Security Design with Interim Public Information*

André Stenzel†

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Abstract

We analyze a strategic security design and trading game as in Dang et al. [7] but with a generalized structure of public information arrival. In the absence of private information, optimal securities are those least affected by interim public information. We provide conditions such that all securities traded in equilibrium consist of multiple imperfect debt tranches. Endogenous tranching obtains in the absence of private information or different risk attitudes and introduces a misalignment in the security designer's incentives: a standard debt tranche minimizes other market participants' incentive to acquire information, but multiple leveraged debt tranches are most robust to public information arrival.

JEL Classification: D84, D86, E51, G14

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†Center for Doctoral Studies in Economics (CDSE), University of Mannheim. Address: Department of Economics, University of Mannheim, L7, 3-5 Room 3.04, E-Mail: andre.stenzel@gess.uni-mannheim.de

1 Introduction

The market for collateralized debt obligations (CDOs) has been at the core of the recent financial crisis. Global CDO issuance ballooned to 520.6 billion US Dollar in 2006 (up from 158 billion in 2004). While issuance decreased significantly to 4.3 billion in 2009 in the wake of the financial crisis, it has started to pick up again, reaching 13 billion in 2011.[1] From these observations and earlier episodes of financial distress it appears that debt and debt-like structures are at the core of these periods of distress, but nonetheless prevail as one of of the major forms of lending in the modern financial system. The structure of assets traded in financial markets has been a matter of interest to researchers and practitioners alike. The economic literature in the past 30 years has focused on the strategic aspects of their design process. One major finding in the literature is that standard debt contracts are optimal with respect to informational concerns in a variety of settings (see e.g. Townsend [21], Gale and Hellwig [13], Innes [16], Nachman and Noe [19]).

One interesting strand of literature has been initiated by Gorton and Pennacchi [14] and later developed by [7]. [14] point out that debt dominates equity with respect to protecting uninformed market participants from exploitation by informed ones. Hence, debt is argued to be less information sensitive than equity. [7] fully endogenize the security design process. In a model of strategic security design where securities are traded, they find that standard debt contracts are optimal. This is due to two features: Standard debt minimizes other market participant's incentive to acquire private information and therefore mitigates adverse selection. Furthermore, it is most robust to interim public information as it minimizes resell value variance.

The present article isolates the public information problem in [7] but generalizes the information structure. We find that the securities most robust to interim public information are composed of debt tranches. While standard debt contracts (SDCs) are composed of a single standard debt tranche and hence in the class of such contracts, we provide conditions for the non-optimality of SDCs. Any security which is not standard debt is leveraged. Leverage increases the incentive of other market participants to acquire private information. Hence, following the analysis by [7], there is a misalignment in the interests of investors: Standard debt contracts minimize the incentive for private information acquisition, whereas leveraged debt tranches - in certain cases even leveraged equity - are most robust to public information arrival.

Furthermore, we identify conditions such that the resulting security traded is composed of multiple imperfect debt tranches. This is a new result in that it is not motivated by different risk attitudes of market participants and explicitly accounts for the issuance of multiple tranches. The literature at large typically obtains a result of tranching where a single debt tranche (or equity tranche) is split off from a cash flow and sold (or held), see for example DeMarzo [9]. [9] shows that an informed investor may exploit her private information best by pooling assets it acquires from an issuer and tranching off a highly liquid standard debt tranche. This tranche can be sold due to its low information sensitivity, with the issuer retaining the resulting equity tranche. Our article, on the other hand, motivates tranching by showing that assets composed

[1] Data: Securities Industry and Financial Markets (SIFMA) press release, 2012.

of multiple debt tranches are most robust to interim public information arrival. The multiple debt tranches are imperfect in the sense that they can not be combined in a single tranche. Hence, the issuer holds on to not one but multiple residual equity tranches. These are kept on the books of the security issuing institution and correspond structurally to the risk retention by sponsors of ABCP conduits as documented by Acharya et al. [1].

The interaction of private and public information effects is fundamental for understanding the performance of real world financial markets, particularly the CDO market in the recent crisis. As [7] point out, adverse information about an asset's value may render it more information sensitive and thus lead to adverse selection and a collapse of trade. The present article does not address this interaction. However, in contrast to their framework, it exhibits a basic tradeoff for the security designer, whose incentives with respect to the public and private information issues may be misaligned. Standard debt is not necessarily optimal with respect to interim public information arrival.

1.1 Relation to the Literature

As highlighted in the introduction, there are multiple channels through which information affects the security design process. Private information is a well-known issue since it may lead to losses when trading with better informed parties. Conversely, if the seller of a security is informed, she faces a lemons problem as in Akerlof [3]. Another important factor is public, i.e. symmetric, information arrival since gains and losses from information may be incurred differently. Specifically, it may be the case that the de facto gains from positive information about an asset's value may not be fully capitalized on due to liquidity constraints of potential trading partners. If the losses from negative information, however, are fully incurred, even symmetric information arrival during the holding period affects liquidity considerations.

Starting with the seminal papers by Diamond and Verrecchia [11], Diamond and Dybvig [10] and [14], a multitude of authors has examined the effects of asymmetric information on financial markets and connected it to the structure of traded assets. [11] focus on information aggregation in markets without financial intermediaries and show that price is not always fully revealing. [10] describe how a financial intermediary can improve the situation of agents who face idiosyncratic uncertainty by providing liquidity irrespective of the state of the world.

The article by [14] takes the issue of informational impact on trading one step further end explicitly models both uninformed and informed traders active in a single market. It shows that a financial intermediary can prevent uninformed traders' losses to insiders who hold private information by issuing them (riskless) debt. Liquidity in this context refers to securities or assets which can be traded without losses to potentially better informed parties. A common feature in the literature is the existence of debt which is motivated by its feature of low information sensitivity. Low information sensitivity refers to the concept of minimizing the value of information. This concept has two main characteristics: On the one hand, in the presence of - potentially exogenous - asymmetric information in the market, gains from exploiting the informational advantage are minimized. On the other hand, if information has to be acquired, incentives for doing so are minimized because the aforementioned exploitation yields minimal profits. This strand of literature, however, takes the existence of debt as given and thus does not explicitly show how debt arises in an endogenous fashion. While it is established that trading with debt as the financial instrument of choice is superior (in terms of preventing losses on behalf of the uninformed party) to other securities, the issue of actual optimality of debt was not studied until recently. It should be noted that in settings with costly state verification or non-verifiable returns, studied for example by [21], [13] and Aghion and Bolton [2], debt is shown to be optimal for issuing a security in a primary market. The asymmetric information in these settings, however, is assumed to be exogenous instead of arising endogenously following choices and actions of the involved agents.

DeMarzo and Duffie [8] analyze a security designer and issuer whose private information results in illiquidity in the sense of a downward sloping demand curve. Standard debt is shown to be optimal under certain conditions, primarily the existence of a uniform worst case, because it minimizes the value of the private information the issuer holds. Furthermore, by retaining the resulting leveraged equity on its books, the issuer gives a signal which lessens the lemons problem. This signal is credible but costly due to the preference for cash over longterm investments. Biais and Mariotti [4] consider an alternative approach to the trading game. They let the issuer commit to a price-quantity menu before private information is observed and the fraction of the security offered is chosen. Analyzing different forms of competition amongst liquidity suppliers, they find that debt is optimally issued because it minimizes the consequences of adverse selection (competitive case) and mitigates the market power of the liquidity supplier (monopolistic case).

The idea is expanded in [7]. Building on Dang [6], who analyzes bargaining with endogenous information acquisition, they show that standard debt is least information sensitive among the class of nondecreasing securities satisfying limited liability and nonnegativity constraints. In this sense, they extend the comparative result from [14]. Furthermore, the paper proposes an explanation for the central involvement of debt in financial crises: The authors argue that a crisis is a collapse of trade after one-sided information acquisition has been triggered. This collapse is due to adverse selection and hence a sharp drop in trading volume.[2] In similar fashion, Yang [22] analyzes a game where information acquisition is no longer rigidly structured but flexible and arrives at the same conclusion: Standard debt contracts minimize incentives to acquire information and therefore maximize liquidity. This result is stronger than that of [7] in that it holds irrespective of the composition of the underlying asset pool, i.e. of the number of assets and the correlation between their returns.

[7] further show that debt is not only optimal with respect to potential private information acquisition but also most robust to public information arrival. In this sense, the incentives of the investor align: standard debt disincentivizes potential trading partners to acquire information and is least sensitive to public

[2] This drop is due to the lemons problem.

information arrival during the holding period. The general idea that information may inhibit efficiency in trading contexts is in line with a literature revolving around the articles by Kaplan [18] and Pagano and Volpin [20]. [18] shows that it can be efficient for a bank to commit to a policy keeping information about its risky assets secret despite being thus forced to offer non-contingent deposit contracts only. [20] in turn show that issuers of assets choose to publish coarse instead of precise ratings to enhance liquidity in the primary market, even though this reduces secondary market liquidity. Both articles have in common that, endogenously, not all available information is utilized. However, these articles assume the existence of debt. Nonetheless, the optimality of debt for trading in both primary and secondary markets critically relies on the structure of the public information which becomes available between the trading periods. By altering the structure, incentives to deviate from debt and instead move towards leveraged securities may come into play, thus misaligning the incentives to minimize potential information production and to minimize the resell value variance.

Farhi and Tirole [12] consider a security trading game with a binary state of nature. In this setup, tranching of securities is feasible only in the sense that they consist of a riskless debt and a pure equity component. They provide conditions under which the *insulation* effect, i.e. the effect that tranching off riskless debt protects this tranche from liquidity risk, outweighs the *trading adjuvant* effect of increasing the likelihood that the risky equity tranche is not sold. Furthermore, irrespective of the relative weight of the two effects, tranching always works against communality of information: Tranching deters information acquisition when it should be encouraged and encourages it when it should be deterred. Hence, even if tranching is superior because the insulation effect outweighs the trading adjuvant effect, it becomes undesirable once information acquisition is endogenized. [12] also extend their framework to a dynamic setting and show that liquidity is self-fulfilling. The expectation of liquidity in future states increases liquidity in the present.

While the conclusion that tranching has socially adverse effects is seemingly opposed to our finding that tranched debt contracts are optimal, it is important to note the differences in the analyses: [12] consider private information and its potential acquisition whereas we are concerned with interim public information arrival. Thus, the the idea that the incentives of security designers are not aligned with respect to the different types of information is in fact corroborated. Furthermore, they analyze a binary outcome space as opposed to a continuum. Our notion of tranching is not implementable in their framework.

The paper proceeds as follows: Section 2 introduces the model as well as key concepts and definitions. Section 3 solves the security design and trading problem after public information has arrived, and Section 4 addresses the security design problem at the initial trading stage. Section 5 further characterizes the optimal (tranched debt) contracts and provides examples. Section 6 discusses extensions of the model and its robustness. Section 7 concludes.

2 The Model

The model is that of [7] without private information acquisition, but with a generalized public information structure. There are three agents in the economy: An institution (called 'bank' or 'issuing institution') B, an investor I and a representative agent M who reflects the market. In the absence of private information and associated information asymmetries, the market is composed of agents willing to transfer utility across periods. The utilities of the agents are additively separable across three time periods $t = 1, t = 2$ and $t = 3$ with U^i being the utility of agent i and

$$\begin{aligned} U^B &= C_1^B + \frac{1}{\phi} C_2^B + C_3^B \\ U^I &= C_1^I + \sigma C_2^I + C_3^I \\ U^M &= C_2^M + C_3^M. \end{aligned}$$

C_t^i denotes the consumption of agent i in period t and $\phi > 1, \sigma > 1$ are parameters reflecting the intertemporal difference in marginal returns to consumption. The bank has (weak) preference for consumption at $t = 1$. Think for example of a preference for undertaking outside investment options which require further cash. I prefers consumption in period 2. Consumption is assumed to occur at the end of a given period. At period 2, M reflects agents willing to transfer consumption and therefore utility from the second into the third period.[3]

The agents' endowments are nonstorable and given as follows: The bank owns a pool of assets with stochastic return X distributed on $\mathbb{I} \subseteq \mathbb{R}_+$ which is due at $t = 3$. For ease of notation, we restrict attention to open intervals of the form $(x_L, x_H) \subset \mathbb{R}_+$, including (x_L, ∞).[4] The inclusion of boundary points (if an upper bound exists) would not alter the results as long as mass points are ruled out. I holds an endowment of ω at $t = 1$, while M holds an endowment of ω_m at $t = 2$. Formally, letting $\omega^i = (\omega_1^i, \omega_2^i, \omega_3^i)$ denote the endowment vector of agent i,[5]

$$\begin{aligned} \omega^B &= (0, 0, X) \\ \omega^I &= (\omega, 0, 0) \\ \omega^M &= (0, \omega_m, 0). \end{aligned}$$

X is stochastic and its payoff is publicly observable at $t = 3$. At $t = 2$, interim information about the distri-

[3] A preference for consumption in period 3 could be introduced and would not alter the analysis performed. However, the intertemporal rate of substitution is normalized to 1 to simplify expressions. Likewise, the issuing institution is set up to be indifferent between consumption at $t = 1$ and $t = 3$ for expositional purposes.

[4] Hence, canonical distributions such as exponential distributions or χ^2-distributions are valid.

[5] The remaining endowments are normalized to 0 for simplicity. Cash endowments at these points in time would lead to first trading them before securitizing the collateral and trading these securities. The problem would remain unchanged.

bution of X arrives. The endowments ω, ω_m are fixed with $\omega > x_L$.[6] The analysis performed in the following sections remains unchanged if ω_m is stochastic, as long as it is independent of the public news. Finally, we assume $\omega < E_f[X]$ to ensure that the whole project cannot be acquired by I at $t = 1$. Note that since the model abstracts from private information concerns, there is no disagreement about the value of the assets involved. Thus, the problem is that of a (limited) number of agents who wish to shift a known amount of consumption intertemporally. Above some threshold value of owned assets, gains from good interim news can no longer be realized because there is no agent willing and/or liquid enough to buy the assets for their underlying value.[7] The problem corresponds to that of Diamond-Dybvig-type models where a limited fraction of the population is patient and therefore willing to shift a limited amount of consumption into the last period by buying assets in the interim period, see for example [10], Jacklin and Bhattacharya [17] and Chari and Jagannathan [5].

In this setup, a public planner can realize gains from trade through a simple reallocation of endowments. For I to consume at $t = 2$, she needs to trade with B at $t = 1$ by buying (parts of) the project. She can then sell shares in the project to M at $t = 2$. When agents trade, they exchange promises contingent on the observable realization of X. These promises are called securities. Throughout this article, securities have to satisfy the following requirements:

Definition 1 *A security is a mapping from a domain $\mathbb{D} \subseteq \mathbb{R}_+$ into the real numbers,*
$s : \mathbb{D} \to \mathbb{R}$,
that satisfies the following restrictions:

- *limited liability: $s(x) \leq x$ for all $x \in \mathbb{D}$*
- *non-negativity: $s(x) \geq 0$ for all $x \in \mathbb{D}$*
- *non-decreasingness: $\forall x_1, x_2 \in \mathbb{D} : x_1 \geq x_2 \Rightarrow s(x_1) \geq s(x_2)$.*

The set of securities $s : \mathbb{D} \to \mathbb{R}$ satisfying these restrictions is denoted $S_{\mathbb{D}}$.

For simplicity, stochastic contracts are ruled out because they make the nondecreasingness requirement hard to evaluate. The nondecreasingness restriction in itself is a standard assumption justified by a moral hazard opportunity of the agent who can throw away output, see for example Innes [16] or Hellwig [15]. Pooling of securities based on different projects, i.e. X_1, X_2, is not included in the model and explicitly ruled out at $t = 2$.[8] However, the random endowment X can be interpreted as a collection of different assets/securities. Similarly, the setup does not rule out tranching. Since agents have constant marginal returns to consumption in any given period, they can be thought of as representing an arbitrarily large number of identical agents who hold an endowment with the aggregate endowment being what is represented in the model. If that is the

[6] If $\omega \leq x_L$, riskless debt with a face value lower or equal to x_L can always be issued. This debt is then unaffected by public information.

[7] Likewise, portfolio considerations may lead potential buyers to not wish to overinvest in the specific security class offered. Hence, they have no incentive to buy the assets at their (conditional) expected value above some threshold.

[8] Allowing pooling of securities depending on correlated underlying payoffs would affect the results. However, the main idea that individual securities should minimize resell value variance subject to the public information still factors into the security design process unless the underlying projects are perfectly negatively correlated. We address this issue in Section 6.

case, any tranching which overall still satisfies limited liability can also be represented by a single contract.[9] In interpreting the results, it is natural to think of the 'optimal' security as composed of different tranches which are sold separately to interchangeable market participants with identical intertemporal substitution rates.

