi(r+\kappa)(\hat{\delta}_s-\delta_s)}{\sigma}\,\frac{dX_s-(r+\kappa)\delta_s\,ds}{\sigma}\right),$$

where $\frac{dX_s-(r+\kappa)\delta_s\,ds}{\sigma}$ represents increments of a Brownian motion under the original measure, under the strategy a, and $\frac{\phi(r+\kappa)(\hat{\delta}_s-\delta_s)}{\sigma}$ is the rate at which the agent's deviation changes the drift of this Brownian motion.

The agent's utility from deviating to the strategy $(1-\phi)a + \phi\hat{a}$ is

$$E^a\left[\int_0^\infty e^{-rt}\xi_t(\phi)(u(c_t) - 1_{t\le\tau}\,h((1-\phi)a_t + \phi\hat{a}_t))\,dt\right]. \tag{14}$$

We would like to differentiate this expression with respect to ϕ at $\phi = 0$. Note that

$$\left.\frac{d\xi_t(\phi)}{d\phi}\right|_{\phi=0} = \int_0^t \frac{(r+\kappa)(\hat{\delta}_s-\delta_s)}{\sigma}\,\frac{dX_s-(r+\kappa)\delta_s\,ds}{\sigma}.$$

Since $\hat{\delta}_s - \delta_s = \int_0^s e^{-\kappa(s-s')}(\hat{a}_{s'} - a_{s'})ds'$, after changing the order of integration and using the definition (11) of ζ_s^t, we obtain

$$\left.\frac{d\xi_t(\phi)}{d\phi}\right|_{\phi=0} = \int_0^t (\hat{a}_s - a_s)\zeta_s^t\,ds. \tag{15}$$

From (14) it follows that the derivative of the agent's expected utility with respect to ϕ is

$$E^a\left[\int_0^\infty e^{-rt}\left(\left.\frac{d\xi_t(\phi)}{d\phi}\right|_{\phi=0}\right)(u(c_t) - 1_{t\le\tau}\,h(a_t))\;dt - \int_0^\tau e^{-rt}(\hat{a}_t - a_t)h'(a_t)\,dt\right].$$

Plugging in (15), and changing the order of integration, this becomes

$$E^a\left[\int_0^\tau e^{-rt}(\hat{a}_t - a_t)\left(\int_t^\infty e^{-r(s-t)}\zeta_t^s\,(u(c_s) - 1_{s\le\tau}\,h(a_s))\;ds - h'(a_t)\right)\;dt\right] =$$

$$E^a\left[\int_0^\tau e^{-rt}(\hat{a}_t - a_t)(\Phi_t - h'(a_t))\,dt\right],$$

where Φ_t is defined by (12).

If the strategy a does not satisfy (13) on a set of positive measure, then let us choose $\hat{a}_t > a_t$ when $\Phi_t > h'(a_t)$ and $\hat{a}_t < a_t$ when $\Phi_t < h'(a_t)$, whenever possible to do this with $\hat{a}_t \geq 0$. Then

$$E^a \left[\int_0^\tau e^{-rt}(\hat{a}_t - a_t)(\Phi_t - h'(a_t))\, dt \right] > 0,$$

and so a deviation to the strategy $(1-\phi)a + \phi\hat{a}$ for sufficiently small ϕ is profitable. ■

Recursive Representation of the Variables (W_t, Φ_t). Next, we derive the stochastic laws of motion that variables W_t and Φ_t have to follow given the contract and the agent's effort strategy. Proposition 2 below provides an if-and-only-if statement: if processes W_t and Φ_t follow (16) and (17), then they satisfy definitions (9) and (12) (or, equivalently, (10)).

Proposition 2 is useful, as it allows us to reduce the principal's problem (5) to an optimal stochastic control problem.[2]

Proposition 2 *Fix a contract* (c, τ) *and a strategy* a*. Then the processes* (W_t, Φ_t) *defined by (9) and (12) follow*

$$dW_t = (rW_t - u(c_t) + 1_{t\leq\tau}\, h(a_t))\, dt + \Delta_t \underbrace{(dX_t - (r+\kappa)\delta_t\, dt)}_{\sigma\, dZ_t} \quad \textit{and} \tag{16}$$

$$d\Phi_t = (r+\kappa)\left(\Phi_t - \Delta_t\right)\, dt + \Gamma_t(dX_t - (r+\kappa)\delta_t\, dt) \tag{17}$$

for some processes Δ *and* Γ *in* L^2.

Conversely, any process W_t *that follows (16) and satisfies the transversality condition* $E_t^a[e^{-rs}W_s] \to 0$ *as* $s \to \infty$ *is the agent's continuation value defined by (9). Likewise, if also the process* Φ_t *follows (17) and the transversality condition* $E_t^a[e^{-(r+\kappa)s}\Phi_s] \to 0$ *as* $s \to \infty$*, then* Φ_t *satisfies (12).*

Proposition 2 has two corollaries. First, the intuitive formula (10) follows from the representation (17).

[2]The representation of Proposition 2 holds for a given contract (c, τ) and effort strategy a regardless of whether the incentive constraints hold.

Corollary 1

$$\underbrace{E_t^a\left[\int_t^\infty e^{-r(s-t)}\zeta_t^s(u(c_s)-1_{s\le\tau}\,h(a_s))\,ds\right]}_{\Phi_t}=E_t^a\left[(r+\kappa)\int_t^\infty e^{-(r+\kappa)(s-t)}\Delta_s\,ds\right].$$

Reducing the Relaxed Problem to Optimal Stochastic Control. The second corollary of Proposition 2 is Theorem 1 below, which allows us to solve the following relaxed problem using optimal stochastic control. Formally, the relaxed problem is to choose a contract (c,τ) together with the agent's strategy a that maximizes

$$E^a\left[\int_0^\tau e^{-rt}a_t\,dt+e^{-r\tau}L-\int_0^\infty e^{-rt}c_t\,dt\right], \tag{18}$$

subject to

$$W_0=E^a\left[\int_0^\infty e^{-rt}(u(c_t)\,dt-1_{t\le\tau}\,h(a_t))\,dt\right]\text{ and }\; a_t=a(\Phi_t)\text{ for all }t\le\tau \tag{19}$$

Theorem 1 *There is a one-to-one correspondence between contracts (c,τ), with effort strategies a, and pairs of controlled processes (W,Φ) that follow (16) and (17) and satisfy the transversality conditions under controls (c,a,Δ,Γ,τ). The controls satisfy the first-order incentive constraints if $a_t=a(\Phi_t)$ for all $t\le\tau$, and the principal's objective is given by (18).*

Proof. Consider a contract (c,τ) and an effort strategy a that satisfy (19), under which both $e^{-rt}(u(c_t)-1_{t\le\tau}h(a_t))$ and $e^{-rt}\zeta_0^t(u(c_t)-1_{t\le\tau}h(a_t))$ are integrable (so the transversality conditions hold). Then, by Proposition 2, processes (W,Φ) defined by (9) and (12) follow the laws of motion (16) and (17) for some processes (Δ,Γ) in L^2.

Conversely, if (W,Φ) follow (16) and (17) under controls (c,a,Δ,Γ,τ), and the transversality conditions hold, then by Proposition 2, (9) and (12) must hold. ■

First-Order Constraints and Full Incentive Compatibility. Of course, the first-order approach is valid only if the solution to the relaxed problem actually ends up satisfying all the incentive constraints (7). The following proposition provides conditions under which a contract, which satisfies the relaxed constraints (13), is fully incentive-compatible, i.e. satisfies (7).

