On Competitive Nonlinear Pricing*

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Abstract

A buyer of a divisible good faces several identical sellers. The buyer's preferences are her private information, and they may directly affect the sellers' profits (common values). Sellers compete by posting menus of nonexclusive contracts, so that the buyer can simultaneously and privately trade with several sellers. We focus on the finite-type case, and we provide a full characterization of pure-strategy equilibria in which sellers post convex tariffs. All equilibria involve linear pricing. When the sellers' cost functions are linear and do not depend on the buyer's type (private values), equilibria exist and trade is efficient. Under common values, or when the sellers' costs are strictly convex, there is a severe form of market breakdown as at most one type of the buyer may actively trade. Moreover equilibria exist only under restrictive conditions.

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1 Introduction

Many markets for goods and services do not restrict in any way the ability of each trader to sign secret, bilateral contracts with different partners. This prevents outside parties from monitoring the whole of a trader's activities. As a consequence, the formation of prices on such nonexclusive markets is by nature a decentralized process, unlike on idealized markets ruled by a Walrasian auctioneer. Bilateral contracts are necessarily incomplete as they only bear on a fraction of each trader's activity. Moreover, bilateral negotiations allow to tailor contracts at will, at odds with contracts that are normalized for quotation. In particular, contracts may be discriminatory, and the balance between supply and demand may be ensured not by a single price, but by nonlinear tariffs. These tariff offers in turn are formulated in a strategic environment in which sellers take into account both the reaction of buyers and the other sellers' offers. The aim of this paper is to understand the formation of prices on nonexclusive markets. In the case of financial markets, our results shed light on the robustness of organized exchanges such as limit-order books to trades that take place "in the dark," outside visible order books. As we will see, the nonexclusive nature of such transactions is a major obstacle to the efficient functioning of these markets.

We study these issues in the context of the following model of trade under uncertainty. There are two commodities, money and a physical good. Trade takes place between a buyer and a finite number of sellers offering this good. The sellers first post possibly nonlinear tariffs expressing how much they ask for any quantity of the good. The buyer then learns her preferences and she decides which quantity to purchase from each seller. There is an arbitrary finite number of states of nature. In each state, the buyer has strictly convex preferences. These preferences are ordered across states according to how much she is willing to trade at the margin, reflecting a strict single-crossing property. As for the sellers, they weakly prefer to sell lower quantities when the buyer is more eager to trade, reflecting a reverse weak single-crossing property. Our model thus encompasses private-value and adverse-selection environments as special cases. In addition, sellers may have constant or increasing marginal costs of serving the buyer in each state of nature.

In this context, we provide a complete characterization of pure-strategy equilibria in which sellers post convex tariffs. Such tariffs can be interpreted as sequences of limit orders, and are natural candidates to consider in nonexclusive models of trade with adverse selection (Biais, Martimort, and Rochet (2000, 2013), Back and Baruch (2013)) or increasing marginal costs (Biais, Foucault, and Salanié (1998)). Importantly, we allow sellers to deviate by posting arbitrary nonconvex tariffs, so as to fit our definition of a nonexclusive market. Our main result is that all equilibria must involve linear pricing. Hence competition in our model is powerful enough to make a single equilibrium price emerge. Sellers then cannot benefit from using nonlinear tariffs. When sellers have constant and state-independent marginal costs, one ends up with a unique equilibrium outcome which is efficient in the strongest sense, as it coincides with the equilibrium outcome of a perfectly competitive market.[1] When there is adverse selection or sellers have increasing marginal costs, linear-price equilibria are such that the buyer trades in at most one state of nature, and does not trade at all in any other state. Hence the market breaks down in a very strong sense. Moreover, in such cases necessary conditions for the existence of an equilibrium are severe. An implication of our analysis is that organized exchanges such as limit-order books can be destabilized by decentralized exchanges such as over-the-counter markets.

Standard analyses of nonexclusive markets take linear pricing as a defining feature of such markets. The opportunity to trade small quantities from several sellers, the argument goes, allows buyers to arbitrage away any nonlinearities in the sellers' tariffs. In line with this intuition, Pauly (1974) analyzed a nonexclusive insurance market in which insurance companies are restricted to post linear tariffs, and showed that equilibria then involve cross-subsidies between sellers' profits across states.[2] Our analysis suggests that these outcomes do not survive when strategic interactions between sellers are explicitly taken into account. The intuition is that due to adverse selection or increasing marginal costs, the sellers face a high demand from the buyer precisely in those states in which the cost of serving her is high. To hedge against this risk, each seller has an incentive to deviate by proposing a limit order specifying the maximal quantity of the good he is ready to trade at the standing price. In these circumstances, linear pricing can be reconciled with nonexclusive competition only if the buyer trades a positive quantity in at most one state. In contrast with this result, Attar, Mariotti, and Salanié (2011) showed that the restriction to linear prices is without loss of generality in a lemons market where an informed seller can trade up to a capacity and all market participants have linear preferences. Cross-subsidies between states can then resist limit-order deviations because, at any given unit price, and depending on the state, the seller is either ready to trade up to the maximum quantity demanded at this price (as long as it does not exceed her capacity) or prefers not to trade at all. By contrast, the informed

[1]The existence of an efficient equilibrium in this Bertrand-like environment with private values is quite straightforward. Still we could not find any previous work showing that no other equilibria with convex tariffs can exist. A similar efficiency result appears in Pouyet, Salanié, and Salanié (2008), albeit in the case of an exclusive market in which the buyer can trade with at most one seller.

[2]The same restriction to linear pricing is postulated in recent analyses of the annuity market, which is nonexclusive in many countries (Rothschild (2007), Sheshinski (2008), and Hosseini (2010)).

buyer in our model has strictly convex preferences and faces no capacity constraint. This implies that, at any given unit price, the buyer typically has different aggregate demands in different states. This in turn gives limit-order deviations their bite and destabilizes linear-price candidate equilibria in which trade takes place in more than one state.

One may then turn to equilibria with nonlinear tariffs, in the hope that they yield more trading under adverse selection or increasing marginal costs. Glosten (1994) proposed a natural candidate in a framework in which the buyer faces an exogenously given tariff and sellers have linear production costs. Specifically, he showed that there is a unique convex tariff that resists entry. This tariff can be interpreted as a generalization of Akerlof (1970) pricing, for marginal quantities. It specifies that each additional quantity above any quantity q is sold at a price equal to the expected cost of serving it, conditional on the fact that the buyer buys at least q. Under single crossing, this amounts to compute an upper-tail expectation, namely, the expectation of the cost given that the buyer is ready to purchase at least q. In each state, the buyer then trades exactly her demand at the tail price. An additional nice property is that by construction such a tariff yields zero profit to the sellers.

In our setting, the question becomes whether we can find convex tariffs for the sellers that once aggregated yield the Glosten (1994) tariff, and such that no seller can profitably deviate by posting another tariff.[3] Suppose that in equilibrium two different types of the buyer, corresponding to two different states of nature, end up trading at two different tail prices. Then there must exist a seller that sells more to the type trading at the highest price than to the other type. Note that when facing this seller, the former type does not want to deviate and choose the quantity traded by the latter type because, when tariffs are convex, optimality conditions imply that all quantities traded by a given type with the sellers are traded at the same price. On the other hand, the seller designs his tariff so as to maximize his expected profit, under incentive-compatibility constraints. Given convex tariffs and single crossing, we show that downward local incentive-compatibility constraints must be binding at the solution of such a problem. But this contradicts the fact that the highest type does not want to mimic the lowest type. Hence in a Glosten-like equilibrium all trades must take place at the same price. Moreover, we show that the above logic also applies to any convex tariff. Therefore, the only equilibria are linear-price equilibria. We are then back to the conclusion that at most one type may trade in equilibrium under adverse selection or increasing marginal costs.

Our results confirm those obtained by Attar, Mariotti, and Salanié (2013). That paper

[3]This study was not performed in Glosten (1994), see the discussion in Glosten (1998).

examines the case with two states of nature, adverse selection, and constant marginal costs in each state. A complete characterization of aggregate equilibrium allocations is provided, with no restriction on equilibrium tariffs. It turns out that all equilibrium allocations can be supported by linear tariffs, with at most one type trading. Focusing on equilibria with convex tariffs, this paper shows that, strikingly, the result that the buyer may trade in at most one state extends to an arbitrary finite number of states. We thus exhibit a new form of market failure, characterized by a dramatic market breakdown that exceeds by far the one first characterized by Akerlof (1970).

On the other hand, our results stand in stark contrast with those obtained in Biais, Martimort, and Rochet (2000), who consider a parametric version of our model with a quasilinear, quadratic utility function for the buyer, and constant marginal costs with adverse selection for the sellers. The main difference is that the set of states is assumed to be continuous, instead of finite as in this paper. This allows Biais, Martimort, and Rochet (2000) to focus on equilibria with strictly convex tariffs.[4] They show that such an equilibrium exists, is unique in this class, and is symmetric across sellers. Moreover the buyer trades in a nontrivial set of states in equilibrium, at a tariff between the perfectly competitive tariff that would obtain under complete information and the monopoly tariff under incomplete information. We thus exhibit in this paper a remarkable discontinuity between the finite-state case and the continuous-state case: the equilibrium characterized in the latter case is not a limit of equilibria in the former case as the number of types grows large.

The paper is organized as follows. Section 2 describes the model. Section 3 states and discusses our central result, the proof of which is outlined in Section 4. Section 5 discusses various extensions of our analysis. Section 6 concludes.

