

# Cautious Expected Utility and the Certainty Effect\*

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## Abstract

One of the most prominently observed behavioral patterns in decision making under risk is the certainty effect: subjects tend to overvalue risk-free prospects, as shown, for example, in the Allais paradoxes. This tendency is captured behaviorally by Negative Certainty Independence (NCI), which requires that if a decision maker prefers the lottery  $p$  over a certain outcome  $x$ , then she will also prefer the mixture of  $p$  with any lottery  $q$  over the (same proportion) mix of  $x$  with  $q$ . We show that continuous and monotone preferences over some interval of prizes satisfy NCI if and only if they can be represented as follows: There exists a set  $\mathcal{W}$  of strictly increasing utility functions over outcomes, such that the value of any lottery  $p$ ,  $V(p)$ , is given by

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)),$$

where  $v^{-1}(\mathbb{E}_p(v))$  is the certainty equivalent of lottery  $p$  calculated using the utility function  $v$ . The decision maker acts as if she were unsure of how risk averse she should be; instead, she has in mind a set of possible utility functions. She then displays a cautious behavior: the value of each lottery is the smallest certainty equivalent using some utility function in the set. We discuss the uniqueness properties of the representation and show how it is related to the notion of a cautious completion of an incomplete preference relation.

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# 1 Introduction

Despite its ubiquitous presence in economic analysis, it is well known that the paradigm of Expected Utility is often violated in choices between risky prospects. While such violations have been documented in many different experiments, one preference pattern emerges as one of the most prominent: the tendency of people to favor certain (risk-free) options even in violation of Expected Utility – the so-called Certainty Effect. This is shown, for example, in the classic Common Ratio Effect, one of the Allais paradoxes, in which subjects face the following two choice problems:

1. A choice between  $A$  and  $B$ , where  $A$  is a degenerate lottery which yields \$3000 for sure and  $B$  is a lottery that yields \$4000 with probability 0.8 and \$0 with probability 0.2.
2. A choice between  $C$  and  $D$ , where  $C$  is a lottery that yields \$3000 with probability 0.25 and \$0 with probability 0.75, and  $D$  is a lottery that yields \$4000 with probability 0.2 and \$0 with probability 0.8.

The typical result, replicated many times, is that the large majority of subjects tend to choose  $A$  in problem 1 and  $D$  in problem 2,<sup>1</sup> in violation of Expected Utility, and in particular of its key postulate, the Independence axiom. To see this, note that prospects  $C$  and  $D$  are the 0.25 : 0.75 mixture of prospect  $A$  and  $B$ , respectively, with the lottery that gives the prize \$0 for sure. This means that the only pairs of choices consistent with Expected Utility are  $(A, C)$  and  $(B, D)$ . Additional evidence of the Certainty Effect includes, among many others, Allais' Common Consequence Effect, as well as the experiments of Cohen and Jaffray (1988), Conlisk (1989), Dean and Ortoleva (2012b), and Andreoni and Sprenger (2012).<sup>2</sup>

Following these observations, Dillenberger (2010) suggests a way to define the Certainty Effect behaviorally, by introducing an axiom, called *Negative Certainty Independence* (NCI), that is designed precisely to capture this tendency. To illustrate, note that in the example above prospect  $A$  is a risk-free option and it is chosen by most subjects over the risky prospect  $B$ . However, once both options are mixed,

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<sup>1</sup>This example is taken from Kahneman and Tversky (1979). Of 95 subjects, 80% choose  $A$  over  $B$ , 65% choose  $D$  over  $C$ , and more than half choose the pair  $A$  and  $D$ . These findings have been replicated many times (see footnote 2).

<sup>2</sup>A comprehensive reference to the evidence on the Certainty Effect can be found in Peter Wakker's annotated bibliography, posted at <http://people.few.eur.nl/wakker/refs/webrfnsc.doc>.

leading to problem 2, the first option is no longer risk-free, and its mixture (option  $C$ ) is now judged worse than the mixture of the other option (option  $D$ ). Intuitively, after the mixture one of the options is no longer risk-free, which reduces its appeal. Following this intuition, axiom NCI states that for any two lotteries  $p$  and  $q$ , any number  $\lambda$  in  $[0, 1]$ , and any lottery  $\delta_x$  that yields the prize  $x$  for sure, if  $p \succ \delta_x$  then  $\lambda p + (1 - \lambda) q \succ \lambda \delta_x + (1 - \lambda) q$ . That is, if the sure outcome  $x$  is not enough to compensate the decision maker (henceforth DM) for the risky prospect  $p$ , then mixing it with any other lottery, thus eliminating its certainty appeal, will not result in the mixture of  $\delta_x$  being more attractive than the corresponding mixture of  $p$ . Since NCI is clearly implied by the Independence axiom, it is always satisfied by Expected Utility. At the same time, it is a strictly weaker postulate, and in particular it allows Independence to fail when the Certainty Effect is present – allowing the DM to favor certainty, but ruling out the converse behavior. For example, the implication of NCI on responses to the Common Ratio questions above is that if the DM chooses the non-degenerate lottery  $B$  in problem 1, then she must also choose  $D$  in problem 2 – in line with the typical pattern of choices. It is easy to see how such property is also satisfied in most other experiments that document the Certainty Effect.

The goal of this paper is to characterize the class of continuous, monotone, and complete preference relations, defined on lotteries over some interval of monetary prizes, that satisfy NCI. That is, we aim to characterize preferences that are consistent with the Certainty Effect, together with very basic rationality postulates. A characterization of all preferences that satisfy NCI could be useful for a number of reasons. First, it provides a way of categorizing some of the existing decision models that can accommodate the Certainty Effect; as we shall see, this class will include many, albeit not all, of them (see Section 6 for details). Second, a representation for a very general class of preferences that allow for the Certainty Effect can help to delimit the potential consequences of such violations of Expected Utility. For example, a statement that a certain type of strategic or market outcomes is not possible for any preferences satisfying NCI would be easier to obtain given a representation theorem.

Our main result is that any continuous, monotone, and complete preference relation over lotteries satisfies NCI if and only if it can be represented as follows: There exists a set  $\mathcal{W}$  of strictly increasing (Bernoulli) utility functions over outcomes, such

that the value of any lottery  $p$ ,  $V(p)$ , is given by

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)),$$

where  $v^{-1}(\mathbb{E}_p(v))$  is the certainty equivalent of lottery  $p$  calculated using the utility function  $v$ .

We call this representation a Cautious Expected Utility representation and interpret it as follows. On the one hand, the DM understands probabilities and takes them at face value. However, she is unsure about how risk averse to be: she does not have one, but many possible utility functions, each of which entails a different risk attitude. She then reacts to this multiplicity using a form of *caution*: she evaluates each lottery according to the lowest possible certainty equivalent corresponding to some  $v$  in  $\mathcal{W}$ .<sup>3</sup> The use of certainty equivalents allows the DM to compare different evaluations of the same lottery, each corresponding to a different utility function, by bringing them to the same “scale” – dollar amounts. Lastly, note that, by definition,  $v^{-1}(\mathbb{E}_{\delta_x}(v)) = x$  for all  $v \in \mathcal{W}$ . Therefore, while the DM acts with caution when evaluating general lotteries, such caution does not play a role when evaluating degenerate ones. This leads to the Certainty Effect.

While the representation theorem discussed above characterizes a complete preference relation, the interpretation of Cautious Expected Utility is linked with the notion of a completion of incomplete preference relations. To wit, consider a DM who has an incomplete preference relation over lotteries, which is well-behaved (that is, a reflexive, transitive, monotone, continuous binary relation that satisfies Independence). Suppose that the DM is asked to choose between two options that the original preference relation is unable to compare; she then needs to choose a completion rule to complete her ranking. While there are many ways to do so, the DM may like to adhere to what we call a Cautious Completion: if the original incomplete preference relation is unable to compare a lottery  $p$  with a degenerate lottery  $\delta_x$ , then in the completion the DM opts for the latter – “when in doubt, go with certainty.”<sup>4</sup> The questions we ask are (i) are Cautious Completions possible?; and (ii) what properties do they have? Theorem 5 shows that not only Cautious Completions are

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<sup>3</sup>As we discuss in more detail in Section 6, this interpretation of cautious behavior is also provided in Cerreia-Vioglio (2009).

<sup>4</sup>We also require the completion of the original preference relation to be transitive, monotone, and to admit certainty equivalents.

always possible, but that they are unique, and that they admit a Cautious Expected Utility representation.<sup>5</sup> That is, we show that Cautious Completions must generate preferences that satisfy NCI – leading to another way to interpret this property.

The remainder of the paper is organized as follows. Section 2 presents the axiomatic structure, states the main representation theorem, and discusses the uniqueness properties of the representation. Section 3 characterizes standard definitions of risk attitude. Section 4 presents the result on the completion of incomplete preference relations. Section 5 studies the class of preference relations that, in addition to our axioms, also satisfy the Betweenness axiom, for which we provide an explicit characterization (as opposed to the implicit one suggested in the literature). Section 6 discusses the related literature. All proofs appear in the Appendices.

## 2 The Model

### 2.1 Framework

Consider a compact interval  $[w, b] \subset \mathbb{R}$  of monetary prizes. Let  $\Delta$  be the set of lotteries (Borel probability measures) over  $[w, b]$ , endowed with the topology of weak convergence. We denote by  $x, y, z$  generic elements of  $[w, b]$  and by  $p, q, r$  generic elements of  $\Delta$ . We denote by  $\delta_x \in \Delta$  the degenerate lottery (Dirac measure at  $x$ ) that gives the prize  $x \in [w, b]$  with certainty. The primitive of our analysis is a binary relation  $\succsim$  over  $\Delta$ . The symmetric and asymmetric parts of  $\succsim$  are denoted by  $\sim$  and  $\succ$ , respectively. The certainty equivalent of a lottery  $p \in \Delta$  is a prize  $x_p \in [w, b]$  such that  $\delta_{x_p} \sim p$ .

We start by imposing the following basic axioms on  $\succsim$ .

**Axiom 1** (Weak Order). *The relation  $\succsim$  is complete and transitive.*

**Axiom 2** (Continuity). *For each  $q \in \Delta$ , the sets  $\{p \in \Delta : p \succsim q\}$  and  $\{p \in \Delta : q \succsim p\}$  are closed.*

**Axiom 3** (Weak Monotonicity). *For each  $x, y \in [w, b]$ ,  $x \geq y$  if and only if  $\delta_x \succsim \delta_y$ .*

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<sup>5</sup>In fact, we will show that they admit a Cautious Expected Utility representation with the same set of utilities that can be used to represent the original incomplete preference relation (in the sense of the Expected Multi-Utility representation of Dubra et al. (2004)).

The three axioms above are standard postulates. Weak Order is a common assumption of rationality. Continuity is a technical assumption, needed to represent  $\succsim$  through a utility function. Finally, under the interpretation of  $\Delta$  as monetary lotteries, Weak Monotonicity simply implies that more money is better than less.

## 2.2 Negative Certainty Independence (NCI)

We now discuss the axiom which is the core assumption of our work. As we have previously mentioned, a bulk of evidence against Expected Utility arises from experiments in which one of the lotteries is degenerate, that is, contains a certain prize for sure. For example, recall Allais' Common Ratio Effect, outlined in the introduction. Subjects choose between  $A$  and  $B$ , where  $A = \delta_{3000}$  and  $B = 0.8\delta_{4000} + 0.2\delta_0$ . They also choose between  $C$  and  $D$ , where  $C = 0.25\delta_{3000} + 0.75\delta_0$  and  $D = 0.2\delta_{4000} + 0.8\delta_0$ . The typical finding is that the majority of subjects tend to *systematically* violate Expected Utility by choosing the pair  $A$  and  $D$ . The next axiom, introduced in Dillenberger (2010), relaxes the vNM-Independence axiom to accommodate such behavior.

**Axiom 4** (Negative Certainty Independence). *For each  $p, q \in \Delta$ ,  $x \in [w, b]$ , and  $\lambda \in [0, 1]$ ,*

$$p \succsim \delta_x \Rightarrow \lambda p + (1 - \lambda)q \succsim \lambda \delta_x + (1 - \lambda)q. \quad (\text{NCI})$$

Axiom NCI states that if the sure outcome  $x$  is not enough to compensate the DM for the risky prospect  $p$ , then mixing it with any other lottery, thus eliminating its certainty appeal, will not result in the mixture of  $x$  being more attractive than the corresponding mixture of  $p$ . Put differently, if we let  $x_{p|\lambda, q}$  be the solution to  $\lambda p + (1 - \lambda)q \sim \lambda \delta_{x_{p|\lambda, q}} + (1 - \lambda)q$ , then the axiom implies that  $x_{p|\lambda, q} \geq x_p$  for all  $p, q \in \Delta$  and  $\lambda \in (0, 1)$ .<sup>6</sup> That is:  $x_p$ , the certainty equivalent of  $p$ , might not be enough to compensate for  $p$  when part of a mixture. This is precisely the Certainty Effect. When applied to the experiment above, NCI only posits that if  $B$  is chosen in the first problem, then  $D$  must be chosen in the second one. In particular, it allows the DM to choose the pair  $A$  and  $D$ , in line with the typical pattern of choice.

Besides capturing the Certainty Effect, NCI has static implications that put additional structure on the shape of preferences over  $\Delta$ . For example, NCI implies Convexity: for each  $p, q \in \Delta$ , if  $p \sim q$  then  $\lambda p + (1 - \lambda)q \succsim q$  for all  $\lambda \in [0, 1]$ .