Proposition 3 *Suppose that the agent's cost of effort is quadratic of the form $h(a) = \theta a^2/2$. Then an effort strategy a under the contract (c, τ) satisfies (7) if it satisfies (13) and also*

$$\Gamma_t \leq \frac{\theta(2\kappa + r)^2}{8(r + \kappa)}. \tag{20}$$

Condition (20) is a bound on the rate Γ_t at which the marginal benefit of effort Φ_t changes with output X_t. If Γ_t is large, then following a reduction in effort, the agent can benefit by lowering effort further as he faces a lower Φ_t.

Note the analogy with options. If the agent's contract is a package of call options on X_t, with downside protection, then the agent's incentives to lower effort depend on how much he can lose. The less downside the agent faces, the quicker the Deltas of the agent's options have to fall as losses occur, i.e. the Gammas of the agent's options are higher.

Several facts are worth noting. First, as $\kappa \to \infty$, the right hand side of (20) becomes infinite. That is, in the standard setting where the agent's effort affects only current output, multi-period deviations are not a concern, as we know from Sannikov (2008). Second, the right hand side of (20) depends on the convexity of the agent's cost of effort θ. If the cost of effort is linear, then the first-order approach fails, as pointed out by Kocherlakota (2004). Third, if the contract derived through a first-order approach (contract A) ends up violating condition (20), it is possible to derive a contract that satisfies the first-order approach (contract B) by imposing (20) as an additional constraint. Then contract A gives an *upper* bound on the profit from an optimal contract, and contract B gives a lower bound. If the two bounds are close, then contract B is approximately optimal.[3]

[3]One may wonder how tight is the bound (20). I conjecture that it is reasonably tight, that is if $\Gamma_t > C$ permanently, for some constant C that may be slightly larger than the right hand side of (20), then the contract is not fully incentive compatible.

4 A Lagrangian Approach to the General Problem.

In this section we develop a Lagrangian approach to the relaxed problem (18) and derive a partial characterization of the optimal contract. The contract can be written in terms of multipliers ν_t and λ_t on the agent's continuation value W_t and his cost of effort Φ_t respectively. The multiplier on W_t maps one-to-one into the agent's compensation, according to the equation[4]

$$c_t = \arg\max_c \ \nu_t u(c) - c. \tag{21}$$

The starting point ν_0 is determined by W_0, and the law of motion of ν_t is given by

$$d\nu_t = \lambda_t \frac{r+\kappa}{\sigma} \frac{dX_t - (\kappa + r)\delta_t\, dt}{\sigma} \qquad \text{with} \tag{22}$$
$$d\lambda_t = 1_{t\leq\tau}\, a'(\Phi_t)(1 - h'(a(\Phi_t))\nu_t)\, dt - \kappa\lambda_t\, dt, \quad \lambda_0 = 0.$$

This characterization is completely explicit about how the agent's compensation depends on the history of shocks, as long as the agent's effort $a(\Phi_t)$ is known for any pair (ν_t, λ_t). Of course, the effort itself depends on the contract, so in this sense this is only a partial characterization. It is possible to find effort as a function of (ν_t, λ_t) by solving a partial differential equation, but that procedure is fairly complicated.

Nevertheless, this partial characterization is immensely informative about the form of the optimal contract. First, since (21) implies that $1/u'(c_t) = \nu_t$, the agent's inverse marginal utility is a martingale. This is the well-known inverse Euler equation (e.g. see Spear and Srivastava (1987)), which characterizes the optimal way to smooth the agent's consumption in environments of dynamic moral hazard. Second, the multiplier λ_t, which determines the sensitivity of ν_t to output, is rising from zero towards a target level, which depends on ν_t. The target level, determined by

$$\bar{\lambda}(\nu_t) = \frac{a'(\Phi_t)}{\kappa}(1 - h'(a(\Phi_t))\nu_t).$$

Keeping $a(\Phi_t)$ fixed, this expression is decreasing in ν_t: as the agent's compensation increases, it gets costlier to compensate him for the cost of effort. After time τ, the multiplier λ_t decays exponentially.

[4]The solution to (21) is $c_t = 0$ if $\nu_t \leq 1/u'(0)$, and is determined by the first-order condition $\nu_t u'(c) = 1$ otherwise.

It is instructive that as $\kappa \to \infty$, the characterization (21)-(22) generates an optimal contract in the standard setting, in which the agent's effort affects only current output. Interestingly, this leads to a different method to find an optimal contract in the setting of Sannikov (2008).

Equation (22) does not explain how the termination time τ is determined. The problem of finding τ is similar to that of an optimal exercise time of an American option. We describe how both τ and effort $a(\Phi_t)$ are determined later in this section, when we present the formal results.

We derive characterization (21)-(22) by backward induction in two steps. First, we address the problem of optimal contract design after time τ. Second, we characterize the optimal contract in the region of employment, together with the optimal choice of termination.

The optimal contract after termination. We allow some of agent's compensation to be paid out after termination. The form of this deferred compensation influences the agent's incentives during employment. A contract that solves the relaxed problem (18) has to give the agent the desired continuation value W_τ and marginal benefit of effort Φ_τ at time τ in the cheapest possible way. Indeed, if we replace the continuation contract after time τ with another contract that has the same values of Φ_τ and W_τ, the agent's marginal incentives during employment remain unchanged.

Formally, the optimal contract after termination solves the following problem (where we set τ to 0 to simplify notation):[5]

$$\max_{c} \quad E\left[\int_0^\infty e^{-rt}(-c_t)\, dt\right] \tag{23}$$

$$\text{s.t.} \quad E\left[\int_0^\infty e^{-rt} u(c_t)\, dt\right] = W_0 \quad \text{and} \quad E\left[\int_0^\infty e^{-rt}\zeta_0^t\, u(c_t)\, dt\right] = \Phi_0.$$

[5]Interestingly, problem (23) also solves a different interesting model, in which the agent puts effort only once at time 0, and his effort determines the unobservable level of fundamentals δ_0. Specifically, suppose the agent's utility is given by

$$E\left[\int_0^\infty e^{-rt} u(c_t)\, dt\right] - H(\delta_0),$$

where H is a convex increasing cost of effort, and fundamentals affect output according to $dX_t = (r+\kappa)\delta_t\, dt + \sigma dZ_t$, where $\delta_t = e^{-\kappa t}\delta_0$. Then the agent's incentive constraint is $H'(\delta_0) = \Phi_0$. A version of this problem has been solved on Hopenhayn and Jarque (2010).

Problem (23) is easy to solve. Letting ν_0 and λ_0 be the multipliers on the two constraints, the Lagrangian is

$$E\left[\int_0^\infty e^{-rt}\left((\nu_0+\zeta_0^t\lambda_0)u'(c_t)-c_t\right)\,dt\right]-\nu_0 W_0-\lambda_0\Phi_0.$$

The first-order condition is

$$c_t=\arg\max_c\;\underbrace{(\nu_0+\zeta_0^t\lambda_0)}_{\nu_t}u(c)-c, \tag{24}$$

where ν_t is the multiplier on the agent's utility at time t. From (11), the laws of motion of the Lagrange multipliers can be expressed as

$$d\nu_t=\lambda_0\,d\zeta_0^t=\underbrace{e^{-rt}\lambda_0}_{\lambda_t}\frac{r+\kappa}{\sigma}\frac{dX_t-(r+\kappa)\delta_t\,dt}{\sigma},\quad\text{and}\quad d\lambda_t=-\kappa\lambda_t\,dt. \tag{25}$$

This corresponds to the solution (22) after time τ.

Proposition 4 characterizes the principal's profit, as well as the correspondence between the multipliers (ν_0,λ_0) and variables (W_0,Φ_0), in problem (23). These relationships are conveniently represented through a single function $\underline{G}(\nu_0,\lambda_0)$, which solves a tractable parabolic partial differential equation (27).