The timing of the game is as follows: At $t = 1$, I makes a take-it-or-leave-it offer to the bank. This offer consists of a security s conditional on the return of X at $t = 3$ which she is willing to buy, and a price p which she pays in exchange. At $t = 2$, a public signal regarding the distribution of X is revealed to all agents. Then, I may make a take-it-or-leave-it offer to agent M. This offer consists of a security $\hat{s}$ conditional on the return of s (and hence of X), and a price $\hat{p}$. We assume that the bargaining power lies in the hand of the investor in both stages. This, coupled with the assumption that marginal utility is constant, is made to isolate the security design process with respect to arriving public information.[10] The second trading stage is only relevant if trade occurred at $t = 1$. Furthermore, the limited liability constraint imposes that $\hat{s}(x) \leq s(x)$ for all $x \in \mathbb{I}$.

Ex ante, X is distributed randomly on $\mathbb{I}$ with density $f(x)$, cumulative distribution function $F(x)$ and finite mean $\int_{\mathbb{I}} x dF(x) < \infty$. To model information arrival, let f be a mixture distribution, i.e. let $\lambda \in (0, 1)$ and

$$f(x) = \lambda f_1(x) + (1 - \lambda) f_2(x) \tag{1}$$

where f_1 and f_2 are strictly positive densities.[11] The public signal arriving at $t = 2$ reveals the true distribution, i.e. whether X is distributed according to f_1 or according to f_2. All distributions are common knowledge, as well as λ. The public information is not verifiable, i.e. securities can not be made contingent on the realization of the public signal. [7] order the underlying distributions by imposing the monotone likelihood ratio property, i.e. that $\frac{f_1(x)}{f_2(x)}$ is monotone in x. We generalize the condition by imposing ordering via first order stochastic dominance:

$$\textbf{FOSD: } \forall x \in (x_L, x_H) : F_1(x) \geq F_2(x). \tag{2}$$

For densities, first order stochastic dominance nests the monotone likelihood ratio property. Hence, if the monotone likelihood ratio property holds, first order stochastic dominance is also satisfied, whereas the reverse is not necessarily true. Note that for nondecreasing securities, first order stochastic dominance implies that

$$\forall s : E_{f_1}[s(x)] \leq E_f[s(x)] \leq E_{f_2}[s(x)]. \tag{3}$$

[9] Consider for example the issuance of two securities contingent on X, $s^1(x)$ and $s^2(x)$ with $s^1(x) + s^2(x) \leq x, \forall x \in (x_L, x_H)$, which are also nondecreasing. Due to the constant marginal returns to consumption and the unique trading partner (see above), this is equivalent to issuing a single security $s(x) = s^1(x) + s^2(x), \forall x \in (x_L, x_H)$, which will still satisfy limited liability and nondecreasingness.

[10] In this particular game, the security design problem remains identical as long as the investor has at least some bargaining power in the first trading stage.

[11] The strict positivity facilitates but does not qualitatively change the analyses. It allows for certain existence and uniqueness statements to be made without accounting for the special case of $f(\cdot)$ being locally zero.

To sum up, this is the timeline of the game in extensive form:

$t = 1.0$: I makes a take-it-or-leave-it offer (s, p) to B

$t = 1.1$: issuer B accepts the contract (s, p) or not

$t = 2.0$: distribution F_i, $i = 1, 2$ is publicly observed

$t = 2.1$: I makes a take-it-or-leave-it offer $(\hat{s}, \hat{p})$ to M

$t = 2.2$: agent M accepts the contract $(\hat{s}, \hat{p})$ or not

$t = 3.0$: x is realized and publicly observed, I is paid $s(x)$, M is paid $\hat{s}(x)$.

2.1 Concepts and Definitions

There are several classes of securities which play an important role in the subsequent analysis. One such class is that of *standard debt contracts*. Standard debt contracts are contracts which pay out according to the limited liability constraint up to their face value; in case the realization of the underlying collateral exceeds this value, the payoff is capped. Formally, the following definition captures this idea.

Definition 2 *A standard debt contract (SDC) on an interval* $\mathbb{I} \subseteq \mathbb{R}_+$ *is given by*

$$s^{SDC}(x; D) = \min\{x, D\}$$

where $D \geq 0$ *is the face value of the debt contract.*

Note that if $\forall x \in \mathbb{I} : D \geq x$, s^{SDC} corresponds to an equity contract

$$s^{SDC}(x; D) = x.$$

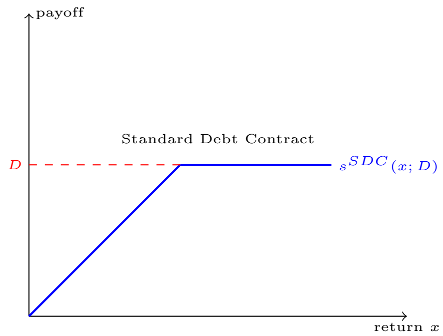


Figure 1: Standard Debt Contract

A second class of securities which is important for our analysis is the class of *leveraged equity contracts*. Leveraged equity contracts only pay out if the payoff of the collateral exceeds a certain threshold (L), but then pay up to the limited liability constraint. Formally, this is captured by the following definition.

Definition 3 *A leveraged equity contract (LE) on an interval $\mathbb{I} \subseteq \mathbb{R}_+$ is given by*

$$s^{LE}(x; L) = x \cdot \mathbf{I}_{x \geq L}$$

where L is the equity cutoff.

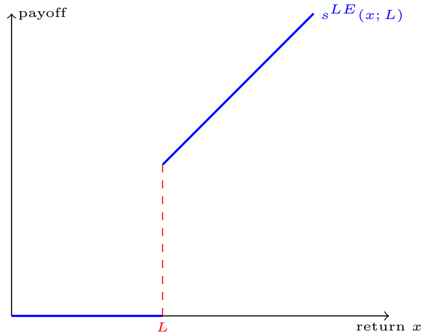


Figure 2: Leveraged Equity Contract

Given $\omega < E_f[x]$ and the strict positivity of densities f_1 and f_2, there exists a unique face value D (a unique equity cutoff L) such that the induced expected value of the standard debt contract (leveraged equity contract) is equal to ω. This is captured by Lemma 1.

Lemma 1 *For any given $\omega < E_f[X]$, there exists a unique $D(\omega)$ such that*

$$E_f[s^{SDC}(x; D(\omega))] = \omega.$$

Likewise, there exists a unique $L(\omega)$ such that

$$E_f[s^{LE}(x; L(\omega))] = \omega.$$

As stated above, the proof follows immediately from the strict positivity of densities and is thus omitted.

Another important class of contracts are what we denote *tranched debt contracts*. Tranched debt contracts are interval-by-interval composed of debt tranches.

Definition 4 *A tranched debt contract is characterized by a strictly increasing sequence $\{x_i\}_{i=1}^N \in (x_L, x_H)$ of points and a strictly increasing sequence $\{D_i\}_{i=1}^N \in \mathbb{R}_+$ of face values where*

$$x_i > D_{i-1} \forall i \geq 2.$$

The tranched debt contract s^{TD} is then characterized by the following payoff structure:

$$s^{TD}(x) = \begin{cases} 0 & \text{for } x < x_1 \\ \min\{x, D_i\} & \text{if } x \in (x_i, x_{i+1}), i < N \\ \min\{x, D_N\} & \text{if } x \in (x_N, x_H) \end{cases}$$

whenever N *is finite, and*

$$s^{TD}(x) = \begin{cases} 0 & \text{for } x < x_1 \\ \min\{x, D_i\} & \text{if } x \in (x_i, x_{i+1}), i < N \\ \min\{x, D\} & \text{if } x \geq \sup_j x_j \end{cases}$$

otherwise, where $D = \sup_j D_j$ *if the supremum exists and* $D = +\infty$ *otherwise.*[12] *At all points of the sequence* $\{x_i\}$*, the payoff is arbitrary but has to be consistent with limited liability and nondecreasingness of* s^{TD}*.*

Tranched debt contracts have payoffs either on the 45 degree line $s(x) = x$, which corresponds to binding limited liability, or on a flat part of the security, up to pointwise deviations with measure 0. Contracts which satisfy the tranched debt requirement are of the following form (here an example with two tranches)

$$s(x) = \begin{cases} 0 \text{ for } x \in (x_L, x_1) \\ \min[x, D_1] \text{ for } x \in [x_1, x_2) \\ \min[x, D_2] \text{ for } x \in [x_2, x_H) \end{cases}$$

where $D_2 > D_1$ and $x_2 > D_1$. This security is the sum of the two tranches:

$$s^{DT_1}(x) = \begin{cases} 0 \text{ for } x \in (x_L, x_1) \\ \min[x, D_1] \text{ for } x \in [x_1, x_H) \end{cases}$$

$$s^{DT_2}(x) = \begin{cases} 0 \text{ for } x \in (x_L, x_2) \\ \min[x, D_2] - D_1 \text{ for } x \in [x_2, x_H). \end{cases}$$

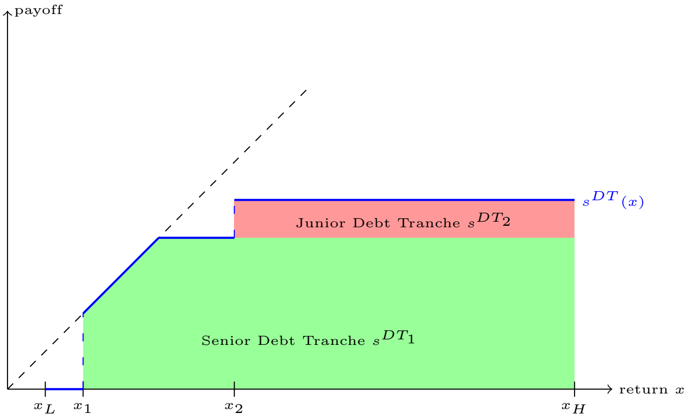


Figure 3: Canonical Tranched Debt Contract

s^{DT_2} in the example is a junior tranche: It pays out only for high realizations of x, and thus only after s^{DT_1}, the senior tranche, has been paid in full.

[12] Hence, the last tranche is a leveraged equity tranche in that case.

From the above definitions, it is clear that any standard debt contract - as well as any leveraged equity contract - is also a tranched debt contract, but the reverse does not hold. Furthermore, the contracts differ with respect to the concept of leverage.

Definition 5 *A non-decreasing security s on the interval (x_L, x_H) includes leverage if*

$$\exists x, \hat{x} \in (x_L, x_H) \text{ such that } s(x) < x \wedge s(x) < s(\hat{x}).$$

Leverage refers to the idea of speculating on high returns. A security is leveraged if it does not pay up to the limited liability constraint for certain values, but then has a higher payoff for higher realizations of the collateral. By increasing the payoff where limited liability is not binding, this dependability on high returns can be mitigated.

The only non-leveraged contracts are standard debt contracts, whereas tranched debt contracts which are not simultaneously standard debt are leveraged. As [7] have shown, leverage leads to higher incentives for private information acquisition.
We will show that tranched debt contracts are optimal with respect to interim public information arrival. This introduces a tradeoff in the investor's incentives: the investor wishes to buy a standard debt contract to avoid private information acquisition by his potential trading partners and prefers tranched debt due to its robustness to interim public information.

3 Security Design and Trading after Information Arrival

To solve for an equilibrium of the game, we first determine the optimal security designed and issued after public information arrival at $t = 2$. At $t = 2$, the bank B and representative market agent M cannot profitably trade. Thus, trade may only occur if I possesses some security s acquired from B at $t = 1$. She may either sell or use this security as collateral for a new security which is offered to M at $t = 2$. Since all information is public, trade can only occur at a price equal to the common conditional expected value of the offered security.[13] Hence, the optimal strategy for I depends on the relation of the updated value of s after public information to the market endowment ω_m.

Lemma 2 *Suppose I holds a security s at $t = 2$ after the arrival of the public information. Any security $\hat{s}$ which is in equilibrium traded to M satisfies:*

(i) If $E[s(x)|f_i] \leq \omega_m$:

$\hat{s}(x) = s(x)$ for all $x \in (x_L, x_H)$

[13] Recall that the bargaining power lies with I and that M is indifferent between consumption at $t = 2$ and $t = 3$.

(ii) If $E[s(x)|f_i] > \omega_m$ *:*

$E[\hat{s}(x)|f_i] = \omega_m$ *and*

$\hat{s}(x) \leq s(x)$ *for all* $x \in (x_L, x_H)$

$\hat{s}$ *is then sold to* M *at the fair price* $\hat{p} = E[\hat{s}|f_i]$.

One particular $\hat{s}(x)$ *which can implement this for* $E[s(x)|f_i] > \omega_m$ *is*

$$\hat{s}(x) = \tau s(x) \text{ for all } x \in (x_L, x_H)$$

where $\tau = \frac{\omega_m}{E[s(x)|f_i]}$.

Lemma 2 follows immediately from the fact that I makes a take-it-or-leave-it offer to M and has preference for consumption at $t = 2$. Intuitively, if the asset is worth weakly less than the market endowment ω_m, it is optimal for I to sell the whole security to maximize consumption at $t = 2$. If the endowment constraint binds, I sells a security that is worth strictly less than $E[s(x)|f_i]$. This new security $\hat{s}$ must satisfy limited liability with respect to s and $E[\hat{s}|f_i] = \omega_m$ to maximize consumption at $t = 2$. In this case, I holds on to the residual security $(s - \hat{s})$ and consumes this remainder - if it is positive - at $t = 3$ after the realization of X becomes observable and $s, \hat{s}$ pay out.

Lemma 2 greatly simplifies the analysis of the security design problem faced by I at $t = 1$. It allows to write the expected utility of I, given that she acquires a security s at $t = 1$, as follows:

$$\begin{aligned} EU(s) &= \omega - E_f[s(x)] + \lambda\left(\sigma \min\left\{E_{f_1}[s(x)], \omega_m\right\} + \max\left\{E_{f_1}[s(x)] - \omega_m, 0\right\}\right) \\ &\quad +(1-\lambda)\left(\sigma \min\left\{E_{f_2}[s(x)], \omega_m\right\} + \max\left\{E_{f_2}[s(x)] - \omega_m, 0\right\}\right) \qquad (4) \\ &= \omega + (\sigma - 1)\left(\lambda \min\left\{E_{f_1}[s(x)], \omega_m\right\} + (1-\lambda)\min\left\{E_{f_2}[s(x)], \omega_m\right\}\right) \qquad (5) \end{aligned}$$

where we have used $E_f[s(x)] = \lambda E_{f_1}[s(x)] + (1-\lambda)E_{f_2}[s(x)]$ and $E_{f_1}[s(x)] \leq E_f[s(x)] \leq \omega$. Hence, utility is weakly increasing in both $E_{f_1}[s(x)]$ and $E_{f_2}[s(x)]$ and it is weakly optimal for I to exhaust her endowment at $t = 1$, i.e. to acquire a security valued $E_f[s(x)] = \omega$ at $t = 1$. Furthermore, of all securities with the same unconditional expected value, i.e. of all s with $E_f[s(x)] = k$, the security with maximal value under bad information induces the (weakly) highest utility for I. This is due to $E_{f_1}[s(x)] \leq E_{f_2}[s(x)]$, which follows from nondecreasingness and F_1, F_2 being ordered by first order stochastic dominance: The endowment constraint binds in the good state if it binds in the bad state, whereas the reverse is not true. These incentives lead to the following observation. It is always an equilibrium of the game to trade solutions to the following equivalent optimization problems at $t = 1$:

$$\textbf{(P1)} \max_{s \in S_{\mathbb{I}}} E_{f_1}[s(x)] \quad \text{s.t.} \quad \lambda E_{f_1}[s(x)] + (1-\lambda) E_{f_2}[s(x)] = \omega$$

or equivalently

$$\textbf{(P2)} \min_{s \in S_{\mathbb{I}}} E_{f_2}[s(x)] \quad \text{s.t.} \quad \lambda E_{f_1}[s(x)] + (1-\lambda) E_{f_2}[s(x)] = \omega.$$

Denote $S_{\mathbb{I},\omega} \equiv \{s \in S_{\mathbb{I}} \text{ such that } E_f[s(x)] = \omega\}$. Then it holds that a solution to

$$\textbf{(P1)} \max_{s \in S_{\mathbb{I},\omega}} E_{f_1}[s(x)]$$

exists (and hence also a solution to **(P2)**).

Proposition 1 *There exists a solution to **(P1)**, i.e.*

$$\exists s^* \in S_{\mathbb{I},\omega} : E_{f_1}[s^*(x)] \geq E_{f_1}[s(x)] \text{ for all } s \in S_{\mathbb{I},\omega}.$$

The proof for Proposition 1 is relegated to the appendix. It can be shown that **(P1)** corresponds to the maximization of a continuous mapping from the convex and closed set $S_{\mathbb{I},\omega}$ into the real numbers. Hence, a maximum is attained on this set.