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<sup>6</sup>We show in Appendix A that our axioms imply that  $\succsim$  preserves First Order Stochastic Dominance and thus for each lottery  $p \in \Delta$  there exists a unique certainty equivalent  $x_p$ .

To see this, note that if we had some  $p, q \in \Delta$  and  $\lambda \in [0, 1]$  such that  $p \sim q$  but  $\lambda p + (1 - \lambda) q \prec q$ , then, by applying NCI twice and noting that  $p \sim q$  implies  $x_p = x_q$ , we would have

$$\delta_{x_q} \sim q \succ \lambda p + (1 - \lambda) q \succcurlyeq \lambda \delta_{x_p} + (1 - \lambda) q \succcurlyeq \lambda \delta_{x_p} + (1 - \lambda) \delta_{x_q} = \delta_{x_q},$$

a contradiction. NCI thus suggests weak preference for randomization between indifferent lotteries. Furthermore, note that by applying again NCI twice we obtain that  $p \sim \delta_x$  implies  $\lambda p + (1 - \lambda) \delta_x \sim p$  for all  $\lambda \in [0, 1]$ , which means neutrality towards mixing a lottery with its certainty equivalent.<sup>7</sup>

NCI also has relevant implications in non-static settings. Dillenberger (2010) demonstrates that in the context of recursive and time neutral, non-expected utility preferences over compound lotteries, NCI is equivalent to an intrinsic aversion to receiving partial information – a property which he termed preferences for one-shot resolution of uncertainty. This property is consistent with the experimental evidence that suggests that individuals are more risk averse when they perceive risk that is gradually resolved over time. In the context of preferences over information structures, Dillenberger (2010) also shows that NCI characterizes all non-expected utility preferences for which, when applied recursively, perfect information is always the most valuable information system.<sup>8</sup> Despite these various potential applications, however, the characterization of preferences that satisfy NCI remained an open question.

## 2.3 Representation Theorem

Before stating our representation theorem, we introduce some notation. We say that a function  $V : \Delta \rightarrow \mathbb{R}$  represents  $\succcurlyeq$  when  $p \succcurlyeq q$  if and only if  $V(p) \geq V(q)$ . Denote by  $\mathcal{U}$  the set of continuous and strictly increasing functions  $v$  from  $[w, b]$  to  $\mathbb{R}$ . We endow  $\mathcal{U}$  with the topology induced by the supnorm. For each lottery  $p$  and function  $v \in \mathcal{U}$ , we denote by  $\mathbb{E}_p(v)$  the expected utility of  $p$  with respect to  $v$ . The certainty equivalent of lottery  $p$  calculated using the utility function  $v$  is thus  $v^{-1}(\mathbb{E}_p(v)) \in [w, b]$ .

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<sup>7</sup>We refer the reader to Section 4 in Dillenberger (2010) for further static implications of NCI. Further discussion on Convexity is provided in Section 6.

<sup>8</sup>In addition, the recursive ambiguity model of Segal (1987) with static preferences that satisfy NCI has also been used recently by Dillenberger and Segal (2012) to address some of the paradoxes reported in Machina (2009, 2012).

**Definition 1.** Let  $\succsim$  be a binary relation on  $\Delta$  and  $\mathcal{W}$  a subset of  $\mathcal{U}$ . The set  $\mathcal{W}$  is a *Cautious Expected Utility representation* of  $\succsim$  if and only if the function  $V : \Delta \rightarrow \mathbb{R}$ , defined by

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta,$$

represents  $\succsim$ . We say that  $\mathcal{W}$  is a *Continuous Cautious Expected Utility representation* if and only if  $V$  is also continuous.

We now present our main representation theorem.

**Theorem 1.** Let  $\Delta$  be the set of lotteries over a compact interval  $[w, b]$  and  $\succsim$  a binary relation on  $\Delta$ . The following statements are equivalent:

- (i) The relation  $\succsim$  satisfies *Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence*;
- (ii) There exists a *Continuous Cautious Expected Utility representation* of  $\succsim$ .

According to a Cautious Expected Utility representation, the DM has a set  $\mathcal{W}$  of possible utility functions. Each of these functions is strictly increasing – i.e., agrees that “more money is better” – and assigns the same value to the best and worst options. Each of these utility functions, however, has a different curvature: it is as if the DM is unsure of how risk averse, or loving, she should be. The DM then reacts to this multiplicity with *caution*: she evaluates each lottery  $p$  by using the most pessimistic of those utilities, i.e., the utility function that returns the lowest *certainty equivalent*. That is, being unsure about her risk attitude, the DM acts conservatively and uses the most cautious criterion at hand. Note that if  $\mathcal{W}$  contains only one element then the model reduces to standard Expected Utility.

An important feature of the representation above is that the DM uses the utility function that minimizes the certainty equivalent of a lottery, instead of just minimizing its expected utility.<sup>9</sup> The reason is that comparing certainty equivalents means bringing each evaluation with each utility function to a unified measure, amounts of money, where a meaningful comparison is possible. To illustrate, suppose that  $[w, b] = [0, 1]$  and, without loss of generality, assume that each  $v \in \mathcal{W}$  is such that

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<sup>9</sup>This alternative model is discussed in Maccheroni (2002). Indeed, this model, despite some similarity, is not designed to capture the Certainty Effect. We discuss the comparison with our model in detail in Section 6.

$v(w) = 0$  and  $v(b) = 1$ . Further suppose that the DM needs to evaluate the binary lottery  $p$ , which pays either zero or one, both equally likely. Since all functions in  $\mathcal{W}$  are normalized, we must have  $\mathbb{E}_p(v) = 0.5$  for all  $v \in \mathcal{W}$ , and hence risk attitudes cannot be reflected in the evaluation of  $p$  according to the minimal expected utility rule. Using the minimal certainty equivalent criterion, on the other hand, provides a meaningful way to selecting the most pessimist view (given  $\mathcal{W}$ ) of  $p$ .

It is then easy to see how the representation in Theorem 1 leads to the Certainty Effect: while the DM acts with caution when evaluating general lotteries, caution does not play a role when evaluating degenerate ones – no matter which utility function is used, the certainty equivalent of a degenerate lottery that yields the prize  $x$  for sure is simply  $x$ . This latter point demonstrates once again why the use of certainty equivalents to make comparisons is an essential component of the representation.

**Example.** Let  $[w, b] \subseteq [0, \infty)$  and  $\mathcal{W} = \{u, v\}$  where

$$u(x) = -\exp(-\beta x), \quad \beta > 0, \quad \text{and} \quad v(x) = x^\alpha, \quad \alpha \in (0, 1).$$

That is,  $u$  (resp.,  $v$ ) displays constant absolute (resp., relative) risk aversion. Furthermore, if the interval  $[w, b]$  is large enough ( $b > \frac{1-\alpha}{\beta}$ ), then  $u$  and  $v$  are not ranked in terms of risk aversion, that is, there exist  $p$  and  $q$  such that the smallest certainty equivalent for  $p$  (resp.,  $q$ ) corresponds to  $u$  (resp.,  $v$ ). This functional form can easily address the Common Ratio Effect from Section 1.<sup>10</sup>

The interpretation of the Cautious Expected Utility representation is different from some of the most prominent existing models of non-expected utility under risk. For example, the common interpretation of the Rank Dependent Utility model of Quiggin (1982) is that the DM knows her utility function but she distorts probabilities. By contrast, in a Cautious Expected Utility representation the DM takes probabilities at face value – she uses them as Expected Utility would prescribe – but she is unsure of how risk averse she should be, and applies caution by using the most

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<sup>10</sup>For example, let  $\alpha = 0.8$  and  $\beta = 0.0002$ . Let  $p = 0.8\delta_{4000} + 0.2\delta_0$ ,  $q = 0.2p + 0.8\delta_0$  and  $r = 0.25\delta_{3000} + 0.75\delta_0$ . Direct calculations show that

$$V(p) = u^{-1}(\mathbb{E}_p(u)) \simeq 2904 < 3000 = V(\delta_{3000}),$$

but

$$V(q) = v^{-1}(\mathbb{E}_q(v)) \simeq 535 > 530 \simeq v^{-1}(\mathbb{E}_r(v)) = V(r).$$

We have  $\delta_{3000} \succ p$  but  $q \succ r$ .

pessimist function in the set. In Section 6, we point out that not only the two models have a different interpretation but they entail stark differences in behavior: the only preference relation that is compatible with both the Rank Dependent and the Cautious Expected Utility models is Expected Utility.

Lastly, we note that the use of the most pessimistic utility in a set is reminiscent of the Maxmin Expected Utility (MEU) model of Gilboa and Schmeidler (1989) under ambiguity, in which the DM has not one, but a set of probabilities, and evaluates acts using the worst probability in the set. Our model can be seen as one possible corresponding model under risk.<sup>11</sup> This analogy with MEU will then be strengthened by our analysis in Section 4, in which we show that both models can be derived from extending incomplete preferences using a similar caution rule.

In the next subsection we will outline the main steps in the proof of Theorem 1. There is one notion that is worth discussing independently, since it plays a major role in the analysis of all subsequent sections. We introduce a *derived* preference relation, denoted  $\succ'$ , which is the largest subrelation of the original preference  $\succ$  that satisfies the Independence axiom. Formally, define  $\succ'$  on  $\Delta$  by

$$p \succ' q \iff \lambda p + (1 - \lambda) r \succ \lambda q + (1 - \lambda) r \quad \forall \lambda \in (0, 1], \forall r \in \Delta. \quad (1)$$

In the context of choice under risk, this derived relation was proposed and characterized by Cerreia-Vioglio (2009). It parallels a notion introduced in the context of choice under ambiguity by Ghirardato et al. (2004) (see also Cerreia-Vioglio et al. (2011)). This binary relation, which contains the comparisons over which the DM abides by the precepts of Expected Utility, is often interpreted as including the comparisons that the DM is confident in making. We refer to  $\succ'$  as the Linear Core of  $\succ$ . Note that, by definition, if the original preference relation  $\succ$  satisfies NCI, then  $p \not\succeq' \delta_x$  implies  $\delta_x \succ p$ . That is, whenever the DM is not confident to declare  $p$  better than the certain outcome  $x$ , the original relation will rank  $\delta_x$  strictly above  $p$ . This intuition will be our starting point in Section 4 where we discuss the idea of Cautious

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<sup>11</sup>This is not the only way in which the intuition of MEU could be applied to risk. On the one hand, the model in Maccheroni (2002) takes a similar approach to ours, but uses the minimum of expected utilities (instead of the minimum of certainty equivalents). Alternatively, one can map the intuition of MEU to risk with a model in which the agent has *one* utility function and multiple probability distortions, the most pessimistic one of which is used by the agent. Dean and Ortleva (2012a) axiomatize this model, showing that it can be derived by a postulate of preference for hedging similar to the one in Schmeidler (1989).

Completion. Lastly, as we will see in Section 2.5,  $\succsim'$  will allow us to identify the uniqueness properties of the set  $\mathcal{W}$  in a Cautious Expected Utility representation.

## 2.4 Proof Sketch of Theorem 1

In what follows we discuss the main intuition of the proof of Theorem 1: a complete proof, which includes the many omitted details, appears in Appendix B. We focus here only on the sufficiency of the axioms for the representation.

*Step 1. Define the Linear Core of  $\succsim$ .* As we have discussed above, we introduce the binary relation  $\succsim'$  on  $\Delta$  defined in (1).

*Step 2. Find the set  $\mathcal{W} \subseteq \mathcal{U}$  that represents  $\succsim'$ .* By Cerreia-Vioglio (2009),  $\succsim'$  is reflexive and transitive (but possibly incomplete), continuous, and satisfies Independence. In particular, there exists a set  $\mathcal{W}$  of continuous functions on  $[w, b]$  that constitutes an Expected Multi-Utility representation of  $\succsim'$ , that is,  $p \succsim' q$  if and only if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}$  (see also Dubra et al. (2004)). Since  $\succsim$  satisfies Weak Monotonicity and NCI,  $\succsim'$  also satisfies Weak Monotonicity. For this reason, the set  $\mathcal{W}$  can be chosen to be composed only of strictly increasing functions.

*Step 3. Representation of  $\succsim$ .* We show that  $\succsim$  admits a certainty equivalent representation, i.e., there exists  $V : \Delta \rightarrow \mathbb{R}$  such that  $V$  represents  $\succsim$  and  $V(\delta_x) = x$  for all  $x \in [w, b]$ .

*Step 4. Relation between  $\succsim$  and  $\succsim'$ .* We note that (i)  $\succsim$  is a completion of  $\succsim'$ , i.e.,  $p \succsim' q$  implies  $p \succsim q$ ; and (ii) for each  $p \in \Delta$  and for each  $x \in [w, b]$ ,  $p \not\succeq' \delta_x$  implies  $\delta_x \succ p$ .

*Step 5. Final step.* We conclude the proof by showing that we must have  $V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$  for all  $p \in \Delta$ . For each  $p$ , find  $x \in [w, b]$  such that  $p \sim \delta_x$ , which means  $V(p) = V(\delta_x) = x$ . First note that we must have  $V(p) = x \leq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ . If this was not the case, then we would have that  $x > v^{-1}(\mathbb{E}_p(v))$  for some  $v \in \mathcal{W}$ , which means  $p \not\succeq' \delta_x$ . But by *Step 2 and Step 4 (ii)*, we would obtain  $\delta_x \succ p$ , contradicting  $\delta_x \sim p$ . Second, we must have  $V(p) = x \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ : if this was not the case, then we would have  $x < \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ . We could then find  $y$  such that  $x < y < \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ , which in turn would yield  $p \succsim' \delta_y$ . By *Step 4 (i)*, we could conclude that  $p \succsim \delta_y \succ \delta_x$ , contradicting  $p \sim \delta_x$ .

