A $\mathbb{B}$-convex Production Model for Evaluating Performance of Firms

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Abstract

Some 30 years ago Charnes, Cooper and Rhodes [7] proposed DEA (Data Envelopement Analysis) as a mean of measuring and evaluating performance of firms. This paper proposes a model for production technologies which differs from the traditional DEA production model. The usual convex framework of the DEA model is replaced by an order theorethical condition: if two input vectors can produce a given output then the maximum coordinatewise of these two vectors can produce that same output. In this model, technologies are dually linked by a min-max cost function that is dual to the Shephard's distance function. Assuming free disposal of outputs these technologies can be completely described and the Shephard's distance function can be given in closed form .

Keywords: min-max cost function, technology, upper semilattice, $\mathbb{B}$-convex sets, non-parametric models, DEA.

1 Introduction

In microeconomic production theory, a technology is characterized by the set of all technically feasible combinations of output and inputs.

Building on the seminal ideas of Farrell [11], Charnes, Cooper and Rhodes [7] proposed to model production technologies using a non parametric approach that does not involve a functional form of the production set. Among other things, they showed how to determine the efficient observed production units in a sample of firms operating on a specific sector of the economy. They termed their model DEA (Data Envelopement Analysis). Banker, Charnes, Cooper [2] extended this approach to the case of variable returns to scale.

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While convexity has been traditionally invoked in Operations Research literature on DEA, its use is sometimes questionable. Prices are often lacking in the public and privates sectors, performance gauging is necessarily limited to technical rather than allocative efficiency.

One of the assumptions of the model developed in this paper is that the least upper bound of a pair of input vectors can produce the upper bound of the outputs they can individually produce. This model allows a dual economic interpretation of technologies through a so called min-max cost function. We prove that the traditional Shephard's distance function is primal to this min-max cost function that is dual to the Shephard's distance function. By replacing the usual convex hull of a subset of $\mathbb{R}^d_+$ by what we call its $\mathbb{B}$-convex hull ($\mathbb{B}$-*convexity* has been introduced by Briec and Horvath in [3]), The Shephard distance function can be explicitely calculated.

The paper unfolds as follows. Notations and the gauges functions associated to a multivalued map are presented in Section 2; gauge and the duality relation between the cogauge and the "max"-support functional of upward sets are presented is Sections 2.2 and 2.3; the classical DEA method for production models is described in Section 3; duality for technologies with strongly disposable inputs is derived in Section 2.3 ; $\mathbb{B}$-convexity is presented in Section 4.1; technologies whose input and output sets are semilattices are described in Sections 4.2 and 4.3 as well as the general form of their Shephard's distance function; the $\mathbb{B}$-convex estimation of a technology is introduced in Section 4.4; the Shephard's distance functions for those technologies are explicitely computed in Section 4.5.

2 Gauge, duality and upward sets

2.1 Notations

The set $\mathbb{R}^d_+ = \{(x_1, \cdots, x_d) \in \mathbb{R}^d : \min\{x_1, \cdots, x_d\} \geq 0\}$ is the positive cone of $\mathbb{R}^d$ and $\mathbb{R}^d_{++}$ the set of those elements $(x_1, \cdots, x_d) \in \mathbb{R}^d$ such that $\min\{x_1, \cdots, x_d\} > 0$. The partial order on $\mathbb{R}^d$ associated to the positive cone $\mathbb{R}^d_+$ is defined, as usual, by $z \leq w$ if $w - z \in \mathbb{R}^d_+$ or, equivalently, $z \leq w$ if, for all $i \in \{1, \cdots, d\}$, $z_i \leq w_i$. If $x \leq y$ we denote by $[x, y]$ the set $\{w \in \mathbb{R}^d : x \leq w \leq y\}$. Given u and v in $\mathbb{R}^d$ the vector $u \vee v$ is the least upper bound of u and v, that is, $(\max\{u_1, v_1\}, \cdots, \max\{u_d, v_d\})$; $\bigvee_{i=1}^k x^i$ stands for the least upper bound of set of vectors $\{x^1, \cdots, x^k\}$ (to avoid confusion between a set of vectors and the coordinates $\{x_1, \cdots, x_d\}$ of a given vector x we will denote "the vector of index i" by x^i).
The norm of a vector $x \in \mathbb{R}^d$ is $\|x\| = \max\{|x_1|, \cdots, |x_d|\}$. The **carrier** of an element u of $\mathbb{R}^m_+$ is the set $\text{car}(u) = \{i : u_i > 0\}$.