As noted above, incentives for I are such that acquiring a solution to **(P1)** at $t = 1$ is an equilibrium strategy. Nonetheless, it may be possible that securities which do not solve **(P1)** can be traded in equilibrium. The following Proposition yields conditions under which this applies only to solutions to **(P1)**.

Proposition 2 *Denote $E_{f_1}[s^*(x)]$ and $E_{f_2}[s^*(x)]$ the expected value of a solution to **(P1)** under bad information and good information respectively. If*

(i) $E_{f_2}[s^*(x)] \geq \omega_m \geq \omega$ *or*

(ii) $E_{f_1}[s^*(x)] \leq \omega_m \leq \omega$

*then only solutions to **(P1)** are traded in equilibrium at $t = 1$. Otherwise, there is multiplicity in the sense that securities with different (state-contingent) expected values may be issued at $t = 1$.*

The proof is relegated to the appendix. There are parameterizations such that the set of securities which may be traded at $t = 1$ in equilibrium consists only of solutions to **(P1)**. These parameterizations capture economically relevant problems: Unless ω_m is very high or very low, the conditions of Proposition 2 are satisfied.

If ω_m is too high, the security design problem becomes less interesting because many securities allow a full realization of gains from trade by never inducing a binding endowment constraint. If ω_m is too low, any security exhausting the constraint in both states captures the realizable surplus. The problem is again less interesting because many securities have this characteristic. Even in cases where the conditions of Proposition 2 do not apply, however, trading solutions to **(P1)** at $t = 1$ constitutes equilibrium behavior. Nonetheless, multiplicity not only in securities (there may be different securities solving **(P1)**) but also in state-contingent expected values arises.

For example, in the case of $E_{f_1}[s^*(x)] > \omega_m$, a security $\hat{s}$ with $E_f[\hat{s}(x)] < \omega$ can be issued at $t = 1$ in equilibrium as long as $E_{f_1}[\hat{s}(x)] \geq \omega_m$, i.e. as long as it exhausts the endowment of M at $t = 2$ in both states.

Henceforth, we will focus on characterizing solutions to **(P1)** and assume that condition (i) or (ii) from Proposition 2 is satisfied.

To illustrate the impact of the generalized information structure in contrast to [7], consider what happens if the two underlying distributions have an identical upper tail. This is not possible under the MLRP restriction except for the trivial case $f_1 = f_2$, i.e. the case without uncertainty, but covered by the assumption of FOSD. If f_1 and f_2 possess a common upper tail, leveraged securities may be traded in equilibrium because they allow for zero value variance, i.e. are not affected by interim public information.

Proposition 3 *If the two distributions f_1 and f_2 have an identical upper tail, i.e. if there exists $\bar{x} \in \mathbb{I}$ such that*

$$\forall x > \bar{x} : f_1(x) = f_2(x) = f(x)$$

and if $\omega \leq \int_{\bar{x}}^{x_H} x dF(x)$,

then any security s^* with

$$s^*(x) = 0 \text{ for } x \in (x_L, \bar{x})$$

and

$$E_f[s^*(x)] = \omega$$

*solves **(P1)**. One such security is the leveraged equity contract*

$$s^{LE}(x; L(\omega)) = x \cdot \mathbf{I}_{x \geq L(\omega)}.$$

The Proposition follows from the fact that specifying positive payoffs only on the common upper tail induces zero resell value variance. Due to (2), securities with zero resell value variance always solve **(P1)**.[14] This yields leverage of the traded securities. Leverage, however, implies an incentive asymmetry for the security designer in the [7] setting: Non-leveraged standard debt is optimal with respect to private information, whereas leveraged securities are perfectly robust to interim public information.

The following section presents the general security design problem faced by I at $t = 1$.

4 The Security Design Problem at $t = 1$

As noted before, securities are optimal and thus designed and traded at $t = 1$ if they are solutions to the problems **(P1)** and **(P2)**.

As in [7], a key role in the analysis is played by the likelihood ratio $\frac{f_1(\cdot)}{f_2(\cdot)}$. The analysis in this section proceeds as follows: First, a solution to **(P1)** is established under specific global requirements on the behavior of the likelihood ratio. Second, it is shown that certain local variations of a security do not affect global nondecreasingness and limited liability while preserving optimality. Third, it is shown that any global problem can be broken down into local ones. These local ones, however, correspond to global problems for which the solution is known. Thus, the general structure of solutions to **(P1)** is obtained and it is established that there always exists an optimal contract composed of debt tranches. Moreover, under certain conditions all optimal contracts satisfy this criterion.

The following Lemma plays an important rule in the subsequent analysis.

Lemma 3 *Consider two disjoint intervals $A, B \subset \mathbb{I}$ and securities s_1, s_2. Suppose that for all $x \in A$, $s_1(x) \geq s_2(x)$ and that for all $x \in B$, $s_1(x) \leq s_2(x)$. If $\exists k \in \mathbb{R}_+$ such that*

$$\begin{aligned}
(i) \quad & \int_A (s_1(x) - s_2(x))dF_1(x) && \leq k \int_A (s_1(x) - s_2(x))dF_2(x) \\
(ii) \quad & \int_B (s_2(x) - s_1(x))dF_1(x) && \geq k \int_B (s_2(x) - s_1(x))dF_2(x) \text{ and} \\
(iii) \quad & \int_{A\cup B} s_1(x)dF(x) && = \int_{A\cup B} s_2(x)dF(x)
\end{aligned}$$

then

$$(iv) \int_{A\cup B} s_1(x)dF_1(x) \leq \int_{A\cup B} s_2(x)dF_1(x).$$

If (i) or (ii) holds strictly, so does (iv).

[14] Typically, such securities do not exist - otherwise, the problem would be trivial. However, they exist for the given restrictions due to the common high tail and upper bound on ω. $L(\omega) \geq \bar{x}$ follows from this upper bound.

Henceforth, all proofs are relegated to the appendix. Lemma 3 states the following: Suppose that two securities have the same expected value on the union of disjoint intervals A, B under the mixture distribution f. Furthermore, suppose that s_1 lies weakly above s_2 on A and s_2 weakly above s_1 on B. If the ratio of the expected values of the difference $s_1 - s_2$ on A under bad information (i.e. f_1 being the true distribution) to that under good information (f_2) is weakly (strictly) lower than the ratio of the expected values of the difference $s_2 - s_1$ on B under bad information to that under good information, then s_2 has a weakly (strictly) higher expected value on $A \cup B$ under good information than s_1. The idea is that unconditionally, decreasing s_1 to s_2 on A and simultaneously increasing s_1 to s_2 on B does not change the expected value. However, of the change in expected value on A, less is attributed to a change in the expected value under bad information than of the change on B. Therefore, s_2 yields a higher overall expected value under bad information and conversely a lower one under good information than s_1.

The following Lemma is a restatement of the [7] Proposition about the optimality of standard debt contracts whenever the monotone likelihood ratio property holds, i.e. whenever $\frac{f_1(x)}{f_2(x)}$ is weakly decreasing in x. The proof differs from that in [7] and is included in the appendix for expositional purposes as it follows the same structure as other proofs throughout this article.

Lemma 4 *Let $\frac{f_1(x)}{f_2(x)}$ be weakly decreasing in x on $\mathbb{I}$. Then one security solving **(P1)** is the standard debt contract*

$$s^{SDC}(x; D(\omega)) = \min\{x, D(\omega)\}.$$

*If $\frac{f_1(x)}{f_2(x)}$ is strictly decreasing, then the standard debt contract $s^{SDC}(x; D(\omega))$ is the (up to pointwise deviations) unique security solving **(P1)**.*

Intuitively, the standard debt contract $s^{SDC}(x; D(\omega))$ with $E_f[s^{SDC}(x; D(\omega))] = \omega$ is optimal because it puts as much of the security payoff on the lower returns of X as possible. By the MLRP, the relative likelihood of payoffs is decreasing in the realized value of the collateral. Therefore, this maximizes expected payoff under bad information, i.e. whenever f_1 is the true distribution.

As argued previously, the assumption of a decreasing monotone likelihood ratio is restrictive. For example, following Proposition 1, for distributions with a common upper tail leveraged contracts are optimal.

Lemma 5 *Let $\frac{f_1(x)}{f_2(x)}$ be weakly increasing in x on $\mathbb{I}$. Then one security solving **(P1)** is the leveraged equity contract*

$$s^{LE}(x, L(\omega)) = x \cdot \mathbf{I}_{x \geq L(\omega)}.$$

*If $\frac{f_1(x)}{f_2(x)}$ is strictly increasing, then the leveraged equity contract $s^{LE}(x, L(\omega))$ is the (up to pointwise deviations) unique security solving **(P1)**.*

Corollary 1 *Let $\frac{f_1(x)}{f_2(x)}$ be weakly increasing in x on $\mathbb{I}$. Then one security solving the modified problem **(P1*)** including an upper bound u*

$$\textbf{\textit{(P1*)}} \; \max_{s \in S_{\mathbb{I}}} E_{f_1}[s(x)] \quad s.t. \quad \lambda E_{f_1}[s(x)] + (1-\lambda) E_{f_2}[s(x)] = \omega$$
$$s(x) \leq u \text{ for all } x \in \mathbb{I},$$

where

$$\omega \leq \int_{\mathbb{I}} \min[x, u] dF(x),$$

is the leveraged debt contract

$$s^{LD}(x; L(\omega), u) = \min\{x, u\} \cdot \mathbf{I}_{x \geq L(\omega)}$$

where $L(\omega)$ is uniquely determined by Lemma 1. If $\frac{f_1(x)}{f_2(x)}$ is strictly increasing, then the leveraged debt contract is the (up to pointwise deviations) unique security solving ***(P1*)****.*

Lemma 4, Lemma 5 and Corollary 2 are statements about how global behavior of $\frac{f_1(\cdot)}{f_2(\cdot)}$ on $\mathbb{I}$ impacts the solution to **(P1)**. If the likelihood ratio $\frac{f_1(\cdot)}{f_2(\cdot)}$ is weakly decreasing, standard debt solves **(P1)**. In the case of an increasing likelihood ratio $\frac{f_1(\cdot)}{f_2(\cdot)}$, leveraged equity or, if an upper bound for the security payoffs is specified, leveraged debt are solutions.[15] Note that an increasing likelihood ratio is inconsistent with $F_1(\cdot)$ being first order stochastically dominated by $F_2(\cdot)$ except for the case of no uncertainty $f_1 = f_2 = f$. It nonetheless is essential for further analysis: If $\frac{f_1(\cdot)}{f_2(\cdot)}$ is locally increasing on some interval $(\xi_1, \xi_2) \subset \mathbb{I}$, the problem on this interval can be transformed into one where the Proposition and/or the following corollary applies. Such a transformation also applies if $\frac{f_1(\cdot)}{f_2(\cdot)}$ is locally decreasing. Hence, Lemma 4 and Lemma 5 provide the foundation for the following two observations:

Lemma 6 *Suppose that $\frac{f_1(x)}{f_2(x)}$ is weakly decreasing in x on $(\underline{x}, \overline{x}) \subset \mathbb{I}$ with $\underline{x} < \overline{x} \leq x_H$. Let s be an optimal security solving* ***(P1)*** *on $(x_L, x_H) \equiv \mathbb{I}$. Denote $e \equiv \int_{\underline{x}}^{\overline{x}} s(x) dF(x)$.*

Define

$$s^*(x) = \begin{cases} s(x) & \text{if } x \notin (\underline{x}, \overline{x}) \\ \hat{s}(x) & \text{if } x \in (\underline{x}, \overline{x}) \end{cases}$$

with

$$\hat{s}(x; D(e)) = \min\{x, D(e)\}$$

[15] The restriction $\omega \leq \int_{\mathbb{I}} \min[x, u] dF(x)$ ensures that there is a security $s \in S_{\mathbb{I}}$ which satisfies $E_f[s(x)] = \omega$ and the constraint $s(x) \leq u$.

where $D(e)$ is the by Lemma 1 unique solution to

$$\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) = e.$$

*s^* is then also a solution to **(P1)**. Furthermore, s^* is globally (i.e. on (x_L, x_H)) nondecreasing and satisfies the tranched debt property on $(\underline{x}, \overline{x})$.*

Lemma 7 *Suppose that $\frac{f_1(x)}{f_2(x)}$ is weakly increasing in x on $(\underline{x}, \overline{x}) \subset \mathbb{I}$ with $\underline{x} < \overline{x} \leq x_H$. Let s be an optimal security solving **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$. Denote $e \equiv \int_{\underline{x}}^{\overline{x}} s(x)dF(x)$.*

Define

$$s^*(x) = \begin{cases} s(x) & \text{if } x \notin (\underline{x}, \overline{x}) \\ \hat{s}(x) & \text{if } x \in (\underline{x}, \overline{x}) \end{cases}$$

with

$$\hat{s}(x) = \begin{cases} s(\underline{x}) & \text{if } x < L \\ \min\{x, D\} & \text{if } x \geq L \end{cases}$$

where

$$D = \sup_{\xi \in (\underline{x}, \overline{x})} s(\xi)$$

and $L(e)$ is the by Lemma 1 unique solution to

$$\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) = e.$$

*s^* is then also a solution to **(P1)**. Furthermore, s^* is globally nondecreasing (i.e. on (x_L, x_H)) and satisfies the tranched debt property on $(\underline{x}, \overline{x})$.*

Intuitively, Lemma 6 and Lemma 7 state that even if a security s is locally inconsistent with the tranched debt property, there exists a security $\hat{s}$ which satisfies the tranched debt property locally. Furthermore, the security s^* which is equal to s everywhere but the local interval $(\underline{x}, \overline{x})$, and equal to $\hat{s}$ on that interval, is globally nondecreasing on $\mathbb{I}$ and satisfies the limited liability and nonnegativity constraints. If s is a solution, it also solves **(P1)**.

We are now able to state the main Proposition of this article: Proposition 4 states that if a solution to **(P1)** exists, there also exists a solution which satisfies the tranched debt property. Furthermore, if the densities $f_1(\cdot)$ and $f_2(\cdot)$ are continuous and never proportional on any interval, any solution s to **(P1)** satisfies the property: In that case, any security designed and issued in equilibrium at $t = 1$ is composed of debt tranches.

Proposition 4 *Let s be a security solving **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$. Then the following statements hold:*

*(i) There exists a valid security s^* which is also a solution to **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$ and satisfies the tranched debt property.*

(ii) If $f_1(x)$ and $f_2(x)$ are continuous and never proportional, i.e. if

$$\forall (\xi_1, \xi_2) \subseteq \mathbb{I} : \forall k \in \mathbb{R}_+ \exists x \in (\xi_1, \xi_2) : f_1 \neq k f_2$$

then s satisfies the tranched debt property.

The nonproportionality condition 4.(ii) requires that the likelihood ratio is never constant on any interval in $\mathbb{I}$. It is satisfied by most canonical distributions, including the class of exponential distributions with different rate parameters λ, the class of χ^2-distributions with different degrees of freedom, and the class of $F(d_1, d_2)$-distributions with fixed d_2 and varying d_1. Furthermore, it is easy to evaluate given parameterizations of f_1 and f_2.

The intuition for the result is as follows: Any interval where a solution s to **(P1)** is inconsistent with the tranched debt property can be decomposed into intervals where $\frac{f_1(\cdot)}{f_2(\cdot)}$ is weakly decreasing and weakly increasing respectively, up to points with measure zero. On these intervals, local changes preserving optimality exist by Lemmata 6 and 7. These changes yield a security s^* which is optimal and composed of debt tranches. Furthermore, if continuity and local nonproportionality of densities hold, any interval where a solution s to **(P1)** is inconsistent with the tranched debt property can be decomposed into intervals where $\frac{f_1(\cdot)}{f_2(\cdot)}$ is strictly decreasing and strictly increasing respectively. On these intervals, however, a local transformation exists which increases the expected payoff under bad information. This would violate optimality of s, thus implying that s must have been composed of debt tranches.

5 Characterization of Tranched Debt Contracts

This section analyzes which tranched debt contracts are optimal under certain conditions and further provides examples of optimal securities given specific parameterizations of f_1, f_2.

The following Proposition provides a condition such that the tranched debt contract solving **(P1)** includes a standard debt tranche.

Proposition 5 *Suppose that for all $x \in \mathbb{I}$ it holds that $F_1(x) > F_2(x)$. Let local nonproportionality and continuity of densities by satisfied. The any solution s to **(P1)** satisfies*

$$\forall x \in \mathbb{I} : s(x) > x_L.$$

Thus, s includes a standard debt tranche.