## 2.5 Uniqueness and Properties of the Set of Utilities

We now turn to discuss the uniqueness properties of a set of utilities  $\mathcal{W}$  in a Cautious Expected Utility representation of  $\succsim$ . To do so, we define the set of *normalized* utility functions  $\mathcal{U}_{\text{nor}} = \{v \in \mathcal{U} : v(w) = 0, v(b) = 1\}$ , and confine our attention to normalized Cautious Expected Utility representation, that is, we further require  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ . This is without loss of generality since, given a Cautious Expected Utility representation of  $\succsim$ , it is immediate to see that we can find another representation which is a subset of  $\mathcal{U}_{\text{nor}}$  by applying standard normalizations. Even with this normalization, note that we are bound to find uniqueness properties only “up to” the closed convex hull: if two sets share the same closed convex hull, then they must generate the same representation, as proved in the following proposition.<sup>12</sup> Denote by  $\overline{\text{co}}(\mathcal{W})$  the closed convex hull of a set  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ . Notice that  $\overline{\text{co}}(\mathcal{W}) \subseteq \mathcal{U}_{\text{nor}}$ .

**Proposition 1.** *If  $\mathcal{W}, \mathcal{W}' \subseteq \mathcal{U}_{\text{nor}}$  are such that  $\overline{\text{co}}(\mathcal{W}) = \overline{\text{co}}(\mathcal{W}')$  then*

$$\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) = \inf_{v \in \mathcal{W}'} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta.$$

Moreover, it is easy to see how  $\mathcal{W}$  will in general not be unique, even up to the closed convex hull, as we can always add redundant utility functions that will never achieve the infimum. In particular, consider any set  $\mathcal{W}$  in a Cautious Expected Utility representation and add to it a function  $\bar{v}$  which is a continuous, strictly increasing, and strictly convex transformation of some other function  $v \in \mathcal{W}$ . The set  $\mathcal{W} \cup \{\bar{v}\}$  will give a Cautious Expected Utility representation of the same preference relation, as the function  $\bar{v}$  will never be used in the representation.<sup>13</sup>

Once we remove these redundant utilities, we can identify a unique (up to the closed convex hull) set of utilities. In particular, for each preference relation that admits a Continuous Cautious Expected Utility representation, there exists a set  $\widehat{\mathcal{W}}$  such that any other Cautious Expected Utility representation,  $\mathcal{W}$ , of these preferences is such that  $\overline{\text{co}}(\widehat{\mathcal{W}}) \subseteq \overline{\text{co}}(\mathcal{W})$ . In this sense  $\widehat{\mathcal{W}}$  is a “minimal” set of utilities. Moreover, the set  $\widehat{\mathcal{W}}$  will have a natural interpretation in our setup: it constitutes a unique (up

<sup>12</sup>This suggest that we could have alternatively assumed, without loss of generality, that the set  $\mathcal{W}$  in Theorem 1 is convex. However, since in applications it is usually easier to work with a finite (and hence not convex) set of utilities, we did not require this property.

<sup>13</sup>That is, there will not be a lottery  $p$  such that  $\inf_{v \in \mathcal{W} \cup \bar{v}} v^{-1}(\mathbb{E}_p(v)) = \bar{v}^{-1}(\mathbb{E}_p(\bar{v})) < \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ .

to the closed convex hull) Expected Multi-Utility representation of the Linear Core  $\succ'$ , the derived preference relation defined in (1). In other words, in terms of uniqueness, if two sets,  $\widehat{\mathcal{W}}_1$  and  $\widehat{\mathcal{W}}_2$  in  $\mathcal{U}_{\text{nor}}$ , constitute a Continuous Cautious Expected Utility representation of  $\succ$  and an Expected Multi-Utility representation of  $\succ'$ , then their closed convex hull must coincide. This is formalized in the following result.

**Theorem 2.** *Let  $\succ$  be a binary relation on  $\Delta$  that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. Then there exists  $\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$  such that*

- (i) *The set  $\widehat{\mathcal{W}}$  is a Continuous Cautious Expected Utility representation of  $\succ$ ;*
- (ii) *If  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$  is a Cautious Expected Utility representation of  $\succ$  then  $\overline{\text{co}}(\widehat{\mathcal{W}}) \subseteq \overline{\text{co}}(\mathcal{W})$ ;*
- (iii) *The set  $\widehat{\mathcal{W}}$  is an Expected Multi-Utility representation of  $\succ'$ , that is,*

$$p \succ' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \widehat{\mathcal{W}}.$$

Moreover,  $\widehat{\mathcal{W}}$  is unique up to the closed convex hull.

### 3 Cautious Expected Utility and Risk Attitudes

In this section we explore the connection between Theorem 1 and standard definitions of risk attitude, and discuss the comparative notion of “more risk averse than”. In Section 3.2, we show how imposing risk aversion allows us to derive additional properties for the set  $\mathcal{W}$ .

#### 3.1 Characterization of Risk Aversion

**Definition 2.** *We say that  $\succ$  is risk averse if  $p \succ q$  whenever  $q$  is a mean preserving spread of  $p$ . Similarly,  $\succ$  is risk seeking if  $q \succ p$  whenever  $q$  is a mean preserving spread of  $p$ .<sup>14</sup>*

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<sup>14</sup>These definitions of risk averse and risk seeking are standard in the literature on non-expected utility. They imply the weaker notions according to which  $\succ$  is risk averse (seeking) if  $p \precs (\succ) \delta_{e(p)}$  for all  $p \in \Delta$ , where  $e(p)$  denotes the expected value of  $p$ .

Let  $\widehat{\mathcal{W}}$  be a set as identified in Theorem 2, that is, a normalized set of utilities that constitutes an Expected Multi-Utility representation of  $\succsim'$  and a Continuous Cautious Expected Utility representation of  $\succsim$ .

**Theorem 3.** *Let  $\succsim$  be a binary relation that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. The following statements are true:*

- (i) *The relation  $\succsim$  is risk averse if and only if each  $v \in \widehat{\mathcal{W}}$  is concave.*
- (ii) *The relation  $\succsim$  is risk seeking if and only if each  $v \in \widehat{\mathcal{W}}$  is convex.*

Theorem 3 shows that the relation found under Expected Utility between the concavity/convexity of the utility function and the risk attitude of the DM holds also for the more general Continuous Cautious Expected Utility model – although it now involves *all* utilities in the set  $\widehat{\mathcal{W}}$ .<sup>15</sup> In turns, this shows that NCI is compatible with many types of risk attitudes. For example, despite the presence of the Certainty Effect, when all utilities are convex the DM would actually be risk seeking.

We now proceed to compare the risk attitude of two individuals. We adopt the following standard notion of comparative risk aversion.

**Definition 3.** *Let  $\succsim_1$  and  $\succsim_2$  be two binary relations on  $\Delta$ . We say that  $\succsim_1$  is more risk averse than  $\succsim_2$  if and only if for each  $p \in \Delta$  and for each  $x \in [w, b]$ ,*

$$p \succsim_1 \delta_x \implies p \succsim_2 \delta_x.$$

There are two channels in a Cautious Expected Utility representation that could lead one DM to be more risk averse than another. First, if  $\widehat{\mathcal{W}}_2 \subseteq \widehat{\mathcal{W}}_1$  then for each  $p \in \Delta$  the certainty equivalent under  $\widehat{\mathcal{W}}_2$  is (weakly) greater than that under  $\widehat{\mathcal{W}}_1$ , while the certain outcomes are, by construction, treated similarly in both, which means that  $\succsim_1$  is more risk averse than  $\succsim_2$ . Second, if each  $v \in \widehat{\mathcal{W}}_2$  is a strictly increasing and convex transformation of some  $\hat{v} \in \widehat{\mathcal{W}}_1$  then  $\succsim_1$  is more risk averse than  $\succsim_2$ . These observations are summarized in the following result:

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<sup>15</sup>The key passage is the “only if” direction in item (i), which shows that to guarantee that  $\succsim$  is risk averse, it is not enough to require members of  $\widehat{\mathcal{W}}$  to have concave parts in relevant segments; instead, they should all be globally concave. Intuitively, this is the case because if  $q$  is a mean preserving spread of  $p$  then we must have  $p \succsim' q$ , which means that  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \widehat{\mathcal{W}}$ . Since any function  $v \in \widehat{\mathcal{W}}$  is strictly increasing, respecting aversion to mean preserving spread implies that any such  $v$  is concave.

**Proposition 2.** *Let  $\succsim_1$  and  $\succsim_2$  be two binary relations on  $\Delta$  that satisfy Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. Then,*

- (i) *If  $\widehat{\mathcal{W}}_2 \subseteq \widehat{\mathcal{W}}_1$  then  $\succsim_1$  is more risk averse than  $\succsim_2$ .*
- (ii) *If each  $v \in \widehat{\mathcal{W}}_2$  is a strictly increasing and convex transformation of some  $\hat{v} \in \widehat{\mathcal{W}}_1$  then  $\succsim_1$  is more risk averse than  $\succsim_2$ .*

### 3.2 Risk Aversion and Compactness of the Set of Utilities

We now show that if we strengthen our Weak Monotonicity axiom and assume that  $\succsim$  satisfies Risk Aversion, that is, it is risk averse as in Definition 2, then we can guarantee that the set of utilities  $\mathcal{W}$  in Theorem 1 is compact. Thus, the infimum in the representation is always achieved and the representation is always continuous.

**Axiom 5** (Monotonicity). *For each  $x, y \in [w, b]$  and  $\lambda \in (0, 1]$*

$$x > y \Rightarrow \lambda \delta_x + (1 - \lambda) \delta_w \succ \lambda \delta_y + (1 - \lambda) \delta_w.$$

By taking  $\lambda = 1$ , it is easy to see that Monotonicity implies Weak Monotonicity.

**Theorem 4.** *Let  $\succsim$  be a binary relation on  $\Delta$ . The following statements are equivalent:*

- (i) *The relation  $\succsim$  satisfies Weak Order, Continuity, Monotonicity, Negative Certainty Independence, and Risk Aversion;*
- (ii) *There exists a compact set  $\mathcal{W}$  in  $\mathcal{U}$  of concave functions such that*

$$p \succsim q \iff \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \geq \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_q(v)).$$

Theorem 4 shows that by imposing Risk Aversion, and strengthening our monotonicity assumption, we can guarantee compactness of the set of utilities in our representation. The following is an intuition of the steps we follow to prove it (restricted to the sufficiency of the axioms). First, we find a suitable representation  $\mathcal{W}$  in  $\mathcal{U}_{\text{nor}} \subseteq \mathcal{U}$ . We show that  $\mathcal{W}$  is relatively compact with respect to the topology of sequential pointwise convergence restricted to  $\mathcal{U}_{\text{nor}}$ ; that is, that any sequence in  $\mathcal{W}$  admits a subsequence which pointwise converges to an element  $v$  in  $\mathcal{U}_{\text{nor}}$ . (See Theorem 6 in

the appendix for details.) The existence of a convergent subsequence with limit  $v$  is granted by the concavity and uniform boundedness of each element of  $\mathcal{W}$  (where concavity, by Theorem 3, is implied by Risk Aversion). It easily follows that  $v$  is concave and, by construction, normalized. We then use the concavity of each element of  $\mathcal{W}$  and Monotonicity to establish that  $v$  is continuous and strictly increasing. We conclude the proof by establishing that the notion of compactness above coincides with that of norm compactness (this is proved in Lemma 2 in the appendix).

## 4 Caution and Completion of Incomplete Preferences

While our analysis thus far has focused on the characterization of a complete preference relation that satisfies NCI (in addition to the other basic axioms), we will now demonstrate that this analysis is deeply related to that of completing an incomplete preference relation over lotteries.

We analyze the following problem. Consider a DM who has an *incomplete* preference relation over the set of lotteries. We can see this relation as representing the comparisons that the DM feels comfortable making. There might be occasions, however, in which the DM is asked to choose among lotteries she cannot compare, and to do this she has to complete her incomplete preference relation. Suppose that the DM wants to do so applying *caution*, i.e., when in doubt between a sure outcome and a lottery, she opts for the sure outcome (the degenerate lottery). Two questions are then natural: (i) which preferences will she obtain after the completion?; and (ii) Which properties will they have?

This analysis parallels the one of Gilboa et al. (2010), who consider an environment with ambiguity instead of risk, although with one minor formal difference: while in Gilboa et al. (2010) both the incomplete relation and its completion are a primitive of the analysis, in our case the primitive is simply the incomplete preference relation over lotteries, and we study the properties of all possible completions of this kind.<sup>16</sup>

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<sup>16</sup>A natural next step is to develop a more general treatment that encompasses both the result in this section and the one in Gilboa et al. (2010), where we start from a preference over acts that admits a Multi-Prior Expected Multi-Utility representation, as in Ok et al. (2012) and Galaabaatar and Karni (2013), and look for Cautious Completions. Riella (2013) shows that this is indeed possible, and proves the corresponding result.