A multivalued map T from from a set X to a set Y assigns to each element x of X a subset $T(x)$ of Y; no special notation will be used for multivalued or single valued maps, both will simply be called maps (from X to Y). As usual a map $T : X \to Y$ will be identified with its graph, that is $\{(x, y) \in X \times Y : y \in T(x)\}$. To an arbitrary map $T : X \to Y$ one associates its inverse T^{-1}; it is the map from Y to X defined by $x \in T^{-1}(y)$ if and only if $y \in T(x)$. The sets $T(x)$ are the **values of T** while the sets $T^{-1}(y)$ are the **fibers of T**. The image of a subset A of X by T is the set $T(A) = \bigcup_{x \in A} T(x)$.

2.2 Gauges and cogauges

For $x \in \mathbb{R}^d_+$ let $\mathbb{R}_+(x) = \{\lambda x : \lambda \geq 0\}$ and $\mathbb{R}_{++}(x) = \{\lambda x : \lambda > 0\}$; if $x \neq 0$ they are, respectively, the halfline and the open halfline through the point x. The **Minkowski gauge** μ_A and **cogauge** ν_A of a nonempty subset A of $\mathbb{R}^d$ are the maps from $\mathbb{R}^d$ to $\mathbb{R}_+ \cup \{+\infty\}$ respectively defined by

$$\mu_A(x) = \inf\{\lambda > 0 : x \in \lambda A\} \text{ and } \nu_A(x) = \sup\{\lambda > 0 : x \in \lambda A\}. \quad (2.1)$$

If $A \cap \mathbb{R}_{++}(x) = \emptyset$ then $\mu_A(x) = \infty$ and $\nu_A(x) = 0$; with this remark one can extend the definition of μ_A and ν_A to the case $A = \emptyset$.

A subset A of $\mathbb{R}^d$ is **radiant** if, for all $0 < \lambda \leq 1$ and all $x \in A$, $\lambda x \in A$; it is **coradiant** if, for all $\lambda \geq 1$ and all $x \in A$, $\lambda x \in A$. A set A is simultaneously radiant and coradiant if and only if, for all $x \in A$, $\mathbb{R}_{++}(x) \subset A$. Such a set is a cone. A closed radiant set is **starshaped**[1], that is, if $\lambda \in [0, 1]$ and $x \in A$ then $\lambda x \in A$.

One can very easily see that the gauge μ_A, respectively the cogauge ν_A, of an arbitrary radiant, respectively coradiant, set A of $\mathbb{R}^d_+$ is positively homogeneous of degree one.

Theorem 2.2.1 (1) *If A is a closed radiant set of $\mathbb{R}^d_+$ then*

$$A = \{x \in \mathbb{R}^d_+ : \mu_A(x) \leq 1\}$$

(2) *If A is a closed coradiant set of $\mathbb{R}^d_+$ then $A = \{x \in \mathbb{R}^d_+ : \nu_A(x) \geq 1\}$.*

This follows from the much more general Propositions 5.1 and 5.6 of [14].

Maps which are cogauges of closed coradiant sets are characterized by the following proposition which can be obtained from Proposition 5.8 of [14].

Theorem 2.2.2 *Given a map $q : \mathbb{R}^d_+ \to [0, \infty]$ let $B = \{x \in \mathbb{R}^d_+ : q(x) \geq 1\}$. Then statements (1) and (2) are equivalent:*

(1) *q is upper semicontinuous, positively homogeneous of degree one and $q \neq 0$;*

(2) *B is a nonempty closed coradiant subset of $\mathbb{R}^d_+$ and $\nu_B = q$.*

[1] Starshaped at 0 would be more precise; we will consider only that particular caes.

Theorem 2.2.1 tells us how T can be reconstructed from either the gauge or the cogauge:

Proposition 2.2.3 (1) *If T has closed and radiant values then, for all $x \in \mathbb{R}_+^m$,*

$$T(x) = \{y \in \mathbb{R}_+^n : \mu_{T(x)}(y) \leq 1\}. \tag{2.2}$$

(2) *If T has closed and coradiant fibers then, for all $y \in \mathbb{R}_+^n$,*

$$T^{-1}(y) = \{x \in \mathbb{R}_+^m : \nu_{T^{-1}(y)}(x) \geq 1\}. \tag{2.3}$$

2.3 Duality and upward sets

A subset A of $\mathbb{R}_+^m$ is **upward** if the following condition holds:

$$\forall x \in A \quad \forall x' \in \mathbb{R}_+^m \quad \big[x \leq x' \Rightarrow x' \in A\big]. \tag{2.4}$$

In section 3, a production model is proposed with strongly disposable inputs. Such an assumption means the input sets are upward. It is therefore appropriate to begin this section with a short analysis of upward subsets of $\mathbb{R}_+^m$. Notice that there is a recent paper [16] which deals with some mathematical concepts used in the present section.