2 The Model

Our model features a buyer who can purchase nonnegative amounts of a divisible good from several sellers. The good is homogeneous, so the buyer only cares about aggregate trade. The possibility of adverse selection plays an important role, as in well-known models of insurance provision, labor supply, or more generally competitive screening.

2.1 The Buyer

[4]Equilibria with strictly convex tariffs do not exist when there are finitely many states. The reason is that otherwise, each type of the buyer would have a unique best response, and no incentive-compatibility constraint would bind, in contradiction with one of our key findings.

The buyer is privately informed of her preferences. Her type may take a finite number of values in the set $\{1, \ldots, I\}$, with positive probabilities m_i such that $\sum_i m_i = 1$. Each type of the buyer only cares about the aggregate quantity $Q \geq 0$ she purchases from the sellers and the aggregate transfer T she makes in return. Type i's preferences over aggregate quantity-transfer bundles (Q, T) are represented by a utility function u_i defined over $\mathbb{R}_+ \times \mathbb{R}$. For each i, u_i is assumed to be continuous and strictly quasiconcave in (Q, T), and strictly decreasing in T. The following strict single-crossing assumption is the main determinant of the buyer's behavior in our model, and is also used throughout the related literature.

Assumption 1 *For all $i < i'$, $Q < Q'$, T, and T',*

$$u_i(Q, T) \leq u_i(Q', T') \text{ implies } u_{i'}(Q, T) < u_{i'}(Q', T').$$

In words, higher types are more eager to increase their purchases than lower types are. At the end of our analysis, we shall also use an additional property that we now introduce. For each $p \in \mathbb{R}$, let $D_i(p)$ be type i's demand at price p, that is, the unique solution to

$$\max_{Q \in \mathbb{R}_+ \cup \{\infty\}} \{u_i(Q, pQ)\}.$$

The continuity and strict quasiconcavity of u_i imply that $D_i(p)$ is uniquely defined and continuous in p. Assumption 1 implies that for each p, $D_i(p)$ is nondecreasing in the buyer's type i. We strengthen this monotonicity property as follows.

Assumption 2 *For all $i < i'$ and $p \in \mathbb{R}$,*

$$0 < D_i(p) < \infty \text{ implies } D_i(p) < D_{i'}(p).$$

A sufficient condition for both Assumptions 1 and 2 to hold is that the marginal rate of substitution $MRS_i(Q, T)$ of the good for money be well defined and strictly increasing in i for all (Q, T).

2.2 The Sellers

There are $K \geq 2$ identical sellers. There are no direct externalities between them: each seller only cares about the quantity $q \geq 0$ he provides the buyer with and the transfer t he receives in return. Such pair (q, t) we call a *trade*. The seller's profits from a trade may depend on the buyer's type. Our key assumption here is a reverse single-crossing property: we impose that each seller weakly prefers to sell lower quantities to higher types. This assumption introduces adverse selection in our model: a higher type is willing to buy more, but faces sellers that are more reluctant to sell.

To allow comparisons with the literature, we represent each seller's preferences over trades (q, t) by a linear profit function: if a seller provides type i with a quantity q and receives a transfer t in return, he earns a profit $t - c_i q$, where c_i is the cost of serving type i. Our reverse single-crossing property can thus be written as follows.

Assumption 3 *For all $i < i'$, $c_i \leq c_{i'}$.*

Assumption 3 is consistent with private-value environments, in which the sellers' cost is independent of the buyer's type, and common-value environments, in which the sellers' cost strictly increases with the buyer's type. Section 5 provides a general definition of the reverse single-crossing property and highlights its role in our analysis.

2.3 Strategies and Equilibrium

The game unfolds as follows:

1. Sellers simultaneously post tariffs, which are mappings $t^k : \mathbb{R}_+ \to \mathbb{R} \cup \{\infty\}$ such that $t^k(0) = 0$. We let $t^k(q) \equiv \infty$ if seller k does not offer the quantity q.

2. After privately learning her type, the buyer purchases a nonnegative quantity q^k from each seller k, for which she pays in total $\sum_k t^k(q^k)$.

A pure strategy for type i is a function s_i that maps any tariff profile $(t^1, \ldots, t^K)$ into a quantity profile $(q^1, \ldots, q^K)$. We let $s = (s_1, \ldots, s_I)$ be the buyer's strategy. To ensure that type i's problem

$$\max_{(q^1,\ldots,q^K)\in\mathbb{R}_+^K} \left\{ u_i\left(\sum_k q^k, \sum_k t^k(q^k)\right)\right\} \tag{1}$$

always has a solution, we require the tariffs t^k to be lower semicontinuous, and the sets $\{q \in \mathbb{R}_+ : t^k(q) < \infty\}$ to be compact. This definition is general enough to allow sellers to offer menus containing a finite number of trades, including the $(0, 0)$ trade. It also allows us to use perfect Bayesian equilibrium as our equilibrium concept.

In line with Biais, Martimort, and Rochet (2000, 2013) and Back and Baruch (2013), we focus on pure-strategy equilibria $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ in which sellers post convex tariffs $\mathfrak{t}^k$ that one can interpret as sequences of limit orders.[5] Two elementary implications of this

[5]By convention, all functions in Gothic letters refer to equilibrium objects.

restriction are worth mentioning at this stage. First, because the utility functions u_i are strictly quasiconcave, any type i has uniquely determined aggregate equilibrium demand Q_i and transfer T_i, which additionally are nondecreasing in i under Assumption 1. Second, convexity of equilibrium tariffs is preserved under aggregation. In particular, suppose that the buyer wishes to trade an aggregate quantity Q^{-k} with the sellers other than k. Then the minimum transfer she has to make in return is

$$\mathfrak{T}^{-k}(Q^{-k}) \equiv \min \left\{ \sum_{k' \neq k} \mathfrak{t}^{k'}(q^{k'}) : q^{k'} \in \mathbb{R}_+ \text{ for all } k' \neq k \text{ and } \sum_{k' \neq k} q^{k'} = Q^{-k} \right\}. \tag{2}$$

The aggregate tariff $\mathfrak{T}^{-k}$ is the infimal convolution of the individual tariffs $\mathfrak{t}^{k'}$ posted by the sellers other than k, and is convex if each of them is convex (Rockafellar (1970)).

3 The Main Result

Our central result is the following theorem.

Theorem 1 *Suppose that Assumptions 1–3 are satisfied, and let* $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ *be an equilibrium with convex tariffs. If some trade takes place in equilibrium, then*

(i) All trades take place at unit price c_I *and each type* i *purchases* $D_i(c_I)$ *in the aggregate.*

(ii) If $D_i(c_I) > 0$, *then* $c_i = c_I$. *Thus each seller earns zero profit on each trade.*

The first insight of Theorem 1 is that nonexclusive competition leads to linear pricing, at least when attention is restricted to equilibria with convex tariffs. This shows the disciplining role of competition in our model: although sellers are allowed to propose arbitrary tariffs, they end up trading at the same price.

From the standard Bertrand undercutting argument, this price cannot be strictly above the highest possible cost c_I. In an equilibrium it cannot lie below neither. If it did, then sellers would want to limit the quantities they sell to the highest types, which they can do by posting a limit order at the equilibrium price with a well-chosen maximum quantity. We then have a tension between zero profits in the aggregate, and the high equilibrium price c_I. In the *pure private-value* case in which the cost c_i is independent of the buyer's type i, this tension is easily relaxed, and we obtain the usual Bertrand result, leading to an efficient outcome. By contrast, in the *pure common-value* case in which the cost c_i is strictly increasing with the buyer's type i, our result implies that only the highest type I may actively trade in equilibrium, whereas all types $i < I$ must be excluded from trade.

This market failure is much more dramatic than in Akerlof (1970) or Rothschild and Stiglitz (1976), as only a single type may actively trade in equilibrium.

Additionally conditions for the existence of an equilibrium are very restrictive: from Theorem 1(ii) one must have $D_i(c_I) = 0$ for all $i < I$ if an equilibrium is to exist at all. Hence the highest type must have preferences different enough from those of other types.

4 Proof Outline

Throughout this section, we suppose the existence of an equilibrium $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ with convex tariffs, and we investigate its properties. Recall that from the viewpoint of seller k the aggregate tariff $\mathfrak{T}^{-k}$ of the sellers other than k can be computed from the tariffs $\mathfrak{t}^{k'}$ as in (2). In turn $\mathfrak{T}^{-k}$ determines how type i evaluates any bundle (q, t) she may trade with seller k through the following indirect utility function

$$\mathfrak{z}_i^{-k}(q, t) \equiv \max_{Q^{-k} \in \mathbb{R}_+} \{u_i(q + Q^{-k}, t + \mathfrak{T}^{-k}(Q^{-k}))\}. \tag{3}$$

Observe that the maximum in (3) is always attained and that the indirect utility functions $\mathfrak{z}_i^{-k}$, when their value is finite, are strictly decreasing in t and continuous in (q, t).[6]

Two types of arguments are used in the proof. Some rely only on the convexity of tariffs and preferences. Because we only assume weak convexity, given a convex function $f : \mathbb{R}_+ \to \mathbb{R}$ we use the notation $\partial f(x)$, $\partial^- f(x)$, and $\partial^+ f(x)$ to denote respectively the subdifferential of f at x, the minimum element of $\partial f(x)$, and the maximum element of $\partial f(x)$. Hence $\partial f(x) = [\partial^- f(x), \partial^+ f(x)]$. Other arguments rely on single-crossing properties, in particular when it comes to examining the buyer's best response to a deviation. Most often the deviations we consider correspond to finite menus, including as many options as there are types. We denote such a menu by $\{(0, 0), \ldots, (q_i, t_i), \ldots\}$.