Proposition 4 *Define*

$$\underline{G}(\nu_0,\lambda_0)=E\left[\int_0^\infty e^{-rt}\left(\nu_t u(c_t)-c_t\right)\,dt\right], \tag{26}$$

when c_t is determined by (24) and (ν_t,λ_t) follow (25). Then $\underline{G}$ solves equation

$$r\underline{G}(\nu,\lambda)=\max_c\;\nu u(c)-c-\kappa\lambda\,\underline{G}_\lambda(\nu,\lambda)+(r+\kappa)^2\frac{\lambda^2}{\sigma^2}\frac{\underline{G}_{\nu\nu}(\nu,\lambda)}{2}. \tag{27}$$

Then $W_0=\underline{G}_\nu(\nu_0,\lambda_0)$, $\Phi_0=\underline{G}_\lambda(\nu_0,\lambda_0)$ and

$$E\left[\int_0^\infty e^{-rt}(-c_t)\,dt\right]=\underline{G}(\nu_0,\lambda_0)-\nu_0W_0-\lambda_0\Phi_0. \tag{28}$$

Proof. Equation (26) is a standard stochastic representation of the solution of the parabolic partial differential equation (27) (see Karatzas and Shreve (1991)). Since $\nu_t = \nu_0 + \zeta_0^t \lambda_0$, differentiating (26) with respect to ν_0 and using the Envelope theorem, we get

$$\underline{G}_\nu(\nu_0, \lambda_0) = E\left[\int_0^\infty e^{-rt} u(c_t)\, dt\right] = W_0. \tag{29}$$

Differentiating with respect to λ_0 we get

$$\underline{G}_\lambda(\nu_0, \lambda_0) = E\left[\int_0^\infty e^{-rt} \zeta_0^t\, u(c_t)\, dt\right] = \Phi_0. \tag{30}$$

Finally, from the stochastic representation (26) itself, we have

$$\underline{G}(\nu_0, \lambda_0) = \nu_0 W_0 + \lambda_0 \Phi_0 + E\left[\int_0^\infty e^{-rt}(-c_t)\, dt\right]. \tag{31}$$

Equation (31), together with (30) and (29), implies (28). ■

Function $\underline{G}$ describes the principal's options regarding how to structure deferred compensation after termination. We will use it to characterize the optimal termination time in the original principal's problem.

Note that it is more convenient to describe the contract after termination via the state variables ν_t and λ_t rather than W_t and Φ_t. The evolution of ν_t and λ_t is described by the simple system (25). In contrast, the laws of motion of W_t and Φ_t are more complex and these variables do not map into payments to the agent in a simple way.

The optimal contract before termination. Before termination, the optimal contract is characterized by the partial differential equation

$$rG = \max_c \; a(G_\lambda) - c + \nu\left(u(c) - h(a(G_\lambda))\right) - \kappa\lambda G_\lambda + (r+\kappa)^2 \frac{\lambda^2}{\sigma^2}\frac{G_{\nu\nu}}{2}. \tag{32}$$

The following proposition shows that as long as equation (32) has an appropriate solution on a subset $\mathcal{R} \subseteq [0,\infty) \times \mathbb{R}$ of the state space, with smooth-pasting conditions

$$G(\nu,\lambda) = \underline{G}(\nu,\lambda) \quad \text{and} \quad \nabla G(\nu,\lambda) = \nabla \underline{G}(\nu,\lambda). \tag{33}$$

on the boundary of $\mathcal{R}$, then the optimal contract is characterized by the laws of motion of the state variables (22), with $\Phi_t = G_\lambda(\nu_t, \lambda_t)$, and τ is the time when (λ_t, ν_t) reach the boundary of the region $\mathcal{R}$.

Proposition 5 *Suppose that function G solves equation (32) on $\mathcal{R} \subseteq [0,\infty)\times\mathbb{R}$ and satisfies the smooth-pasting conditions (33) on the boundary. Then, as long as transversality conditions hold,*

$$W_t = G_\nu(\nu_t, \lambda_t), \quad \Phi_t = G_\lambda(\nu_t, \lambda_t)$$

and the principal's continuation payoff is $G(\nu_t, \lambda_t) - \nu_t W_t - \lambda_t \Phi_t$, in the contract defined by (22).

In addition, if on $\mathcal{R}$, $G(\lambda,\nu) \geq \underline{G}(\lambda,\nu)$ and the Hessian of G is positive definite, and outside $\mathcal{R}$,

$$r\underline{G} \geq \max_c \; a(\underline{G}_\lambda) - c + \nu\left(u(c) - h(a(\underline{G}_\lambda))\right) - \kappa\lambda\underline{G}_\lambda + (r+\kappa)^2\frac{\lambda^2}{\sigma^2}\frac{\underline{G}_{\nu\nu}}{2} \quad (34)$$

and the Hessian of $\underline{G}$ is positive definite, then the contract is optimal.

5 Environments with Significant Noise.

This section focuses on environments, in which it is possible to expose the agent to only a small fraction of project risk due to noise, yet the benefits of giving the agent even small exposure to project risk can be significant. One situation in practice that matches these assumptions well is executive compensation. As firm value is typically a much larger quantity than CEO wealth, CEOs can have significant exposure to firm risk, but small pay-performance sensitivity. Jensen and Murphy (1990) and Murphy (1999) estimate that average CEO wealth increases by only \$3.25 to \$5 for each \$1000 increase in shareholder value. Yet, presumably, a well-designed contract with even small pay-performance sensitivity can add significantly to shareholder value.

In environments with a lot of noise, it is difficult to identify the agent's effort. Consider the example of CEO compensation. If a CEO of a \$10-billion dollar firm can add \$500 million a year (i.e. 5%) to shareholder value by increasing effort, this value is economically significant. However, if the volatility of firm value is 30%, then effort is extremely difficult to identify. Over t years, we are trying to identify a difference in firm return of 5% when the standard error is $30\%/\sqrt{t}$. Effort identification leads to a large probability of type I and II errors, and to motivate effort, the contract has to expose the agent to a significant amount of risk. These are environments where the moral hazard problem is large.

It turns out that the optimal contract in environments with significant noise is extremely tractable. In fact, we get a closed-form map from the performance signal to optimal compensation. Also, the problem of optimal termination time reduces to the optimal exercise time of an American option. Below, after we formalize the assumption of large noise and characterize the optimal contract. The characterization is a special case of the general case in Section 5. However, to justify the optimal contract in this special environment, we construct a different simpler proof.

Let the agent's cost effort $h(a)$ be a C^2 increasing and convex function that satisfies $h(0) = h'(0) = 0$, and denote $\theta = h''(0)$. Effort adds to firm fundamentals according to

$$d\delta_t = (Ka_t - \kappa\delta_t)\, dt,$$

where $K > 0$ is a constant, and fundamentals affect the firm's output according to

$$dX_t = (r + \kappa)\delta_t + \sigma dZ_t.$$

The principal's profit takes the form

$$\delta_0 + E^a \left[\int_0^\tau e^{-rt} K a_t\, dt + e^{-r\tau} L - \int_0^\infty e^{-rt} c_t\, dt \right].$$

To capture environments where signal is noisy, but reasonable pay-performance sensitivity adds value, we take $\sigma \to \infty$ and $K \to \infty$ at appropriate rates. If σ is large, then it is only feasible to motivate effort levels a_t close to the agent's preferred level of 0. However, if K is also large, then even these effort levels have significant economic benefit. We focus on the limit in which $K^2/\sigma = \psi$ stays constant.

The following proposition provides a very simple expression for the principal's profit, given a contract (c, τ). The expression is very convenient, as it provides a way to compute the principal's profit directly from compensation, without solving for the agent's effort first.