A symmetrical definition is useful to model strongly disposable outputs. A subset B of $\mathbb{R}_+^m$ is **downward** if the following condition holds:

$$\forall y \in B \quad \forall y' \in \mathbb{R}_+^m \quad \big[y \geq y' \Rightarrow y' \in B\big]. \tag{2.5}$$

A detailed analysis of **downward** sets is proposed in [13].

Lemma 2.3.1 below gives an analytic characterization of closed upward subsets of $\mathbb{R}_+^m$. First, let us introduce the following notation[2]: for $(w, x) \in \mathbb{R}_+^m$ let

$$\langle w, x\rangle = \max_{1 \leq i \leq m} w_i x_i \tag{2.6}$$

and for $A \subset \mathbb{R}_+^m$ let $\sigma_A^{\max} : \mathbb{R}_+^m \to [0, \infty]$ be the map defined as follows:

$$\sigma_A^{\max}(w) = \inf_{x \in A} \langle w, x\rangle \tag{2.7}$$

if $A \neq \emptyset$ and $\sigma_A^{\max}(w) = +\infty$ otherwise.

For all $u \in \mathbb{R}_+^m$ let $\mathbb{K}(u) = \{v \in \mathbb{R}_+^m : \langle u, v\rangle > 0\}$ and $\mathbb{K}_1(u) = \{v \in \mathbb{R}_+^m : \langle u, v\rangle \geq 1\}$.

[2] $\langle w, x\rangle_{\max}$, or even $\langle w, x\rangle_{\max,m}$ would be more precise, but the lack of subscript should not be a cause of confusion since the meaning of $\langle w, x\rangle$ will remain constant throughout this paper.

Lemma 2.3.1 *For all non empty closed subsets A of $\mathbb{R}^m_+ \backslash \{0\}$ the following statements are equivalent.*

(1) *A is an upward subset of $\mathbb{R}^m_+$;*

(2) *for all $x \in \mathbb{R}^m_+ \backslash A$ there exists $\hat{w} \in \mathbb{R}^m_{++}$ such that $\langle \hat{w}, x \rangle < \sigma^{\max}_A(\hat{w})$;*

(3) *$A = \{x \in \mathbb{R}^m_+ : \forall w \in \mathbb{E}(x)\ \langle w, x \rangle \geq \sigma^{\max}_A(w)\}$ where $\mathbb{E}(x)$ can be any one of the following sets: $\mathbb{R}^m_{++}$, $\mathbb{R}^m_+$, $\mathbb{K}(x)$, $\mathbb{K}_1(x)$.*

Proof (1) Assume that A is closed and upward and that $x \notin A$. Then $\inf_{a \in A} \max_{1 \leq i \leq n} |x_i - a_i| > 0$ and therefore one can choose $\eta > 0$ such that $\hat{x} = (x_1 + \eta, \cdots, x_m + \eta) \notin A$. Let $\hat{w} = \hat{x}^{-1}$. We obviously have $\langle \hat{w}, x \rangle \leq 1$. From $\hat{x} \notin A$ and A being upward we have $[0, \hat{x}] \cap A = \emptyset$, that is, for all $a \in A$ there exists at least one index j such that $a_j > \hat{x}_j$ or, equivalently, $\langle \hat{w}, a \rangle > 1$ and therefore $\sigma^{\max}_A(\hat{w}) \geq 1$. This proves (2). Assume that (2) holds then

$$\mathbb{R}^m_+ \backslash A \ \subset \bigcup_{w \in \mathbb{R}^m_{++}} \{x \in \mathbb{R}^m_+ : \langle w, x \rangle < \sigma^{\max}_A(w)\}.$$

The reverse inclusion trivially holds. This proves (3) for $\mathbb{E}(x) = \mathbb{R}^m_{++}$ and $\mathbb{E}(x) = \mathbb{R}^m_+$, since $\mathbb{R}^m_{++} \subset \mathbb{R}^m_+$.