Finally, we say that individual quantities are nondecreasing if, given a family of tariffs, the quantities q_i^k traded by each type i with each seller k are such that for any k and $i < I$ one has $q_i^k \leq q_{i+1}^k$.

4.1 The Buyer's Behavior

Consider first the buyer's choice problem when she faces an arbitrary family of convex tariffs. When these tariffs are strictly convex, the buyer clearly has a unique best response, with individual quantities that are nondecreasing in her type. On the other hand, when some

[6] The last statement follows from Berge's maximum theorem (Aliprantis and Border (2006, Theorem 17.31)).

tariffs are affine with the same slope on some intervals of quantities, then the buyer may have multiple best responses. Still we can show the following result.

Lemma 1 *Let $(t^1, \ldots, t^k)$ be a family of convex tariffs. Then the buyer has a best response to $(t^1, \ldots, t^k)$ with nondecreasing individual quantities.*

The proof of Lemma 1 introduces some notations and additional results that will be used later on. It only relies on convexity, by showing the existence of a best response with individual quantities that are comonotonic with aggregate quantities.

Consider next the choice problem faced by the buyer in her relationship with any seller k, fixing the equilibrium tariffs $\mathfrak{t}^{k'}$ of the sellers other than k. From these tariffs one can build $\mathfrak{T}^{-k}$ as in (2), and $\mathfrak{z}_i^{-k}$ as in (3). The convexity of the aggregate tariff $\mathfrak{T}^{-k}$ crucially implies that the indirect utility functions $\mathfrak{z}_i^{-k}$ inherit a weak single-crossing property from the primitive utility functions u_i.

Lemma 2 *For all k, $i < i'$, $q \leq q'$, t, and t',*

$$\mathfrak{z}_i^{-k}(q,t) \leq \mathfrak{z}_i^{-k}(q',t') \quad \textit{implies} \quad \mathfrak{z}_{i'}^{-k}(q,t) \leq \mathfrak{z}_{i'}^{-k}(q',t'), \tag{4}$$

$$\mathfrak{z}_i^{-k}(q,t) < \mathfrak{z}_i^{-k}(q',t') \quad \textit{implies} \quad \mathfrak{z}_{i'}^{-k}(q,t) < \mathfrak{z}_{i'}^{-k}(q',t'). \tag{5}$$

In words, higher types are more eager to buy higher quantities from a given seller. As an application, suppose that seller k deviates and posts an arbitrary tariff t^k. From the viewpoint of seller k, type i's maximization problem amounts to

$$\max_{q^k \in \mathbb{R}_+} \{\mathfrak{z}_i^{-k}(q^k, t^k(q^k))\}. \tag{6}$$

Given Lemma 2, it follows from standard monotone-comparative-statics considerations that there exists for each i a solution to (6) that is nondecreasing in i. Lemma 2 therefore complements Lemma 1: if all tariffs but the k^{th} one are convex, then there exists a best response of the buyer such that the quantities traded with seller k are nondecreasing in her type. This property, which plays a central role in our analysis, suggests that the restriction to convex equilibria allows one to make use of standard screening techniques.

4.2 How the Sellers Can Break Ties

We now consider the behavior of a single seller k, in a situation in which all other sellers post their equilibrium tariffs $\mathfrak{t}^{k'}$. Suppose that seller k deviates to a menu $\{(0,0), \ldots, (q_i, t_i), \ldots\}$.

For each type i of the buyer to select the trade (q_i, t_i) in this menu, it must be that the following incentive-compatibility and individual-rationality constraints hold for all i and i':

$$\mathfrak{z}_i^{-k}(q_i, t_i) \geq \mathfrak{z}_i^{-k}(q_{i'}, t_{i'}), \tag{7}$$

$$\mathfrak{z}_i^{-k}(q_i, t_i) \geq \mathfrak{z}_i^{-k}(0, 0). \tag{8}$$

These constraints are not sufficient to ensure that each type i will choose to trade (q_i, t_i) after the deviation. Indeed, a given type may be indifferent between two trades, thus creating some ties. The following result shows that, as long as he sticks to nondecreasing quantities, seller k can secure the profit he would obtain if he could break ties in his favor. Define

$$\mathfrak{V}^k(\mathfrak{t}^{-k}) \equiv \sup \left\{ \sum_{i=1}^{I} m_i(t_i - c_i q_i) \right\} \tag{9}$$

over all menus $\{(0,0), \ldots, (q_i, t_i), \ldots\}$ that satisfy (7)–(8) for all i and i', and that have nondecreasing quantities $q_{i+1} \geq q_i$ for all $i < I$.

Lemma 3 *In an equilibrium $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ with convex tariffs, seller k's profit is no less than $\mathfrak{V}^k(\mathfrak{t}^{-k})$.*

Any seller k can thus control the quantities he trades with the buyer if, given the other sellers' tariffs, he deviates to an incentive-compatible menu that displays nondecreasing quantities. This last requirement is not a direct consequence of (7)–(8), given that the buyer's preferences only satisfy the *weak* single-crossing property characterized in Lemma 2. Indeed, this requirement is likely to be costly because, given Assumption 3, any seller would prefer to sell less to higher types. However, it cannot be dispensed with as the buyer always has a best response with nondecreasing quantities. Therefore, $\mathfrak{V}^k(\mathfrak{t}^{-k})$ is the highest payoff that seller k may expect by deviating, if he faces a buyer who systematically selects a best response with nondecreasing quantities.

The proof for Lemma 3 goes as follows. Consider a menu of trades that verifies the constraints in the Lemma, and suppose that two consecutive types i and $i + 1$ are both indifferent between their trade and the other type's trade. Then seller k can modify his menu by pooling both types on the same trade. Under Assumption 3, because $q_i \leq q_{i+1}$ this can be done without reducing the profits on the right-hand side of (9). This first step is key to the proof, as it shows that between two neighboring types only one incentive-compatible constraint can be binding. The proof then shows that seller k can slightly perturb the transfers in the menus so as to make all the relevant incentive-compatibility constraints slack. Hence the buyer has a unique best response, which guarantees that seller k gets the profit on the right-hand side of (9).

4.3 Equilibria with Nondecreasing Quantities

The above results suggest that we first focus on equilibria with nondecreasing individual quantities, that is $q_i^k \leq q_{i+1}^k$ for all k and $i < I$. In this section, we characterize these equilibria. We then show in Section 4.4 that the latter restriction on the buyer's behavior actually is inconsequential.

So suppose that such an equilibrium $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ exists. The equilibrium trades of seller k then verify all the constraints in program (9). An immediate consequence of Lemma 3 is thus that these trades must be solution to this program, and that the equilibrium profit of seller k is equal to $\mathfrak{V}^k(\mathfrak{t}^{-k})$. Considering program (9), it is clear that for each type i at least one constraint must bind, for, otherwise, one could slightly increase t_i. Our next result relies on Lemma 2 to determine which constraints are binding.

Lemma 4 *Let* $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ *be an equilibrium with convex tariffs and nondecreasing individual quantities. Then, for any seller* k, *if for some* i *the equilibrium trades of type* i *are such that the individual-rationality constraint (8) is slack, one has* $i \geq 2$ *and the incentive-compatibility constraint (7) for* $i' = i - 1$ *binds.*

Therefore, from the perspective of each seller, the individual-rationality constraint binds at the bottom, or more generally for all types below a threshold, and the downward local incentive-compatibility constraints bind for all other types. This result is reminiscent of those obtained under monopolistic screening, with the difference that they are formulated in terms of the indirect utility functions $\mathfrak{z}_i^{-k}$ instead of the primitive utility functions u_i. Under monopolistic screening, the aim is to characterize Pareto-optimal allocations, which implies that ties are broken in the most favorable way to the monopolist.[7] In our competitive setting, Lemma 3 offers a condition under which the seller can break ties as desired, namely, that quantities are nondecreasing. This allows us to proceed without introducing further restrictions on the buyer's behavior.

Our next result builds on Lemma 4 to show that equilibria with convex tariffs and nondecreasing quantities actually feature linear pricing if trade takes place at all.

Lemma 5 *Let* $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ *be an equilibrium with convex tariffs and nondecreasing individual quantities such that some trade takes place in equilibrium. Then there exists* $p \in \mathbb{R}$ *such that all trades take place at unit price* p, *and each type* i *purchases* $D_i(p)$ *in the aggregate.*

[7] See Hellwig (2010) for a complete treatment of the monopolistic case under weak single-crossing and private values, and Chade and Schlee (2012) for a simpler approach to the common values case.

The proof of Lemma 5 goes as follows. When sellers offer convex tariffs, every best response of each type i is such that she buys the last unit of the good at some price p_i, independently of the sellers she trades with. Because the corresponding aggregate quantity is nondecreasing in the type, it is easily shown that one must have $p_i \geq p_{i-1}$. Consider now an equilibrium, and suppose that type i trades at a price $p_i > p_{i-1}$. Clearly, it is not optimal for type i to mimic type $i-1$ and trade the quantity q^k_{i-1} with seller k, as this would imply trading at a marginal price different from p_i. Hence the downward local incentive constraint from type i to type $i-1$ cannot bind. A fortiori, it is not optimal for type i to trade a zero quantity with seller k. Hence the individual rationality constraint of type i cannot bind. But these results contradict Lemma 4.

We now show that each equilibrium trade must yield zero profit to the seller who makes it. The intuition is simple. Under linear pricing, sellers collectively have to share a risky demand $D_i(p)$. Under Assumption 2, we know that $D_I(p) > D_{I-1}(p)$ if some trade takes place at all, so the price p must be high enough to convince some of the sellers to provide additional quantities to the highest type. In fact, one must have $p \geq c_I$, otherwise a seller could deviate by posting a limit order with unit price p and maximum quantity q^k_{I-1}. On the other hand, aggregate profits cannot be positive, by a standard Bertrand argument. Because c_I is the highest possible cost, we get the following result.