Since we analyze an incomplete preference relation, the analysis in this section requires a slightly stronger notion of continuity, called Sequential Continuity.<sup>17</sup>

**Axiom 6** (Sequential Continuity). *Let  $\{p_n\}_{n \in \mathbb{N}}$  and  $\{q_n\}_{n \in \mathbb{N}}$  be two sequences in  $\Delta$ . If  $p_n \rightarrow p$ ,  $q_n \rightarrow q$ , and  $p_n \succsim q_n$  for all  $n \in \mathbb{N}$  then  $p \succsim q$ .*

In the rest of the section, we assume that  $\succsim'$  is a reflexive and transitive (though potentially incomplete) binary relation over  $\Delta$ , which satisfies Sequential Continuity, Weak Monotonicity, and Independence. We look for completions of  $\succsim'$  which exhibit a degree of caution as formalized in the following definition.

**Definition 4.** *Let  $\succsim'$  be a binary relation on  $\Delta$ . We say that the relation  $\hat{\succsim}$  is a Cautious Completion of  $\succsim'$  if and only if the following hold:*

1. *The relation  $\hat{\succsim}$  satisfies Weak Order, Weak Monotonicity, and for each  $p \in \Delta$  there exists  $x \in [w, b]$  such that  $p \sim \delta_x$ ;*
2. *For each  $p, q \in \Delta$ , if  $p \succsim' q$  then  $p \hat{\succsim} q$ ;*
3. *For each  $p \in \Delta$  and  $x \in [w, b]$ , if  $p \not\succeq' \delta_x$  then  $\delta_x \hat{\succ} p$ .*

Point 1 imposes few minimal requirements of rationality on  $\hat{\succsim}$ , most notably, the existence of a certainty equivalent for each lottery  $p$ . Weak Monotonicity will imply that this certainty equivalent is unique. In point 2, we assume that the relation  $\hat{\succsim}$  extends  $\succsim'$ . Finally, point 3 requires that such a completion of  $\succsim'$  is done with caution.<sup>18</sup>

**Theorem 5.** *If  $\succsim'$  is a reflexive and transitive binary relation on  $\Delta$  that satisfies Sequential Continuity, Weak Monotonicity, and Independence then  $\succsim'$  admits a unique Cautious Completion  $\hat{\succsim}$  and there exists a set  $\mathcal{W} \subseteq \mathcal{U}$  such that for all  $p, q \in \Delta$*

$$p \hat{\succ} q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}$$

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<sup>17</sup>This notion coincides with our previous Continuity axiom if the binary relation is complete and transitive. (This means that we could have equivalently replaced Continuity with Sequential Continuity in our previous analysis.) However, this is no longer true when the binary relation is incomplete, as it is the case in this section.

<sup>18</sup>It is immediate to see that the condition in point 3 is the translation, in a context of choice under risk, of the assumption of Default to Certainty of Gilboa et al. (2010).

and

$$p \hat{\succ} q \iff \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_q(v)).$$

Moreover,  $\mathcal{W}$  is unique up to the closed convex hull.

Theorem 5 shows that, given a binary relation  $\succ'$  which satisfies all the tenets of Expected Utility except completeness, not only a Cautious Completion  $\hat{\succ}$  is always possible, but it is also unique. Most importantly, such completion  $\hat{\succ}$  admits a Cautious Expected Utility representation, using the same set of utilities as in the Expected Multi-Utility Representation of the original preference  $\succ'$ . This suggests one possible origin of violations of Expected Utility, as reflected in Allais-type behavior: subjects might be unable to compare some of the available options and, when asked to extend their ranking, they do so by being cautious – generating the Certainty Effect, and the representation in Theorem 1.

Finally, Theorem 5 strengthens the link between the Cautious Expected Utility model and the Maxmin Expected Utility model of Gilboa and Schmeidler (1989): Gilboa et al. (2010) show the latter could be derived as a completion of an incomplete preference over acts by applying a form of caution according to which, when in doubt, the DM chooses a constant act (a lottery); similarly, here we derive the Cautious Expected Utility model by extending an incomplete preference over lotteries using a form of caution according to which, when in doubt, the DM chooses a risk-free lottery.

## 5 NCI and Betweenness Preferences: an Explicit Representation

In this section we discuss an important special case of the preference relations studied in Theorem 1: preference relations that satisfy both NCI and Betweenness: a broad class that includes Expected Utility as well as Gul (1991)'s model of Disappointment Aversion. We show that for preferences in this class, we can provide an *explicit* characterization that links the Cautious Expected Utility model with the implicit characterizations derived in Dekel (1986), Chew (1989), and Gul (1991).

Suppose that in addition to our four axioms, Weak Order, Continuity, Weak Monotonicity, and NCI, the binary relation  $\succ$  also satisfies the following axiom.

**Axiom 7** (Betweenness). *For each  $p, q \in \Delta$  and  $\lambda \in [0, 1]$ ,  $p \succ q$  (resp.,  $p \sim q$ ) implies  $p \succ \lambda p + (1 - \lambda)q \succ q$  (resp.,  $p \sim \lambda p + (1 - \lambda)q \sim q$ ).*

Betweenness implies neutrality toward randomization among equally-good lotteries. Dekel (1986) and Chew (1989) establish that a continuous, monotone, and complete preference relation that satisfy Betweenness admits the following representation: there exists a local-utility function  $u : [w, b] \times [0, 1] \rightarrow [0, 1]$ , continuous in both arguments, strictly increasing in the first argument, that satisfies  $u(w, v) = 0 = 1 - u(b, v)$  for all  $v \in [0, 1]$  and such that  $p \succ q$  if and only if  $V(p) \geq V(q)$ , where  $V(p)$  is defined implicitly as the unique  $v \in [0, 1]$  that solves:

$$\int_{[w, b]} u(x, v) dp(x) = v.$$

This is referred to as an implicit representation.<sup>19</sup> We will now show that if the preference relation also satisfies NCI, then an explicit characterization is possible, in the form of a Cautious Expected Utility representation that involves precisely the set of utilities identified in the implicit representation.

To see this, suppose  $\succ$  admits the implicit representation above. For each  $v \in [0, 1]$ , let  $u_v(x) := u(x, v)$ . Proposition 4 in Dillenberger (2010) establishes that  $\succ$  also satisfies NCI if and only if for each  $p \in \Delta$  and  $v \in [0, 1]$ , we have that  $\mathbb{E}_p(u_v) \geq u_v(x_p)$ , or

$$u_v^{-1}(\mathbb{E}_p(u_v)) \geq x_p.$$

Since, by the implicit representation, we also have that  $V(p) = \mathbb{E}_p(u_{V(p)}) = u_{V(p)}(x_p)$ , we can conclude that

$$\inf_{v \in [0, 1]} u_v^{-1}(\mathbb{E}_p(u_v)) = x_p,$$

leading to an explicit utility representation in the form of a Cautious Expected Utility representation with  $\mathcal{W} = \{u_v(\cdot)\}_{v \in [0, 1]}$ . Furthermore, it can be shown that this set  $\mathcal{W}$  is one of the sets identified by Theorem 2, which means that it also constitutes an Expected Multi-Utility representation of  $\succ'$ .

One of the most prominent examples of preference relations that satisfy Betweenness is Gul (1991)'s model of Disappointment Aversion. For some parameter  $\beta \in (-1, \infty)$  and a strictly increasing function  $\phi : [w, b] \rightarrow \mathbb{R}$ , local utility function is

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<sup>19</sup>It should be noted that Dekel's assumption of monotonicity is stronger than ours. We refer the interested reader to Dekel (1986).

given by

$$u_{v,\beta}(x) = \begin{cases} \frac{\phi(x)+\beta v}{1+\beta} & \phi(x) > v \\ \phi(x) & \phi(x) \leq v \end{cases}$$

(see Gul (1991)). In most applications, attention is confined to the case where  $\beta > 0$ , which corresponds to “Disappointment Aversion” (the case of  $\beta \in (-1, 0)$  is referred to “Elation Seeking”, and the model reduces to Expected Utility when  $\beta = 0$ ). Artstein-Avidan and Dillenberger (2011) show that Gul’s preferences satisfy NCI if and only if  $\beta \geq 0$ . Combining this result with the observation above about the general Betweenness class immediately yields the following explicit characterization for Disappointment Averse preference relations (the proof is omitted).

**Proposition 3.** *If  $\succsim$  is a Disappointment Averse preference relation with  $\beta \geq 0$  then for each  $p \in \Delta$*

$$x_{p|\beta} = \inf_{v \in [\phi(w), \phi(b)]} u_{v,\beta}^{-1}(\mathbb{E}_p(u_{v,\beta})).$$

It is then easy to see that in the case of  $\beta \in (-1, 0)$ , which violates NCI, a certainty-equivalent representation can be achieved by replacing the inf with the sup.

## 6 Related Literature

Dillenberger (2010) introduces NCI and discusses its implication in dynamic settings, but does not provide a utility representation. Dillenberger and Erol (2011) provide an example of a continuous, monotone, and complete preference relation over simple lotteries that satisfies NCI but not Betweenness (see Section 5). Our paper generalized this example and, more importantly, provide a complete characterization of all binary relations that satisfy NCI (in addition to the other three basic assumptions).

Cerreia-Vioglio (2009) characterizes the class of continuous and complete preference relations that satisfy Convexity, that is,  $p \sim q$  implies  $\lambda p + (1 - \lambda) q \succsim q$  for all  $\lambda \in (0, 1)$ . Loosely speaking, Cerreia-Vioglio shows that there exists a set  $\mathcal{V}$  of normalized Bernoulli utility functions, and a real function  $U$  on  $\mathbb{R} \times \mathcal{V}$ , such that preferences are represented by

$$V(p) = \inf_{v \in \mathcal{V}} U(\mathbb{E}_p(v), v).$$

Using this representation, Cerreia-Vioglio interprets Convexity as a behavioral prop-

erty that captures a preference for hedging; such preferences may arise in the face of uncertainty about the value of outcomes, future tastes, and/or the degree of risk aversion. He also suggests the choice of the minimal certainty equivalent as a criterion to resolve uncertainty about risk attitudes and as a completion procedure. In Section 2 we showed that NCI implies Convexity, which means that the preferences we study in this paper are a subset of those studied by Cerreia-Vioglio. Indeed, this is apparent also from Theorem 1: our preferences correspond to the special case in which  $U(\mathbb{E}_p(v), v) = v^{-1}(\mathbb{E}_p(v))$ . Furthermore, our representation theorem establishes that NCI is the *exact* strengthening of convexity needed to characterize the minimum certainty equivalent criterion.

A popular generalization of Expected Utility, designed to explain the behavior observed in the Allais paradoxes, is the Rank Dependent Utility (RDU) model of Quiggin (1982). According to this model, individuals weight probability in a non-linear way. Specifically, if we order the prizes in the support of the lottery  $p$ , with  $x_1 < x_2 < \dots < x_n$ , then the functional form for RDU is:

$$V(p) = u(x_n)f(p(x_n)) + \sum_{i=1}^{n-1} u(x_i)[f(\sum_{j=i}^n p(x_j)) - f(\sum_{j=i+1}^n p(x_j))]$$

where  $f : [0, 1] \rightarrow [0, 1]$  is strictly increasing and onto, and  $u : [w, b] \rightarrow \mathbb{R}$  is increasing. If  $f(p) = p$  then RDU reduces to Expected Utility. If  $f$  is convex, then larger weight is given to inferior outcomes; this corresponds to a pessimistic probability distortion suitable to explain the Allais paradoxes. Apart from the different interpretation of RDU compared to our Caution Expected Utility representation, as discussed in Section 2.3, the two models have completely different behavioral implications; Dillenberger (2010) demonstrates that the only RDU preference relations that satisfy NCI are Expected Utility. That is, RDU is generically incompatible with NCI.<sup>20</sup>

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<sup>20</sup>The observation that NCI does not imply probabilistic distortion becomes relevant in experiments similar to the one reported in Conlisk (1989). Conlisk studies the robustness of Allais-type behavior to boundary effects. He considers a slight perturbation of prospects similar to  $A, B, C$  and  $D$  (as in Section 1), so that (i) each of the new prospects,  $A', B', C'$  and  $D'$ , yields all three prizes with strictly positive probability, and (ii) in the resulting “displaced Allais question” (namely choosing between  $A'$  and  $B'$  and then choosing between  $C'$  and  $D'$ ), the only pattern of choice that is consistent with expected utility is either the pair  $A'$  and  $C'$  or the pair  $B'$  and  $D'$ . Although violations of expected utility become significantly less frequent and are no longer systematic (a result that supports the claim that violations can be explained by the certainty effect), a nonlinear probability function predicts that this increase in consistency would be the result of fewer subjects choosing  $A'$  over  $B'$ , and not because more subjects choose  $C'$  over  $D'$ . In fact, the latter occurred,

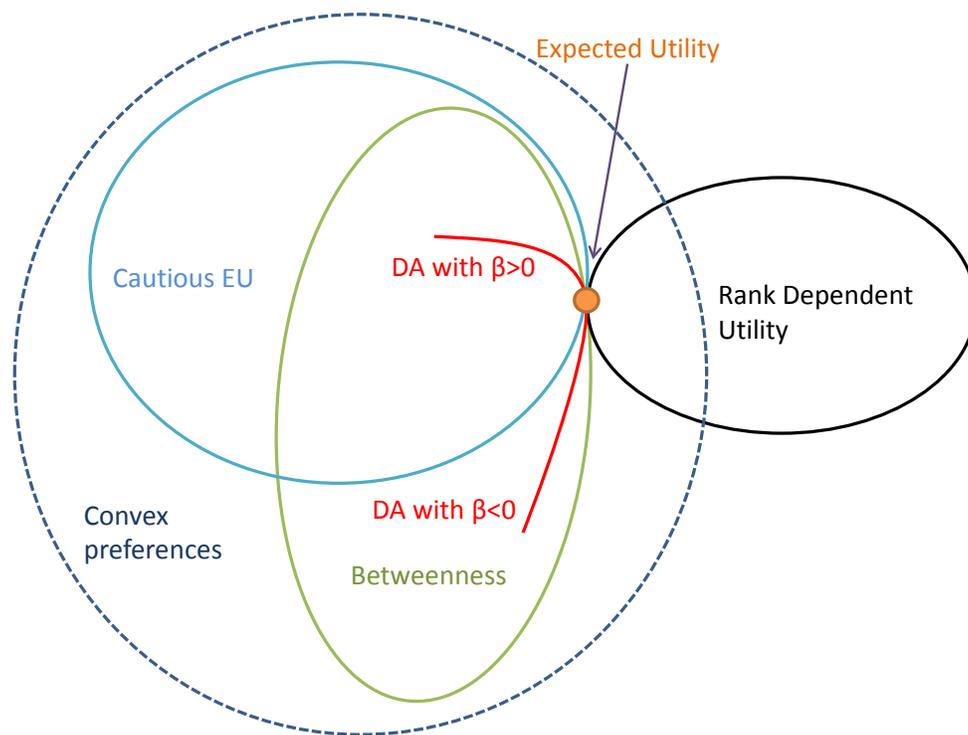


Figure 1: Cautious Expected utility and other models

We have already discussed how Gul (1991)'s model of Disappointment Aversion (denoted DA in Figure (1)) belong to our class if and only if  $\beta \geq 0$  and how the preferences in our class neither nest, nor are nested in, those that satisfy Betweenness (see Section 5).