The set defined defined in (3) is clearly closed and upward. We have shown that (1), (2) and (3) for $\mathbb{E}(x) = \mathbb{R}^m_{++}$ and $\mathbb{E}(x) = \mathbb{R}^m_+$ are equivalent.

To obtain the remaining equivalences notice that if $\langle w, x \rangle > 0$ then $\hat{w} = \langle w, x \rangle^{-1} w \in \mathbb{K}_1(x)$ and use the homogeneity of σ_A and of $w \mapsto \langle w, x \rangle$. □

Corollary 2.3.2 *If A is a closed non empty upward subset of $\mathbb{R}^m_+ \backslash \{0\}$ then*

$$A = \left\{ x \in \mathbb{R}^m_+ : \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{\sigma^{\max}_A(w)} \geq 1 \right\} \tag{2.8}$$

and also

$$A = \left\{ x \in \mathbb{R}^m_+ : \sup_{\substack{w \geq 0 \\ \langle w, x \rangle = 1}} \sigma^{\max}_A(w) \leq 1 \right\} \tag{2.9}$$

Proof Since $0 \notin A$, we have $\sigma^{\max}_A(w) > 0$ for all $w \neq 0$. Equation 2.9 is a direct consequence of 2.8 (which does not need a proof), of the homogeneity of the functions involved and of $\langle w_x, x \rangle = 1$ if $\langle w, x \rangle > 0$ and $w_x = \langle w, x \rangle^{-1} w$.□

A set A_0 is a generating family of the upward set $A \subset \mathbb{R}^m_+$ if, for all $x \in \mathbb{R}^m_+$, $x \in A$ if and only if there exists $a \in A_0$ such that $a \leq x$.

Proposition 2.3.3 *If A_0 is a generating family of the non empty upward set $A \subset \mathbb{R}^m_+ \backslash \{0\}$ then*

$$\nu_A(x) = \sup_{a \in A_0} \min_{i \in car(a)} \left(\frac{x_i}{a_i} \right) \tag{2.10}$$

and

$$\sigma_A^{\max}(w) = \inf_{x \in A_0} \max_{i \in car(w)} w_i x_i. \tag{2.11}$$

Proof Since $0 \notin A$, for all $a \in A_0$ $\mathrm{car}(a) \neq \emptyset$. Let $q(x)$ be the expression on the right hand side of 2.10. It is a continuous positively homogeneous map that is not identically 0. The inequality $x \geq a$ is equivalent to, for all $i \in \mathrm{car}(a)$, $x_i \geq a_i$. From Theorem 2.2.1, q is ν_A. Equation 2.11 trivially follows from the definition of $\sigma_A^{\max}$ and of $\langle w, x \rangle \geq \langle w, a \rangle$ if $x \geq a$. $\square$

Corollary 2.3.4 (Duality) *If A is a closed and upward subset of $\mathbb{R}_+^m$ then, for all $x, w \in \mathbb{R}_+^m \setminus \{0\}$,*

$$\nu_A(x) = \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{\sigma_A^{\max}(w)} = \Big[\sup_{\substack{w \geq 0 \\ \langle w, x \rangle = 1}} \sigma_A^{\max}(w) \Big]^{-1} \tag{2.12}$$

and

$$\sigma_A^{\max}(w) = \inf_{x \in \mathbb{K}_1(w)} \frac{\langle w, x \rangle}{\nu_A(x)} = \Big[\sup_{\substack{w \geq 0 \\ \langle w, x \rangle = 1}} \nu_A(x) \Big]^{-1}. \tag{2.13}$$

Proof Suppose that $A = \emptyset$. Since for all $x, w \in \mathbb{R}_+^m \setminus \{0\}$, $\nu_A(x) = 0$ and $\sigma_A^{\max}(w) = +\infty$, equations (2.12) and (2.13) hold true. Suppose that A is a nonempty set. If $0 \in A$ then $A = \mathbb{R}_+^m$. By definition $\nu_A(x) = +\infty$ and since σ_A is identically 0, the result is immediate. If $0 \notin A$ then the first part of equation follows from Corollary 2.3.2 and Theorem 2.2.1; the second part from the homogeneity of the maps involved.