Proposition 6 *In the limit as $\sigma \to \infty$ and $K \to \infty$ while $K^2/\sigma = \psi$, the principal's profit under the contract (c, τ) is given by*

$$E \left[\int_0^\infty e^{-rt} \left(\frac{\psi}{\theta} \frac{r + \kappa}{\kappa} \left(\int_0^t \hat{\lambda}_s\, dZ_s \right) u(c_t) - c_t \right) dt + e^{-r\tau} L \right], \tag{35}$$

where

$$\hat{\lambda}_0 = 0, \quad d\hat{\lambda}_t = \kappa(1_{t\le\tau} - \hat{\lambda}_t)\,dt.$$

Proof. With parameter K, the agent's incentive constraints are given by

$$h'(a_t) = \frac{dW_t}{d\delta_t}\frac{d\delta_t}{da_t} = \Phi_t K,$$

where

$$\Phi_t = E_t^a\left[\int_t^\infty e^{-r(s-t)}\zeta_t^s(u(c_s) - 1_{s\le\tau}\,h(a_s))\,ds\right], \quad \zeta_t^s = \int_t^s e^{-\kappa(s'-t)}\frac{r+\kappa}{\sigma}dZ_{s'}.$$

When a is close to zero, the cost of effort has a quadratic approximation $h(a) = \theta a^2/2$, and the incentive constraints can be written as

$$Ka_t = \underbrace{\frac{K^2}{\theta\sigma}}_{\psi/\theta} E_t^a\left[\int_t^\infty e^{-r(s-t)}\sigma\zeta_t^s u(c_s)\,ds\right]. \tag{36}$$

Note that the term $h(a_s)$ drops put, as $h(a_s) \to 0$ as $a_s \to 0$.

Then the principal's profit is

$$E\left[\int_0^\tau e^{-rt}Ka_t\,dt + e^{-r\tau}L - \int_0^\infty e^{-rt}c_t\,dt\right] =$$

$$E\left[\frac{\psi}{\theta}\int_0^\tau e^{-rt}\int_t^\infty e^{-r(s-t)}\sigma\zeta_t^s u(c_s)\,ds\,dt + e^{-r\tau}L - \int_0^\infty e^{-rt}c_t\,dt\right] =$$

$$E\left[\int_0^\infty e^{-rt}\left(\left(\frac{\psi}{\theta}\int_0^{\min(t,\tau)}\sigma\zeta_s^t\,ds\right)u(c_t) - c_t\right)\,dt + e^{-r\tau}L\right] =$$

$$E\left[\int_0^\infty e^{-rt}\left(\frac{\psi(r+\kappa)}{\theta\kappa}\left(\int_0^t \hat{\lambda}_s\,dZ_s\right)u(c_t) - c_t\right)\,dt + e^{-r\tau}L\right]. \tag{37}$$

To obtain the last expression, we used integration by parts and the relationship

$$\int_0^{\min(t,\tau)}\frac{\sigma\zeta_s^t\,ds}{r+\kappa} = \int_0^{\min(t,\tau)}\int_s^t e^{-\kappa(s'-s)}dZ_{s'}ds = \int_0^t \underbrace{\int_0^{\min(s',\tau)} e^{-\kappa(s'-s)}\,ds}_{\hat{\lambda}_{s'}/\kappa}\,dZ_{s'}.$$

■

The ratio ψ/θ in equation (35) has an intuitive interpretation. Consider hypothetically that $du(c_t) = r\hat{\Delta}\, dZ_t$ permanently, so that $dW_t = \hat{\Delta}\, dZ_t$. Then $\Delta_t = dW_t/dX_t = (dW_t/dZ_t)/(dX_t/dZ_t) = \hat{\Delta}/\sigma$, $\Phi_t = \hat{\Delta}/\sigma$ and (36) implies that $Ka_t = \psi/\theta\hat{\Delta}$. That is, ψ/θ is the value that the agent creates per unit of time when the sensitivity of his payoff to the Brownian motion Z_t is permanently set to $\hat{\Delta}$. In other words, it is the marginal value created from the agent's exposure to the normalized signal $dZ_t = (dX_t - (r+\kappa)\delta_t)/\sigma$.

Expression (35) leads to a closed-form expression for the agent's compensation, as a function of past performance history, in the optimal contract. If ν_0 is the multiplier on the agent's required utility constraint (6), then the principal wants to solve

$$\max_{c,\tau} E\left[\int_0^\infty e^{-rt}\left(\left(\nu_0 + \frac{\psi}{\theta}\frac{r+\kappa}{\kappa}\int_0^t \hat{\lambda}_s\, dZ_s\right)u(c_t) - c_t\right)\, dt + e^{-r\tau}L\right]. \quad (38)$$

Therefore, optimal payments c_t solve

$$\max_c \nu_t u(c) - c, \quad \text{where} \quad \nu_t = \nu_0 + \frac{\psi}{\theta}\frac{r+\kappa}{\kappa}\int_0^t \hat{\lambda}_s\, dZ_s.$$

The problem of finding the optimal termination times reduces to the optimal exercise time of an American option. Denote

$$f(\nu) = \max_c \nu u(c) - c,$$

and consider a security that pays a payoff flow of $f(\nu_t)$, where

$$d\nu_t = \frac{\psi}{\theta}\frac{r+\kappa}{\kappa}\hat{\lambda}_t\, dZ_t,$$

and $d\hat{\lambda}_t = \kappa(1-\hat{\lambda}_t)\, dt$ before time τ and $d\hat{\lambda}_t = -\kappa\hat{\lambda}_t\, dt$ after time τ. Then the optimal termination time τ is the stopping time that maximizes the value of this security.

The standard theory of real options implies that after time τ, the value function for the optimization problem (38) is defined by

$$r\underline{G}(\nu,\hat{\lambda}) = f(\nu) - \kappa\hat{\lambda}\,\underline{G}_{\hat{\lambda}}(\nu,\hat{\lambda}) + \left(\frac{\psi}{\theta}\frac{r+\kappa}{\kappa}\right)^2 \hat{\lambda}^2\,\frac{\underline{G}_{\nu\nu}(\nu,\hat{\lambda})}{2}. \quad (39)$$

Before termination, on the region of employment $\mathcal{R} \subset \mathbb{R} \times [0,1]$, the value function must solve

$$rG(\nu, \hat{\lambda}) = f(\nu) + \kappa(1-\hat{\lambda})\, G_{\hat{\lambda}}(\nu, \hat{\lambda}) + \left(\frac{\psi}{\theta}\frac{r+\kappa}{\kappa}\right)^2 \hat{\lambda}^2\, \frac{G_{\nu\nu}(\nu, \hat{\lambda})}{2}, \tag{40}$$

and satisfy the smooth-pasting conditions

$$G(\nu, \hat{\lambda}) = \underline{G}(\nu, \hat{\lambda}) + L \quad \text{and} \quad \nabla G(\nu, \hat{\lambda}) = \nabla \underline{G}(\nu, \hat{\lambda})$$

on the boundary of $\mathcal{R}$. In addition, to ensure that the stopping time τ is optimal, function $\underline{G}(\nu, \hat{\lambda})$ must satisfy

$$r\underline{G}(\nu, \hat{\lambda}) \geq f(\nu) + \kappa(1-\hat{\lambda})\, \underline{G}_{\hat{\lambda}}(\nu, \hat{\lambda}) + \left(\frac{\psi}{\theta}\frac{r+\kappa}{\kappa}\right)^2 \hat{\lambda}^2\, \frac{\underline{G}_{\nu\nu}(\nu, \hat{\lambda})}{2}$$

outside $\mathcal{R}$.[6]

Since ν_t is the multiplier on the agent's utility, it follows immediately that the agent's continuation payoff is given by

$$W_t = \begin{cases} G_\nu(\nu_t, \hat{\lambda}_t) \text{ for } t \leq \tau \\ \underline{G}_\nu(\nu_t, \hat{\lambda}_t) \text{ for } t > \tau \end{cases} \tag{41}$$

Equations (39) and (40) are standard parabolic equations, which can be solved by a finite difference scheme. Equation (39) can be solved in the direction of increasing $\hat{\lambda}$, starting from the boundary $\underline{G}(0, \nu) = f(\nu)$. Because $\hat{\lambda}$ decreases deterministically with time after termination, values of $\underline{G}(\hat{\lambda}, \nu)$ with higher $\hat{\lambda}$ depend on those with lower $\hat{\lambda}$. Similarly, equation (40) together with the optimal termination boundary can be solved in the direction of decreasing $\hat{\lambda}$, because $\hat{\lambda}_t$ increases deterministically over time until termination.