Figure (1) summarizes our discussion thus far about the relationship between the various models.<sup>21</sup>

Maccheroni (2002) (see also Chatterjee and Krishna (2011)) derives a utility function over lotteries of the following form: there exists a set of utilities over outcomes  $\mathcal{T}$ , such that the value of every lottery  $p$  is the lowest Expected Utility, calculated

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which is consistent with NCI.

<sup>21</sup>Chew and Epstein (1989) show that there is no intersection between RDU and Betweenness other than expected utility. Whether or not RDU satisfies convexity depends on the curvature of the distortion function  $f$ ; in particular, concave  $f$  implies convexity. In addition to disappointment aversion with negative  $\beta$ , an example of preferences that satisfy betweenness but do not satisfy NCI is Chew (1983)'s model of weighted utility.

with respect to members of  $\mathcal{T}$ , that is,

$$V(p) = \min_{v \in \mathcal{T}} \mathbb{E}_p(v).$$

Maccheroni’s interpretation of this functional form, according to which “the most pessimist of her selves gets the upper hand over the others” is closely related to our idea that “the DM acts conservatively and uses the most cautious criterion at hand.” In addition, both models satisfy Convexity. Despite these similarities, the two models are very different. First, Maccheroni’s model cannot (and it was not meant to) address the Certainty Effect. Second, one of Maccheroni (2002)’s axioms is Best Outcome Independence, which states that for each  $p, q \in \Delta$  and each  $\lambda \in (0, 1)$ ,  $p \succcurlyeq q$  if and only if  $\lambda p + (1 - \lambda) \delta_b \succcurlyeq \lambda q + (1 - \lambda) \delta_b$ . Not only this axiom is conceptually distinct from NCI, but it can also be shown that along with the other axioms, the combination of NCI and Best Outcome Independence implies the Independence axiom. Lastly, the example we discuss in Section 2.3 illustrates how comparing the expected utilities calculated using different utility functions – instead of their certainty equivalents – does not fully capture cautious risk attitude when evaluating lotteries.

Schmidt (1998) develops a static model of Expected Utility with certainty preferences, modeled in a way that is very close to NCI. In his model, the value of any non-degenerate lottery  $p$  is  $\mathbb{E}_p(u)$ , whereas the value of the degenerate lottery  $\delta_x$  is  $v(x)$ . The certainty effect is captured by requiring  $v(x) > u(x)$  for all  $x$ . Schmidt’s model violates both Continuity and Monotonicity, while in this paper we confine our attention to preferences that satisfy both properties.

Dean and Ortoleva (2012a) present a model which, when restricted to preferences over lotteries, generalizes pessimistic Rank Dependent Utility.<sup>22</sup> In their model, the DM has a single utility function and a set of pessimistic probability distortions; she then evaluates each lottery using the most pessimistic of these distortions. This property is derived by an axiom, Hedging, that captures the intuition of preference for hedging of Schmeidler (1989), but applies it to preferences over lotteries. Dean and Ortoleva (2012a) show how this axiom captures the behavior in the Allais paradoxes. The exact relation between NCI and Hedging remains an open question.

Finally, we have already discussed how our Theorem 5 is related the result in

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<sup>22</sup>The general model is defined over Anscombe-Aumann acts, and is designed to capture both the Allais and the Ellsberg paradoxes. When there is only one state of the world, however, the model reduces to a model on preferences over lotteries.

Gilboa et al. (2010) (see Section 4).

## Appendix A: Preliminary Results

We begin by proving some preliminary results that will be useful for the proofs of the main results in the text. In the sequel, we denote by  $C([w, b])$  the set of all real valued continuous functions on  $[w, b]$ . Unless otherwise specified, we endow  $C([w, b])$  with the topology induced by the supnorm. We denote by  $\Delta = \Delta([w, b])$  the set of all Borel probability measures endowed with the topology of weak convergence. We denote by  $\Delta_0$  the subset of  $\Delta$  which contains just the elements with finite support. Since  $[w, b]$  is closed and bounded,  $\Delta$  is compact with respect to this topology and  $\Delta_0$  is dense in  $\Delta$ . Given a binary relation  $\succsim$  on  $\Delta$ , we define an auxiliary binary relation  $\succsim'$  on  $\Delta$  by

$$p \succsim' q \iff \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r \quad \forall \lambda \in (0, 1], \forall r \in \Delta.$$

**Lemma 1.** *Let  $\succsim$  be a binary relation on  $\Delta$  that satisfies Weak Order. The following statements are true:*

1. *The relation  $\succsim$  satisfies Negative Certainty Independence if and only if for each  $p \in \Delta$  and for each  $x \in [w, b]$*

$$p \succsim \delta_x \implies p \succsim' \delta_x. \quad (\text{Equivalently } p \not\succsim' \delta_x \implies \delta_x \succ p.)$$

2. *If  $\succsim$  satisfies also Negative Certainty Independence then  $\succsim$  satisfies Weak Monotonicity if and only if for each  $x, y \in [w, b]$*

$$x \geq y \iff \delta_x \succsim' \delta_y,$$

*that is,  $\succsim'$  satisfies Weak Monotonicity.*

**Proof of Lemma 1.** It follows from the definition of  $\succsim'$ .

We define

$$\begin{aligned}\mathcal{V}_{in} &= \{v \in C([w, b]) : v \text{ is increasing}\}, \\ \mathcal{V}_{inco} &= \{v \in C([w, b]) : v \text{ is increasing and concave}\}, \\ \mathcal{U} &= \mathcal{V}_{s-in} = \{v \in C([w, b]) : v \text{ is strictly increasing}\}, \\ \mathcal{U}_{nor} &= \{v \in C([w, b]) : v(b) - 1 = 0 = v(w)\} \cap \mathcal{V}_{s-in}.\end{aligned}$$

Consider a binary relation  $\succ^*$  on  $\Delta$  such that

$$p \succ^* q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W} \quad (2)$$

where  $\mathcal{W}$  is a subset of  $C([w, b])$ . Define  $\mathcal{W}_{\max}$  as the set of all functions  $v \in C([w, b])$  such that  $p \succ^* q$  implies  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ . Define also  $\mathcal{W}_{\max-nor}$  as the set of all functions  $v \in \mathcal{U}_{nor}$  such that  $p \succ^* q$  implies  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ . Clearly, we have that  $\mathcal{W}_{\max-nor} = \mathcal{W}_{\max} \cap \mathcal{U}_{nor}$  and  $\mathcal{W}_{\max-nor}, \mathcal{W} \subseteq \mathcal{W}_{\max}$ .

**Proposition 4.** *Let  $\succ^*$  be a binary relation represented as in (2) and such that  $x \geq y$  if and only if  $\delta_x \succ^* \delta_y$ . The following statements are true:*

1.  $\mathcal{W}_{\max}$  and  $\mathcal{W}_{\max-nor}$  are convex and  $\mathcal{W}_{\max}$  is closed;
2.  $\emptyset \neq \mathcal{W}_{\max-nor}$ ;
3.  $\mathcal{W}_{\max} \subseteq \mathcal{V}_{in}$ ,  $\emptyset \neq \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ , and  $cl(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}) = \mathcal{W}_{\max}$ ;
4.  $p \succ^* q$  if and only if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for each  $v \in \mathcal{W}_{\max-nor}$ ;
5. If  $\mathcal{W}$  is a convex subset of  $\mathcal{U}_{nor}$  that satisfies (2) then  $cl(\mathcal{W}) = cl(\mathcal{W}_{\max-nor})$ .

**Proof of Proposition 4.** 1. Consider  $v_1, v_2 \in \mathcal{W}_{\max-nor}$  (resp.,  $v_1, v_2 \in \mathcal{W}_{\max}$ ) and  $\lambda \in (0, 1)$ . Since both functions are continuous and strictly increasing (resp., continuous), it follows that  $\lambda v_1 + (1 - \lambda)v_2$  is continuous and strictly increasing (resp., continuous). Since both functions are such that  $v_1(w) = 0 = v_2(w)$  and  $v_1(b) = 1 = v_2(b)$ , we have that

$$(\lambda v_1 + (1 - \lambda)v_2)(w) = \lambda v_1(w) + (1 - \lambda)v_2(w) = 0$$

and

$$(\lambda v_1 + (1 - \lambda)v_2)(b) = \lambda v_1(b) + (1 - \lambda)v_2(b) = 1.$$

Finally, if  $p \succ^* q$  then  $\mathbb{E}_p(v_1) \geq \mathbb{E}_q(v_1)$  and  $\mathbb{E}_p(v_2) \geq \mathbb{E}_q(v_2)$ . This implies that

$$\begin{aligned} \mathbb{E}_p(\lambda v_1 + (1 - \lambda)v_2) &= \lambda \mathbb{E}_p(v_1) + (1 - \lambda) \mathbb{E}_p(v_2) \\ &\geq \lambda \mathbb{E}_q(v_1) + (1 - \lambda) \mathbb{E}_q(v_2) = \mathbb{E}_q(\lambda v_1 + (1 - \lambda)v_2), \end{aligned}$$

proving that  $\mathcal{W}_{\max-\text{nor}}$  and  $\mathcal{W}_{\max}$  are convex. Next, consider  $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}_{\max}$  such that  $v_n \rightarrow v$ . It is immediate to see that  $v$  is continuous. Moreover, if  $p \succ^* q$  then  $\mathbb{E}_p(v_n) \geq \mathbb{E}_q(v_n)$  for all  $n \in \mathbb{N}$ . By passing to the limit, we obtain that  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ , that is, that  $v \in \mathcal{W}_{\max}$  thus proving the closure of the latter.

2. By (Dubra et al., 2004, Proposition 3), it follows that there exists  $\hat{v} \in C([w, b])$  such that

$$\begin{aligned} p \sim^* q &\implies \mathbb{E}_p(\hat{v}) = \mathbb{E}_q(\hat{v}) \\ &\text{and} \\ p \succ^* q &\implies \mathbb{E}_p(\hat{v}) > \mathbb{E}_q(\hat{v}). \end{aligned}$$

By assumption, we have that  $x \geq y$  if and only if  $\delta_x \succ^* \delta_y$ . This implies that  $x \geq y$  if and only if  $\hat{v}(x) \geq \hat{v}(y)$ , proving that  $\hat{v}$  is strictly increasing. Since  $\hat{v}$  is strictly increasing, by taking a positive and affine transformation,  $\hat{v}$  can be chosen to be such that  $\hat{v}(w) = 0 = 1 - \hat{v}(b)$ . It is immediate to see that  $\hat{v} \in \mathcal{W}_{\max-\text{nor}}$ .

3. By definition of  $\mathcal{W}_{\max}$ , we have that if  $p \succ^* q$  then  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}_{\max}$ . On the other hand, by assumption and since  $\mathcal{W} \subseteq \mathcal{W}_{\max}$ , we have that if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}_{\max}$  then  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}$  which, in turn, implies that  $p \succ^* q$ . In other words,  $\mathcal{W}_{\max}$  satisfies (2) for  $\succ^*$ . By assumption, we can thus conclude that

$$x \geq y \implies \delta_x \succ^* \delta_y \implies \mathbb{E}_{\delta_x}(v) \geq \mathbb{E}_{\delta_y}(v) \quad \forall v \in \mathcal{W}_{\max} \implies v(x) \geq v(y) \quad \forall v \in \mathcal{W}_{\max},$$

proving that  $\mathcal{W}_{\max} \subseteq \mathcal{V}_{in}$ . By point 2 and since  $\mathcal{W}_{\max-\text{nor}} \subseteq \mathcal{W}_{\max}$ , we have that  $\emptyset \neq \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ . Since  $\mathcal{W}_{\max} \cap \mathcal{V}_{s-in} \subseteq \mathcal{W}_{\max}$  and the latter is closed, we have that  $cl(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}) \subseteq \mathcal{W}_{\max}$ . On the other hand, consider  $\hat{v} \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$  and

$v \in \mathcal{W}_{\max}$ . Define  $\{v_n\}_{n \in \mathbb{N}}$  by

$$v_n = \frac{1}{n} \dot{v} + \left(1 - \frac{1}{n}\right) v \quad \forall n \in \mathbb{N}.$$

Since  $v, \dot{v} \in \mathcal{W}_{\max}$  and the latter set is convex, we have that  $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}_{\max}$ . Since  $\dot{v}$  is strictly increasing and  $v$  is increasing,  $v_n$  is strictly increasing for all  $n \in \mathbb{N}$ , proving that  $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ . Since  $v_n \rightarrow v$ , it follows that  $v \in cl(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in})$ , proving that  $\mathcal{W}_{\max} \subseteq cl(\mathcal{W}_{\max} \cap \mathcal{V}_{s-in})$  and thus the opposite inclusion.