To establish 2.13 notice that by homogeneity equation 2.12 can also be written as

$$\nu_A(x) = \inf_{\substack{w \geq 0 \\ \langle w, x \rangle > 0}} \frac{\langle w, x \rangle}{\sigma_A^{\max}(w)}$$

from which we have

$$\sigma_A^{\max}(w) \leq \inf_{\substack{w \geq 0 \\ \langle w, x \rangle > 0}} \frac{\langle w, x \rangle}{\nu_A(x)}.$$

let $(a^n)_{n \in \mathbb{N}}$ be a sequence of elements of A such that $\lim_{n \to \infty} \langle w, a^n \rangle = \sigma_A^{\max}(w)$. From $\nu_A(a^n) \geq 1$ and $\langle w, a^n \rangle > 0$ we obtain $\inf_{\substack{w \geq 0 \\ \langle w, x \rangle > 0}} \frac{\langle w, x \rangle}{\nu_A(x)} \leq \langle w, a^n \rangle$. $\square$

Corollary 2.3.4 could also have been obtained directly from Proposition 2.3.3. Notice that if A and B are two subsets of $\mathbb{R}_+^m$ satisfying the conditions of Corollary 2.3.4, then $\sigma_A^{\max} = \sigma_B^{\max}$ if and only if $A = B$. One can go a bit further to establish a link between the support function of A and the cogauge of its dual defined by $A^* = \bigcap_{x \in A} \mathbb{K}_1(x)$.

Corollary 2.3.5 *If A is a closed nonempty and upward subset of $\mathbb{R}^m_+$. Suppose that $0 \notin A$. Then,*

$$\nu_{A^*} = \sigma_A^{\max} \quad \textit{and} \quad \nu_A = \sigma_{A^*}^{\max}. \tag{2.14}$$

Proof By definition, $\mathbb{K}_1(x) = \{w \in \mathbb{R}^m_+ : \langle w, x\rangle \geq 1\}$. Using the fact that $\nu_{A^*} = \mu_{\mathbb{R}^m_+ \setminus A^*}$, we have:

$$\nu_{A^*}(w) = \inf \Big\{\lambda > 0 : \frac{w}{\lambda} \in \mathbb{R}^m_+ \setminus \bigcap_{x \in A} \mathbb{K}_1(x)\Big\}.$$

Hence, one trivially has:

$$\nu_{A^*}(w) = \inf \Big\{\lambda > 0 : \frac{w}{\lambda} \in \bigcup_{x \in A} \big(\mathbb{R}^m_+ \setminus \mathbb{K}_1(x)\big)\Big\}.$$

Consequently,

$$\nu_{A^*}(w) = \inf_{x \in A} \inf \big\{\lambda > 0 : \frac{w}{\lambda} \in \mathbb{R}^m_+ \setminus \mathbb{K}_1(x)\big\},$$

which can be rewritten:

$$\nu_{A^*}(w) = \inf_{x \in A} \inf \big\{\lambda > 0 : \langle \frac{w}{\lambda}, x\rangle < 1\big\} = \inf_{x \in A} \langle w, x\rangle = \sigma_A^{\max}(w).$$

The proof of the second statement is similar. □

There are cases when ν_A can be simply and explicitely computed, as Lemma 2.3.6 below shows.

Lemma 2.3.6 *Let $\varphi_j : \mathbb{R}^m_+ \to \mathbb{R}_+$, $j \in \{1, \cdots, n\}$, be upper semicontinuous and positively homogeneous maps and let $A = \bigcap_{j=1}^{n} \big\{x \in \mathbb{R}^m_+ : r_j \leq \varphi_j(x)\big\}$. Assume that $A \neq \emptyset$ and $0 \notin A$. Then, for all $x \in \mathbb{R}^m_+ \setminus \{0\}$*

$$\nu_A(x) = \min_{r_j \neq 0} \frac{\varphi_j(x)}{r_j} \tag{2.15}$$

Proof Since $0 \notin A$, $\{j : r_j \neq 0\} \neq \emptyset$. Obviously

$$A = \bigcap_{r_j \neq 0} \big\{x \in \mathbb{R}^m_+ : \frac{\varphi_j(x)}{r_j} \geq 1\big\} = \big\{x \in \mathbb{R}^m_+ : q(x) \geq 1\big\}$$

where $q(x)$ stands for the right hand side of 2.15. Since A is not empty q is not identically zero, it is also upper semicontinuous and positively homogeneous. By Theorem 2.2.1, $q = \nu_A$. □

3 Production models

In this section we present the basic concepts of production theory in a non parametric context.