---

[6] These value function are defined as the following expectations, under the appropriate laws of motion of ν_t and $\hat{\lambda}_t$:

$$\underline{G}(\nu_t, \hat{\lambda}_t) = E_t\left[\int_t^\infty e^{-r(s-t)} f(\nu_s)\, ds\right] \quad \text{and}$$

$$G(\nu_t, \hat{\lambda}_t) = \max_\tau\; E_t\left[\int_t^\tau e^{-r(s-t)} f(\nu_s)\, ds + e^{-r(\tau-t)}(L + \underline{G}(\nu_\tau, \hat{\lambda}_\tau))\right].$$

6 Application: CEO Compensation.

The tractability of settings with large noise easily extends to more general environments. In particular, in this section we apply the insights of Section 5 to the application of optimal executive compensation. Suppose that the scale of the firm is not constant, but instead follows the geometric Brownian motion

$$dV_t = gV_t\, dt + \sigma V_t\, dZ_t.$$

The agent's actions $a_s \in [0, \infty)$, $s \in [0, \infty)$, affect the variable δ_t, unobservable by the market, according to equation

$$d\delta_t = \int_0^t e^{-\kappa(t-s)} a_s\, ds, \tag{42}$$

and has the impact on firm value at time t given by

$$C_1 \delta_t V_t^{1-\alpha}, \tag{43}$$

where C_1 is a constant. Because $\alpha \in (0, 1)$, for large firms effort has a small percentage effect but a large absolute effect on firm value. Moreover, due to volatility, it gets harder to detect effort for large firms.

In order to solve the principal's problem, we have to identify the marginal impact of the agent's effort on firm value and on the agent's utility.

From (42) and (43), effort a_t adds to firm value at rate

$$a_t E_t \left[\int_t^\infty e^{-(r+\kappa)(s-t)} V_s^{1-\alpha}\, ds \right] = \frac{a_t V_t^{1-\alpha}}{r + \kappa - (1-\alpha)g + \alpha(1-\alpha)\sigma^2/2}, \tag{44}$$

since the law of motion of $V_s^{1-\alpha}$ is

$$dV_t^{1-\alpha} = \left((1-\alpha)g - \frac{(1-\alpha)\alpha\sigma^2}{2} \right) V_t^{1-\alpha}\, dt + (1-\alpha)\sigma\, V_t^{1-\alpha}\, dZ_t.$$

At the same time, the marginal impact of effort on the agent's utility is

$$E_t \left[\int_t^\infty C_1 e^{-(r+\kappa)(s-t)} V_s^{1-\alpha} \Delta_s\, ds \right], \tag{45}$$

where $\Delta_t = dW_t/dV_t$ is the sensitivity of the agent's payoff to firm value. Another way to express the quantity (45) is by focusing on the impact of the agent's effort on the probability of future paths of V_t, which is summarized by

$$\zeta_t^s = \int_t^s \frac{C_1 e^{-\kappa(s'-t)} V_{s'}^{1-\alpha}}{\sigma V_{s'}} dZ_{s'}, \quad \text{where} \quad dZ_{s'} = \frac{dV_{s'} - gV_{s'}\, ds'}{\sigma V_{s'}}.$$

Then the marginal impact of effort on the agent's utility can also be expressed as

$$E_t \left[\int_t^\infty e^{-r(s-t)} \zeta_t^s u(c_s)\, ds \right].$$

The agent's incentive constraint takes the form

$$a_t = C_2 E_t \left[\int_t^\infty e^{-r(s-t)} \zeta_t^s u(c_s)\, ds \right],$$

where C_2 is another constant that depends on the agent's cost of effort.

From (44) and (45), the total impact of the agent's effort on firm value is

$$E\left[\int_0^\tau e^{-rt} \frac{a_t V_t^{1-\alpha}}{r + \kappa - (1-\alpha)g + \alpha(1-\alpha)\sigma^2/2}\, dt \right] =$$

$$E\left[\int_0^\tau e^{-rt} \frac{V_t^{1-\alpha}}{r + \kappa - (1-\alpha)g + \alpha(1-\alpha)\sigma^2/2} C_2 \int_t^\infty e^{-r(s-t)} \zeta_t^s u(c_s)\, ds\, dt \right] =$$

$$E\left[\int_0^\infty e^{-rt} \left(\frac{C_2}{r + \kappa - (1-\alpha)g + \alpha(1-\alpha)\sigma^2/2} \int_0^{\min(t,\tau)} V_s^{1-\alpha}\, \zeta_s^t\, ds \right) u(c_t)\, dt \right].$$

Furthermore,

$$\int_0^{\min(t,\tau)} V_s^{1-\alpha} \zeta_s^t\, ds = C_1 \int_0^{\min(t,\tau)} \int_s^t V_s^{1-\alpha} V_{s'}^{-\alpha} e^{-\kappa(s'-s)} \frac{dZ_{s'}}{\sigma}\, ds =$$

$$\frac{C_1}{\sigma^2} \int_0^t V_{s'}^{-\alpha} \underbrace{\int_0^{\min(s',\tau)} e^{-\kappa(s'-s)} V_s^{1-\alpha} ds}_{\hat{\lambda}_{s'}} \frac{dV_{s'} - gV_{s'}\, ds'}{V_{s'}},$$

where

$$d\hat{\lambda}_t = (V_t^{1-\alpha} - \kappa \hat{\lambda}_t)\, dt \quad \text{for } t \le \tau \quad \text{and} \qquad d\hat{\lambda}_t = -\kappa \hat{\lambda}_t\, dt \quad \text{for } t \ge \tau.$$

If ν_0 is the initial multiplier on the agent's utility, then c_t must maximize

$$\nu_t u(c_t) - c_t, \quad \text{where} \quad \nu_t = \nu_0 + C_3 \int_0^t \frac{\hat{\lambda}_s}{V_s^{1+\alpha}} \, (dV_s - gV_s \, ds). \tag{46}$$

We can make several observations. First, the term $dV_s - gV_s\,ds$ has a direct meaning in practice: it is the change in the firm's enterprise value, adjusted for the expected return. Second, since $\hat{\lambda}_s = O(V_s^{1-\alpha})$, it follows that the absolute volatility of ν_t is on the order of $V_s^{1-2\alpha}$. If $\alpha < 1/2$, then the agent's pay-performance sensitivity has to increase as the firm gets larger. While it gets harder to estimate the agent's effort as the firm gets larger, the importance of effort gets larger. If $\alpha < 1/2$, then the latter force dominates the former. Third, the sensitivity of the agent's pay to performance tends to increase with the agent's tenure as $\hat{\lambda}_s$ rises. Fourth, the volatility of ν_t increases with the past values of the firm, and decreases with the firm's current value. That is, the volatility of ν_t falls when firm value suddenly goes up, as in this case the stock price is less informative about CEO effort in this model. In contrast, if firm value falls, then stock price becomes a more informative signal of firm value, and so the volatility of ν_t rises. Of course, in this case the agent's pay-performance sensitivity can only rise so far before the agent's limited liability constraint protects the agent.