$\forall x \in \mathbb{I} : s(x) > x_L$ implies inclusion of a standard debt tranche because local nonproportionality establishes that any solution is composed of debt tranches. The condition $F_1(x) > F_2(x)$ on the interior of $\mathbb{I}$ is satisfied by most canonical distributions, including the aforementioned exponential distributions, χ^2-distributions and F-distributions. Furthermore, as can be seen in Example 2 below, if $F_1(x) = F_2(x)$ for some $x \in \mathbb{I}$, a security perfectly robust to public information can be constructed for ω below a certain upper bound.

The final Proposition yields a condition for the non-optimality of standard debt contracts.

Proposition 6 *Denote $D(\omega)$ the face value of the standard debt contract $s^{SDC}(x; D(\omega))$ with $E_f[s^{SDC}(x; D(\omega))] = \omega$. Let $G(x) \equiv \frac{1-F_1(x)}{1-F_2(x)}$. Suppose that f_1, f_2 are continuous.*
If

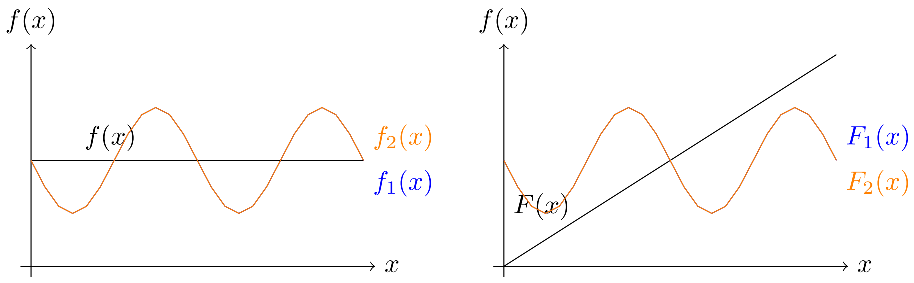
$$(i)\ \exists \xi \in (D(\omega), x_H) : G(\xi) > \inf_{x \in (x_L, D(\omega)]} \frac{f_1(x)}{f_2(x)}$$

*then $s^{SDC}(x; D(\omega))$ is not a solution to **(P1)**. Hence, the optimal contract necessarily involves leverage.*

Propositions 4, 5 and 6 yield the following insight: There are conditions under which the optimal security is composed of debt tranches, includes a standard debt tranche, but is not simultaneously a standard debt contract. Therefore, the optimal security is composed of multiple, leveraged debt tranches. In the following, Example 1 discusses the analysis of a case where such a structure is optimal.

5.1 Examples

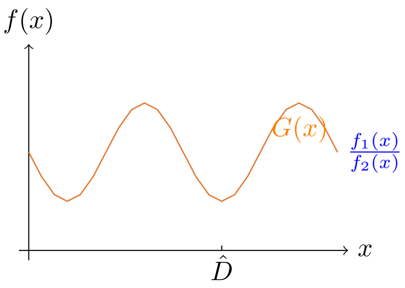
Example 1:
Consider the following densities f_1, f_2 and the associated CDFs:



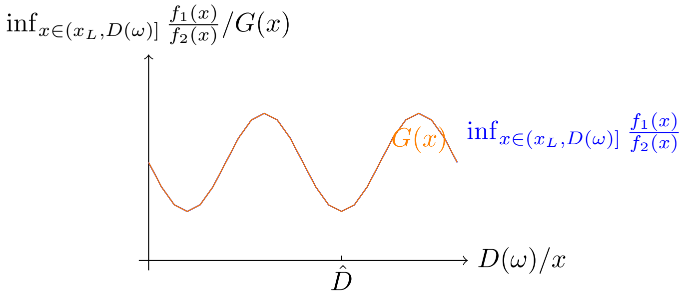
(a) Densities (b) Cumulative Distribution Functions

Figure 4: Densities and CDFs for Example 1

In Example 1, it is straightforward to see that local nonproportionality of the densities is satisfied. Hence, the optimal security is a tranched debt contract by Proposition 4. Furthermore, $F_1(x) > F_2(x)$ for all x. Proposition 5 applies and a standard debt tranche is included in the optimal contract. Since a standard debt tranche is included, it remains to be checked whether the optimal contract is indeed a standard debt contract or whether it involves multiple tranches. Proposition 6 identifies a sufficient condition for non-optimality of a standard debt contract which can be evaluated using the following illustration.



(a) LR and $G(\cdot)$-function



(b) $G(\cdot)$-function vs. infimum of LR

Figure 5: Likelihood Ratio and $G(\cdot)$-function for Example 1

For $D(\omega)$ and therefore ω large enough, $\exists \xi \in (D, x_H) : G(\xi) > \inf_{x\in(x_L,D(\omega)]} \frac{f_1(x)}{f_2(x)}$. Hence, multiple tranches are optimal. This holds for $D(\omega) > \hat{D}$. For those $D(\omega)$ and thus ω, the optimal tranched debt contract s^{TD} will have a structure as indicated in Figure 5.[16] Note the endogenous residual equity tranche(s) which will not be traded but remain(s) on the books of the institution emitting the security.

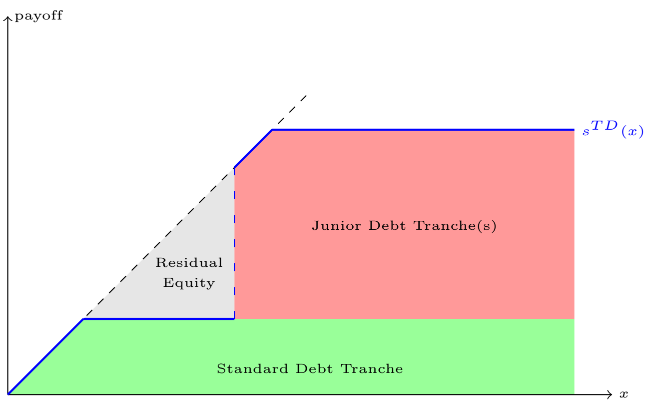
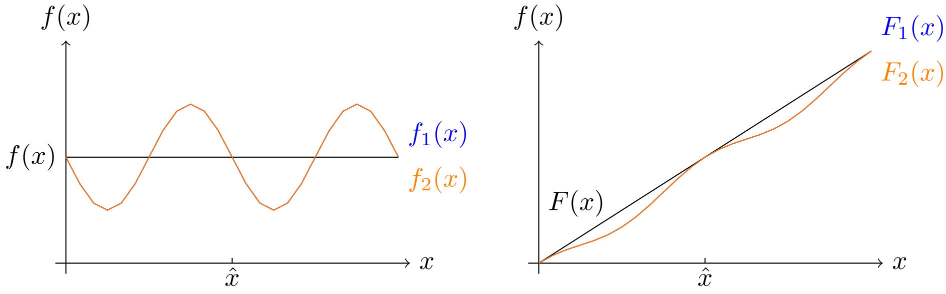


Figure 6: Tranched Debt Contract with Standard Debt Tranche

Example 2:

Consider the following densities f_1, f_2 and the associated CDFs:

[16]The number of junior debt tranches is not specified; at least one junior tranched debt is included in the optimal security by Proposition 4.



(a) Densities (b) Cumulative Distribution Functions

Figure 7: Densities and CDFs for Example 1

At $\hat{x}$, $F_1(\hat{x}) = F_2(\hat{x}) = F(\hat{x})$. Therefore, any debt tranche s^{DT} with

$$s^{DT}(x; D) = \begin{cases} 0 & \text{if } x < \hat{x} \\ D & \text{if } x \geq \hat{x}, \end{cases}$$

where $D \leq \hat{x}$, is perfectly robust to interim public information, i.e.

$$E_f[s^{DT}(x; D)] = E_{f_1}[s^{DT}(x; D)] = E_{f_2}[s^{DT}(x; D)].$$

With ω small enough, i.e. $\omega \leq \int_{\hat{x}}^{x_H} \hat{x} dF(x)$, there exists a unique $D(\omega)$ which solves $E_f[s^{DT}(x; D(\omega)] = \omega$. Because $F_1(\xi) > F_2(\xi)$ for all $\xi \in (x_L, x_H)\backslash\hat{x}$, the security $s^{DT}(x; D(\omega))$ is the unique security which offers perfect robustness. Hence, the optimal security is composed of a single leveraged debt tranche. Again, a residual equity tranche is kept on the books of the issuing institution (along with a leveraged equity tranche).

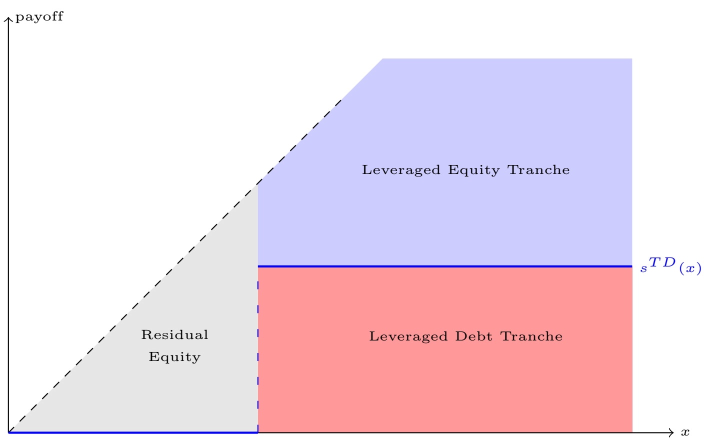


Figure 8: Single Leveraged Debt Tranche

5.2 Explaining the Residual Equity Tranches

One prediction of the model are residual equity tranches (see the above examples). The residual equity tranches are the non-traded parts of the initially owned collection of assets, X, and are hence kept on the books of the issuing institution. They do not appear to be traded explicitly in financial markets. However, we argue that they structurally correspond to the risk retention by sponsors of ABCP conduits.

[1] analyze the use of conduits, particularly asset-backed commercial paper (ABCP) conduits, in the early phase of the financial crisis of 2007-2009. They document that sponsors of conduits, especially of single seller conduits, retained significant risk when endowing conduits with assets. Extendible notes guarantees and guarantees via structured investment vehicles (SIV) lead to partial insurance of the conduit's investors. Hence, the conduit's sponsor retained the risk of the conduit's assets - assets which it originally endowed the conduit with. Full credit and full liquidity guarantees went even further and in effect provided full risk insurance.

To see the correspondence to the residual equity tranches, suppose that a sponsor endows its conduit with a debt tranche (either junior or senior debt). The conduit uses this debt tranche as collateral to secure the asset-backed commercial paper it issues. If the sponsor is (partially) covering the conduit's risk, in particular the risk of a deterioration of the conduit's asset values - i.e. the value of the debt tranche the conduit was endowed with - this may give rise to the sponsoring institution being liable for the residual equity tranche.

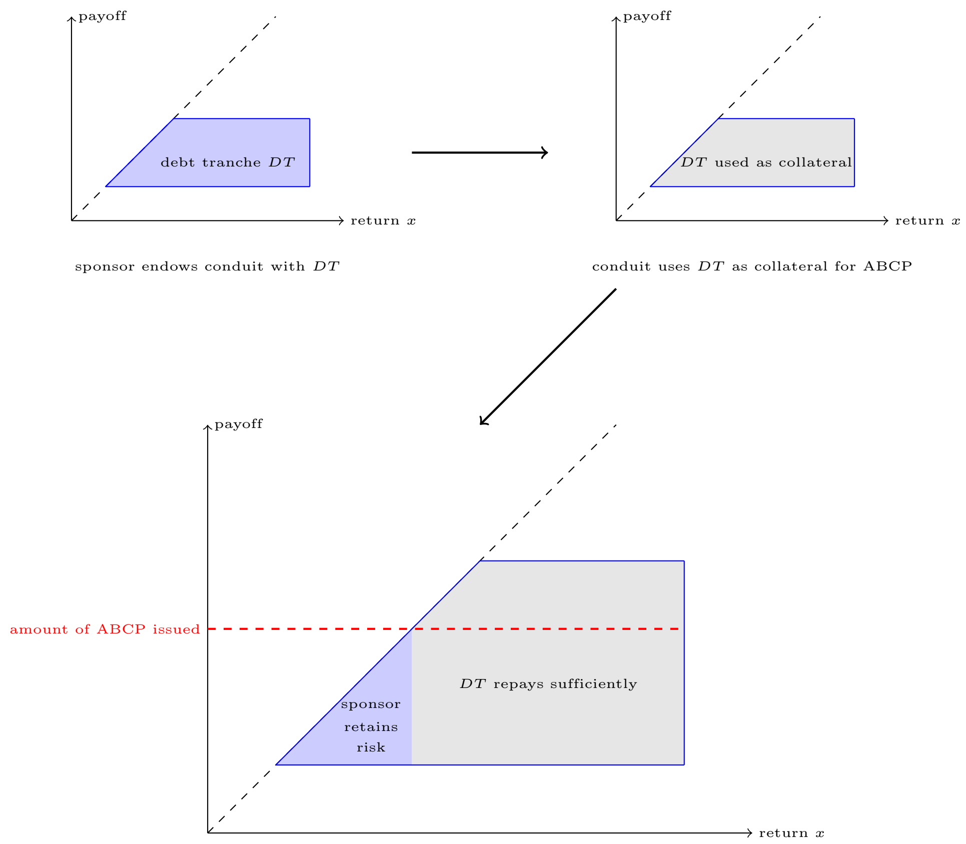


Figure 9: Process of Securitization via Conduit

If the debt tranche is used as collateral for the asset backed commercial paper issued by the conduit, a deterioration of the conduit's asset values corresponds to a realization of states of the world where the debt tranche pays off less than the value of the issued ABCP. This issue is particularly prominent if the face value of the endowment is the base for the issued commercial paper, ignoring the 'risk' that the debt tranche itself may not pay in full. This is depicted in Figure 10. If the sponsor provides insurance to the conduit via guarantees, it implicitly keeps this risk on the books. This structurally corresponds to holding on to (parts of) the residual equity tranche.

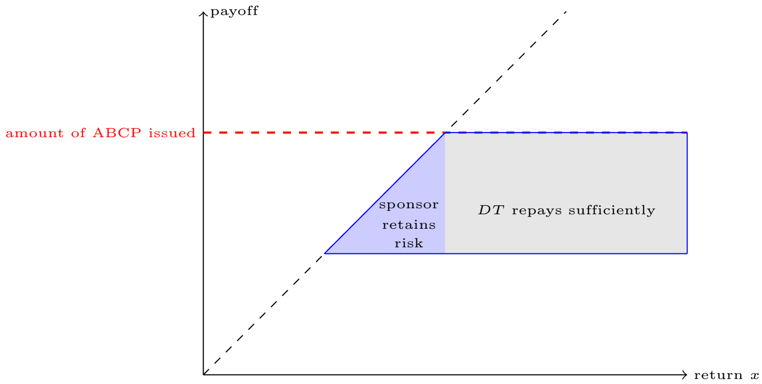


Figure 10: Face value of debt tranche equals amount of ABCP

6 Robustness and Potential Extensions

There are several ways the presented analysis can be modified and extended. Naturally, it is of interest to analyze the case of more than two underlying distributions. One way to model this goes back to [7]. There, the binary state of the world forms the baseline but the public signal does not reveal which distribution is the true one but instead the 'updated' probability of the true distribution being f_1. Hence, the signal reveals λ. This does not change the analysis presented in the previous sections.

Furthermore, it is of interest to address pooling. Suppose that the issuing institution holds not a single collection of assets with uncertain return X but explicitly consider the case where there are multiple assets $X_1, X_2, \cdots, X_N$ which are affected by the same signal. In this case, it can again be established that for any given bundle, tranched debt is optimal if first order stochastic dominance is satisfied not just for the solitary assets but also for the bundle. However, first order stochastic dominance does not need to be satisfied, for example if the returns of the assets are negatively correlated.

This leads to another interesting avenue to explore: How are the results affected if the ordering by first order stochastic dominance does not hold and f_1, f_2 (and corresponding F_1, F_2) are arbitrary densities with full support? It turns out that there always exists an optimal security which either is a tranched debt contract or a convex combination of two tranched debt contracts offering perfect robustness to the public signal.

In the following subsections, we address these three modifications of the model in detail.

6.1 Multiple underlying distributions

Consider first the modification as in [7]. Let f_1, f_2 be such that $F_1(x) \geq F_2(x)$ for all $x \in \mathbb{I}$. However, let the public signal now reveal λ, i.e. let $f(x) = \hat{\lambda} f_1(x) + (1-\hat{\lambda}) f_2(x)$ be the distribution prior to the interim information arrival and let $f(x|\lambda) = \lambda f_1(x) + (1-\lambda) f_2(x)$, with λ being the public information arriving at the interim stage.