Lemma 6 *Let* $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ *be an equilibrium with convex tariffs and nondecreasing individual quantities such that some trade takes place in equilibrium. Then, for* p *defined as in Lemma 5, we have* $p = c_i = c_I$ *for any type* i *who actively trades.*

The proof of this result, unlike that of Lemmas 1 to 5, relies on Assumption 2. If we relax it, we can still prove that in equilibrium the types who trade are exactly those above a threshold i_0, and that the equilibrium price is $\mathbf{E}[c_i \,|\, i \geq i_0]$. Moreover, these types must demand exactly the same aggregate quantity at that price, implying in most setups that there is only one such type $i_0 = I$, leaving Theorem 1 unaffected.

4.4 Other Equilibrium Outcomes

It follows from Lemmas 5 and 6 that the conclusions of Theorem 1 hold in the case of equilibria with nondecreasing individual quantities. To complete the proof of Theorem 1, we now show how to turn any equilibrium with convex tariffs into an equilibrium with the same tariffs, but now with nondecreasing quantities.

So let $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ be an equilibrium with convex tariffs. Let $\mathfrak{v}^k$ be the equilibrium profits of seller k. Lemma 3 offered a lower bound $\mathfrak{V}^k(\mathfrak{t}^{-k})$ for this profit. We can build another lower bound by imposing in program (9) the additional constraint that the transfers t_i must be computed using the equilibrium schedule $\mathfrak{t}^k$. So define

$$\underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \equiv \sup \left\{ \sum_{i=1}^{I} m_i[\mathfrak{t}^k(q_i) - c_i q_i] \right\} \tag{10}$$

over all $(q_1, \ldots, q_I) \in \mathbb{R}_+^I$ that satisfy

$$\mathfrak{z}_i^{-k}(q_i, \mathfrak{t}^k(q_i)) \geq \mathfrak{z}_i^{-k}(q_{i'}, \mathfrak{t}^k(q_{i'})), \tag{11}$$

$$\mathfrak{z}_i^{-k}(q_i, \mathfrak{t}^k(q_i)) \geq \mathfrak{z}_i^{-k}(0, 0), \tag{12}$$

and such that $q_{i+1} \geq q_i$ for all $i < I$. By Lemma 3, we therefore have

$$\mathfrak{v}^k \geq \mathfrak{V}^k(\mathfrak{t}^{-k}) \geq \underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \tag{13}$$

for all k. Now, recall from Lemma 1 that the buyer has at least one best response with nondecreasing individual quantities. Choose one such best response, and let $\mathfrak{v}'^k$ be the resulting profit for seller k. Because the corresponding trades for seller k verify the constraints in the above program, one must have

$$\underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \geq \mathfrak{v}'^k \tag{14}$$

for all k. Finally, given the convexity of the tariffs $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$, the aggregate quantities Q_i and the aggregate transfers T_i are the same for any best response of the buyer. Due to the linearity of the sellers' profits, we get $\sum_k \mathfrak{v}^k = \sum_i m_i[T_i - c_i Q_i] = \sum_k \mathfrak{v}'^k$. Using the inequalities (13)–(14), we finally obtain $\mathfrak{v}^k = \mathfrak{V}^k(\mathfrak{t}^{-k}) = \underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) = \mathfrak{v}'^k$ for all k.

This proves in particular that, in any equilibrium, each seller k earns $\mathfrak{V}^k(\mathfrak{t}^{-k})$. Therefore, no seller can get more than the profit he could secure by sticking to nondecreasing quantities. If we now specify that the buyer's strategy must select nondecreasing quantities whenever possible, it is easily understood that with this new strategy we have built an equilibrium with nondecreasing individual quantities. This last result is proven more formally in the Appendix.

Lemma 7 *If $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ is an equilibrium with convex tariffs, then there exists a strategy $\hat{\mathfrak{s}}$ for the buyer such that $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \hat{\mathfrak{s}})$ is an equilibrium with nondecreasing individual quantities that yields the same profit to each seller.*

Note that the aggregate equilibrium quantities Q_i and the indirect utility functions $\mathfrak{z}_i^{-k}$ are the same in the initial and the final equilibrium. Combining Lemmas 5, 6, and 7 then shows that Theorem 1 applies to all equilibria with convex tariffs.

5 Extensions

So far, we have assumed that sellers have constant and possibly type-dependent marginal costs. An examination of the proof of Lemmas 1–5 reveals that we can handle much more general cases. We now endow each seller k with a profit function $v_i^k(q,t)$, which we take to be continuous and strictly increasing in t, and such that the following generalized reverse single-crossing assumption holds.

Assumption 4 *For all k, $i < i'$, $q < q'$, t, and t',*

$$v_i^k(q,t) \geq v_i^k(q',t') \quad \textit{implies} \quad v_{i'}^k(q,t) \geq v_{i'}^k(q',t').$$

Each seller k therefore weakly prefers to sell lower quantities to higher types. Then the following result holds.

Corollary 1 *Under Assumptions 1–2 and 4, any equilibrium with convex tariffs and nondecreasing individual quantities such that some trade takes place in equilibrium displays linear pricing: there exists $p \in \mathbb{R}$ such that all trades take place at unit price p, and each type i purchases $D_i(p)$ in the aggregate.*

Extending this result to equilibria with quantities that may be decreasing requires some additional structure. Assume that each seller's cost of providing type i with a quantity q is $c_i(q)$, where $c_i : \mathbb{R}_+ \to \mathbb{R}_+$ is now a strictly convex cost function, with $c_i(0) = 0$. In this setting, the analogue of Assumption 4 can be stated in terms of the one-sided derivatives of these cost functions.

Assumption 5 *For all $i < i'$ and $q < q'$,*

$$\partial^- c_{i'}(q') \geq \partial^+ c_i(q).$$

Assumption 5 is consistent with private-value and common-value environments. Theorem 1 generalizes as follows.

Theorem 2 *Suppose that Assumptions 1–2 and 5 are satisfied, and let $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ be an equilibrium with convex tariffs. If some trade takes place in equilibrium, then there exists $p \in \mathbb{R}$ solution to*

$$p \in \partial c_I\left(\frac{D_I(p)}{K}\right) \tag{15}$$

and such that:

(i) *All trades take place at unit price p and each type i purchases $D_i(p)$ in the aggregate, and $D_i(p)/K$ from each seller.*

(ii) *Only type I actively trades in equilibrium:*

$$D_1(p) = \ldots = D_{I-1}(p) = 0 < D_I(p). \tag{16}$$

When there is a single type I, this result states that any equilibrium is competitive in the sense that the equilibrium price equalizes type I's demand and the sum of the sellers' supplies. Equilibrium outcomes are hence first-best efficient, as in the case of linear costs. The introduction of multiple types does not affect this property, the only change being that all types below I must demand a zero quantity at the equilibrium price.

The structure of the proof of Theorem 2 is similar to that of Theorem 1. First, given Corollary 1, one has to show that the result holds for all equilibria with convex tariffs and nondecreasing individual quantities.

Lemma 8 *Let $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ be an equilibrium with convex tariffs and nondecreasing individual quantities such that some trade takes place at price p in equilibrium. Then p satisfies (15)–(16).*

The result that no trade may take place except perhaps at the top of the buyer's type distribution now holds whether or not the environment features common values. As in the linear cost case, sellers collectively have to share a risky demand $D_i(p)$, but under convex costs the precise sharing now matters. Under Assumption 2, we know that $D_I(p) > D_{I-1}(p)$, so the price p must be high enough to convince some of the sellers to provide additional quantities to the highest type. In fact, one must have $p \geq \partial^- c_I(q_I^k)$ for all k, otherwise seller k could deviate by posting a limit order with a unit price p and a maximum quantity slightly below q_I^k. But, at such a high price, sellers are willing to sell high quantities to lower types, which is consistent with equilibrium only if all these types demand a zero quantity.

To complete the proof of Theorem 2, there thus only remains to show that the restriction to equilibria with nondecreasing individual quantities is innocuous. To this end, consider an equilibrium $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \mathfrak{s})$ with convex tariffs. Denote by $\mathfrak{v}^k$ the equilibrium profits of seller k. Replacing the linear cost functions in the definitions (9) and (10) of $\mathfrak{V}^k(\mathfrak{t}^{-k})$ and $\underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$ by the now convex cost functions, we formally get the lower bound (13) for the profits $\mathfrak{v}^k$. On the other hand, the sum of these profits cannot exceed the value they would reach if the buyer were to break ties in favor of the coalition of sellers. Formally, define

$$\mathfrak{V}^0(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \equiv \sup \left\{ \sum_k \sum_i m_i [\mathfrak{t}^k(q_i) - c_i(q_i)] \right\} \tag{17}$$

over all $(q_1, \ldots, q_I) \in \mathbb{R}_+^I$ that satisfy (11)–(12). Note that we do not impose the constraint that quantities be nondecreasing. We thus have

$$\mathfrak{V}^0(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \geq \sum_k \mathfrak{v}^k. \tag{18}$$

But the program (17) defining $\mathfrak{V}^0(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$ can be simplified into

$$\inf \left\{ \sum_k \sum_i m_i c_i(q_i) \right\}$$

under the same constraints, as the aggregate transfer chosen by the buyer is uniquely defined given the tariffs. The proof of Lemma 1 shows that such a risk-sharing problem admits a solution with nondecreasing individual quantities: this is the efficient manner to share risk. Let $\mathfrak{v}'^k$ be the associated profit for seller k; note that $\sum_k \mathfrak{v}'^k = \mathfrak{V}^0(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$. Moreover, in such a solution, each seller k trades a family of quantities that are nondecreasing, and thus his associated profit $\mathfrak{v}'^k$ must be no more than $\underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$. Summarizing, we get from (13) and (18) that

$$\sum_k \underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \geq \sum_k \mathfrak{v}'^k = \mathfrak{V}^0(\mathfrak{t}^1, \ldots, \mathfrak{t}^K) \geq \sum_k \mathfrak{v}^k \geq \sum_k \mathfrak{V}^k(\mathfrak{t}^{-k}) \geq \sum_k \underline{\mathfrak{V}}^k(\mathfrak{t}^1, \ldots, \mathfrak{t}^K),$$

and thus these inequalities are in fact equalities. In particular, this implies for every k that $\mathfrak{v}^k = \mathfrak{V}^k(\mathfrak{t}^{-k})$. We can then apply Lemma 7 without changes.