I have not discussed optimal termination policy in the application, but plan to do so in a future version of the paper.

7 Conclusions.

This paper wants to enhance our understanding of environments where the agent's actions can have delayed consequences. If a contract is thought of as a derivative on project value, which pays in the units of utility to the agent, and *Delta* is the sensitivity of derivative value to the performance signal, then the agent's incentives on the margin are captured by a discounted expectation of future contract *Deltas*. These first-order incentive constraints sufficient if contract the discounted expectation of future contract *Gammas* is not too large. The first-order incentive constraints alone allow us to frame the problem of finding an optimal contract as an optimal stochastic control problem. We partially characterize a solution to this problem using the method of Lagrange multipliers.

The optimal contract becomes particularly tractable under the assumption that the signal about the agent's performance is very noisy. In this case, also, and the first-order approach holds automatically. In these settings, the map from the signal about the agent's performance to the agent's compensation is determined in closed form. The problem of determining the optimal termination time is a real options problem. Moreover, the tractability of settings with large noise easily extends to other environments. This paper develops the application of optimal executive compensation, in which the value of the firm follows a geometric Brownian motion.

The intuitive optimal contract is based on two variables: ν_t that fully determines the agent's pay flow and λ_t that determines the sensitivity of ν_t to the performance signal. Generally, λ_t rises towards a target level during the agent's tenure and falls exponentially to 0 after the agent is fired. The agent is paid only when $\nu_t > 0$. When $\nu_t \leq 0$, the contract is "out of the money:" variable ν_t still adjusts to performance, but the agent is no longer paid. When the contract is sufficiently far out of money, the agent is fired, but he may still get paid after termination if signals after termination are positive.

A few natural follow-up research questions come to mind. There are several technical issues connected with the general problem. First, it is not clear how often the solution of the relaxed problem satisfies the sufficient incentive-compatibility condition (20). It would be useful to find conditions on the primitive of the setting, which guarantee that first-order approach works. Alternatively, one can try to solving the principal's problem by imposing (20) as an additional constraint, and then compare how restrictive this condition is. Then, the solution would give a lower bound on the principal's profit in the original problem (5), while the solution to the relaxed problem would give an upper bound. Second, technical issues remain regarding the solution of the partial differential equation that characterizes the optimal contract in a general setting. Generally speaking, these technical issues are a question of mathematics that applies to a broad class of optimal stochastic control problems. The relevant question here is one of existence of an appropriate solution G with a positive definite Hessian, which satisfies the relevant smooth-pasting conditions. The verification argument presented in this paper assumes that a function G satisfying these condition exists.

Third, the contract implied by settings with large noise is very intuitive, and it is natural to conjecture that this contract does well more generally. Of course, this contract ignores the agent's cost of effort function. However, it may still be beneficial to estimate how well this contract does, compared to the optimal contract, in specific examples. Fourth, for the executive compensation application, it is interesting to calibrate the model and develop a natural implementation using options and/or deferred stock grants. For calibration, the empirical estimates of pay-performance sensitivities and the impact of firm size on the level of compensation will be useful. The agent's coefficient of risk aversion is another parameter that is likely to affect the optimal contract. Fifth, the assumption of large noise leads to tractability not just in settings where the impact of the agent's effort on future output is exponentially delaying. In particular, in many settings, the impact of the agent's effort is not felt until some time in the future. It would be interesting to investigate how this alternative assumption would affect the optimal contract. It seems natural that the coefficient λ_t will still converge to a target level, but will stay near 0 during the initial period when the agent cannot have a strong effect on output.

Appendix.

Proof of Proposition 2. The statement that the process W_t satisfies (9) if and only if it follows (16) and satisfies the appropriate transversality condition is standard. See example Proposition 1 in Sannikov (2008).

Concerning Φ_t, first, let us show that if Φ is defined by (12), then there exists a process Γ in L^2 such that (17) holds. Note that the process

$$\bar{\Phi}_t = \int_0^t e^{-rs}\zeta_0^s(u(c_s)-1_{s\leq\tau}h(a_s))ds+e^{-rt}\underbrace{E_t^a\left[\int_t^\infty e^{-r(s-t)}\zeta_0^s\,(u(c_s)-1_{s\leq\tau}\,h(a_s))\,ds\right]}_{\zeta_0^t W_t+e^{-\kappa t}\Phi_t \text{ since } \zeta_0^s=\zeta_0^t+e^{-\kappa t}\zeta_t^s}$$

is a martingale. By the Martingale Representation Theorem, there exists a process Y in L^2 such that

$$d\bar{\Phi}_t = e^{-(r+\kappa)t}\,Y_t\sigma\,dZ_t. \tag{47}$$

Differentiating $\bar{\Phi}_t$ with respect to t, using (11), we get

$$d\bar{\Phi}_t = \underbrace{e^{-rt}\zeta_0^t\,(u(c_t)-1_{t\leq\tau}\,h(a_t))\,dt - re^{-rt}\zeta_0^t W_t\,dt + e^{-rt}\zeta_0^t dW_t}_{e^{-rt}\zeta_0^t\Delta_t\sigma dZ_t} + e^{-rt}W_t\,d\zeta_0^t$$

$$+e^{-(r+\kappa)t}(r+\kappa)\Delta_t\,dt-(r+\kappa)e^{-(r+\kappa)t}\Phi_t\,dt+e^{-(r+\kappa)t}d\Phi_t. \qquad (48)$$

Combining (47) and (48), we get

$$d\Phi_t=(r+\kappa)\left(\Phi_t-\Delta_t\right)\,dt+e^{\kappa t}\zeta_0^t\Delta_t\sigma\,dZ_t+e^{\kappa t}W_t\,d\zeta_0^t+Y_t\sigma\,dZ_t.$$

Letting $\Gamma_t=e^{\kappa t}\zeta_0^t\Delta_t+W_t\frac{r+\kappa}{\sigma^2}+Y_t$, we get the desired representation (17).

The representation (17) also implies that

$$\hat{\Phi}_t^s=(r+\kappa)\int_t^s e^{-(r+\kappa)s'}\Delta_{s'}\,ds'+e^{-(r+\kappa)(s-t)}\Phi_s \qquad (49)$$

is a martingale. Therefore, using the transversality condition,

$$\Phi_t=\hat{\Phi}_t^t=\lim_{s\to\infty}E[\hat{\Phi}_t^s]=E\left[(r+\kappa)\int_0^\infty e^{-(r+\kappa)s'}\Delta_{s'}\,ds'\right],$$

and so Corollary 1 holds (i.e. Φ_t also satisfies (10)).

Second, conversely, if a process Φ_t follows (17) for some Γ in L^2, then $\hat{\Phi}_t^s$ defined by (49) is a martingale. If in addition the transversality condition holds, then Φ_t satisfies (10), and by Corollary 1, also satisfies (12). ■

Proof of Proposition 3. Denote by δ the level of fundamentals under the original strategy, and by $\hat{\delta}$, under a possible deviation strategy $\hat{a}$. We claim that after the agent deviated from time 0 until time t, his future expected payoff is bounded from above by

$$\hat{W}_t(\hat{\delta}_t)=W_t+\Phi_t(\hat{\delta}_t-\delta_t)+L(\hat{\delta}_t-\delta_t)^2, \qquad (50)$$

where the constant L will be specified below. Then it follows immediately that when $\hat{\delta}_t=\delta_t$, the agent's continuation payoff is bounded from above by W_t, which is also the payoff he receives by following strategy a. Thus, if the bound (50) is valid, then the the full set of incentive-compatiblity constraints (7) holds.