In that case the analysis remains unchanged and the optimal contract is of the tranched debt form. To see this, consider a solution s^* to **(P1)** and any other security s. Since s^* solves **(P1)**, it holds that

$$E_{f_1}[s(x)] \leq E_{f_1}[s^*(x)] \leq E_{f_2}[s^*(x)] \leq E_{f_2}[s(x)]$$

Hence, for any realization λ, it holds that

$$\begin{aligned} U(s|\lambda) &= \omega + (\sigma - 1)\left(\lambda \min[E_{f_1}[s(x)], \omega_m] + (1-\lambda) \min[E_{f_2}[s(x)], \omega_m]\right) \\ &\leq \omega + (\sigma - 1)\left(\lambda \min[E_{f_1}[s^*(x)], \omega_m] + (1-\lambda) \min[E_{f_2}[s^*(x)], \omega_m]\right) \\ &= U(s^*|\lambda). \end{aligned}$$

This is due to the fact that if the endowment constraint binds under f_1, it also does so under f_2 by first order stochastic dominance and nondecreasingness of s, s^*.[17] Since this holds for any λ, the expected utility of s^* ex ante is higher than that of s. Thus, the analysis performed in the previous sections holds true and the characterization of solutions to **(P1)** remains the matter of interest.

6.2 Pooling of multiple assets

For simplicity, consider the case where the issuing institution ex ante owns two assets with uncertain return X_1, X_2. Further suppose that X_1 is distributed according to $f(x_1) = \lambda f_1(x_1) + (1-\lambda) f_2(x_1)$ and X_2 according to $g(x_2) = \lambda g_1(x_2) + (1-\lambda) g_2(x_2)$. Let the public signal reveal the true distribution in the sense that X_1 is distributed according to f_1 and X_2 according to g_1 with probability λ. With probability $(1-\lambda)$, the true distribution of X_1 is f_2 and that of X_2 is g_2. By pooling the two assets, i.e. by considering the asset $X \equiv X_1 + X_2$, this asset is distributed according to $h(x) = \lambda h_1(x) + (1-\lambda) h_2(x)$.

If $F_1(x_1) \geq F_2(x_1)$ and $G_1(x_2) \geq G_2(x_2)$ for all x_1, x_2 in the respective support, i.e. if the ordering of the distributions by first order stochastic dominance applies in the same direction (for both assets, the signal is bad with probability λ and good with probability $(1-\lambda)$), then it follows that $H_1(x) \geq H_2(x)$ for all $x \in \mathbb{I}$. Hence, irrespective of whether the assets are pooled or not, the analysis performed holds and the optimal contract will be a tranched debt contract. It nonetheless needs to be evaluated separately whether pooling is

[17] The detailed proof of this claim follows that of Proposition 2 and is omitted. Intuitively, the higher value of s^* under bad information allows to capture a (weakly) larger surplus from trading in those cases where the endowment constraint does not bind for s under bad information.

beneficial or not; this in particular will depend on how the endowments of the liquidity shifters (agents M) covary with the pooling decision.

However, it is possible that the assets' returns are negatively correlated in the sense that good news for one asset is bad news for the other. Consider for example the case that $F_1(x_1) \geq F_2(x_1)$ and $G_1(x_2) \leq G_2(x_2)$. In this case, there is no clear ordering with respect to the distributions of the pooled asset. This directly leads to our last modification.

6.3 Arbitrary F_1, F_2 without ordering

Suppose that f_1, f_2 are arbitrary densities and that first order stochastic dominance does not hold. Fix $\omega_m = \omega$ for simplicity and recall that first order stochastic dominance was only employed to reduce the game to solving the problem **(P1)**. Reconsider

$$\textbf{(P1)} \max_{s \in S_{\mathbb{I}}} E_{f_1}[s(x)] \quad \text{s.t.} \quad \lambda E_{f_1}[s(x)] + (1-\lambda) E_{f_2}[s(x)] = \omega$$

and denote s a solution to **(P1)**. Note that all solutions have the same expected values in all states. Furthermore, let

$$\textbf{(P1')} \max_{s \in S_{\mathbb{I}}} E_{f_2}[s(x)] \quad \text{s.t.} \quad \lambda E_{f_1}[s(x)] + (1-\lambda) E_{f_2}[s(x)] = \omega$$

and let s' be a solution to **(P1')**. By the analysis performed in this paper, it is known that there always exist s, s' which are composed of debt tranches. There are the following cases:

	$E_{f_1}[s(x)] \geq \omega$	$E_{f_1}[s(x)] < \omega$
$E_{f_2}[s'(x)] \geq \omega$	Case 1	Case 2
$E_{f_2}[s'(x)] < \omega$	Case 3	Case 4

Case 4 is impossible as $E_{f_2}[s'(x)] < \omega$ implies $E_{f_1}[s(x)] \geq E_{f_1}[s'(x)] > \omega$. In Case 1, it follows that a convex combination s^* of s and s' can be constructed which has perfect robustness to public information, i.e. $E_f[s^*(x)] = E_{f_1}[s^*(x)] = E_{f_2}[s^*(x)] = \omega$. Furthermore, Case 2 corresponds to a case where the endowment constraint ω_m may only bind if the true distribution is f_2. If that is the case, the analysis presented in the previous sections carries through and s will be an optimal security. In Case 3, the role of f_1 and f_2 is reversed: s' is optimal, f_1 can be considered good information and f_2 bad information.

This classification allows the statement that irrespective of any ordering imposed on f_1 and f_2, there always exists an optimal security which is either composed of debt tranches or which is perfectly robust to interim public information and can be constructed as the convex combination of two tranched debt contracts. Hence, even in the case of pooling where first order stochastic dominance no longer holds, tranched debt contracts play a pivotal role in designing the optimal security.

7 Conclusion

This article has built upon the analysis by [7] to analyze a security design problem where public information arrives between trading periods. In the absence of private information acquisition, a security is optimal if it is least sensitive to interim public information: Since 'gains' from good interim information can not be fully capitalized upon whereas 'losses' from bad information are fully incurred, an optimal security has maximal value in the bad information state and correspondingly minimal value after good information has arrived in the interim period.

If good information and bad information can be differentiated according to an ordering imposed by the monotone likelihood ratio property, standard debt is optimal as shown by [7]. However, this ordering is restrictive. Whenever the arriving information only affects e.g. the low tail of the distribution of returns of the underlying collateral, a more general information structure seems prudent. We model such a generalization by imposing ordering by first order stochastic dominance which nests the previous analysis.

As a result an optimal security can be seen as a composition of debt tranches. Contracts with a tranched debt form range from the standard debt contract to leveraged equity and multiple tranches of different seniorities. We identify conditions under which the optimal security includes, but is not limited to a standard debt tranche.

The results provide an explanation for the endogenous occurrence of tranches which are frequently seen in financial markets. This explanation is new in that it is independent of private information. Instead, the tranches arise because they allow to put the most weight on those returns of the underlying collateral which are more likely to be met even after 'bad' information arrived. If the monotone likelihood ratio criterion on the underlying distributions fails to hold, it no longer is necessarily optimal to put the most weight of the payoffs on the lowest realized returns, i.e. to issue standard debt. Instead, it may even be optimal to trade leveraged equity. Furthermore, our notion of tranching differs from that in the literature. We provide an explanation for designing securities composed of multiple, imperfect tranches, whereas the literature typically focuses on (pooling and) tranching off a single standard debt tranche and holding on to the residual equity. Following our results, if the initially traded contract is composed of different debt tranches, the issuer will always hold on to multiple residual equity tranches.

The residual equity tranches are a specific prediction of the model. While they are not explicitly traded in financial markets, we argue that they arise from the use of conduits in the ABCP market as documented by [1]. Specifically, a sponsor who endows its conduit with a debt tranche, and who retains the risk of the conduit through guarantees, implicitly keeps the risk that the debt tranche does not pay up to face value on the books. This structurally corresponds to the residual equity tranches which arise endogenously in our model.

Lastly, there are implications for the relation between private and public information concerns. While private information acquisition is abstracted from in our framework, it is clear from [7] that leveraged contracts do not minimize other market participant's incentives to acquire private information. Quite to the contrary, the possible case of leveraged equity maximizes these incentives. Furthermore, while tranching is beneficial with respect to the interim public information arrival, [12] show that it works against communality of information when looking at the private information problem. Hence, there exists a tradeoff for the security designer: With the more general structure of public information, tranches involving leverage will be optimal in terms of creating a security robust to interim public information. Nonetheless, a standard debt contract would minimize the other market participants' desire to acquire private information. How this tradeoff plays out is an interesting avenue to explore in the future. Intuitively, for very large costs of private information acquisition, the public information issue should dominate and leveraged debt tranches should be issued.

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Appendix: Proof of Lemmata and Propositions

Proposition 1 *There exists a solution to* ***(P1)****, i.e.*

$$\exists s^* \in S_{\mathbb{I},\omega} : E_{f_1}[s^*(x)] \geq E_{f_1}[s(x)] \textit{ for all } s \in S_{\mathbb{I},\omega}.$$

Proof:
Existence is established in the following way: **(P1)** corresponds to the optimization of $E_{f_1}[s(x)]$ over the set $S_{\mathbb{I},\omega} \equiv \{s \in S_{\mathbb{I}} \text{ such that } E_f[s(x)] = \omega\}$. The objective $E_{f_1}[s(x)]$ corresponds to a mapping $h : S_{\mathbb{I},\omega} \to \mathbb{R}$ where $h(s) = E_{f_1}[s(x)]$. By showing that $S_{\mathbb{I},\omega}$ is convex and closed and h is continuous, existence of a maximum of h on the set $S_{\mathbb{I},\omega}$ follows.

To see that $S_{\mathbb{I},\omega}$ is convex, consider $s_1, s_2 \in S_{\mathbb{I},\omega}$. It has to be shown that

$$\forall \alpha \in [0,1] : \alpha s_1 + (1-\alpha)s_2 \in S_{\mathbb{I},\omega}. \tag{6}$$

Note that:

- nondecreasingness of s_1, s_2 implies that $\alpha s_1 + (1-\alpha)s_2$ is nondecreasing as well
- $(\forall x \in \mathbb{I} : s_1(x) \leq x, s_2(x) \leq x) \Rightarrow \alpha s_1(x) + (1-\alpha)s_2(x) \leq x,$ for all $x \in \mathbb{I}$
  hence, $\alpha s_1 + (1-\alpha)s_2$ satisfies limited liability
- $(\forall x \in \mathbb{I} : s_1(x) \geq 0, s_2(x) \geq 0) \Rightarrow \alpha s_1(x) + (1-\alpha)s_2(x) \geq 0,$ for all $x \in \mathbb{I}$
  hence, $\alpha s_1 + (1-\alpha)s_2$ satisfies nonnegativity
- $E_f[\alpha s_1(x) + (1-\alpha)s_2(x)] = \alpha E_f[s_1(x)] + (1-\alpha)E_f[s_2(x)] = \omega.$

Thus, (6) follows.

Before establishing continuity of h and closedness of $S_{\mathbb{I},\omega}$, we first need to define a distance metric on the room $S_{\mathbb{I},\omega}$. Let $s_1, s_2 \in S_{\mathbb{I},\omega}$. Define

$$d(s_1, s_2) \equiv \sup_{x \in \mathbb{I}} |s_1(x) - s_2(x)|. \tag{7}$$

It follows that

$$\begin{aligned}
\forall s_1, s_2, s_3 \in S_{\mathbb{I},\omega} : \quad & d(s_1, s_2) \geq 0 && (8)\\
& d(s_1, s_2) = 0 \quad \Leftrightarrow (\forall x \in \mathbb{I} : s_1(x) = s_2(x)) \Leftrightarrow s_1 = s_2 && (9)\\
\forall x \in \mathbb{I} : |s_1(x) - s_2(x)| \quad & \leq |s_1(x) - s_3(x)| + |s_3(x) - s_2(x)| &&\\
\Rightarrow \sup\nolimits_{x \in \mathbb{I}} |s_1(x) - s_2(x)| \quad & \leq \sup_{x \in \mathbb{I}} |s_1(x) - s_3(x)| + \sup_{x \in \mathbb{I}} |s_3(x) - s_2(x)|. && (10)
\end{aligned}$$

Hence, (7) constitutes a metric on $S_{\mathbb{I},\omega}$. Therefore,

$$s_i \xrightarrow{i \to \infty} s \Leftrightarrow d(s_i, s) \xrightarrow{i \to \infty} 0. \tag{11}$$

Closedness follows from the fact that for any converging sequence $s_i \xrightarrow{i \to \infty} s$ where $s_i \in S_{\mathbb{I},\omega}$ for all i, the limit s is contained in $S_{\mathbb{I},\omega}$. We will show that $s \in S_{\mathbb{I},\omega}$ by establishing that it satisfies limited liability, nondecreasingness, nonnegativity and has an expected value $E_f[s(x)] = \omega$.

First, $E_f[s(x)] = \omega$ is established. Suppose $E_f[s(x)] > \omega$ (the contradiction for $E_f[s(x)] < \omega$ works analogously). Let $\delta \equiv E_f[s(x)] - \omega$. Then $s_i \xrightarrow{i \to \infty} s$ implies for any $\epsilon > 0, \epsilon < \delta$:

$$\exists N : \forall i \geq N : s_i(x) \geq s(x) - \epsilon \text{ for all } x \in \mathbb{I}. \tag{12}$$

Hence, for all $i \geq N$ it follows that

$$E_f[s_i(x)] \geq E_f[s(x)] - \epsilon > E_f[s(x)] - \delta = \omega. \tag{13}$$

This is a contradiction to $s_i \in S_{\mathbb{I},\omega}$.
Next, suppose that s does not satisfy limited liability, i.e. that $s(\xi) > \xi$ for some $\xi \in \mathbb{I}$. By $s_i \xrightarrow{i\to\infty} s$, this implies that

$$\exists N : \forall i \geq N : s_i(\xi) > \xi. \tag{14}$$

This contradicts $s_i \in S_{\mathbb{I},\omega}$ for $i \geq N$. In the same manner, nonnegativity of s is established.

Finally, suppose that s violates nondecreasingness, i.e. that

$$\exists x_1, x_2 \in \mathbb{I} \text{ such that } x_1 < x_2 \wedge s(x_1) > s(x_2). \tag{15}$$

However, s_i is nondecreasing for all i. Hence, $s_i(x_1) \leq s_i(x_2)$ for all i. Since $s_i \xrightarrow{i\to\infty} s$, a contradiction again follows. $s(x_1) > s(x_2)$ requires $s_i(x_1) > s_i(x_2)$ for all $i \geq N$ for some N.

Lastly, continuity of h is established. Recall

$$h(s) = E_{f_1}[s(x)] = \int_{x_L}^{x_H} s(x)dF_1(x). \tag{16}$$

Now take $s_i \xrightarrow{i\to\infty} s$. It follows that

$$\begin{aligned} \lim_{i\to\infty} h(s_i) &= \lim_{i\to\infty} \int_{x_L}^{x_H} s_i(x)dF_1(x) \\ &= \int_{x_L}^{x_H} \lim_{i\to\infty} s_i(x)dF_1(x) \\ &= \int_{x_L}^{x_H} s(x)dF_1(x) \\ &= h(s), \end{aligned} \tag{17}$$

which yields continuity of h.
We have thus established that $S_{\mathbb{I},\omega}$ is convex and closed and that h is continuous. Hence, $h(s) = E_{f_1}[s(x)]$ attains a maximum on $S_{\mathbb{I},\omega}$ and a solution to **(P1)** exists. ∎

Proposition 2 *Denote $E_{f_1}[s^*(x)]$ and $E_{f_2}[s^*(x)]$ the expected value of any solution to* ***(P1)*** *under bad information and good information respectively. If*

(i) $E_{f_2}[s^*(x)] \geq \omega_m \geq \omega$ *or*

(ii) $E_{f_1}[s^*(x)] \leq \omega_m \leq \omega$

then only solutions to ***(P1)*** *are traded in equilibrium at $t = 1$. Otherwise, there is multiplicity in the sense that securities with different (state-contingent) expected values may be issued at $t = 1$.*

Proof:

Let $E_{f_1}[s^*(x)]$ be the expected value of a solution s^* to **(P1)** after bad information. By construction of **(P1)**, all solutions have the same state-contingent expected values. Recall (5), i.e. the expected utility of I from acquiring a security s at $t = 1$:

$$EU(s) = \omega + (\sigma - 1)\left(\lambda \min[E_{f_1}[s(x)], \omega_m] + (1-\lambda)\min[E_{f_2}[s(x)], \omega_m]\right).$$

Consider any security s which could be traded at $t = 1$, i.e. which satisfies

$$E_f[s(x)] \leq \omega = E_f[s^*(x)], \tag{18}$$

and which is not a solution to **(P1)**. It needs to hold that

$$E_{f_1}[s(x)] < E_{f_1}[s^*(x)]. \tag{19}$$

If $E_{f_1}[s(x)] \geq E_{f_1}[s^*(x)]$, a contradiction would be obtained in the sense that s^* is not a solution or that s is a solution to **(P1)**.