Appendix

Proof of Lemma 1. Recall that given a family $(t^1, \ldots, t^K)$ of convex tariffs, the aggregate equilibrium demand Q_i of type i is uniquely defined and nondecreasing in i. Given Q_i, type i's utility-maximization problem (1) reduces to minimizing total payment for Q_i:

$$\min \left\{ \sum_k t^k(q^k) : q^k \in \mathbb{R}_+ \text{ for all } k \text{ and } \sum_k q^k = Q_i \right\}.$$

This is a convex program, so that by the Kuhn–Tucker theorem one can associate to any solution $(q^1, \ldots, q^K)$ a Lagrange multiplier p_i such that $p_i \in \partial t^k(q^k)$ for all k. If there are two different solutions $(q^1, \ldots, q^K)$ and $(q'^1, \ldots, q'^K)$ with different multipliers $p_i < p'_i$, then because each tariff is convex one obtains $q^k \leq q'^k$ for all k, and because both solutions sum to the same Q_i they must be identical, a contradiction. This shows that two different solutions must share the same p_i. Consequently one can associate to each type i a price p_i such that whatever the solution $(q^1, \ldots, q^K)$ to type i's problem, one has $p_i \in \partial t^k(q^k)$ for all k. Moreover, by the same argument as above, p_i is nondecreasing in i.

For each i and each k, one can thus build the nonempty set $\{q : p_i \in \partial t^k(q)\}$. Let $\underline{s}_i^k$ be its minimum element, and let $\overline{s}_i^k$ be its maximum. Both $\underline{s}_i^k$ and $\overline{s}_i^k$ are nondecreasing in i. The interval $[\underline{s}_i^k, \overline{s}_i^k]$ is in fact the set of quantities that are provided by seller k at a marginal price equal to p_i. If this interval is nontrivial, then t^k is affine over it, with slope p_i. Consequently solutions $(q_i^1, \ldots, q_i^K)$ to type i payment minimization problem must verify

$$\sum_k q_i^k = Q_i \text{ and } \underline{s}_i^k \leq q_i^k \leq \overline{s}_i^k \text{ for all } k, \tag{19}$$

and these conditions are in fact sufficient, as all tariffs have the same slope p_i for quantities in these intervals. Our problem thus reduces to find a family of nondecreasing quantities verifying (19). We in fact prove a stronger result, which will be useful for future reference. Choose a family of strictly convex functions $(f_1, \ldots, f_I)$, and consider the following family of problems, indexed by i:

$$\min \left\{ \sum_k f_i(q_i^k) \right\}$$

subject to (19). By strict convexity of the functions f_i, each such problem admits a unique solution. We show below that the family of these solutions must display nondecreasing individual quantities. This naturally implies the existence of a family with nondecreasing individual quantities verifying (19), and shows the lemma.

To do so, proceed by contradiction and suppose that a family of solutions has $q_i^k > q_{i+1}^k$, for some k and $i < I$. Under (19), this implies

$$\underline{s}_i^k \le \underline{s}_{i+1}^k \le q_{i+1}^k < q_i^k \le \overline{s}_i^k \le \overline{s}_{i+1}^k. \tag{20}$$

Because the intervals for i and $i+1$ have a nontrivial intersection, it must be that $p_i = p_{i+1}$. Therefore, for any seller k' we have $\underline{s}_i^{k'} = \underline{s}_{i+1}^{k'}$ and $\overline{s}_i^{k'} = \overline{s}_{i+1}^{k'}$. Moreover, because $q_i^k > q_{i+1}^k$ and $Q_i \le Q_{i+1}$, we know that there exists $k' \neq k$ such that $q_i^{k'} < q_{i+1}^{k'}$. Using the equalities we have just shown, this implies

$$\underline{s}_i^{k'} = \underline{s}_{i+1}^{k'} \le q_i^{k'} < q_{i+1}^{k'} \le \overline{s}_i^{k'} = \overline{s}_{i+1}^{k'}. \tag{21}$$

Given (20)–(21), one can slightly reduce q_i^k and increase $q_i^{k'}$ by the same amount, so that (19) is still verified. Because $(q_i^1, \ldots, q_i^K)$ is assumed to minimize $\sum_k f_i(q_i^k)$, it must be that at the margin $-\partial^- f_i(q_i^k) + \partial^+ f_i(q_i^{k'}) \ge 0$. Because f_i is strictly convex, this implies that $q_i^k \le q_i^{k'}$. Alternatively, one could slightly increase q_{i+1}^k, and reduce $q_{i+1}^{k'}$ by the same amount. Once more, it must be that at the margin $\partial^+ f_{i+1}(q_{i+1}^k) - \partial^- f_{i+1}(q_{i+1}^{k'}) \ge 0$. Because f_{i+1} is strictly convex, this implies that $q_{i+1}^{k'} \le q_{i+1}^k$.

Overall we thus have shown that $q_i^k \le q_i^{k'} < q_{i+1}^{k'} \le q_{i+1}^k$, in contradiction with our assumption that $q_i^k > q_{i+1}^k$. This concludes the proof. ■

Proof of Lemma 2. Fix some k, $i < I$, $q < q'$, t, and t'. Let $\mathfrak{T}(Q) \equiv t + \mathfrak{T}^{-k}(Q - q)$, defined for $Q \ge q$. Similarly, let $\mathfrak{T}'(Q) \equiv t' + \mathfrak{T}^{-k}(Q - q')$, defined for $Q \ge q'$. According to (3), computing $\mathfrak{z}_i^{-k}(q,t)$ amounts to maximize $u_i(Q, \mathfrak{T}(Q))$ with respect to $Q \ge q$. Let $Q_i \ge q$ be the solution to this problem; it is unique as u_i is strictly quasiconcave and strictly decreasing in aggregate transfers, and $\mathfrak{T}(Q)$ is convex in Q. Similarly, computing $\mathfrak{z}_i^{-k}(q', t')$ amounts to maximize $u_i(Q, \mathfrak{T}'(Q))$ with respect to $Q \ge q'$. Let $Q_i' \ge q'$ be the unique solution to this problem. Suppose that

$$\mathfrak{z}_i^{-k}(q,t) < \mathfrak{z}_i^{-k}(q',t') \tag{22}$$

and let $i' > i$. Because $Q_{i'} \ge q$ is an admissible candidate in the problem that defines $\mathfrak{z}_i^{-k}(q,t)$, we must have

$$u_i(Q_{i'}, \mathfrak{T}(Q_{i'})) \le \mathfrak{z}_i^{-k}(q,t) < \mathfrak{z}_i^{-k}(q',t') = u_i(Q_i', \mathfrak{T}'(Q_i')).$$

Suppose first that $Q_{i'} < Q_i'$. Using Assumption 1, we get

$$\mathfrak{z}_{i'}^{-k}(q,t) = u_{i'}(Q_{i'}, \mathfrak{T}(Q_{i'})) < u_{i'}(Q_i', \mathfrak{T}'(Q_i')) \le \mathfrak{z}_{i'}^{-k}(q',t'),$$

where the last inequality stems from the fact that $Q'_i \geq q'$ is an admissible candidate in the problem that defines $\mathfrak{z}_{i'}^{-k}(q', t')$. This shows (5) in this case.

Otherwise we have $Q_{i'} \geq Q'_i \geq q'$. Then $Q_{i'}$ is an admissible candidate in the problem that defines $\mathfrak{z}_{i'}^{-k}(q', t')$, and we get

$$u_{i'}(Q_{i'}, \mathfrak{T}'(Q_{i'})) \leq \mathfrak{z}_{i'}^{-k}(q', t').$$

If $\mathfrak{T}'(Q_{i'}) < \mathfrak{T}(Q_{i'})$, we obtain

$$\mathfrak{z}_{i'}^{-k}(q, t) = u_{i'}(Q_{i'}, \mathfrak{T}(Q_{i'})) < u_{i'}(Q_{i'}, \mathfrak{T}'(Q_{i'})) \leq \mathfrak{z}_{i'}^{-k}(q', t'),$$

which shows (5) in this case.