3.1 The basic framework

A **production technology** is the graph $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ of a map from $\mathbb{R}_+^m$ to $\mathbb{R}_+^n$; the elements $x = (x_1, \cdots, x_m)$ of $\mathbb{R}_+^m$ are the **inputs of the technology** and the elements $y = (y_1, \cdots, y_n)$ of $\mathbb{R}_+^n$ are its **outputs**. The map $x \mapsto T(x)$ is the **output map**; its inverse is the **input map**. The set $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ is the set of all feasible input-output vectors:

$$T = \left\{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : x \text{ can produce } y\right\}.$$

To a given a map $T : \mathbb{R}_+^m \to \mathbb{R}_+^n$ one associates two single valued maps D_i and D_o from $\mathbb{R}_+^m \times \mathbb{R}_+^n \to [0, \infty]$, the so called output and input Shephard's distance functions, defined as follows:

$$D_o(x, y) = \mu_{T(x)}(y) \text{ and } D_i(x, y) = \nu_{T^{-1}(y)}(x). \tag{3.1}$$

Under suitable conditions $D_i(x, y)$ can be interpreted as a **measure of efficiency** of the input vector x given the output vector y. For example, under the conditions of part (2) of Proposition 2.2.3, if $D_i(x, y) < 1$ then x cannot produce y and if $D_i(x, y) > 1$ then there exists a $\lambda < 1$ such that $(\lambda x, y) \in T$; all the nonzero coordinates of λx are strictly smaller than those of x and the output y can be produced from the input λx. If $D_i(x, y) = 1$ then, given the output y, the input vector x is efficient.

There are some standard assumptions that are usually made on the production technology, that T is not empty and closed, for example (see Shephard [15] for a taxonomy); some are purely mathematical others have a natural interpretation.

NFL (there is no free lunch)

$$\forall y \in \mathbb{R}_+^n \quad (0, y) \in T \text{ implies that } y = 0.$$

NFL means that a positive output cannot be obtained from a null input vector.

IS (inputs are strongly disposable)

$$\forall y \in \mathbb{R}_+^n \quad T^{-1}(y) \text{ is an upward set.}$$

IS can also be interpreted as a condition on the output map since it is clearly equivalent to the monotonicity of T, that is, if $x \leq x'$ then $T(x) \subset T(x')$;

(IS) implies that, for all $y \in \mathbb{R}^n_+$, $T^{-1}(y)$ is coradial. It can also be written as

$$\forall x \in \mathbb{R}^m_+ \quad T\big([0,x]\big) = T(x). \tag{3.2}$$

The interpretation is obvious: if a given output y can be produced from the input x then it can also be produced from a larger input.

OS (outputs are strongly disposable)

$$\forall x \in \mathbb{R}^m_+ \quad T(x) \text{ is a downward set.}$$

OS can be seen as a condition on the input map since it clearly says that T^{-1} is decreasing, that is, if $y' \leq y$ then $T^{-1}(y) \subset T^{-1}(y')$. It implies that T has radial images.

Condition OS says that less output can always be produced with the same input.

TS (the technology is strongly disposable)

$$\forall (x,y) \in T \quad \text{if} \quad x \leq x' \quad \text{and} \quad 0 \leq y' \leq y \quad \text{then} \quad (x',y') \in T.$$

TS is obviously the conjunction of IS and OS.

Using the so called free disposal cone, that is $K = \mathbb{R}^m_+ \times (-\mathbb{R}^n_+)$, TS can be written as $T = \big(T + K\big) \cap \big(\mathbb{R}^m_+ \times \mathbb{R}^n_+\big)$.

Other commonly found conditions are :

(**CR** $\forall \lambda \geq 0$), (**ND** $\forall \lambda \geq 1$); (**NI** $\forall \lambda \in [0,1]$) $\quad \lambda T \subset T$. [3]

Clearly, these conditions imply, in the language of the previous section, that T is, respectively, a cone, coradial, radial.

3.2 Estimating the technology from a given data

To estimate the efficiency of each of the input-output vectors of a given a finite set $A = \big\{(x^1, y^1), \cdots, (x^l, y^l)\big\} \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ of observed input-output vectors the data set A is embedded in a technology T for which D_i can possibly be used as a measure of efficiency and preferably actually computed, using for example mathematical programming techniques .