Consider the process

$$\hat{V}_t=\int_0^t e^{-rs}(u(c_s)-1_{s\le\tau}h(\hat{a}_s))\,ds+e^{-rt}\hat{W}_t(\hat{\delta}_t)$$

under the deviation strategy $\hat{a}$, so that $d\hat{\delta}_t=(\hat{a}_t-\kappa\hat{\delta}_t)\,dt$, $\hat{\delta}_0=\delta_0$. To prove that the bound (50) is valid, it is enough to show that $\hat{V}$ is a supermartingale. Indeed, then

$$\hat{V}_t\ge E_t[\hat{V}_\infty] \quad\Rightarrow\quad \hat{W}_t(\hat{\delta}_t)\ge E_t\left[\int_t^\infty e^{-r(s-t)}(u(c_s)-1_{s\le\tau}h(\hat{a}_s))\,ds\right].$$

Differentiating $\hat{V}_t$ with respect to t, we find that

$$e^{rt}\frac{d\hat{V}_t}{dt} = (u(c_t) - 1_{t\leq\tau}h(\hat{a}_t))\,dt - r\,\overbrace{\left(W_t + \Phi_t(\hat{\delta}_t - \delta_t) + L(\hat{\delta}_t - \delta_t)^2\right)}^{\hat{W}_t(\hat{\delta}_t)}\,dt$$

$$+(rW_t - u(c_t) + 1_{t\leq\tau}h(a_t))\,dt + \Delta_t\,(dX_t - (r+\kappa)\delta_t\,dt) +$$

$$(r+\kappa)(\Phi_t - \Delta_t)(\hat{\delta}_t - \delta_t)dt + \Gamma_t(dX_t - (r+\kappa)\delta_t dt)(\hat{\delta}_t - \delta_t) + \Phi_t(\hat{a}_t - a_t - \kappa(\hat{\delta}_t - \delta_t))dt$$

$$+2L(\hat{\delta}_t - \delta_t)(\hat{a}_t - a_t - \kappa(\hat{\delta}_t - \delta_t))\,dt.$$

Using the fact that $dX_t = (r+\kappa)\hat{\delta}_t\,dt + \sigma\,dZ_t$ the drift of $\hat{V}_t$ is e^{-rt} times

$$-\frac{\theta}{2}(\hat{a}_t - a_t)^2 + ((r+\kappa)\Gamma_t - (2\kappa + r)L)(\hat{\delta} - \delta_t)^2 + 2L(\hat{\delta}_t - \delta_t)(\hat{a}_t - a_t)$$

where we used $h(a_t) - h(\hat{a}_t) = h'(a_t)(a_t - \hat{a}_t) - \frac{\theta}{2}(a_t - \hat{a}_t)^2$ and $\Phi_t = h'(a_t)$ (and we have to set $a_t = \hat{a}_t$ if $t > \tau$.) Now, set $L = \frac{\theta}{4}(2\kappa + r)$. Then as long as $\Gamma_t \leq \frac{(2\kappa+r)^2\theta}{8(r+\kappa)}$,

$$(r+\kappa)\Gamma_t - (2\kappa + r)L \leq \frac{(2\kappa + r)^2\theta}{8} - (2\kappa + r)L = -2L^2/\theta.$$

Then the drift of $\hat{V}_t$ is less than or equal to e^{-rt} times

$$-\frac{\theta}{2}\left(\hat{a}_t - a_t - (\kappa + r/2)(\hat{\delta} - \delta_t)\right)^2 \leq 0.$$

■

Proof of Proposition 5. First, let us show that if G solves (32) on $\mathcal{R} \subseteq [0,\infty) \times \mathbb{R}$ and satisfies the smooth-pasting conditions (33) on the boundary, then $W_t = G_\nu(\nu_t, \lambda_t)$, $\Phi_t = G_\lambda(\nu_t, \lambda_t)$ and the principal's continuation payoff is $G(\nu_t, \lambda_t) - \nu_t W_t - \lambda_t \Phi_t$ in the contract defined by (22).

Differentiating (32) with respect to ν and using the Envelope Theorem, we get

$$rG_\nu - u(c) + h(a(G_\lambda)) = (a'(G_\lambda)\,(1 - \nu h'(a(G_\lambda))) - \kappa\lambda)\,G_{\nu\lambda} + (r+\kappa)^2\frac{\lambda^2}{\sigma^2}\frac{G_{\nu\nu\nu}}{2}. \quad (51)$$

Note that the right hand side of (51) represents the drift of the process $G_\nu(\nu_t, \lambda_t)$ when (ν_t, λ_t) follow (22), and $G_\nu(\nu_\tau, \lambda_\tau) = \underline{G}_\nu(\nu_\tau, \lambda_\tau)$ is the agent's continuation value at time τ by Proposition 4. Therefore, as long as the transversality condition holds, Proposition 2 implies that $G_\nu(\nu_t, \lambda_t)$ is the agent's continuation value W_t under the effort strategy $\{a(G_\lambda(\nu_t, \lambda_t))\}$.

Similarly, differentiating (32) with respect to λ and using the Envelope Theorem, we get

$$(r+\kappa)\left(G_\lambda - (r+\kappa)\frac{\lambda}{\sigma^2}G_{\nu\nu}\right) = \left(a'(G_\lambda)\left(1 - \nu h'(a(G_\lambda))\right) - \kappa\lambda\right) G_{\lambda\lambda} + (r+\kappa)^2 \frac{\lambda^2}{\sigma^2}\frac{G_{\lambda\nu\nu}}{2}. \tag{52}$$

Using Ito's lemma, $(r+\kappa)\frac{\lambda}{\sigma^2}G_{\nu\nu}(\nu_t, \lambda_t) = \Delta_t$, so the drift of $G_\lambda(\nu_t, \lambda_t)$ is $(r+\kappa)(G_\lambda(\nu_t, \lambda_t) - \Delta_t)$. Also, $G_\lambda(\nu_\tau, \lambda_\tau) = \underline{G}_\lambda(\nu_\tau, \lambda_\tau) = \Phi_\tau$ by Proposition 4. Therefore, as long as the transversality condition holds, Proposition 2 implies that $\Phi_t = G_\lambda(\nu_t, \lambda_t)$ under the effort strategy $\{a(G_\lambda(\nu_t, \lambda_t))\}$.

Finally, subtracting ν times (51) and λ times (52) from (32), we get

$$r(G - \nu G_\nu - \lambda G_\lambda) = a(G_\lambda) - c + \left(a'(G_\lambda)\left(1 - \nu h'(a(G_\lambda))\right) - \kappa\lambda\right) \underbrace{\left(-\nu G_{\nu\lambda} - \lambda G_{\lambda\lambda}\right)}_{\frac{\partial(G - \nu G_\nu - \lambda G_\lambda)}{\partial\lambda}}$$

$$+\frac{1}{2}(r+\kappa)^2\frac{\lambda^2}{\sigma^2}\underbrace{\left(-G_{\nu\nu} - \nu G_{\nu\nu\nu} - \lambda G_{\lambda\nu\nu}\right)}_{\frac{\partial^2(G - \nu G_\nu - \lambda G_\lambda)}{\partial\nu^2}}. \tag{53}$$

Hence, the process

$$\bar{F}_t = \int_0^t e^{-rs}\left(a_s - c_s\right)\, ds + e^{-rt}(G(\nu_t, \lambda_t) - \nu_t W_t - \lambda_t \Phi_t).$$

is a martingale. Since

$$\bar{F}_t = E_t[\bar{F}_\tau] = \int_0^t e^{-rs}\left(a_s - c_s\right)\, ds +$$

$$e^{-rt}E_t\left[\int_t^\tau e^{-r(s-t)}\left(a_s - c_s\right)\, ds + e^{-r(\tau - t)}(\underline{G}(\nu_\tau, \lambda_\tau) - \nu_\tau W_\tau - \lambda_\tau \Phi_\tau)\right],$$

where $\underline{G}(\nu_\tau, \lambda_\tau) - \nu_\tau W_\tau - \lambda_\tau\Phi_\tau$ is the principal's continuation payoff at time τ by Proposition 4, it follows that $G(\nu_t, \lambda_t) - \nu_t W_t - \lambda_t\Phi_t$ is the principal's continuation payoff in the contract defined by (22).