The only remaining case is when $Q_{i'} \geq Q'_i \geq q'$ and $\mathfrak{T}'(Q_{i'}) \geq \mathfrak{T}(Q_{i'})$. Note that because $q < q'$ and $\mathfrak{T}^{-k}$ is convex, $\mathfrak{T}'(Q) - \mathfrak{T}(Q)$ is nonincreasing in Q for $Q \geq q'$. Because $Q_{i'} \geq Q'_i \geq q'$, we get $\mathfrak{T}'(Q'_i) \geq \mathfrak{T}(Q'_i)$ and thus, as $Q'_i \geq q' > q$ is an admissible candidate in the problem that defines $\mathfrak{z}_i^{-k}(q, t)$,

$$\mathfrak{z}_i^{-k}(q, t) \geq u_i(Q'_i, \mathfrak{T}(Q'_i)) \geq u_i(Q'_i, \mathfrak{T}'(Q'_i)) = \mathfrak{z}_i^{-k}(q', t'),$$

in contradiction with (22). Hence we have shown (5). The proof of (4) follows by continuity. Indeed, assume that $\mathfrak{z}_i^{-k}(q, t) = \mathfrak{z}_i^{-k}(q', t')$. Then, for each $\varepsilon > 0$, $\mathfrak{z}_i^{-k}(q, t + \varepsilon) < \mathfrak{z}_i^{-k}(q, t)$ and thus $\mathfrak{z}_{i'}^{-k}(q, t + \varepsilon) < \mathfrak{z}_{i'}^{-k}(q, t)$ from (5). Because $\mathfrak{z}_{i'}^{-k}$ is continuous, one can take limits as ε goes to zero to obtain (4). The result follows. ■

Proof of Lemma 3. The proof consists of two steps.

Step 1 Pick a menu $\mu = \{(0, 0), \ldots, (q_i, t_i), \ldots\}$ that satisfies the incentive-compatibility and individual-rationality constraints (7)–(8) for all i and i', and that has nondecreasing quantities $q_{i+1} \geq q_i$ for all $i < I$. We build a new menu $\mu' = \{(0, 0), \ldots, (q'_i, t'_i), \ldots\}$ by applying the following algorithm. At each step $n \geq 0$ of the algorithm, let $\mu^{(n)} = \{(0, 0), \ldots, (q_i^{(n)}, t_i^{(n)}), \ldots\}$ be the current menu, with $\mu^{(0)} \equiv \mu$ by convention. If there exists $i < I$ such that $q_i^{(n)} < q_{i+1}^{(n)}$ and the following local incentive-compatibility constraints both bind:

$$\begin{aligned} \mathfrak{z}_i^{-k}\big(q_i^{(n)}, t_i^{(n)}\big) &= \mathfrak{z}_i^{-k}\big(q_{i+1}^{(n)}, t_{i+1}^{(n)}\big), \\ \mathfrak{z}_{i+1}^{-k}\big(q_{i+1}^{(n)}, t_{i+1}^{(n)}\big) &= \mathfrak{z}_{i+1}^{-k}\big(q_i^{(n)}, t_i^{(n)}\big), \end{aligned}$$

then take the smallest such i, $i^{(n)}$, and pool types $i^{(n)}$ and $i^{(n)} + 1$ on the same trade $\big(q_{i^{(n)}}^{(n+1)}, t_{i^{(n)}}^{(n+1)}\big) = \big(q_{i^{(n)}+1}^{(n+1)}, t_{i^{(n)}+1}^{(n+1)}\big)$ equal to either $\big(q_{i^{(n)}}^{(n)}, t_{i^{(n)}}^{(n)}\big)$ or $\big(q_{i^{(n)}+1}^{(n)}, t_{i^{(n)}+1}^{(n)}\big)$ according to the maximum value it gives to the profit on the $(i^{(n)}, i^{(n)}+1)$ pair

$$m_{i^{(n)}}\left[t^{(n+1)}_{i^{(n)}} - c_{i^{(n)}}q^{(n+1)}_{i^{(n)}}\right] + m_{i^{(n)}+1}\left[t^{(n+1)}_{i^{(n)}} - c_{i^{(n)}+1}q^{(n+1)}_{i^{(n)}}\right].$$

Otherwise, the algorithm stops, and $\mu' \equiv \mu^{(n)}$. Note that the algorithm stops in a finite number of steps as there are finitely many types. Moreover, applying the algorithm only affects the way ties are broken. Therefore, the menu μ' remains incentive compatible and individually rational. Moreover, by construction, it has nondecreasing quantities $q'_{i+1} \geq q'_i$ for all $i < I$. Finally, at each step of the algorithm, seller k's profit cannot be decreased. Indeed, the algorithm is active at step $n \geq 0$ only if $q^{(n)}_{i^{(n)}} < q^{(n)}_{i^{(n)}+1}$. In that case, either

$$t^{(n)}_{i^{(n)}} - c^{(n)}_{i^{(n)}}q^{(n)}_{i^{(n)}} < t^{(n)}_{i^{(n)}+1} - c^{(n)}_{i^{(n)}}q^{(n)}_{i^{(n)}+1}$$

and then seller k's profit is increased by pooling $i^{(n)}$ and $i^{(n)}+1$ on $\left(q^{(n)}_{i^{(n)}+1}, t^{(n)}_{i^{(n)}+1}\right)$, or

$$t^{(n)}_{i^{(n)}} - c^{(n)}_{i^{(n)}}q^{(n)}_{i^{(n)}} \geq t^{(n)}_{i^{(n)}+1} - c^{(n)}_{i^{(n)}}q^{(n)}_{i^{(n)}+1},$$

so that, as $c_{i^{(n)}} \leq c_{i^{(n)}+1}$ by Assumption 3 and $q^{(n)}_{i^{(n)}} < q^{(n)}_{i^{(n)}+1}$ by construction, seller k's profit cannot be decreased by pooling $i^{(n)}$ and $i^{(n)}+1$ on $\left(q^{(n)}_{i^{(n)}}, t^{(n)}_{i^{(n)}}\right)$. As a result,

$$\sum_{i=1}^{I} m_i(t'_i - c_i q'_i) \geq \sum_{i=1}^{I} m_i(t_i - c_i q_i), \tag{23}$$

that is, seller k's profit under μ' is as least as large as under μ.

Step 2 We may now proceed to the second step of the proof. Let $\varepsilon > 0$ be given. We are going to modify transfers $(t'_1, \ldots, t'_I)$ into transfers $(t''_1, \ldots, t''_I)$ such that the menu $\mu'' = \{(0,0), \ldots, (q'_i, t''_i), \ldots\}$ satisfies the following incentive-compatibility and individual-rationality constraints for all i and i':

$$\mathfrak{z}_i^{-k}(q'_i, t''_i) \geq \mathfrak{z}_i^{-k}(q'_{i'}, t''_{i'}), \tag{24}$$

$$\mathfrak{z}_i^{-k}(q'_i, t''_i) \geq \mathfrak{z}_i^{-k}(0,0), \tag{25}$$

where now these inequalities are *strict* as soon as, respectively, $q'_i \neq q'_{i'}$ and $q'_i \neq 0$. Moreover, we will perform this modification in such a way that transfers remain almost the same:

$$t''_i \geq t'_i - \varepsilon. \tag{26}$$

Suppose this modification performed. Then for each $\varepsilon > 0$ seller k could deviate to the menu μ''. Because of the above properties, each type i must then choose to trade (q'_i, t''_i) with seller k. Hence by playing so seller k can secure a profit

$$\sum_{i=1}^{I} m_i(t''_i - c_i q'_i) \geq \sum_{i=1}^{I} m_i(t'_i - c_i q'_i) - \varepsilon \geq \sum_{i=1}^{I} m_i(t_i - c_i q_i) - \varepsilon,$$

where the first and second inequalities follow from (26) and (23). As ε can be made arbitrarily small, this shows that seller k's equilibrium profit is at least (9), and the result follows.

To conclude the proof, there only remains to modify the transfers as announced above. We now turn to this task. Because the quantities $(q'_1, \ldots, q'_I)$ are given, and because the functions $\mathfrak{z}_i^{-k}$ are continuous and strictly decreasing in transfers, we can define two families of (extended) real-valued functions γ_i^k and δ_i^k for $i < I$ such that, for each t,

$$\mathfrak{z}_i^{-k}(q'_i, t) = \mathfrak{z}_i^{-k}(q'_{i+1}, \gamma_i^k(t)) \text{ and } \mathfrak{z}_{i+1}^{-k}(q'_i, t) = \mathfrak{z}_{i+1}^{-k}(q'_{i+1}, \delta_i^k(t)). \tag{27}$$

Here $\gamma_i^k(t) = -\infty$ or $\delta_i^k(t) = -\infty$ by convention if there exists no solution to the relevant equation, which may occur for t below some threshold. Both γ_i^k and δ_i^k are continuous and strictly increasing where they are finite. If $q'_i = q'_{i+1}$, then clearly $\gamma_i^k(t) = \delta_i^k(t) = t$. If $q'_i < q'_{i+1}$, then, according to (4) along with the fact that the functions $\mathfrak{z}_i^{-k}$ are strictly decreasing in transfers, $\gamma_i^k(t) \leq \delta_i^k(t)$ for all $t \geq 0$. Finally, by construction of the menu μ', if $q'_i < q'_{i+1}$, then $\gamma_i^k(t'_i) \leq t'_{i+1} \leq \delta_i^k(t'_i)$ with at least one strict inequality. (One may have $\gamma_i^k(t'_i) = -\infty$, but $\delta_i^k(t'_i)$ is necessarily finite.)