$E_{f_1}[s(x)] > E_{f_1}[s^*(x)]$ implies that $\hat{s}$ with $\hat{s}(x) \geq s(x)$ at all $x \in \mathbb{I}$ exists, where $E_f[\hat{s}(x)] = \omega$ and $E_{f_1}[\hat{s}(x)] > E_{f_1}[s^*(x)]$. This violates s^* being a solution and follows from the strict positivity of densities.

If $E_{f_1}[s(x)] = E_{f_1}[s^*(x)]$ and $E_f[s(x)] = \omega$, then s would be a solution, whereas $E_{f_1}[s(x)] = E_{f_1}[s^*(x)]$, $E_f[s(x)] < \omega$ and strict positivity of densities again yields existence of $\hat{s}$ with

$$\begin{aligned} & \hat{s}(x) \geq s(x) \text{ for all } x \in \mathbb{I} \\ \wedge \quad & \exists \mathbb{A} \subseteq \mathbb{I} : \hat{s}(x) > s(x) \forall x \in \mathbb{A}, \int_{\mathbb{A}} 1 dF(x) > 0 \\ \wedge \quad & E_f[\hat{s}(x)] = \omega. \end{aligned} \tag{20}$$

Hence, $E_{f_1}[\hat{s}(x)] > E_{f_1}[s^*(x)]$ due to $\hat{s}(x) > s(x) \forall x \in \mathbb{A}$. Thus, (19) needs to hold as otherwise a contradiction is obtained.

If condition (i) or (ii) is satisfied, $E_{f_2}[s^*(x)] \geq \omega_m$.[18] Therefore, using (19),

$$\begin{aligned} EU(s) &= \omega + (\sigma - 1)\left[\lambda \min[E_{f_1}[s(x)], \omega_m] + (1-\lambda)\min[E_{f_2}[s(x)], \omega_m]\right] \\ &\leq \omega + (\sigma - 1)\left[\lambda \min[E_{f_1}[s(x)], \omega_m] + (1-\omega_m)\right] \\ &= \omega + (\sigma - 1)\left[\lambda E_{f_1}[s(x)] + (1-\lambda)\omega_m)\right] \\ &< \omega + (\sigma - 1)\left[\lambda E_{f_1}[s^*(x)] + (1-\lambda)\omega_m)\right] \\ &= EU(s^*). \end{aligned} \tag{21}$$

Hence we have established that only solutions to **(P1)** are traded in equilibrium at $t = 1$ if (i) or (ii) holds. ∎

If $\omega_m < \omega$ and $E_{f_1}[s^*(x)] > \omega_m$, then multiplicity arises in the sense that securities which may be traded in equilibrium at $t = 1$ differ in their expected values across states. Specifically, any security s which satisfies $E_{f_1}[s] \geq \omega_m$ (and hence also $E_{f_2}[s] \geq E_{f_1}[s] \geq \omega_m$ by first order stochastic dominance) along with $E_f[s(x)] \leq \omega$ may be traded in equilibrium at $t = 1$. These securities have in common that they fully exhaust the trading capacity which is limited by the endowment constraint ω_m.

If $\omega \leq E_{f_2}[s^*(x)] < \omega_m$, any security s with $E_f[s(x)] = \omega$ and $E_{f_2}[s(x)] \leq \omega_m$ may also be issued. Even though those securities have a higher value after good interim information than solutions to **(P1)**, they still do not induce a binding endowment constraint ω_m. Gains from trade are fully realized, i.e. the security acquired at $t = 1$ is fully sold irrespective of the interim public information.

Lemma 3 *Consider two disjoint intervals $A, B \subset \mathbb{I}$ and securities s_1, s_2. Suppose that for all $x \in A$, $s_1(x) \geq s_2(x)$ and that for all $x \in B$, $s_1(x) \leq s_2(x)$. If $\exists k \in \mathbb{R}_+$ such that*

[18] For condition (ii) this follows from $E_{f_2}[s^*(x)] \geq \omega$ and $\omega \geq \omega_m$.

$$\begin{array}{rll} (i) & \int_A (s_1(x)-s_2(x))dF_1(x) & \leq k\int_A (s_1(x)-s_2(x))dF_2(x) \\ (ii) & \int_B (s_2(x)-s_1(x))dF_1(x) & \geq k\int_B (s_2(x)-s_1(x))dF_2(x) \text{ and} \\ (iii) & \int_{A\cup B} s_1(x)dF(x) & = \int_{A\cup B} s_2(x)dF(x) \end{array}$$

then

$$(iv)\int_{A\cup B} s_1(x)dF_1(x) \leq \int_{A\cup B} s_2(x)dF_1(x).$$

If (i) or (ii) holds strictly, so does (iv).

Proof:
The proof is straightforward and done here for (i) or (ii) holding strictly. If both hold weakly, the same steps yield the result with the weak inequality for (iv). Fix k and suppose without loss of generality that (i) holds strictly, i.e.

$$\int_A (s_1(x)-s_2(x))dF_1(x) < k\int_A (s_1(x)-s_2(x))dF_2(x). \tag{22}$$

Recall (1), i.e. $f(x) = \lambda f_1(x) + (1-\lambda)f_2(x)$. Now it holds that

$$\begin{array}{rcl} \int_B (s_2(x)-s_1(x))dF_1(x) & \geq & k\int_B (s_2(x)-s_1(x))dF_2(x) \\ \wedge \int_A (s_1(x)-s_2(x))dF_1(x) & < & k\int_A (s_1(x)-s_2(x))dF_2(x) \end{array} \tag{23}$$

Thus, since

$$\begin{array}{rc} & (\lambda + k(1-\lambda))\int_A (s_1(x)-s_2(x))dF_1(x) \\ = & \lambda\int_A (s_1(x)-s_2(x))dF_1(x) + k(1-\lambda)\int_A (s_1(x)-s_2(x))dF_1(x) \\ < & \lambda\int_A (s_1(x)-s_2(x))dF_1(x) + (1-\lambda)\int_A (s_1(x)-s_2(x))dF_2(x) \\ = & \lambda\int_B (s_2(x)-s_1(x))dF_1(x) + (1-\lambda)\int_B (s_2(x)-s_1(x))dF_2(x) \\ \leq & \lambda\int_B (s_2(x)-s_1(x))dF_1(x) + k(1-\lambda)\int_B (s_2(x)-s_1(x))dF_1(x) \\ = & (\lambda + k(1-\lambda))\lambda\int_B (s_2(x)-s_1(x))dF_1(x) \end{array} \tag{24}$$

it holds that

$$\int_A (s_1(x)-s_2(x))dF_1(x) < \int_B (s_2(x)-s_1(x))dF_1(x) \tag{25}$$

and therefore

$$\begin{array}{c} \int_{A\cup B} (s_1(x)-s_2(x))dF_1(x) < 0 \\ \Leftrightarrow \int_{A\cup B} s_1(x)dF_1(x) < \int_{A\cup B} s_2(x)dF_1(x). \end{array} \tag{26}$$

This concludes the proof. ∎

Lemma 4 *Let $\frac{f_1(x)}{f_2(x)}$ be weakly decreasing in x on $\mathbb{I}$. Then one security solving **(P1)** is the standard debt contract*

$$s^{SDC}(x; D(\omega)) = \min\{x, D(\omega)\}.$$

*If $\frac{f_1(x)}{f_2(x)}$ is strictly decreasing, then the standard debt contract $s^{SDC}(x; D(\omega))$ is the (up to pointwise deviations) unique security solving **(P1)**.*

Proof:
First note that $D(\omega)$ is unique by Lemma 1. Now consider the standard debt contract $s^{SDC}(x; D(\omega))$ and any contract s with $E_f[s(x)] = \omega$ and $s \neq s^{SDC}(x; D(\omega))$ in the sense that they differ on a subset of $\mathbb{I}$ with positive measure. Formally, consider s such that, letting

$$\Delta \equiv \left\{x \in \mathbb{I} \text{ such that } s(x) \neq s^{SDC}(x; D(\omega))\right\}, \text{ it holds that } \int_{\Delta} 1dF(x) > 0. \tag{27}$$

To show that the standard debt contract solves **(P1)**, it is sufficient to show that for all such s, $E_{f_1}[s(x)] \leq E_{f_1}[s^{SDC}(x; D(\omega))]$ and equivalently $E_{f_2}[s(x)] \geq E_{f_2}[s^{SDC}(x; D(\omega))]$.
Since $s^{SDC}(x; D(\omega))$ is a standard debt contract and $E_f[s(x)] = E_f[s^{SDC}(x; D(\omega))]$, it holds that

$$\begin{aligned} \exists \hat{x} \in \mathbb{I} \text{ s.t.} \quad & s^{SDC}(x; D(\omega)) \geq s(x) \quad \text{if } x < \hat{x} \\ & s^{SDC}(x; D(\omega)) \leq s(x) \quad \text{if } x > \hat{x}. \end{aligned} \tag{28}$$

There exists a point $\hat{x}$ such that $s^{SDC}(x; D(\omega))$ lies weakly above s when $x < \hat{x}$, with the roles reversed for $x > \hat{x}$. Furthermore,

$$\int_{x_L}^{\hat{x}} (s^{SDC}(x; D(\omega)) - s(x))dF(x) = \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x; D(\omega)))dF(x) \tag{29}$$

$$\begin{aligned} \Leftrightarrow \quad & \lambda \int_{x_L}^{\hat{x}} (s^{SDC}(x; D(\omega)) - s(x))dF_1(x) + (1-\lambda) \int_{x_L}^{\hat{x}} (s^{SDC}(x; D(\omega)) - s(x))dF_2(x) \\ = \quad & \lambda \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x; D(\omega)))dF_1(x) + (1-\lambda) \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x; D(\omega)))dF_2(x). \end{aligned} \tag{30}$$

Let $\frac{f_1(\hat{x})}{f_2(\hat{x})} \equiv k$ and recall that $\frac{f_1(\cdot)}{f_2(\cdot)}$ is weakly decreasing. It follows that

$$\begin{aligned} \forall \xi \in \mathbb{I}, \xi \leq \hat{x}: \quad & \frac{f_1(\xi)}{f_2(\xi)} && \geq k \\ \Leftrightarrow \quad & f_1(\xi) && \geq k f_2(\xi) \\ \Rightarrow \quad & [s^{SDC}(\xi; D(\omega)) - s(\xi)]f_1(\xi) && \geq k[s^{SDC}(\xi; D(\omega)) - s(\xi)]f_2(\xi). \end{aligned} \tag{31}$$

Hence,

$$\int_{x_L}^{\hat{x}} [s^{SDC}(x; D(\omega)) - s(x)]dF_1(x) \geq k \int_{x_L}^{\hat{x}} [s^{SDC}(x; D(\omega)) - s(x)]dF_2(x). \tag{32}$$

By the same argument,

$$\int_{\hat{x}}^{x_H} [s(x) - s^{SDC}(x; D(\omega))]dF_1(x) \leq k \int_{\hat{x}}^{x_H} [s(x) - s^{SDC}(x; D(\omega))]dF_2(x). \tag{33}$$

By Lemma 3, it thus holds that

$$\int_{x_L}^{x_H} s^{SDC}(x; D(\omega))dF_1(x) \geq \int_{x_L}^{x_H} s(x)dF_1(x). \tag{34}$$

This implies that $s^{SDC}(x; D(\omega))$ is indeed a solution to **(P1)** if $\frac{f_1(\cdot)}{f_2(\cdot)}$ is nonincreasing.

The proof for unique optimality of $s^{SDC}(x; D(\omega))$ whenever $\frac{f_1(\cdot)}{f_2(\cdot)}$ is strictly decreasing follows the same approach. It needs to be noted that if $s \neq s^{SDC}(x; D(\omega))$ in the above sense, there exists $\epsilon > 0$ such that, letting $\mathbb{A} \equiv \{x \in \mathbb{I} \backslash U_\epsilon(\hat{x})$ such that $s(x) \neq s^{SDC}(x; D(\omega))\}$,

$$\int_{\mathbb{A}} 1dF(x) > 0. \tag{35}$$

s and $s^{SDC}(x;D(\omega))$ need to be different with positive measure outside of an ϵ-neighborhood around $\hat{x}$. If this were not the case, s would be equal to $\hat{s}$ almost everywhere.[19] Now the same construction as in the previous approach can be utilized, but with

$$k_1 \equiv \frac{f_1(\hat{x}+\epsilon)}{f_2(\hat{x}+\epsilon)} > k = \frac{f_1(\hat{x})}{f_2(\hat{x}} > \frac{f_1(\hat{x}-\epsilon)}{f_2(\hat{x}-\epsilon)} \equiv k_2 \tag{36}$$

as reference points. This, coupled with $\int_{\mathbb{A}} 1 dF(x) > 0$, allows to establish that

$$\begin{aligned}\int_{x_L}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF_2(x) &\leq \frac{1}{\lambda k_2 + (1-\lambda)} \int_{x_L}^{\hat{x}-\epsilon} (s^{SDC}(x;D(\omega)) - s(x)) dF(x) \\ &+ \frac{1}{\lambda k + (1-\lambda)} \int_{\hat{x}-\epsilon}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF(x) \\ &\leq \frac{1}{\lambda k + (1-\lambda)} \int_{x_L}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF(x)\end{aligned} \tag{37}$$

and

$$\begin{aligned}\int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF_2(x) &\geq \frac{1}{\lambda k + (1-\lambda)} \int_{\hat{x}}^{\hat{x}+\epsilon} (s(x) - s^{SDC}(x;D(\omega))) dF(x) \\ &+ \frac{1}{\lambda k_1 + (1-\lambda)} \int_{\hat{x}+\epsilon}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF(x) \\ &\geq \frac{1}{\lambda k + (1-\lambda)} \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF(x)\end{aligned} \tag{38}$$

need to hold. Finally, by (35),

$$\int_{\hat{x}+\epsilon}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF(x) > 0 \vee \int_{x_L}^{\hat{x}-\epsilon} (s^{SDC}(x;D(\omega)) - s(x)) dF(x) > 0. \tag{39}$$

Hence, (37) or (38) needs to hold strictly, i.e.

$$\begin{aligned}\int_{x_L}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF_2(x) &< \frac{1}{\lambda k + (1-\lambda)} \int_{x_L}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF(x) \\ \wedge \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF_2(x) &\geq \frac{1}{\lambda k + (1-\lambda)} \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF(x) \\ &\vee\end{aligned} \tag{40}$$

$$\begin{aligned}\int_{x_L}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF_2(x) &\leq \frac{1}{\lambda k + (1-\lambda)} \int_{x_L}^{\hat{x}} (s^{SDC}(x;D(\omega)) - s(x)) dF(x) \\ \wedge \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF_2(x) &> \frac{1}{\lambda k + (1-\lambda)} \int_{\hat{x}}^{x_H} (s(x) - s^{SDC}(x;D(\omega))) dF(x)\end{aligned} \tag{41}$$

Thus, by Lemma 3,

$$E_{f_2}[s^{SDC}(x;D(\omega))] < E_{f_2}[s(x)] \Leftrightarrow E_{f_1}[s^{SDC}(x;D(\omega))] > E_{f_1}[s(x)] \tag{42}$$

needs to hold whenever $\frac{f_1(\cdot)}{f_2(\cdot)}$ is strictly decreasing. $s^{SDC}(x;D(\omega))$ is therefore the unique solution to **(P1)** up to pointwise deviations. ∎

---

[19] This follows from a simple proof by contradiction.