4. By assumption, we have that there exists a subset  $\mathcal{W}$  of  $C([w, b])$  such that  $p \succ^* q$  if and only if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}$ . By point 3 and its proof, we can replace first  $\mathcal{W}$  with  $\mathcal{W}_{\max}$  and then with  $\mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ . Consider  $v \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ . Since  $v$  is strictly increasing, there exist (a unique)  $\gamma_1 > 0$  and  $\gamma_2 \in \mathbb{R}$  such that  $\bar{v} = \gamma_1 v + \gamma_2$  is continuous, strictly increasing, and satisfies  $\bar{v}(w) = 0 = 1 - \bar{v}(b)$ . For each  $v \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}$ , it is immediate to see that  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  if and only if  $\mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v})$ . Define  $\bar{\mathcal{W}} = \{\bar{v} : v \in \mathcal{W}_{\max} \cap \mathcal{V}_{s-in}\}$ . Notice that  $\bar{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$ . From the previous part, we can conclude that  $p \succ^* q$  if and only if  $\mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v})$  for all  $\bar{v} \in \bar{\mathcal{W}}$ . It is also immediate to see that  $\bar{\mathcal{W}} \subseteq \mathcal{W}_{\max-nor}$ . By construction of  $\mathcal{W}_{\max-nor}$ , notice that

$$p \succ^* q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max-nor}.$$

On the other hand, since  $\bar{\mathcal{W}} \subseteq \mathcal{W}_{\max-nor}$ , we have that

$$\mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\max-nor} \implies \mathbb{E}_p(\bar{v}) \geq \mathbb{E}_q(\bar{v}) \quad \forall \bar{v} \in \bar{\mathcal{W}} \implies p \succ^* q.$$

We can conclude that  $\mathcal{W}_{\max-nor}$  represents  $\succ^*$ .

5. Consider  $v \in \mathcal{W}$ . By assumption,  $v$  is a strictly increasing and continuous function on  $[w, b]$  such that  $v(w) = 0 = 1 - v(b)$ . Moreover, since  $\mathcal{W}$  satisfies (2), it follows that  $p \succ^* q$  implies that  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ . This implies that  $v \in \mathcal{W}_{\max-nor}$ . We can conclude that  $\mathcal{W} \subseteq \mathcal{W}_{\max-nor}$ , that is,  $cl(\mathcal{W}) \subseteq cl(\mathcal{W}_{\max-nor})$ . In order to prove the opposite inclusion, we argue by contradiction. Assume that there exists  $v \in cl(\mathcal{W}_{\max-nor}) \setminus cl(\mathcal{W})$ . Since  $v \in cl(\mathcal{W}_{\max-nor})$ , we have that  $v(w) = 0 = 1 - v(b)$ . By (Dubra et al., 2004, p. 123–124) and since both  $\mathcal{W}$  and  $\mathcal{W}_{\max-nor}$  satisfy (2), we

also have that

$$cl \left( cone(\mathcal{W}) + \{\theta 1_{[w,b]}\}_{\theta \in \mathbb{R}} \right) = cl \left( cone(\mathcal{W}_{\max-nor}) + \{\theta 1_{[w,b]}\}_{\theta \in \mathbb{R}} \right).$$

We can conclude that  $v \in cl \left( cone(\mathcal{W}) + \{\theta 1_{[w,b]}\}_{\theta \in \mathbb{R}} \right)$ . Observe that there exists  $\{\hat{v}_n\}_{n \in \mathbb{N}} \subseteq cone(\mathcal{W}) + \{\theta 1_{[w,b]}\}_{\theta \in \mathbb{R}}$  such that  $\hat{v}_n \rightarrow v$ . By construction and since  $\mathcal{W}$  is convex, there exist  $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq [0, \infty)$ ,  $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}$ , and  $\{\theta_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$  such that  $\hat{v}_n = \lambda_n v_n + \theta_n 1_{[w,b]}$  for all  $n \in \mathbb{N}$ . It follows that

$$0 = v(w) = \lim_n \hat{v}_n(w) = \lim_n \{\lambda_n v_n(w) + \theta_n 1_{[w,b]}(w)\} = \lim_n \theta_n$$

and

$$1 = v(b) = \lim_n \hat{v}_n(b) = \lim_n \{\lambda_n v_n(b) + \theta_n 1_{[w,b]}(b)\} = \lim_n \{\lambda_n + \theta_n\}.$$

This implies that  $\lim_n \theta_n = 0 = 1 - \lim_n \lambda_n$ . Without loss of generality, we can thus assume that  $\{\lambda_n\}_{n \in \mathbb{N}}$  is bounded away from zero, that is, that there exists  $\varepsilon > 0$  such that  $\lambda_n \geq \varepsilon > 0$  for all  $n \in \mathbb{N}$ . Since  $\{\theta_n\}_{n \in \mathbb{N}}$  and  $\{\hat{v}_n\}_{n \in \mathbb{N}}$  are both convergent, both sequences are bounded, that is, there exists  $k > 0$  such that

$$\|\hat{v}_n\| \leq k \text{ and } |\theta_n| \leq k \quad \forall n \in \mathbb{N}.$$

It follows that

$$\begin{aligned} \varepsilon \|v_n\| &\leq \lambda_n \|v_n\| = \|\lambda_n v_n\| = \|\lambda_n v_n + \theta_n 1_{[w,b]} - \theta_n 1_{[w,b]}\| \\ &\leq \|\lambda_n v_n + \theta_n 1_{[w,b]}\| + \|-\theta_n 1_{[w,b]}\| \\ &\leq \|\lambda_n v_n + \theta_n 1_{[w,b]}\| + |\theta_n| \\ &\leq \|\hat{v}_n\| + |\theta_n| \leq 2k \quad \forall n \in \mathbb{N}, \end{aligned}$$

that is,  $\|v_n\| \leq \frac{2k}{\varepsilon}$  for all  $n \in \mathbb{N}$ . We can conclude that

$$\begin{aligned} \|v - v_n\| &= \|v - \hat{v}_n + \hat{v}_n - v_n\| \leq \|v - \hat{v}_n\| + \|\hat{v}_n - v_n\| \\ &= \|v - \hat{v}_n\| + \|\lambda_n v_n + \theta_n 1_{[w,b]} - v_n\| \\ &\leq \|v - \hat{v}_n\| + |\lambda_n - 1| \|v_n\| + |\theta_n| \\ &\leq \|v - \hat{v}_n\| + |\lambda_n - 1| \frac{2k}{\varepsilon} + |\theta_n| \quad \forall n \in \mathbb{N}. \end{aligned}$$

Passing to the limit, it follows that  $v_n \rightarrow v$ , that is,  $v \in cl(\mathcal{W})$ , a contradiction. ■

We next provide a characterization of  $\succ'$  which is due to Cerreia-Vioglio (2009). Here, it is further specialized to the particular case where  $\succ$  satisfies Weak Monotonicity and Negative Certainty Independence on top of Weak Order and Continuity. Before proving the statement, we need to introduce a piece of terminology. We will say that  $\succ''$  is an integral stochastic order if and only if there exists a set  $\mathcal{W}'' \subseteq C([w, b])$  such that

$$p \succ q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}''.$$

**Proposition 5.** *Let  $\succ$  be a binary relation on  $\Delta$  that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. The following statements are true:*

- (a) *There exists a set  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$  such that  $p \succ' q$  if and only if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}$ .*
- (b) *For each  $p, q \in \Delta$  if  $p \succ' q$  then  $p \succ q$ .*
- (c) *If  $\succ''$  is an integral stochastic order that satisfies (b) then  $p \succ'' q$  implies  $p \succ' q$ .*
- (d) *If  $\succ''$  is an integral stochastic order that satisfies (b) and such that  $\mathcal{W}''$  can be chosen to be a subset of  $\mathcal{U}_{\text{nor}}$  then  $\overline{\text{co}}(\mathcal{W}) \subseteq \overline{\text{co}}(\mathcal{W}'')$ .*

**Proof of Proposition 5.** (a). By (Cerreia-Vioglio, 2009, Proposition 22), there exists a set  $\mathcal{W} \subseteq C([w, b])$  such that  $p \succ' q$  if and only if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}$ . By Lemma 1, we also have that  $x \geq y$  if and only if  $\delta_x \succ' \delta_y$ . By point 4 of Proposition 4, if  $\succ^* = \succ'$  then  $\mathcal{W}$  can be chosen to be  $\mathcal{W}_{\text{max-nor}}$ .

(b), (c), and (d). The statements follow from Cerreia-Vioglio (2009). ■

Next proposition clarifies what is the relation between our assumption of Weak Monotonicity and First Order Stochastic Dominance. Given two binary relations,  $p$  and  $q$ , we write  $p \succ_{FSD} q$  if and only if  $p$  dominates  $q$  with respect to First Order Stochastic Dominance.

**Proposition 6.** *If  $\succsim$  is a binary relation on  $\Delta$  that satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence then*

$$p \succsim_{FSD} q \implies p \succsim q.$$

**Proof of Proposition 6.** By Proposition 5, there exists  $\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$  such that

$$p \succsim' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \widehat{\mathcal{W}}.$$

By Proposition 5 and since  $\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}} \subseteq \mathcal{V}_{\text{in}}$ , it follows that

$$\begin{aligned} p \succsim_{FSD} q &\implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{V}_{\text{in}} \\ &\implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \widehat{\mathcal{W}} \\ &\implies p \succsim' q \implies p \succsim q, \end{aligned}$$

proving the statement. ■

We conclude this appendix with two results that explore the topological properties of a subset  $\mathcal{W}$  in  $\mathcal{U}_{\text{nor}}$ .

**Lemma 2.** *Let  $\mathcal{W}$  be a subset of  $\mathcal{U}_{\text{nor}}$ . The following statements are equivalent:*

- (i)  $\mathcal{W}$  is compact with respect to the topology of sequential pointwise convergence;
- (ii)  $\mathcal{W}$  is norm compact.

**Proof of Lemma 2.** (i) implies (ii). Consider  $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}$ . By assumption, there exists  $\{v_{n_k}\}_{k \in \mathbb{N}} \subseteq \{v_n\}_{n \in \mathbb{N}}$  and  $v \in \mathcal{W}$  such that  $v_{n_k}(x) \rightarrow v(x)$  for all  $x \in [w, b]$ . By (Aliprantis and Burkinshaw, 1998, pag. 79) and since  $v$  is a continuous function and each  $v_{n_k}$  is increasing, it follows that this convergence is uniform, proving the statement.

(ii) implies (i). It is trivial. ■

**Theorem 6.** *Let  $V : \Delta \rightarrow \mathbb{R}$  and  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$  be such that*

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta.$$

If each element of  $\mathcal{W}$  is concave,  $V$  is continuous and such that for each  $x, y \in [w, b]$  and for each  $\lambda \in (0, 1]$

$$x > y \implies V(\lambda\delta_x + (1 - \lambda)\delta_w) > V(\lambda\delta_y + (1 - \lambda)\delta_w) \quad (3)$$

then  $\mathcal{W}$  is relatively compact with respect to the topology of sequential pointwise convergence restricted to  $\mathcal{U}_{\text{nor}}$ .