Given $\varepsilon > 0$, we can recursively construct a family of strictly positive real numbers $(\varepsilon_1, \ldots, \varepsilon_I)$ as follows. Let $\varepsilon_I \equiv \varepsilon$. Then, for each $i < I$, consider $\varepsilon_{i+1} > 0$ as given. If $q'_i = q'_{i+1}$, choose ε_i such that $0 < \varepsilon_i < \varepsilon_{i+1}$, which is clearly feasible. If $q'_i < q'_{i+1}$, choose ε_i such that $0 < \varepsilon_i < \varepsilon_{i+1}/2$ and such that

$$\gamma_i^k(t) < \delta_i^k(t) \text{ and } \gamma_i^k(t) - \frac{\varepsilon_{i+1}}{2} < t'_{i+1} < \delta_i^k(t) + \frac{\varepsilon_{i+1}}{2} \tag{28}$$

for all t that satisfy $|t - t'_i| < \varepsilon_i$. This is feasible because if $\varepsilon_{i+1} > 0$, all these properties hold for $t = t'_i$, and because the functions γ_i^k and δ_i^k are continuous at t'_i. Observe that the family $(\varepsilon_1, \ldots, \varepsilon_I)$ is strictly increasing.

We now recursively construct a family of transfers $(t''_1, \ldots, t''_I)$ such that $|t''_i - t'_i| < \varepsilon_i$ for all i. Set $t''_1 \equiv t'_1$ if $q'_1 = 0$, and set $t''_1 \equiv t'_1 - \varepsilon_1/2$ otherwise. Note that $|t''_1 - t'_1| < \varepsilon_1$. Suppose next that $|t''_i - t'_i| < \varepsilon_i$ for some $i < I$, and define t''_{i+1} as follows:

(i) If $q'_{i+1} = q'_i$, set $t''_{i+1} \equiv t''_i$. Note that because $t'_{i+1} = t'_i$ in this case, we then have $|t''_{i+1} - t'_{i+1}| = |t''_i - t'_i| < \varepsilon_i < \varepsilon_{i+1}$, as required.

(ii) If $q'_i < q'_{i+1}$, then, as $|t''_i - t'_i| < \varepsilon_i$ by assumption, we know from the first part of (28) that $\gamma_i^k(t''_i) < \delta_i^k(t''_i)$. Choose any $\hat{\varepsilon}$ such that $0 < \hat{\varepsilon} < \min\{\varepsilon_{i+1}/2, \delta_i^k(t''_i) - \gamma_i^k(t''_i)\}$, and consider the following three subcases. If $t'_{i+1} \geq \delta_i^k(t''_i)$, set $t''_{i+1} \equiv \delta_i^k(t''_i) - \hat{\varepsilon}$. If $t'_{i+1} \leq \gamma_i^k(t''_i)$, set $t''_{i+1} \equiv \gamma_i^k(t''_i) + \hat{\varepsilon}$. Otherwise, set $t''_{i+1} \equiv t'_{i+1}$. The second part of

(28) ensures that in each of these three subcases $|t''_{i+1} - t'_{i+1}| < \hat{\varepsilon} + \varepsilon_{i+1}/2 < \varepsilon_{i+1}$, as required.

By construction, we have $\gamma_i^k(t''_i) < t''_{i+1} < \delta_i^k(t''_i)$ if $q'_i < q'_{i+1}$. This shows that the local incentive-compatibility constraints

$$\mathfrak{z}_i^{-k}(q'_i, t''_i) \geq \mathfrak{z}_i^{-k}(q'_{i+1}, t''_{i+1}),$$
$$\mathfrak{z}_{i+1}^{-k}(q'_{i+1}, t''_{i+1}) \geq \mathfrak{z}_{i+1}^{-k}(q'_i, t''_i)$$

are satisfied, with strict inequalities if $q'_i < q'_{i+1}$. Similarly, our choice of t''_1 ensures that the individual rationality constraint for $i = 1$,

$$\mathfrak{z}_1^{-k}(q'_1, t''_1) \geq \mathfrak{z}_1^{-k}(0, 0),$$

is also a strict inequality if $q'_1 \neq 0$. Given the single-crossing property (4)–(5), a standard argument can be used to establish that this set of local constraints implies that the menu μ'' satisfies the incentive-compatibility and individual-rationality constraints (24)–(25) for all i and i', with strict inequalities as soon as, respectively, $q'_i \neq q'_{i'}$ and $q'_i \neq 0$. Finally, for each i we have $|t''_i - t'_i| < \varepsilon_i < \varepsilon_I = \varepsilon$, which yields (26). The result follows. ■

Proof of Lemma 4. Fix a seller k, and suppose by way of contradiction that

$$\mathfrak{z}_j^{-k}(q_j^k, \mathfrak{t}^k(q_j^k)) > \mathfrak{z}_j^{-k}(q_{j-1}^k, \mathfrak{t}^k(q_{j-1}^k)) \text{ and } \mathfrak{z}_j^{-k}(q_j^k, \mathfrak{t}^k(q_j^k)) > \mathfrak{z}_j^{-k}(0, 0) \tag{29}$$

for some $j \geq 2$. Because the quantities $(q_1^k, \ldots, q_I^k)$ are given, we can define a family of (extended) real-valued functions δ_i^k for $i < I$ as in (27). Using this notation, the first inequality in (29) equivalently says that $\mathfrak{t}^k(q_j^k) < \delta_{j-1}^k(\mathfrak{t}^k(q_{j-1}^k))$. Now, choose ε such that $0 < \varepsilon < \delta_{j-1}^k(\mathfrak{t}^k(q_{j-1}^k)) - \mathfrak{t}^k(q_j^k)$ and define a family of transfers $(t'_1, \ldots, t'_I)$ as follows: for $i < j$, set $t'_i \equiv \mathfrak{t}^k(q_i^k)$; for $i = j$, set $t'_j \equiv \mathfrak{t}^k(q_j^k) + \varepsilon$; for $i > j$, define recursively $t'_i \equiv \max\{\mathfrak{t}^k(q_i^k), \delta_{i-1}^k(t'_{i-1})\}$.

The menu $\{(0,0), \ldots, (q_i^k, t'_i), \ldots\}$ has three noticeable features:

(i) It has the same nondecreasing quantities and higher—sometimes strictly higher—transfers as the equilibrium allocation.

(ii) It satisfies the incentive-compatibility constraints (7). This is obvious for types $i < j$, because their transfers are unchanged, whereas the transfers of types $i' \geq j$ are (weakly) increased. As for type j, observe that she cannot be better off mimicking type $j - 1$ because $t'_j = \mathfrak{t}^k(q_j^k) + \varepsilon$, $t'_{j-1} = \mathfrak{t}^k(q_{j-1}^k)$, and ε has been chosen so that

$\mathfrak{z}_j^{-k}(q_j^k, \mathfrak{t}^k(q_j^k) + \varepsilon) > \mathfrak{z}_j^{-k}(q_{j-1}^k, \mathfrak{t}^k(q_{j-1}^k))$. Using the fact that the quantities $(q_1^k, \ldots, q_j^k)$ are nondecreasing along with the single-crossing property (4), it follows that type j cannot be better off mimicking any type $i < j - 1$ either. Finally, type j cannot be better off mimicking any type $i > j$. Indeed, suppose by way of contradiction that $i > j$ is the first type such that $\mathfrak{z}_j^{-k}(q_j^k, t'_j) < \mathfrak{z}_j^{-k}(q_i^k, t'_i)$. Then $\mathfrak{z}_j^{-k}(q_{i-1}^k, t'_{i-1}) = \mathfrak{z}_j^{-k}(q_j^k, t'_j) < \mathfrak{z}_j^{-k}(q_i^k, t'_i)$ so that, from $q_{i-1}^k \leq q_i^k$ along with the single-crossing property (5), we have $\mathfrak{z}_i^{-k}(q_{i-1}^k, t'_{i-1}) < \mathfrak{z}_i^{-k}(q_i^k, t'_i)$. But $\mathfrak{z}_i^{-k}(q_i^k, t'_i) \leq \mathfrak{z}_i^{-k}(q_i^k, \delta_{i-1}^k(t'_{i-1}))$, so that $\mathfrak{z}_i^{-k}(q_{i-1}^k, t'_{i-1}) < \mathfrak{z}_i^{-k}(q_i^k, \delta_{i-1}^k(t'_{i-1}))$, in contradiction with the definition (27) of δ_{i-1}^k. The claim follows. The proof that no type $i > j$ can be better off mimicking any type $i' \neq i$ is similar, and is therefore omitted.

(iii) It satisfies the individual-rationality constraints (8). This is obvious for types $i < j$, because their trade is unchanged. Thus in particular $\mathfrak{z}_{j-1}^{-k}(q_{j-1}^k, t'_{j-1}) \geq \mathfrak{z}_i^{-k}(0, 0)$. The single-crossing property (4) then implies that $\mathfrak{z}_{i'}^{-k}(q_{j-1}^k, t'_{j-1}) \geq \mathfrak{z}_{i'}^{-k}(0, 0)$ for all $i' \geq j$, from which the claim follows as the menu $\{(0, 0), \ldots, (q_i^k, t'_i), \ldots\}$ is incentive compatible.

Given (i)–(iii), we can apply Lemma 3 to conclude that this menu must give seller k at most his equilibrium profit. This however contradicts (i) above. The contradiction establishes that for each $j \geq 2$, at least one inequality in (29) must bind. The proof that the individual rationality constraint (8) binds at $i = 1$ is similar, and is therefore omitted. ■

Proof of Lemma 5. As shown in Lemma 1, there exists a nondecreasing sequence of prices $(p_1, \ldots, p_I)$ such that $p_i \in \partial \mathfrak{t}^k(q^k)$ for all k and all best responses $(q^1, \ldots, q^K)$ of type i. In fact, those best responses are exactly the quantities verifying (19) given the equilibrium tariffs $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$.