Lemma 5 *Let $\frac{f_1(x)}{f_2(x)}$ be weakly increasing in x on $\mathbb{I}$. Then one security solving **(P1)** is the leveraged equity contract*

$$s^{LE}(x, L(\omega)) = x \cdot \mathbf{I}_{x \geq L(\omega)}.$$

*If $\frac{f_1(x)}{f_2(x)}$ is strictly increasing, then the leveraged equity contract $s^{LE}(x, L(\omega))$ is the (up to pointwise deviations) unique security solving **(P1)**.*

Proof:
This proof is almost identical to the one of Lemma 4. $L(\omega)$ is uniquely determined by Lemma 1. Take any security $s \neq s^{LE}(x; L(\omega))$, in the sense that they are different on a subset of $\mathbb{I}$ with positive measure, with $E[s(x)] = \omega = E[s^{LE}(x; L(\omega))]$. Note that by construction of $s^{LE}(x; L(\omega))$,

$$\begin{aligned} \exists \hat{x} \in \mathbb{I} \text{ s.t.} \quad & s^{LE}(x; L(\omega)) \leq s(x) \quad \text{if } x \leq \hat{x} \\ & s^{LE}(x; L(\omega)) \geq s(x) \quad \text{if } x \geq \hat{x}. \end{aligned} \tag{43}$$

Now by repeating the previous analysis, but with an increasing likelihood ratio, it holds that for $k \equiv \frac{f_1(\hat{x})}{f_2(\hat{x})}$:

$$\int_{x_L}^{\hat{x}} [s(x) - s^{LE}(x; L(\omega))]dF_1(x) \quad \leq \quad k \int_{x_L}^{\hat{x}} [s(x) - s^{LE}(x; L(\omega))]dF_2(x) \tag{44}$$

$$\int_{\hat{x}}^{x_H} [s^{LE}(x; L(\omega)) - s(x)]dF_1(x) \quad \geq \quad k \int_{\hat{x}}^{x_H} [s^{LE}(x; L(\omega)) - s(x)]dF_1(x). \tag{45}$$

Recalling $E_f[s(x)] = E_f[s^{LE}(x; L(\omega))]$, applying Lemma 3 yields

$$\int_{x_L}^{x_H} s(x)dF_1(x) \leq \int_{x_L}^{x_H} s^{LE}(x; L(\omega))dF_1(x). \tag{46}$$

This yields the desired result that $s^{LE}(x; L(\omega))$ solves **(P1)**. Furthermore, as in Proposition 2, this inequality can be established to hold strictly whenever $\frac{f_1(\cdot)}{f_2(\cdot)}$ is strictly increasing. Thus, $s^{LE}(x; L(\omega))$ is the unique solution to **(P1)** (up to pointwise deviations). The corollary follows because the observation about the intermediate point $\hat{x}$ extends to s^{LD} in the modified problem **(P1*)**.

$$\begin{aligned} \exists \hat{x} \in \mathbb{I} \text{ s.t.} \quad & s^{DE}(x) \leq s(x) \quad \text{if } x \leq \hat{x} \\ & s^{DE}(x) \geq s(x) \quad \text{if } x \geq \hat{x}. \end{aligned} \tag{47}$$

Repeating the same steps as in the proof of the Proposition yields the corollary; $\omega \leq \int_{\mathbb{I}} \min[x, u]dF(x)$ ensures that a security subject to the restrictions exists. ∎

Lemma 6 *Suppose that $\frac{f_1(x)}{f_2(x)}$ is weakly decreasing in x on $(\underline{x}, \overline{x}) \subset \mathbb{I}$ with $\underline{x} < \overline{x} \leq x_H$. Let s be an optimal security solving **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$. Denote $e \equiv \int_{\underline{x}}^{\overline{x}} s(x)dF(x)$.*

Define

$$s^*(x) = \begin{cases} s(x) & \text{if } x \notin (\underline{x}, \overline{x}) \\ \hat{s}(x) & \text{if } x \in (\underline{x}, \overline{x}) \end{cases}$$

with

$$\hat{s}(x; D(e)) = \min\{x, D(e)\}$$

where $D(e)$ is the by Lemma 1 unique solution to

$$\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) = e.$$

*s^* is then also a solution to **(P1)**. Furthermore, s^* is globally (i.e. on (x_L, x_H)) nondecreasing and satisfies the tranched debt property on $(\underline{x}, \overline{x})$.*

Proof:
First, global nondecreasingness is established. Since $D(e)$ solves

$$\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) = e, \tag{48}$$

$D(e) \geq \inf_{\xi \in (\underline{x}, \overline{x})} s(\xi)$ has to hold as otherwise

$$\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) < \int_{\underline{x}}^{\overline{x}} s(x)dF(x) = e. \tag{49}$$

However, by nondecreasingness of s this implies $D(e) \geq s(\underline{x})$. Likewise, $D(e) \leq \sup_{\xi \in (\underline{x}, \overline{x})} s(\xi)$ has to hold as otherwise

$$\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) > \int_{\underline{x}}^{\overline{x}} s(x)dF(x) = e. \tag{50}$$

Thus, $D(e) \leq s(\overline{x})$, which with $D(e) \geq s(\underline{x})$ and the nondecreasingness of $\hat{s}$ by construction yields global nondecreasingness of s^*. Next, it needs to be shown that s^* solves **(P1)** on $\mathbb{I}$.
First note that s and s^* have the same expected value outside of $(\underline{x}, \overline{x})$, i.e.

$$\int_{\xi \notin (\underline{x}, \overline{x})} s(\xi)dF_1(\xi) = \int_{\xi \notin (\underline{x}, \overline{x})} s^*(\xi)dF_1(\xi), \tag{51}$$

since s^* and s are equal at all those points. Consider now the following modified problem on $(\underline{x}, \overline{x})$:

$$\textbf{(P1mod)} \max_{t \in S_{(\underline{x}, \overline{x})}} E_{g_1}[t(x)] \quad \text{s.t.} \quad \lambda E_{g_1}[t(x)] + (1-\lambda)E_{g_2}[t(x)] = e$$
$$0 \leq t(x) \leq x \text{ for all } x$$

where

$$g_1(x) = f_1(x) \cdot \frac{1}{F_1(\overline{x}) - F_1(\underline{x})} \tag{52}$$
$$g_2(x) = f_2(x) \cdot \frac{1}{F_2(\overline{x}) - F_2(\underline{x})} \tag{53}$$

with associated cdf on $(\underline{x}, \overline{x})$:

$$G_1(x) = [F_1(x) - F_1(\underline{x})] \cdot \frac{1}{F_1(\overline{x}) - F_1(\underline{x})} \tag{54}$$
$$G_2(x) = [F_2(x) - F_2(\underline{x})] \cdot \frac{1}{F_2(\overline{x}) - F_2(\underline{x})}. \tag{55}$$

In this formulation, since $\frac{f_1(\cdot)}{f_2(\cdot)}$ is weakly decreasing on $(\underline{x}, \overline{x})$, so is $\frac{g_1(\cdot)}{g_2(\cdot)} = \frac{f_1(\cdot)}{f_2(\cdot)} \cdot \frac{F_2(\overline{x}) - F_2(\underline{x})}{F_1(\overline{x}) - F_1(\underline{x})}$. This implies that **(P1mod)** corresponds to a problem where Lemma 4 applies. Hence, $\hat{s}$ solves **(P1mod)**. This in turn implies that

$$
\begin{aligned}
& \textstyle\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dG_1(x) && \geq \int_{\underline{x}}^{\overline{x}} s(x)dG_1(x) \\
\Rightarrow \quad & \tfrac{1}{F_1(\overline{x})-F_1(\underline{x})} \textstyle\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dG_1(x) && \geq \frac{1}{F_1(\overline{x})-F_1(\underline{x})} \int_{\underline{x}}^{\overline{x}} s(x)dG_1(x) \\
\Rightarrow \quad & \textstyle\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF_1(x) && \geq \int_{\underline{x}}^{\overline{x}} s(x)dF_1(x). \qquad (56)
\end{aligned}
$$

Thus, if s is a solution to **(P1)** on $\mathbb{I}$, $\hat{s}$ has to be a solution as well. With (51) it follows that

$$
\int_{x_L}^{x_H} s^*(x)dF_1(x) \geq \int_{x_L}^{x_H} s(x)dF_1(x). \quad \blacksquare \qquad (57)
$$

Lemma 7 *Suppose that $\frac{f_1(x)}{f_2(x)}$ is weakly increasing in x on $(\underline{x}, \overline{x}) \subset \mathbb{I}$ with $\underline{x} < \overline{x} \leq x_H$. Let s be an optimal security solving **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$. Denote $e \equiv \int_{\underline{x}}^{\overline{x}} s(x)dF(x)$.*

Define

$$
s^*(x) = \begin{cases} s(x) & \text{if } x \notin (\underline{x}, \overline{x}) \\ \hat{s}(x) & \text{if } x \in (\underline{x}, \overline{x}) \end{cases}
$$

with

$$
\hat{s}(x) = \begin{cases} s(\underline{x}) & \text{if } x < L \\ \min\{x, D\} & \text{if } x \geq L \end{cases}
$$

where

$$
D = \sup_{\xi \in (\underline{x}, \overline{x})} s(\xi)
$$

and $L(e)$ is the by Lemma 1 unique solution to

$$
\int_{\underline{x}}^{\overline{x}} \hat{s}(x)dF(x) = e.
$$

*s^* is then also a solution to **(P1)**. Furthermore, s^* is globally nondecreasing (i.e. on (x_L, x_H)) and satisfies the tranched debt property on $(\underline{x}, \overline{x})$.*

Proof:
First, nondecreasingness of s^* on (x_L, x_H) is established. On $(x_L, \underline{x}]$ and $[\overline{x}, x_H)$, s^* is nondecreasing by virtue of being equal to the nondecreasing s. On $(\underline{x}, \overline{x})$, $\hat{s}$ is nondecreasing by construction. Finally $s(\underline{x}) \leq s(x) \leq s(\overline{x})$ for all $x \in (\underline{x}, \overline{x})$ as $D = \sup_{\xi \in (\underline{x}, \overline{x})} s(\xi) \leq s(\overline{x})$.

Hence, s^* is globally nondecreasing. Since

$$
\int_{\xi \notin (\underline{x}, \overline{x})} s(\xi)dF_1(\xi) = \int_{\xi \notin (\underline{x}, \overline{x})} s^*(\xi)dF_1(\xi) \qquad (58)
$$

as s^* and s coincide at all those points, it is sufficient to show

$$
\int_{\xi \in (\underline{x}, \overline{x})} s(\xi)dF_1(\xi) \leq \int_{\xi \in (\underline{x}, \overline{x})} s^*(\xi)dF_1(\xi) \qquad (59)
$$

to establish s^* as a solution to **(P1)**. This follows from the corollary to Lemma 5. Note that $e \equiv \int_{\underline{x}}^{\overline{x}} s(x)dF(x) \leq \int_{(\underline{x}, \overline{x})} \min\{x, D\}$ by construction of D. The corollary thus establishes that $\hat{s}$ solves **(P1mod*)** on $(\underline{x}, \overline{x})$, where

$$\textbf{(P1mod*)} \max_{t \in S_{(\underline{x},\overline{x})}} E_{g_1}[t(x)] \quad \text{s.t.} \quad \lambda E_{g_1}[t(x)] + (1-\lambda) E_{g_2}[t(x)] = e$$
$$t(x) \leq D \text{ for all } x \in (\underline{x}, \overline{x}),$$

and

$$g_1(x) = f_1(x) \cdot \frac{1}{F_1(\overline{x}) - F_1(\underline{x})} \tag{60}$$
$$g_2(x) = f_2(x) \cdot \frac{1}{F_2(\overline{x}) - F_2(\underline{x})}. \tag{61}$$

Thus, s^* solves **(P1)** on $\mathbb{I}$ and is indeed nondecreasing, as well as compliant with the limited liability constraint. ∎

Proposition 4 *Let s be a security solving **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$. Then the following statements hold:*

*(i) There exists a valid security s^* which is also a solution to **(P1)** on $(x_L, x_H) \equiv \mathbb{I}$ and satisfies the tranched debt property.*

(ii) If $f_1(x)$ and $f_2(x)$ are continuous and never proportional, i.e. if

$$\forall (\xi_1, \xi_2) \subseteq \mathbb{I} : \forall k \in \mathbb{R}_+ \exists x \in (\xi_1, \xi_2) : f_1 \neq k f_2$$

then s satisfies the tranched debt property.

Proof:

Before turning to the proofs of 4(i) and 4(ii), introduce the following notation: Let TD be the set of largest disjoint intervals where s is consistent with the tranched debt property. Further, let NTD be the set of largest disjoint intervals where s is inconsistent with the property. Denote NTD_i the set of largest disjoint intervals in NTD where $\frac{f_1(\cdot)}{f_2(\cdot)}$ is weakly increasing and let NTD_d be the set of largest disjoint intervals where $\frac{f_1(\cdot)}{f_2(\cdot)}$ is weakly decreasing within the complementary set to NTD_i with respect to NTD. Lastly, let P denote the set of points not in TD and NTD, but in $\mathbb{I}$.

Formally,

$$
\begin{aligned}
TD \equiv & \quad \{(\xi_1, \xi_2) \subset \mathbb{I} | s \text{ satisfies the tranched debt property on } (\xi_1, \xi_2) \\
& \wedge \forall \epsilon > 0 : s \text{ does not satisfy the tranched debt property on } (\xi_1 - \epsilon, \xi_1) \text{ or } (\xi_2, \xi_2 + \epsilon)\} \\
NTD \equiv & \quad \{(\xi_1, \xi_2) \subset \mathbb{I} | s \text{ does not satisfy the tranched debt property on } (\xi_1, \xi_2) \\
& \wedge \forall \epsilon > 0 : s \text{ satisfies the tranched debt property on } (\xi_1 - \epsilon, \xi_1) \text{ or } (\xi_2, \xi_2 + \epsilon)\} \\
NTD_i \equiv & \quad \{(\xi_1, \xi_2) \subset NTD | \tfrac{f_1(\cdot)}{f_2(\cdot)} \text{ is weakly increasing in } \xi \text{ on } (\xi_1, \xi_2) \\
& \wedge \forall \epsilon > 0 : \tfrac{f_1(\cdot)}{f_2(\cdot)} \text{ is not weakly increasing in } \xi \text{ on } (\xi_1 - \epsilon, \xi_2) \text{ and } (\xi_1, \xi_2 + \epsilon)\} \\
NTD_d \equiv & \quad \{(\xi_1, \xi_2) \subset NTD \backslash NTD_i | \tfrac{f_1(\cdot)}{f_2(\cdot)} \text{ is weakly decreasing in } \xi \text{ on } (\xi_1, \xi_2) \\
& \wedge \forall \epsilon > 0 : \tfrac{f_1(\cdot)}{f_2(\cdot)} \text{ is not weakly decreasing in } \xi \text{ on } (\xi_1 - \epsilon, \xi_2) \text{ and } (\xi_1, \xi_2 + \epsilon)\} \\
NTD_p \equiv & \quad NTD \backslash \left(NTD_i \cup NTD_d \right) \\
P \equiv & \quad \mathbb{I} \backslash \left(TD \cup NTD \right).
\end{aligned} \tag{62}
$$

By construction,

$$TD \cup NTD_i \cup NTD_d \cup NTD_p \cup P = \mathbb{I} \tag{63}$$

and

$$\int_P 1 dF(x) = 0 = \int_{NTD_p} 1 dF(x). \tag{64}$$

Now consider 4(i). Take any interval $(\xi_1, \xi_2) \in NTD_i$. Consider the security

$$\hat{s}(x) = \begin{cases} s(x) & \text{if } x \notin (\xi_1, \xi_2) \\ s(\xi_1) & \text{if } x \in (\xi_1, L] \\ \min\{x, \sup_{\xi \in (\xi_1, \xi_2)} s(\xi)\} & \text{if } x \in (L, \xi_2) \end{cases} \tag{65}$$

where L solves

$$\int_{\xi_1}^{\xi_2} s(\xi) dF(\xi) = \int_{\xi_1}^{\xi_2} \hat{s}(\xi) dF(\xi). \tag{66}$$

By Lemma 7, $\hat{s}(x)$ is consistent with the tranched debt property on (ξ_1, ξ_2) and $s^* = s(x) + (\hat{s}(x) - s(x))\mathbf{I}_{x \in (\xi_1, \xi_2)}$ also solves **(P1)**. Likewise, for any $(\psi_1, \psi_2) \in NTD_d$, the security

$$\hat{s}(x) = \begin{cases} s(x) & \text{if } x \notin (\psi_1, \psi_2) \\ \min\{x, D\} & \text{otherwise} \end{cases} \tag{67}$$

where D solves

$$\int_{\psi_1}^{\psi_2} s(\psi) dF(\psi) = \int_{\psi_1}^{\psi_2} \hat{s}(\psi) dF(\psi) \tag{68}$$

is consistent with the tranched debt property on (ψ_1, ψ_2) and $s^* = s(x) + (\hat{s}(x) - s(x))\mathbf{I}_{x \in (\xi_1, \xi_2)}$ also solves **(P1)** by Lemma 6. Since the above statements hold for any $(\xi_1, \xi_2) \in NTD_i$ and for any $(\psi_1, \psi_2) \in NTD_d$, repeated use of the above local modification on all intervals in NTD_i and NTD_d yields a security s^* which is consistent with the tranched debt property on

$$NTD_i \cup NTD_d \cup TD.$$

With (63) and (64) this implies that s^* is consistent with the tranched debt property on $\mathbb{I}$ and solves **(P1)**. This concludes the proof for 4(i).