**Proof of Theorem 6.** We start by proving an ancillary claim.

*Claim:* For each  $\varepsilon > 0$  there exists  $\delta \in (0, b - w)$  such that for each  $v \in \mathcal{W}$

$$v(w + \delta) < \varepsilon.$$

*Proof of the Claim.*

By contradiction, assume that there exists  $\bar{\varepsilon} > 0$  such that for each  $\delta \in (0, b - w)$  there exists  $v_\delta \in \mathcal{W}$  such that  $v_\delta(w + \delta) \geq \bar{\varepsilon}$ . In particular, for each  $k \in \mathbb{N}$  such that  $\frac{1}{k} < b - w$  there exists  $v_k \in \mathcal{W}$  such that  $v_k(w + \frac{1}{k}) \geq \bar{\varepsilon}$ . Define  $\lambda_k \in [0, 1]$  for each  $k > \frac{1}{b-w}$  to be such that

$$\lambda_k v_k(b) + (1 - \lambda_k) v_k(w) = \lambda_k = v_k\left(w + \frac{1}{k}\right) \geq \bar{\varepsilon} > 0. \quad (4)$$

Define  $p_k = \lambda_k \delta_b + (1 - \lambda_k) \delta_w$  for all  $k > \frac{1}{b-w}$ . Without loss of generality, we can assume that  $\lambda_k \rightarrow \lambda$ . Notice that  $\lambda \geq \bar{\varepsilon} > 0$ . Define  $p = \lambda \delta_b + (1 - \lambda) \delta_w$ . It is immediate to see that  $p_k \rightarrow p$ . By (4) and by definition of  $V$ , it follows that

$$w \leq V(p_k) \leq v_k^{-1}(\mathbb{E}_{p_k}(v_k)) = w + \frac{1}{k} \quad \forall k > \frac{1}{b-w}.$$

Since  $V$  is continuous and by passing to the limit, we have that

$$V(\lambda\delta_b + (1 - \lambda)\delta_w) = V(p) = w = V(\lambda\delta_w + (1 - \lambda)\delta_w),$$

a contradiction with  $V$  satisfying (3). □

Consider  $\{v_n\}_{n \in \mathbb{N}} \subseteq \mathcal{W}$ . Observe that, by construction,  $\{v_n\}_{n \in \mathbb{N}}$  is uniformly bounded. By (Rockafellar, 1970, Theorem 10.9), there exists  $\{v_{n_k}\}_{k \in \mathbb{N}} \subseteq \{v_n\}_{n \in \mathbb{N}}$  and  $v \in \mathbb{R}^{(w,b)}$  such that  $v_{n_k}(x) \rightarrow v(x)$  for all  $x \in (w, b)$ . Since  $v_{n_k}([w, b]) = [0, 1]$

for all  $k \in \mathbb{N}$ ,  $v$  takes values in  $[0, 1]$ . Define  $\bar{v} : [w, b] \rightarrow [0, 1]$  by

$$\bar{v}(w) = 0, \bar{v}(b) = 1, \text{ and } \bar{v}(x) = v(x) \quad \forall x \in (w, b).$$

Since  $\{v_{n_k}\}_{k \in \mathbb{N}} \subseteq \mathcal{W}$ , we have that  $v_{n_k}(x) \rightarrow \bar{v}(x)$  for all  $x \in [w, b]$ . It is immediate to see that  $\bar{v}$  is increasing and concave. We are left to show that  $\bar{v} \in \mathcal{U}_{\text{nor}}$ , that is,  $\bar{v}$  is continuous and strictly increasing. By (Rockafellar, 1970, Theorem 10.1) and since  $\bar{v}$  is finite and concave, we have that  $\bar{v}$  is continuous at each point of  $(w, b)$ . We are left to check continuity at the extrema. Since  $\bar{v}$  is increasing, concave, and such that  $\bar{v}(w) = 0 = \bar{v}(b) - 1$ , we have that  $\bar{v}(x) \geq \frac{x-w}{b-w}$  for all  $x \in [w, b]$ . It follows that  $1 \geq \limsup_{x \rightarrow b^-} \bar{v}(x) \geq \liminf_{x \rightarrow b^-} \bar{v}(x) \geq \lim_{x \rightarrow b^-} \frac{x-w}{b-w} = 1$ , proving continuity at  $b$ . We next show that  $\bar{v}$  is continuous at  $w$ . By the initial claim, for each  $\varepsilon > 0$  we have that there exists  $\delta > 0$  such  $v_{n_k}(w + \delta) < \frac{\varepsilon}{2}$  for all  $k \in \mathbb{N}$ . Since  $\bar{v}(w) = 0$ ,  $\bar{v}$  is increasing, and the pointwise limit of  $\{v_{n_k}\}_{k \in \mathbb{N}}$ , we have that for each  $x \in [w, w + \delta)$

$$\begin{aligned} |\bar{v}(x) - \bar{v}(w)| &= |\bar{v}(x)| = \bar{v}(x) \leq \bar{v}(w + \delta) \\ &= \lim_k v_{n_k}(w + \delta) \leq \frac{\varepsilon}{2} < \varepsilon, \end{aligned}$$

proving continuity at  $w$ . We are left to show that  $\bar{v}$  is strictly increasing. We argue by contradiction. Assume that  $\bar{v}$  is not strictly increasing. Since  $\bar{v}$  is increasing, continuous, concave, and such that  $\bar{v}(w) = 0 = \bar{v}(b) - 1$ , there exists  $x \in (w, b)$  such that  $\bar{v}(x) = 1$ . Define  $\{\lambda_k\}_{k \in \mathbb{N}} \subseteq [0, 1]$  such that  $\lambda_k v_{n_k}(b) + (1 - \lambda_k) v_{n_k}(w) = \lambda_k = v_{n_k}(x)$ . Since  $\bar{v}$  is the pointwise limit of  $\{v_{n_k}\}_{k \in \mathbb{N}}$ , it follows that  $\lambda_k \rightarrow 1$ . Define  $p_k = \lambda_k \delta_b + (1 - \lambda_k) \delta_w$  for all  $k \in \mathbb{N}$ . It is immediate to see that  $p_k \rightarrow \delta_b$ . Thus, we also have that for each  $k \in \mathbb{N}$

$$V(p_k) \leq v_{n_k}^{-1}(\mathbb{E}_{p_k}(v_{n_k})) \leq x.$$

Since  $V$  is continuous and by passing to the limit, we have that  $x < b = V(\delta_b) \leq x$ , a contradiction. ■

## Appendix B: Proof of the Results in the Text

**Proof of Theorem 1.** Before starting, we point out that in proving (i) implies (ii), we will prove the existence of a Continuous Cautious Expected Utility representation

$\mathcal{W}$  which is convex and normalized, that is, a subset of  $\mathcal{U}_{\text{nor}}$ . This will turn out to be useful in the proofs of other results in this section. The normalization of  $\mathcal{W}$  will play no role in proving (ii) implies (i).

(i) implies (ii). We proceed by steps.

*Step 1. There exists a continuous certainty equivalent utility function  $V : \Delta \rightarrow \mathbb{R}$ .*

*Proof of the Step.*

Since  $\succsim$  satisfies Weak Order and Continuity, there exists a continuous function  $\bar{V} : \Delta \rightarrow \mathbb{R}$  such that  $\bar{V}(p) \geq \bar{V}(q)$  if and only if  $p \succsim q$ . By Weak Monotonicity, we have that

$$x \geq y \iff \delta_x \succsim \delta_y \iff \bar{V}(\delta_x) \geq \bar{V}(\delta_y). \quad (5)$$

Next, observe that  $\delta_b \succsim_{FSD} q \succsim_{FSD} \delta_w$  for all  $q \in \Delta$ . By Proposition 6 and since  $\succsim$  satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence, this implies that

$$\delta_b \succsim q \succsim \delta_w \quad \forall q \in \Delta. \quad (6)$$

Consider a generic  $q \in \Delta$  and the sets

$$\begin{aligned} \{\delta_x : \delta_x \succsim q\} &= \{p \in \Delta : p \succsim q\} \cap \{\delta_x\}_{x \in [w, b]} \\ &\text{and} \\ \{\delta_x : q \succsim \delta_x\} &= \{p \in \Delta : q \succsim p\} \cap \{\delta_x\}_{x \in [w, b]}. \end{aligned}$$

By (6), Continuity, and (Aliprantis and Border, 2005, Theorem 15.8), both sets are nonempty and closed. Since  $\succsim$  satisfies Weak Order, it follows that the sets

$$\{x \in [w, b] : \delta_x \succsim q\} \text{ and } \{x \in [w, b] : q \succsim \delta_x\}$$

are nonempty, closed, and their union coincides with  $[w, b]$ . Since  $[w, b]$  is connected, there exists an element  $x_q$  in their intersection. In other words, there exists  $x_q \in [w, b]$  such that  $\delta_{x_q} \sim q$ . Since  $q$  was chosen to be generic and by (5) and (6), such element is unique and we further have that

$$\bar{V}(\delta_b) \geq \bar{V}(q) = \bar{V}(\delta_{x_q}) = \bar{V}(q) \geq \bar{V}(\delta_w) \quad \forall q \in \Delta. \quad (7)$$

Next, define  $f : [w, b] \rightarrow \mathbb{R}$  by  $f(x) = \bar{V}(\delta_x)$  for all  $x \in [w, b]$ . By (5), (Aliprantis and Border, 2005, Theorem 15.8), and (7),  $f$  is strictly increasing, continuous, and such that  $f([w, b]) = \bar{V}(\Delta)$ . It follows that  $V : \Delta \rightarrow \mathbb{R}$  defined by  $u = f^{-1} \circ \bar{V}$  is a well defined continuous function such that  $p \succsim q$  if and only if  $V(p) \geq V(q)$  and  $V(\delta_x) = x$  for all  $x \in [w, b]$ , proving the statement.  $\square$

*Step 2.*  $\succsim'$  is represented by a set  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ , that is,

$$p \succsim' q \iff v^{-1}(\mathbb{E}_p(v)) \geq v^{-1}(\mathbb{E}_q(v)) \quad \forall v \in \mathcal{W}. \quad (8)$$

*Proof of the Step.*

It follows by point (a) of Proposition 5. Recall that  $\mathcal{W}$  can be chosen to be  $\mathcal{W}_{\text{max-nor}}$  for  $\succsim'$ .  $\square$

*Step 3.* For each  $p \in \Delta$  we have that  $\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \in [w, b]$ .

*Proof of the Step.*

Fix  $p \in \Delta$ . By construction, we have that  $b \geq v^{-1}(\mathbb{E}_p(v)) \geq w$  for all  $v \in \mathcal{W}$ . It follows that  $c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$  is a real number in  $[w, b]$ .

*Step 4.* For each  $p \in \Delta$  we have that

$$V(p) \leq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)).$$

*Proof of the Step.*

Fix  $p \in \Delta$ . By Step 3,  $c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$  is a real number in  $[w, b]$ . Since  $V(\Delta) = [w, b]$ , if  $c = b$  then we have that  $V(p) \leq b = c$ . Otherwise, pick  $d$  such that  $b > d > c$ . Since  $d > c$ , we have that there exists  $\tilde{v} \in \mathcal{W}$  such that

$$\tilde{v}^{-1}(\mathbb{E}_p(\tilde{v})) < d = \tilde{v}^{-1}(\mathbb{E}_{\delta_d}(\tilde{v})).$$

By Step 2, it follows that  $p \not\sucsim' \delta_d$ . By Lemma 1, this implies that  $\delta_d \succ p$ , that is,  $V(p) < V(\delta_d) = d$ . Since  $d$  was chosen to be generic and strictly greater than  $c$ , we have that  $V(p) \leq c$ , proving the statement.  $\square$

*Step 5.* For each  $p \in \Delta$  we have that

$$V(p) \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)).$$

*Proof of the Step.*

Fix  $p \in \Delta$ . By Step 3,  $c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$  is a real number in  $[w, b]$ . By construction, we have that

$$v^{-1}(\mathbb{E}_p(v)) \geq c = v^{-1}(\mathbb{E}_{\delta_c}(v)) \quad \forall v \in \mathcal{W}.$$

By Step 2, it follows that  $p \succneq' \delta_c$ . By Proposition 5 point (b), this implies that  $p \succneq \delta_c$ , that is,  $V(p) \geq V(\delta_c) = c$ , proving the statement.  $\square$

The implication follows from Steps 1, 2, 4, and 5.

(ii) implies (i). Assume there exists a set  $\mathcal{W} \subseteq \mathcal{U}$  such that  $V : \Delta \rightarrow \mathbb{R}$ , defined by

$$V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta,$$

is a continuous utility function for  $\succneq$ . Since  $\succneq$  is represented by a continuous utility function, it follows that it satisfies Weak Order and Continuity. By construction, it is also immediate to see that  $V(\delta_x) = x$  for all  $x \in [w, b]$ . In light of this fact, Weak Monotonicity follows immediately. Finally, consider  $p \in \Delta$  and  $x \in [w, b]$ . Assume that  $p \succneq \delta_x$ . It follows that for each  $\lambda \in [0, 1]$  and for each  $q \in \Delta$

$$\begin{aligned} v^{-1}(\mathbb{E}_p(v)) &\geq V(p) \geq V(\delta_x) = x = v^{-1}(\mathbb{E}_{\delta_x}(v)) && \forall v \in \mathcal{W} \\ \implies \mathbb{E}_p(v) &\geq \mathbb{E}_{\delta_x}(v) && \forall v \in \mathcal{W} \\ \implies \mathbb{E}_{\lambda p + (1-\lambda)q}(v) &\geq \mathbb{E}_{\lambda \delta_x + (1-\lambda)q}(v) && \forall v \in \mathcal{W} \\ \implies v^{-1}(\mathbb{E}_{\lambda p + (1-\lambda)q}(v)) &\geq v^{-1}(\mathbb{E}_{\lambda \delta_x + (1-\lambda)q}(v)) && \forall v \in \mathcal{W} \\ \implies V(\lambda p + (1-\lambda)q) &\geq V(\lambda \delta_x + (1-\lambda)q) \\ \implies \lambda p + (1-\lambda)q &\succneq \lambda \delta_x + (1-\lambda)q, \end{aligned}$$

proving that  $\succneq$  satisfies Negative Certainty Independence.  $\blacksquare$

**Proof of Proposition 1.** Consider  $\mathcal{W}$  and  $\mathcal{W}'$  in  $\mathcal{U}_{\text{nor}}$  such that  $\overline{\text{co}}(\mathcal{W}) = \overline{\text{co}}(\mathcal{W}')$ . Notice first that if both  $\mathcal{W}$  and  $\mathcal{W}'$  are convex, the proposition follows trivially. To prove the proposition, it will therefore suffice to show that for each  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$  we have that

$$\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) = \inf_{v \in \text{co}(\mathcal{W})} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta.$$