As a preliminary result, let us show that there is no type i such that $Q_i > 0$ and $q_i^k = \overline{s}_i^k$ for all k. Indeed, these conditions imply that type i has a unique best response that exhausts all available supply at the marginal price p_i. Moreover, because $Q_i > 0$ there exists at least one seller k such that $q_i^k > 0$. For any such k, the individual-rationality constraint (8) of type i is slack in equilibrium, because type i has a unique best response. This implies from Lemma 4 that $i \geq 2$, and that the incentive-compatibility constraint constraint (7) for $i' = i - 1$ binds in equilibrium. As type i has a unique best response, it must thus be that $q_i^k = q_{i-1}^k$. This equality also holds for those k such that $q_i^k = 0$, as individual quantities are nondecreasing by assumption. Overall we have shown that $i \geq 2$, and that type $i - 1$ also exhausts supply at price p_i, so that $p_{i-1} = p_i$, $q_{i-1}^k = \overline{s}_{i-1}^k = \overline{s}_i^k$ for all k, and $Q_{i-1} = Q_i > 0$.

We can then iterate the reasoning until reaching type 1, in contradiction with $i \geq 2$.

We can now complete the proof of Lemma 5. Suppose that for some $i > 1$ we have $p_i > p_{i-1}$. Then $\underline{s}_i^k \geq \overline{s}_{i-1}^k \geq q_{i-1}^k$ for all k. If for k the individual-rationality constraint (8) of type i binds in equilibrium, then it must be that $\underline{s}_i^k = 0$, and therefore that $q_{i-1}^k = 0 = \overline{s}_{i-1}^k$. If for k the individual-rationality constraint (8) of type i is slack in equilibrium, then from Lemma 4 the incentive-compatibility constraint constraint (7) for $i' = i - 1$ must bind in equilibrium, which implies $\underline{s}_i^k = q_{i-1}^k = \overline{s}_{i-1}^k$. Hence for all k we have $q_{i-1}^k = \overline{s}_{i-1}^k$. Applying our preliminary result, we have shown that $p_i > p_{i-1}$ implies that $Q_{i-1} = 0$. Because the aggregate quantity is nondecreasing, it must thus be that all trades take place at the same price p. Moreover, once more applying our preliminary result, a type i who actively trades cannot exhaust the aggregate supply at price p, and thus can freely choose her most preferred quantity at price p, which is $D_i(p)$. The result follows. ■

Proof of Lemma 6. Any seller k could deviate to a menu that would allow types $i < I$ to buy the equilibrium quantity q_i^k at price p, whereas type I would be asked to buy only q_{I-1}^k at price p. Such an offer is incentive compatible and individually rational, with nondecreasing quantities. From Lemma 3, the variation in the deviator's profit must be at most zero,

$$(p - c_I)(q_{I-1}^k - q_I^k) \leq 0.$$

Summing on k yields $(p - c_I)[D_{I-1}(p) - D_I(p)] \leq 0$, and under Assumption 3 this implies that $p \geq c_I$. On the other hand, if aggregate profits are strictly positive then, because the functions D_i are continuous, any buyer k could claim almost all profits for himself by charging a uniform unit price slightly below p. So aggregate profits are zero, and under $p \geq c_I$ we get that $p = c_i = c_I$ for any type i who actively trades. The result follows. ■

Proof of Lemma 7. From Lemma 1 we know that the buyer has a best response $\boldsymbol{q} \in \mathbb{R}_+^{IK}$ with nondecreasing individual quantities. Construct a strategy $\hat{\mathfrak{s}}$ for the buyer as follows:

(i) When the buyer faces the tariff profile $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$, she trades the quantities in $\boldsymbol{q}$.

(ii) When the buyer faces a unilateral deviation $(t^k, \mathfrak{t}^{-k})$ from the tariff profile $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K)$ by seller k, we know as a consequence of Lemma 2 that there exists a solution to (6) that is nondecreasing in i. Given the tariffs $(t^k, \mathfrak{t}^{-k})$, this corresponds to a solution to the buyer's utility-maximization problem that has nondecreasing quantities for seller k. Let $\hat{\mathfrak{s}}$ select this solution.

(iii) In all other cases, the strategy $\hat{\mathfrak{s}}$ simply selects a best response.

It remains to show that $(\mathfrak{t}^1, \ldots, \mathfrak{t}^K, \hat{\mathfrak{s}})$ is an equilibrium. First, note that for $\hat{\mathfrak{s}}$ is indeed a best response for the buyer. Second, we have shown in the text that each seller k earns exactly $\mathfrak{V}^k(\mathfrak{t}^{-k})$. Third, if seller k were to deviate, then according to $\hat{\mathfrak{s}}$ the buyer would react by trading with k a family of nondecreasing quantities. Then the resulting profit cannot exceed $\mathfrak{V}^k(\mathfrak{t}^{-k})$, by definition of this threshold. The result follows. ■

Proof of Corollary 1. Lemmas 1–2 remain valid, as they are only concerned with the buyer's behavior, and not with the sellers' profits. Our tie-breaking result, Lemma 3, needs obvious modifications in the definition of $\mathfrak{V}^k(\mathfrak{t}^{-k})$, which becomes the supremum of

$$\sum_{i=1}^{I} m_i v_i^k(q_i, t_i)$$

over the relevant set. Step 1 in the proof of Lemma 3 can easily be extended thanks to Assumption 4, as this condition is enough to ensure that when $q_i < q_{i+1}$ a seller cannot loose by pooling both types on either (q_i, t_i) or (q_{i+1}, t_{i+1}). Indeed, one will choose to pool both types on (q_{i+1}, t_{i+1}) when

$$v_i^k(q_i, t_i) \leq v_i^k(q_{i+1}, t_{i+1}).$$

Otherwise, the reverse inequality holds, and by applying Assumption 4 we get

$$v_{i+1}^k(q_i, t_i) \geq v_{i+1}^k(q_{i+1}, t_{i+1}),$$

which allows to pool both types on (q_i, t_i) without reducing profits. Step 2 of the proof requires no modification, as it only relies on modifications of transfers, leaving quantities and hence costs unaffected. Lemma 4 then follows immediately: the proof is the same as in the constant-marginal-cost case. Finally Lemma 5 also goes through. The result follows. ■

Proof of Lemma 8. Fix an equilibrium with convex tariffs and nondecreasing individual quantities in which all trades take place at price p and accordingly each type i purchases $D_i(p)$ in the aggregate. We first prove that (16) holds if at least one type actively trades in equilibrium, that is, if $D_I(p) > 0$. Fix some k and let

$$C_1 = \{(q_1, \ldots, q_I) \in \mathbb{R}_+^I : \text{there exists } (t_1, \ldots, t_I) \text{ such that (7)–(8) hold for all } (i, i')\},$$
$$C_2 = \{(q_1, \ldots, q_I) \in \mathbb{R}_+^I : q_{i+1} \geq q_i \text{ for all } i < I\},$$
$$C_3 = [0, D_1(p)] \times \ldots \times [0, D_I(p)].$$

Seller k's equilibrium profit $\mathfrak{v}^k$ satisfies

$$\mathfrak{v}^k = \max\left\{\sum_{i=1}^{I} m_i[pq_i - c_i(q_i)] : (q_1, \ldots, q_I) \in C_1 \cap C_2 \cap C_3\right\} \tag{30}$$

for all k, where the equality in (30) follows from Lemma 3 given that the equilibrium has nondecreasing individual quantities. In particular, $\mathfrak{v}^k$ is no larger than

$$\overline{\mathfrak{v}}^k \equiv \max\left\{\sum_{i=1}^{I} m_i[pq_i - c_i(q_i)] : (q_1, \ldots, q_I) \in C_2 \cap C_3\right\}. \tag{31}$$

The relaxed problem (31) is compact and strictly convex, and therefore admits a unique solution $(q_1^k, \ldots, q_I^k)$. Using the strict convexity of the cost functions c_i and Assumption 5, we get that for each $i < I$, $q_i^k < q_{i+1}^k$ implies that $q_i^k = D_i(p)$. As a result, we have $q_i^k = \min\{q_I^k, D_i(p)\}$ for all i. Consider now the menu $\{(0,0), \ldots, (q_i^k, pq_i^k), \ldots\}$. By the single-crossing property (4), this menu satisfies the incentive-compatibility and individual-rationality constraints (7)–(8); moreover, it has nondecreasing quantities at most equal to $D_I(p)$. Thus $(q_1^k, \ldots, q_I^k) \in C_1 \cap C_2 \cap C_3$. Because the above menu yields a profit $\overline{\mathfrak{v}}^k$ to seller k, we get from (30)–(31) that $\mathfrak{v}^k = \overline{\mathfrak{v}}^k$ and, by the strict concavity of the common objective function in (30)–(31), that $(q_1^k, \ldots, q_I^k)$ is the unique solution to (30). Because the equilibrium quantities traded by seller k must solve (30), we thus get that they are exactly given by $(q_1^k, \ldots, q_I^k)$. Moreover, because $(q_1^k, \ldots, q_I^k)$ solves (31), which is independent of k, so must be $(q_1^k, \ldots, q_I^k)$, which we can thus write as $(q_1, \ldots, q_I)$. All sellers hence trade the same quantities with each type in equilibrium, so that $Kq_i = D_i(p)$ for all i. As $q_i = \min\{q_I, D_i(p)\}$ for all i, two cases may now arise. If $q_i = D_i(p)$, then necessarily $D_i(p) = 0$. Alternatively, if $q_i < D_i(p)$, then $q_i = q_I$ and thus $Q_i = D_I(p)$. By Assumption 2, (16) must thus hold as soon as $D_I(p) > 0$, as claimed. To prove that (15) must then hold, simply observe by (31) that $q_I = D_I(p)/K > 0$ maximizes $pq - c_I(q)$. The result follows. ■

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