Following the work by Farrell [11], Charnes, Cooper and Rhodes [7] introduced the DEA model. Under a constant return to scale assumption this nonparametric technology is defined by

$$T_c^+ = \Big\{(x,y) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+ : \exists (x',y') \in Cc(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\Big\} \tag{3.3}$$

[3] CR stands for Constant Return to scale, ND for Non Decreasing return to scale and NI for Non Increasing return to scale.

which can also be written as $T_c^+ = \big[Cc(A) + K\big] \cap \big[\mathbb{R}_+^m \times \mathbb{R}_+^n\big]$[4]. It is not hard to see that T_c^+ is the smallest closed and convex technology containing A for which (TS) and (CR) hold, that is the, the smallest closed convex cone of $\mathbb{R}_+^m \times \mathbb{R}_+^n$ containing A for which T is increasing and T^{-1} is decreasing.

Following Banker, Charnes, Cooper [2], the production technology can also be defined as the weakly monotonic convex hull of the observations, that is

$$T_v^+ = \Big\{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in Co(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\Big\}. \tag{3.4}$$

Equivalently, $T_v^+ = \big[Co(A) + K\big] \cap \big[\mathbb{R}_+^m \times \mathbb{R}_+^n\big]$.

This discussion summarizes the so called DEA method (Data Envelopement Analysis) for which $D_i(x, y)$ is usually computed using linear programming techniques.

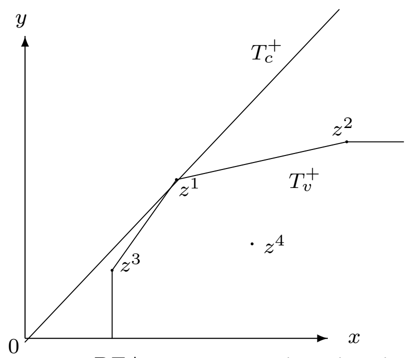

Figure 3.1 DEA non parametric estimation

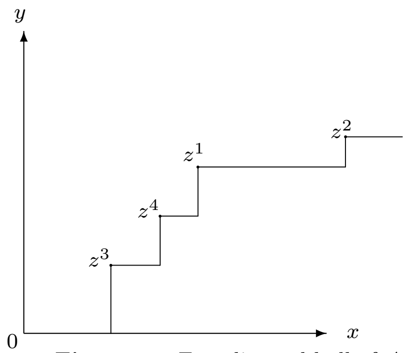

Figure 3.2 Free disposal hull of A

There are models in which convexity is relaxed. A classical example is given by the FDH approach introduced and developed in [9] (FDH stands for "Free Disposal Hull"). The technology is the smallest set containing the data and satisfying a free disposal assumption.

Such production models are said to be nonparametric because because they do not rely on an *a priori* functional specification of the production frontier.

In Section 4.4 the data set A will be embedded in a technology whose construction is similar to that of the DEA model with the difference that the convex hull of A will be replaced by a non convex set. As we will see, the Shephard's distance is effectively computable.

[4] The convex hull and the conical convex hull of a set A are denoted $Co(A)$ and $Cc(A)$ respectively.

4 $\mathbb{B}$-convex production technologies

4.1 $\mathbb{B}$-convexity

In this section we focus on a particular class of path-connected semi-lattice.

A subset A of $\mathbb{R}^d_+$ is a **semilattice**[5] if, for all u and v in A, $u \vee v \in A$. The output sets of a technology T are semilattices if, whenever an input x can produce y^1 and y^2, that is $(x, y^1) \in T$ and $(x, y^2) \in T$, then it can produce $\big(\max\{y^1_1, y^2_1\}, \cdots, \max\{y^1_n, y^2_n\}\big)$. This section describes technologies and their Shepard's distance functions under the assumption that outputs are strongly disposable and input sets are connected semilattices.

This we do by considering the case where the input sets $T^{-1}(y)$ are $\mathbb{B}$-convex. An assumption of $\mathbb{B}$-convexity means that technology obeys two basic properties. First it is endowed with a upper semilattice structure: the least upper bound of two input vectors allows to produce the least upper bound of the output vectors they can individually produce. This upper semilattice structure stands in place of the additive structure inherited from the traditional convexity assumption. Moreover $\mathbb{B}$-convexity implies that the production vectors are divisible as in the usual convexity. We have mentioned above that a $\mathbb{B}$-convex technology is a path-connected upper semilattice. From an economical viewpoint connexity is important because it allows the possibility of continuously transforming a production technique.