Consider  $p \in \Delta$ . It is immediate to see that

$$\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \geq \inf_{v \in co(\mathcal{W})} v^{-1}(\mathbb{E}_p(v)).$$

Conversely, consider  $\bar{v} \in co(\mathcal{W})$ . It follows that there exists  $\{v_i\}_{i=1}^n \subseteq \mathcal{W}$  and  $\{\lambda_i\}_{i=1}^n \subseteq [0, 1]$  such that

$$\sum_{i=1}^n \lambda_i = 1 \text{ and } \sum_{i=1}^n \lambda_i v_i = \bar{v}.$$

Define  $x \in [w, b]$  and  $\{x_i\}_{i=1}^n \subseteq [w, b]$  by  $x = \bar{v}^{-1}(\mathbb{E}_p(\bar{v}))$  and  $x_i = v_i^{-1}(\mathbb{E}_p(v_i))$  for all  $i \in \{1, \dots, n\}$ . By contradiction, assume that  $x < \min_{i \in \{1, \dots, n\}} x_i$ . Since  $\{v_i\}_{i=1}^n \subseteq \mathcal{W} \subseteq \mathcal{V}_{s-in}$ , we have that

$$\mathbb{E}_p(\bar{v}) = \mathbb{E}_p\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i \mathbb{E}_p(v_i) = \sum_{i=1}^n \lambda_i v_i(x_i) > \sum_{i=1}^n \lambda_i v_i(x) = \bar{v}(x),$$

that is,  $x = \bar{v}^{-1}(\mathbb{E}_p(\bar{v})) > x$ , a contradiction. This implies that

$$\bar{v}^{-1}(\mathbb{E}_p(\bar{v})) = x \geq \min_{i \in \{1, \dots, n\}} x_i = \min_{i \in \{1, \dots, n\}} v_i^{-1}(\mathbb{E}_p(v_i)) \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)).$$

Since  $\bar{v}$  was chosen to be generic in  $co(\mathcal{W})$ , we can conclude that

$$\bar{v}^{-1}(\mathbb{E}_p(\bar{v})) \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \quad \forall \bar{v} \in co(\mathcal{W}),$$

proving that  $\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \leq \inf_{v \in co(\mathcal{W})} v^{-1}(\mathbb{E}_p(v))$  and thus the statement. ■

**Proof of Theorem 2.** By the proof of Theorem 1 (Steps 1, 2, 4, and 5), we have that there exists a set  $\widehat{\mathcal{W}} \subseteq \mathcal{U}_{\text{nor}}$  such that

$$p \succ' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \widehat{\mathcal{W}} \quad (9)$$

and such that  $V : \Delta \rightarrow \mathbb{R}$ , defined by

$$V(p) = \inf_{v \in \widehat{\mathcal{W}}} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta, \quad (10)$$

is a continuous utility function for  $\succ'$ . This proves points (i) and (iii). Next con-

sider a subset  $\mathcal{W}$  of  $\mathcal{U}_{\text{nor}}$  such that the function  $V : \Delta \rightarrow \mathbb{R}$  defined by  $V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$  for all  $p \in \Delta$  represents  $\succsim$ . Define  $\succsim''$  by

$$p \succsim'' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}.$$

It is immediate to see that if  $p \succsim'' q$  then  $p \succsim q$ . By point (d) of Proposition 5, this implies that  $\overline{\text{co}}(\widehat{\mathcal{W}}) \subseteq \overline{\text{co}}(\mathcal{W})$ , proving point (ii).

Finally, consider two sets  $\widehat{\mathcal{W}}_1$  and  $\widehat{\mathcal{W}}_2$  in  $\mathcal{U}_{\text{nor}}$  that satisfy (9) and (10). By point 5 of Proposition 4, it follows that  $\overline{\text{co}}(\widehat{\mathcal{W}}_1) = \overline{\text{co}}(\widehat{\mathcal{W}}_2)$ .  $\blacksquare$

**Proof of Theorem 3.** We just prove point (i) since point (ii) follows by an analogous argument. Given  $p \in \Delta$ , we denote by  $e(p)$  its expected value. We say that  $p \succsim_{MPS} q$  if and only if  $q$  is a mean preserving spread of  $p$ .<sup>23</sup> Recall that  $\succsim$  is risk averse if and only if  $p \succsim_{MPS} q$  implies  $p \succsim q$ . Assume that  $\succsim$  is risk averse. Let  $p, q \in \Delta_0$ .<sup>24</sup> Since  $\Delta_0$  is dense in  $\Delta$  and  $\succsim$  satisfies Weak Order and Continuity, we have that

$$\begin{aligned} p \succsim_{MPS} q &\implies \lambda p + (1 - \lambda)r \succsim_{MPS} \lambda q + (1 - \lambda)r \quad \forall \lambda \in (0, 1], \forall r \in \Delta_0 \\ &\implies \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r \quad \forall \lambda \in (0, 1], \forall r \in \Delta_0 \\ &\implies \lambda p + (1 - \lambda)r \succsim \lambda q + (1 - \lambda)r \quad \forall \lambda \in (0, 1], \forall r \in \Delta \\ &\implies p \succsim' q. \end{aligned}$$

This implies that

$$p \succsim_{MPS} q \implies p \succsim' q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \widehat{\mathcal{W}}.$$

We can conclude that each  $v$  in  $\widehat{\mathcal{W}}$  is concave. For the other direction, assume that

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<sup>23</sup>Recall that, by Rothschild and Stiglitz (1970), if  $p$  and  $q$  are elements of  $\Delta_0$  and  $q$  is a mean preserving spread of  $p$  then  $p$  and  $q$  have the same mean and they give the same probability to each point in their support with the exception of four ordered points  $x_1 < x_2 < x_3 < x_4$ . There the following relations hold:

$$\begin{aligned} q(x_1) - p(x_1) &= p(x_2) - q(x_2) \geq 0 \\ &\text{and} \\ q(x_4) - p(x_4) &= p(x_3) - q(x_3) \geq 0. \end{aligned}$$

<sup>24</sup>Recall that  $\Delta_0$  is the subset of  $\Delta$  which contains just the elements with finite support.

each  $v$  in  $\widehat{\mathcal{W}}$  is concave. Since  $\widehat{\mathcal{W}} \subseteq \mathcal{V}_{inco}$ , we have that

$$\begin{aligned} p \succ_{MPS} q &\implies e(p) = e(q) \text{ and } \mathbb{E}_p(v) \geq \mathbb{E}_q(v) && \forall v \in \mathcal{V}_{inco} \\ &\implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v) && \forall v \in \widehat{\mathcal{W}} \implies p \succ' q \implies p \succ q, \end{aligned}$$

proving that  $\succ$  is risk averse. ■

**Proof of Theorem 4.** (i) implies (ii). It is immediate to see that Monotonicity implies Weak Monotonicity. By the proof of Theorem 1 and the definition of  $\succ'$ , we have that the set  $\mathcal{W}_{\max-nor} \subseteq \mathcal{U}_{nor}$  is such that the function  $V : \Delta \rightarrow \mathbb{R}$ , defined by

$$V(p) = \inf_{v \in \mathcal{W}_{\max-nor}} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta, \quad (11)$$

is a continuous utility function for  $\succ$ . Since  $\succ$  satisfies Monotonicity,  $V$  is such that for each  $x, y \in [w, b]$  and for each  $\lambda \in (0, 1]$

$$x > y \implies V(\lambda \delta_x + (1 - \lambda) \delta_w) > V(\lambda \delta_y + (1 - \lambda) \delta_w).$$

By definition of  $\mathcal{W}_{\max-nor}$ , it is also immediate to check that  $\mathcal{W}_{\max-nor}$  is closed under the topology of sequential pointwise convergence restricted to  $\mathcal{U}_{nor}$ . By Theorem 6, it follows that  $\mathcal{W}_{\max-nor}$  is compact in the same topology. This yields compactness in the topology of sequential pointwise convergence. By Lemma 2, this implies that  $\mathcal{W}_{\max-nor}$  is also compact with respect to the topology induced by the supnorm. We can conclude that the inf in (11) is attained and so the statement follows.

(ii) implies (i). Consider  $V : \Delta \rightarrow \mathbb{R}$  defined by

$$V(p) = \min_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \quad \forall p \in \Delta.$$

By hypothesis,  $V$  is well defined and it represents  $\succ$ . Since  $\mathcal{W}$  is compact, we have that  $V$  is continuous. By Theorem 1, this implies that  $\succ$  satisfies Weak Order, Continuity, Weak Monotonicity, and Negative Certainty Independence. By the same arguments contained in the proof of Theorem 3 and since each  $v \in \mathcal{W}$  is concave, we have that  $\succ$  satisfies Risk Aversion. Next, consider  $p, q \in \Delta$  such that  $p \succ_{FSD} q$ . Consider also  $v \in \mathcal{W}$  such that  $V(p) = v^{-1}(\mathbb{E}_p(v))$ . Since  $v$  is strictly increasing, we have that  $V(p) = v^{-1}(\mathbb{E}_p(v)) > v^{-1}(\mathbb{E}_q(v)) \geq V(q)$ , proving that  $\succ$  satisfies Strict First Order Stochastic Dominance and so, in particular, Monotonicity. ■

**Proof of Theorem 5.** Let  $\succsim'$  be a reflexive and transitive binary relation on  $\Delta$  that satisfies Sequential Continuity, Weak Monotonicity, and Independence. We first prove the existence of a Cautious Completion. In doing this, we show that this completion has a Cautious Expected Utility representation. By Dubra et al. (2004), there exists a set  $\mathcal{W} \subseteq C([w, b])$  such that  $p \succsim' q$  if and only if  $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$  for all  $v \in \mathcal{W}$ . By Proposition 4, without loss of generality, we can assume that  $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}} \subseteq \mathcal{U}$ .

Next define the preference relation  $\hat{\succsim}$  as

$$p \hat{\succsim} q \iff \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_q(v)). \quad (12)$$

Notice that  $\hat{\succsim}$  is well defined and it satisfies Weak Order, Weak Monotonicity, and clearly for each  $p \in \Delta$  there exists  $x \in [w, b]$  such that  $p \hat{\sim} \delta_x$ . Next, we show  $\hat{\succsim}$  is a completion of  $\succsim'$ . Since each  $v \in \mathcal{W}$  is strictly increasing, we have that

$$p \succsim' q \iff v^{-1}(\mathbb{E}_p(v)) \geq v^{-1}(\mathbb{E}_q(v)) \quad \forall v \in \mathcal{W}. \quad (13)$$

This implies that if  $p \succsim' q$  then  $\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \geq \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_q(v))$ , that is, if  $p \succsim' q$  then  $p \hat{\succsim} q$ . Finally, let  $x$  be an element of  $[w, b]$  and  $p \in \Delta$  such that  $p \not\sucsim' \delta_x$ . By (13), it follows that there exists  $\tilde{v} \in \mathcal{W}$  such that  $\tilde{v}^{-1}(\mathbb{E}_{\delta_x}(\tilde{v})) = x > \tilde{v}^{-1}(\mathbb{E}_p(\tilde{v}))$ . By (12), this implies that  $\inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_{\delta_x}(v)) = x > \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ , hence  $\delta_x \hat{\succ} p$ . This concludes the proof of the existence of a Cautious Completion with the desired properties.

We are left with proving uniqueness. Let  $\succsim^\circ$  be a Cautious Completion of  $\succsim'$ . By point 1 of Definition 4,  $\succsim^\circ$  satisfies Weak Order, Weak Monotonicity, and for each  $p \in \Delta$  there exists  $x \in [w, b]$  such that  $p \sim^\circ \delta_x$ . This implies that there exists  $V : \Delta \rightarrow \mathbb{R}$  such that  $V$  represents  $\succsim^\circ$  and  $V(\delta_x) = x$  for all  $x \in [w, b]$ . Moreover, we have that  $V(\Delta) = [w, b]$ . Let  $p \in \Delta$ . Define  $c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v)) \in [w, b]$ . If  $c = b$  then  $V(p) \leq b = c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ . If  $c < b$  then for each  $d \in (c, b)$  there exists  $\tilde{v} \in \mathcal{W}$  such that  $\tilde{v}^{-1}(\mathbb{E}_{\delta_d}(\tilde{v})) = d > \tilde{v}^{-1}(\mathbb{E}_p(\tilde{v}))$ , yielding that  $p \not\sucsim' \delta_d$ . By point 3 of Definition 4, we can conclude that  $\delta_d \succ^\circ p$ , that is,  $d = V(\delta_d) > V(p)$ . Since  $d$  was arbitrarily chosen in  $(c, b)$ , it follows that  $V(p) \leq c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ . Finally, by definition of  $c$  and (13), we have that  $v^{-1}(\mathbb{E}_p(v)) \geq c = v^{-1}(\mathbb{E}_{\delta_c}(v))$  for all  $v \in \mathcal{W}$ , that is,  $p \succsim' \delta_c$ . By point 2 of Definition 4, it follows that  $p \succsim^\circ \delta_c$ , that is,  $V(p) \geq V(\delta_c) = c = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$ . In other words, we have shown that

$V(p) = \inf_{v \in \mathcal{W}} v^{-1}(\mathbb{E}_p(v))$  for all  $p \in \Delta$ . By (12) and since  $V$  represents  $\succcurlyeq^\circ$ , we can conclude that  $\succcurlyeq^\circ = \hat{\succcurlyeq}$ , proving the statement.  $\blacksquare$

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