Next, we will show that under any alternative contract, for which $W_0 = G_\nu(\nu_0, \lambda_0)$ and $\Phi_0 = G_\lambda(\nu_0, \lambda_0)$, the principal's profit is bounded from above by $G(\nu_0, \lambda_0) - \nu_0 W_0 - \lambda_0 \Phi_0$. The key step in the argument is showing that the process $\bar{F}_t$ is a supermartingale for appropriate processes (ν_t, λ_t) chosen to match the law of motion of (W_t, Φ_t) under the alternative contract.

Lemma 1 *Consider an alternative contract, characterized by controls* (c, Δ, Γ) *and termination time* τ*, and denote by* W *and* Φ *the state variables under those controls (see Theorem 1). Define* $G(\nu, \lambda) = \underline{G}(\nu, \lambda)$ *outside* $\mathcal{R}$*. If the Hessian of* G *is positive definite, then there exist processes*

$$d\nu_t = \mu_t^\nu \, dt + \sigma_t^\nu \, dZ_t \quad \text{and} \quad d\lambda_t = \mu_t^\lambda \, dt + \sigma_t^\lambda \, dZ_t \tag{54}$$

such that $W_t = G_\nu(\nu_t, \lambda_t)$ *and* $\Phi_t = G_\lambda(\nu_t, \lambda_t)$ *for* $t \le \tau$*.*

Proof. We would like to make sure that there are processes σ_t^ν, σ_t^λ, μ_t^ν and μ_t^λ such that the laws of motion of $G_\nu(\nu_t, \lambda_t)$ and $G_\lambda(\nu_t, \lambda_t)$ are identical to those of W_t and Φ_t. To match volatilities, let σ_t^ν and σ_t^λ be determined from equations

$$\underbrace{\begin{bmatrix} G_{\nu\nu} & G_{\nu\lambda} \\ G_{\lambda\nu} & G_{\lambda\lambda} \end{bmatrix}}_{H(G)} \begin{bmatrix} \sigma_t^\nu \\ \sigma_t^\lambda \end{bmatrix} = \begin{bmatrix} \Delta_t \sigma \\ \Gamma_t \sigma \end{bmatrix}, \tag{55}$$

which follow from Ito's lemma. The solutions exist and are unique because $H(G)$, the Hessian of G, is invertible.

Similarly, to match drifts, let μ_t^ν and μ_t^λ be determined from equations

$$H(G) \begin{bmatrix} \mu_t^\nu \\ \mu_t^\lambda \end{bmatrix} + \ldots = \begin{bmatrix} rW_t - u(c_t) + h(a(\Phi_t)) \\ (r + \kappa)(\Phi_t - \Delta_t) \end{bmatrix},$$

where "..." stand for terms that depend on the volatilities of ν_t and λ_t and not the drifts. Again, the solution exists because the Hessian of G is invertible. ■

In order to prove that the alternative contract cannot be superior to the contract defined in Proposition 5, we will first show that the drift of the process $\bar{F}_t$ defined above is non-positive when ν_t and λ_t follow (54).

Using Ito's lemma and the laws of motion of W_t and Φ_t, the drift of $G(\lambda_t, \nu_t) - \nu_t W_t - \lambda_t \Phi_t$ is

$$G_\nu \mu_t^\nu + G_\lambda \mu_t^\lambda + \frac{1}{2}[\sigma_t^\nu \ \sigma_t^\lambda] H(G) \begin{bmatrix} \sigma_t^\nu \\ \sigma_t^\lambda \end{bmatrix} - \mu_t^\nu W_t - \mu_t^\lambda \Phi_t - [\sigma_t^\nu \ \sigma_t^\lambda] \begin{bmatrix} \Delta_t \sigma \\ \Gamma_t \sigma \end{bmatrix}$$

$$-\nu_t(rW_t - u(c_t) + h(a_t)) - \lambda_t(r+\kappa)(\Phi_t - \Delta_t) =$$

$$-\frac{1}{2}[\sigma_t^\nu \ \sigma_t^\lambda]H(G)\begin{bmatrix}\sigma_t^\nu \\ \sigma_t^\lambda\end{bmatrix} - \nu_t(rW_t - u(c_t) + h(a_t)) - \lambda_t(r+\kappa)(\Phi_t - \Delta_t),$$

where we used (55). Without loss of generality, we can assume that $c_t = \chi(\nu_t)$, which maximizes the drift of $\bar{F}_t$.

For comparison, when λ_t and ν_t follow (22) then the drift of $G(\nu_t, \lambda_t) - \nu_t W_t - \lambda_t \Phi_t$ is

$$-(r+\kappa)^2\frac{\lambda_t^2}{\sigma^2}\frac{G_{\nu\nu}}{2} - \nu_t(rW_t - u(c_t) + h(a_t)) - \lambda_t(r+\kappa)\left(\Phi_t - (r+\kappa)\frac{\lambda_t}{\sigma^2}G_{\nu\nu}\right),$$

which, according to (53), leads to a drift of $\bar{F}_t$ of zero in $\mathcal{R}$ and negative outside $\mathcal{R}$, by (34).

Now, the difference between the drift of $\bar{F}_t$ under the alternative contract contract and under the contract, in which λ_t and ν_t follow (54), is e^{-rt} times

$$-\frac{1}{2}[\sigma_t^\nu \ \sigma_t^\lambda]\, H(G)\begin{bmatrix}\sigma_t^\nu \\ \sigma_t^\lambda\end{bmatrix} + \lambda_t(r+\kappa)\underbrace{\frac{\sigma_t^\nu G_{\nu\nu} + \sigma_t^\lambda G_{\nu\lambda}}{\sigma}}_{\Delta_t} - \frac{1}{2}(r+\kappa)^2\frac{\lambda_t^2}{\sigma^2}G_{\nu\nu} =$$

$$-\frac{1}{2}[\sigma_t^\nu - (r+\kappa)\lambda_t/\sigma, \ \ \sigma_t^\lambda]\, H(G)\begin{bmatrix}\sigma_t^\nu - (r+\kappa)\lambda_t/\sigma \\ \sigma_t^\lambda\end{bmatrix} \leq 0,$$

since the matrix $H(G)$ is positive definite. Hence, the drift of $\bar{F}_t$ under the alternative contract cannot be greater than that under the contract, in which λ_t and ν_t follow (22), so it must be negative. In other words, $\bar{F}_t$ is a supermartingale.

Hence,

$$\bar{F}_0 = G(\nu_0, \lambda_0) - \nu_0 W_0 - \lambda_0 \Phi_0 \geq E[\bar{F}_\tau] =$$

$$E\left[\int_0^\tau e^{-rs}\,(a_s - c_s)\ ds + e^{-r\tau}(\underline{G}(\nu_\tau, \lambda_\tau) - \nu_\tau W_\tau - \lambda_\tau \Phi_\tau)\right]$$

$$\geq E\left[\int_0^\tau e^{-rs} a_s\, ds - \int_0^\infty e^{-rs} c_s\, ds\right],$$

where we used Proposition 4 for the last inequality.[7] Therefore, the contract, in which λ_t and ν_t follow (22), is optimal. ■

[7] The transversality condition $\liminf E[1_{t<\tau} e^{-rt}(G(\lambda_t, \nu_t) - \lambda_t W_t - \nu_t \Phi_t)] \geq 0$ needs to hold in order to extend the supermartingale $\bar{F}$ to time τ.

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