For 4(ii), suppose that s is not consistent with the tranched debt property on $\mathbb{I}$. This implies that

$$\int_{NTD_i \cup NTD_d} 1dF(x) > 0. \tag{69}$$

Suppose for the remainder of the proof that $\int_{NTD_i} 1dF(x) > 0$.[20] By the continuity of f_1, f_2 and local nonproportionality, i.e.

$$\forall (\xi_1, \xi_2) \subseteq \mathbb{I} : \forall k \in \mathbb{R}_+ \exists x \in (\xi_1, \xi_2) : f_1 \neq k f_2, \tag{70}$$

it follows that $\frac{f_1(\cdot)}{f_2(\cdot)}$ is strictly increasing on any $(\xi_1, \xi_2) \in NTD_i$.[21] Now take any $(\xi_1, \xi_2) \in NTD_i$. Consider the following modified problem on (ξ_1, ξ_2):[22]

$$\textbf{(Pmod)} \max_{t \in S_{(\xi_1, \xi_2)}} E_{g_1}[t(x)] \quad \text{s.t.} \quad \lambda E_{g_1}[t(x)] + (1-\lambda) E_{g_2}[t(x)] = \int_{\xi_1}^{\xi_2} s(x) dG(x)$$
$$t(x) \leq \sup_{\xi \in (\xi_1, \xi_2)} s(\xi)$$

where

$$g_1(x) = f_1(x) \cdot \frac{1}{F_1(\overline{x}) - F_1(\underline{x})} \tag{71}$$

$$g_2(x) = f_2(x) \cdot \frac{1}{F_2(\overline{x}) - F_2(\underline{x})}. \tag{72}$$

By Corollary 2 to Lemma 5, the leveraged debt security

$$s^{LD}(x) = \min\{x, \sup_{\xi \in (\xi_1, \xi_2)} s(\xi)\} \cdot \mathbf{I}_{x \in [L, \xi_2)} \tag{73}$$

with L chosen such that

$$\int_{\xi_1}^{\xi_2} s^{LD}(x) dG(x) = \int_{\xi_1}^{\xi_2} s(x) dG(x) \tag{74}$$

$$\Leftrightarrow \int_{\xi_1}^{\xi_2} s^{LD}(x) dF(x) = \int_{\xi_1}^{\xi_2} s(x) dF(x) \tag{75}$$

is the unique (up to pointwise deviations) solution to **(Pmod)** on (ξ_1, ξ_2). Hence, since s is not consistent with the tranched debt property on (ξ_1, ξ_2) and can thus not be leveraged debt there,

$$E_{g_1}[s^{LD}(x)] = \int_{\xi_1}^{\xi_2} s^{LD}(x) dG_1(x) > \int_{\xi_1}^{\xi_2} s(x) dG_1(x) = E_{g_1}[s(x)]. \tag{76}$$

Therefore,

$$\int_{\xi_1}^{\xi_2} s^{LD}(x) dF_1(x) > \int_{\xi_1}^{\xi_2} s(x) dF_1(x) \tag{77}$$

and thus

$$\int_{\mathbb{I}} \hat{s}(x) dF_1(x) > \int_{\mathbb{I}} s(x) dF_1(x) \tag{78}$$

where

$$\hat{s}(x) = \begin{cases} s(x) & \text{if } x \notin (\xi_1, \xi_2) \\ s^{LD}(x) & \text{otherwise.} \end{cases} \tag{79}$$

[20] The proof for $\int_{NTD_d} 1dF(x) > 0$ is similar and utilizes a contradiction built with the use of Lemma 4.

[21] Weakly increasing and not strictly is ruled out by the nonproportionality, which would be necessary for a locally constant $\frac{f_1(\cdot)}{f_2(\cdot)}$.

[22] On (ξ_1, ξ_2) refers to $g_1(x) = g_2(x) = 0$ for $x \notin (\xi_1, \xi_2)$.

However, by Lemma 7 this $\hat{s}(x)$ is a valid security, i.e. $\hat{s}$ satisfies limited liability and nondecreasingness.By construction

$$\int_{\mathbb{I}} \hat{s}(x)dF(x) = \int_{\mathbb{I}} s(x)dF(x) = \omega. \tag{80}$$

Hence, optimality of s is violated. Thus, the assumption that s is not consistent with the tranched debt property has to be false. This concludes the proof by contradiction. ∎

Proposition 5 *Suppose that for all $x \in \mathbb{I}$ it holds that $F_1(x) > F_2(x)$. Let local nonproportionality and continuity of densities by satisfied. The any solution s to **(P1)** satisfies*

$$\forall x \in \mathbb{I} : s(x) > x_L.$$

Thus, s includes a standard debt tranche.

Proof:
The proof is a proof by contradiction. Note first that since local nonproportionality holds, any solution to **(P1)** will have to satisfy the tranched debt property. Consider some security s with $E_f[s(x)] = \omega$ which is consistent with the tranched debt property, but does not satisfy $\forall x \in \mathbb{I} : s(x) > x_L$. As such,

$$\exists \hat{x} \in (x_L, x_H) : s(x) > x_L \Leftrightarrow x \geq \hat{x}. \tag{81}$$

We need to show that s cannot be a solution to **(P1)**.

First note that for s to be a solution to **(P1)**, $s(x) \geq x_L$ for all $x \in (x_L, x_H)$ is necessary. This corresponds to Lemma 4 of [4]. To see this, suppose otherwise. Increasing the payoff to x_L at all values x increases the payoff in both states by the same value ($x \geq x_L$ with probability 1). Furthermore, any security s with $E_f[s(x)] = \omega > x_L$ has $E_{f_1}[s(x)] < E_{f_2}[s(x)]$ by $F_1(x) > F_2(x)$ for all $x \in (x_L, x_H)$. Hence, increasing the payoff at all $\xi \in (x_L, x_H)$ where $s(x) < x_L$ and decreasing the payoff proportionally at all other points such that the expected payoff E_f remains unchanged increases the expected payoff under bad information. Thus, s cannot solve **(P1)**.

Having established that $s(x) \geq x_L$ for all x, denote $\{x_1, D_1\}$ the pair characterizing starting point and face value of the first tranche with a face value larger than x_L, $D_1 > x_L$. Since s specifies a payoff larger than x_L whenever $x \geq \hat{x} > x_L$ for some $\hat{x} > x_L$, it has to be the case that $x_1 > x_L$. Suppose that s consists of a finite number N of tranches and denote x_N, D_N the starting point and face value of the last debt tranche. Denote

$$\frac{1 - F_1(x_N)}{1 - F_2(x_N)} \equiv k < 1 \tag{82}$$

$$F_1(x_1) - F_2(x_1) \equiv c > 0 \tag{83}$$

$$D_N - D_{N-1} \equiv d > 0 \tag{84}$$

Since $F_1(x_L) = F_2(x_L) = 0$, it has to hold that

$$\exists \epsilon > x_L \text{ such that } \forall x_L < \delta < \epsilon : \frac{\int_{x_L}^{x_H} \min\{x, \delta\} dF_1(x)}{\int_{x_L}^{x_H} \min\{x, \delta\} dF_2(x)} > k, \tag{85}$$

i.e. that for all sufficiently small face values δ the corresponding standard debt tranche pays out at least k times as much (in expectation) after bad information as after good information. Existence of $\epsilon > x_L$ is due to the fact that the k is bounded away from 1, whereas the initial tranche can be constructed with a ratio arbitrarily close to 1 due to strict positivity of densities and $F_1(x_L) = F_2(x_L) = 0$. Take some such ϵ and consider $D_1 > \gamma > x_L$ where $\gamma < \epsilon$ and $\kappa \equiv \int_{x_L}^{x_1} \min\{x, \gamma\} dF(x) < d[1 - F(x_N)]$. The contract s^* with

$$s^*(x) = \begin{cases} \min\{x, \gamma\} & \text{for } x < x_1 \\ \min\{x, D_N\} - \frac{\kappa}{1-F(x_N)} & \text{for } x \in (x_N, x_H) \\ s(x) & \text{otherwise} \end{cases} \tag{86}$$

can be shown to be nondecreasing, to satisfy limited liability, to have an expected value $E_f[s^*(x)] = \omega$ and to satisfy $E_{f_1}[s^*(x)] > E_{f_1}[s(x)]$, thus violating the supposed optimality of s. First, $E_f[s^*(x)] = \omega$ follows from the definition of s^* and $\kappa \equiv \int_{x_L}^{x_1} \min\{x, \gamma\} dF(x)$. Next, nondecreasingness stems from $\kappa < d[1 - F(x_N)]$ and $\gamma < D_1$. Finally,

$$\frac{\int_{x_L}^{x_H} \min\{x, \gamma\} dF_1(x)}{\int_{x_L}^{x_H} \min\{x, \gamma\} dF_2(x)} > k \tag{87}$$

$$\Rightarrow \frac{\int_{x_L}^{x_H} \min\{x, \gamma\} dF_1(x) - \gamma[1 - F_2(x_1) - c]}{\int_{x_L}^{x_H} \min\{x, \gamma\} dF_2(x) - \gamma[1 - F_2(x_1)]} > k = \frac{1 - F_1(x_N)}{1 - F_2(x_N)} \tag{88}$$

$$\Rightarrow \int_{x_L}^{x_H} \min\{x, \gamma\} dF_1(x) - \gamma[1 - F_2(x_1) - c] - \frac{\kappa}{1 - F(x_N)}[1 - F_1(x_N)] > 0 \tag{89}$$

$$\Rightarrow \int_{x_L}^{x_H} s^*(x) dF_1(x) > \int_{x_L}^{x_H} s(x) dF_1(x). \tag{90}$$

To illustrate this construction, consider the following graph:

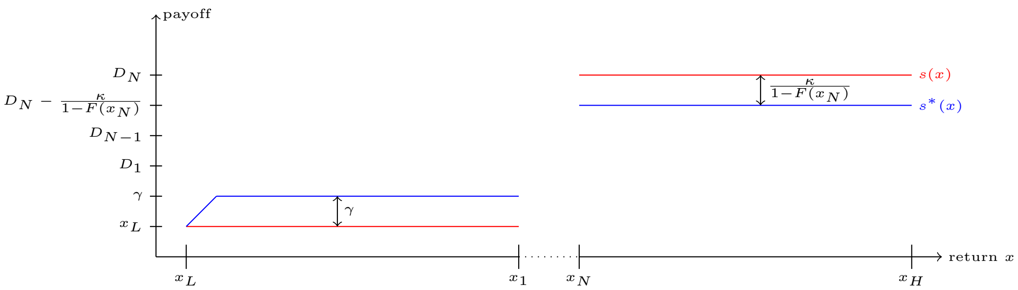

Illustration: Inclusion of Standard Debt Tranche

s^* is different from s in that it includes a standard debt tranche with face value γ and correspondingly decreases the payoff of the last tranche by $\frac{\kappa}{1-F(x_N)}$, thus ensuring that the expected values $E_f[s^*(x)] = E_f[s(x)] = \omega$ remain unchanged. The decrease of the face value of the most junior tranche characterized by $\{x_N, D_N\}$ affects the expected values of the security in the bad and good state with ratio k. By construction, the inclusion of the standard debt tranche with face value γ increases the expected values in the good and bad state with a larger proportion. Since the unconditional expected values of s and s^* are identical, this implies that $E_{f_1}[s^*(x)] > E_{f_1}[s(x)]$ and thus that s cannot have been optimal.

If s consists of an infinite number of debt tranches, there has to exist at least one tranche where the ratio is bounded away from 1, i.e. where (letting j denote the tranche) $\frac{F_1(x_{j+1}) - F_1(x_j)}{F_2(x_{j+1}) - F_2(x_j)} \leq k < 1$. If no such tranche existed, first order stochastic dominance would be violated. The remaining construction is then as above for the last tranche. This concludes the proof for (81). Coupled with the observation that any solution to **(P1)** has to be tranched debt, the only securities satisfying the requirement are tranched debt contracts including a standard debt tranche. ∎

Proposition 6 *Denote $D(\omega)$ the face value of the standard debt contract $s^{SDC}(x; D(\omega))$ with $E_f[s^{SDC}(x; D(\omega))] = \omega$. Let $G(x) \equiv \frac{1-F_1(x)}{1-F_2(x)}$. Suppose that f_1, f_2 are continuous.*

If

$$\text{(i) } \exists \xi \in (D(\omega), x_H) : G(\xi) > \inf_{x \in (x_L, D(\omega)]} \frac{f_1(x)}{f_2(x)}$$

then $s^{SDC}(x; D(\omega))$ *is not a solution to* ***(P1)****. Hence, the optimal contract necessarily involves leverage.*

Proof:
It can be established that a contract s with $E_f[s(x)] = E_f[s^{SDC}(x; D(\omega))] = \omega$ and $E_{f_1}[s(x)] > E_{f_1}[s^{SDC}(x; D(\omega))]$ exists whenever (i) holds. Consider some such ξ. By continuity of f_1, f_2, if $G(\xi) > \inf_{x \in (x_L, D(\omega)]} \frac{f_1(x)}{f_2(x)}$, there has to exist an ϵ-neighborhood around some $\hat{x} \in (x_L, D(\omega))$ such that

$$\forall x \in U_\epsilon(\hat{x}) : \frac{f_1(x)}{f_2(x)} < G(\xi) \tag{91}$$

where $x_L < \hat{x} - \epsilon < \hat{x} < \hat{x} + \epsilon < D(\omega)$.
Consider the following security:

$$s(x) = \begin{cases} x & \text{for } x \in (x_L, \hat{x} - \epsilon) \\ \hat{x} - \epsilon & \text{for } x \in [\hat{x} - \epsilon, \hat{x} + \epsilon] \\ \min\{x, D(\omega)\} & \text{for } x \in (\hat{x} + \epsilon, \xi) \\ D(\omega) + \kappa & \text{for } x \in (\xi, x_H) \end{cases} \tag{92}$$

where $\kappa(1 - F(\xi)) = \int_{U_\epsilon(\hat{x})} [x - (\hat{x} - \epsilon)] dF(x)$ and ϵ is chosen sufficiently small such that $\kappa \leq \xi - D$.

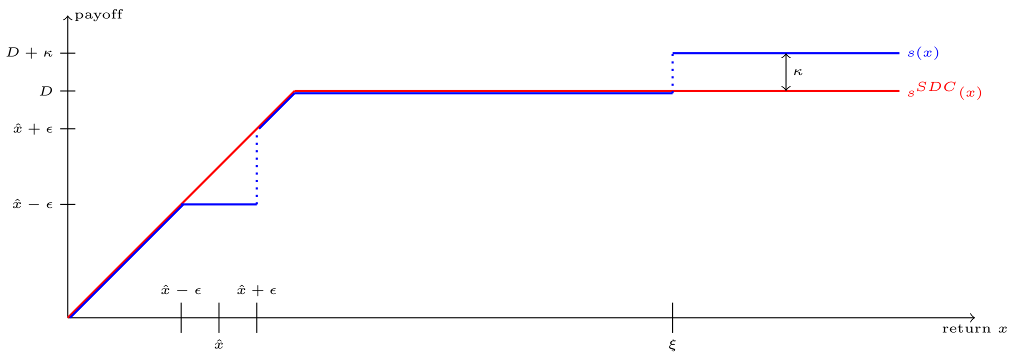

Illustration: Inclusion of Junior Debt Tranche

By construction, it holds that $E_f[s(x)] = E_f[s^{SDC}(x; D(\omega))]$. Thus, since $\frac{f_1(x)}{f_2(x)} < G(\xi)$ for all $x \in U_\epsilon(\hat{x})$,

$$\frac{\int_{U_\epsilon(x)} (s^{SDC}(x; D(\omega)) - s(x)) dF_1(x)}{\int_{U_\epsilon(x)} (s^{SDC}(x; D(\omega)) - s(x)) dF_2(x)} < \kappa G(\xi) = \kappa \frac{1 - F_1(\xi)}{1 - F_2(\xi)} \tag{93}$$

$$\Rightarrow \frac{\int_{U_\epsilon(x)} (s^{SDC}(x; D(\omega)) - s(x)) dF_1(x)}{\int_{U_\epsilon(x)} (s^{SDC}(x; D(\omega)) - s(x)) dF_2(x)} < \frac{\int_\xi^{x_H} (s(x) - s^{SDC}(x; D(\omega)))) dF_1(x)}{\int_\xi^{x_H} (s(x) - s^{SDC}(x; D(\omega))) dF_2(x)} \tag{94}$$

$$\Rightarrow \int_{x_L}^{x_H} s^{SDC}(x; D(\omega)) dF_1(x) < \int_{x_L}^{x_H} s(x) dF_1(x) \tag{95}$$

where the last implication follows from Lemma 3 and the fact that s and $s^{SDC}(x; D(\omega))$ are identical outside of $U_\epsilon(\hat{x})$ and (ξ, x_H). $s^{SDC}(x; D(\omega))$ cannot be optimal. Intuitively, the above construction does not affect the unconditional expected value. However, the decrease at all points in $U_\epsilon(\hat{x})$ yields a decrease in both the expected value under bad and good information. Of the overall change in expected value, less is attributed to the expected value under bad information (as compared to that under good information) than in the increase of the payoff by κ at all points in (ξ, x_H). Thus, since the unconditional expected value changes exactly offset each other, s has a higher expected value under bad information than $s^{SDC}(x; D(\omega))$. Hence, $s^{SDC}(x; D(\omega))$ cannot solve **(P1)**. ∎