We say that a subset C of $\mathbb{R}^d_+$ is a **$\mathbb{B}$-convex set** if,

$$\forall\, (z^1, z^2, t) \in C \times C \times [0,1] \quad tz^1 \vee z^2 \in C. \tag{4.1}$$

Remark 4.1.1 *A $\mathbb{B}$-convex set C is a semilattice (take $t = 1$) and it is connected.*

Remark 4.1.2 *If C is a $\mathbb{B}$-convex subset of $\mathbb{R}^m_+ \times \mathbb{R}^n_+$ then its projections on $\mathbb{R}^m_+$ and $\mathbb{R}^n_+$ are also $\mathbb{B}$-convex.*

Remark 4.1.3 *If C is an upward subset of $\mathbb{R}^m_+$ then it is $\mathbb{B}$-convex, since $\forall\, (z^1, z^2, t) \in C \times C \times [0,1]$ we have $tz^1 \vee z^2 \geq z^2$ implies that $tz^1 \vee z^2 \in C$.*

For an arbitrary finite subset $A = \{z^1, \cdots, z^l\}$ of $\mathbb{R}^d_+$ let

$$\mathbb{B}(A) = \Big\{ \bigvee_{k=1}^{l} t_k z^k : (t_1, \cdots, t_l) \in [0,1]^l \text{ and } \max\{t_1, \cdots, t_l\} = 1 \Big\}. \tag{4.2}$$

$\mathbb{B}(A)$ is the **$\mathbb{B}$-convex hull** of A; it is clearly compact and path connected, and therefore connected.

Examples of $\mathbb{B}$-convex sets are depicted in Figures 4.4.4 and 4.4.5.

[5] An upper semilattice would be more precise.

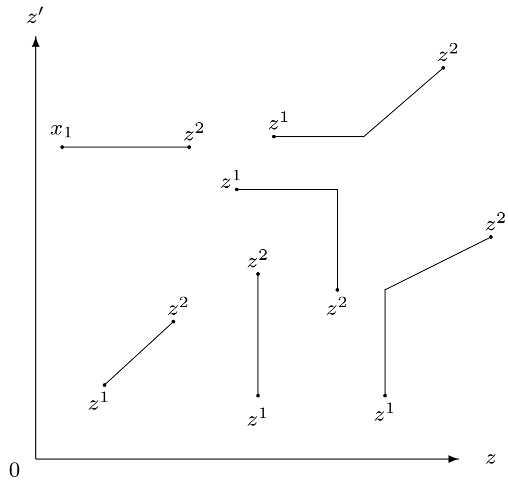

Figure 4.4.4 $\mathbb{B}$-convex hulls of 2 points.

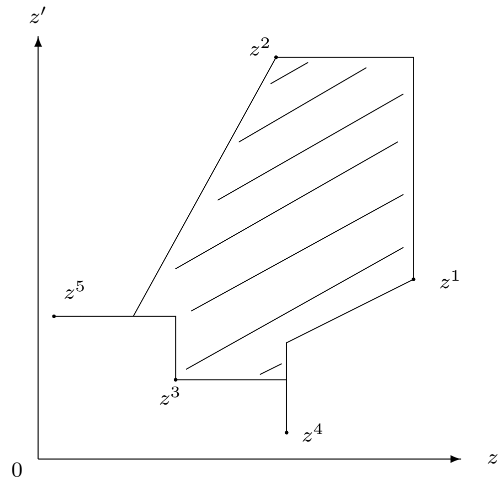

Figure 4.4.5 $\mathbb{B}$-convex hull of 5 points.

Notice that Remark 4.1.3 implies that the continuity properties of the cogauge of an upward set can be deduced from [4]. More details on $\mathbb{B}$-convex sets can be found in Briec and Horvath [3], Briec, Horvath and Rubinov [4] and Adilov and Rubinov [1].

4.2 Technologies whose the input set is semilattice

Notice that remark 4.1.3 means that the upward sets are a special case of $\mathbb{B}$-convex sets. Consequently, an input set satisfying a free disposal assumption is $\mathbb{B}$-convex.Given a technology $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$, the **min-max cost function**

associated to T is the map $C^{\max}$ from $\mathbb{R}^n_+ \times \mathbb{R}^m_+$ to $\mathbb{R}_+ \cup \{+\infty\}$ defined by

$$C^{\max}(w, y) = \sigma^{\max}_{T^{-1}(y)}(w). \tag{4.3}$$

The following proposition is then immediate.

Proposition 4.2.1 *Let $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ be a technology such that, for all $y \in \mathbb{R}^n_+$, $T^{-1}(y)$ is closed. Then IS holds if and only if, for all $y \in \mathbb{R}^n_+$,*

$$T^{-1}(y) = \big\{x \in \mathbb{R}^m_+ : \forall w \in \mathbb{R}^m_+ \ \langle w, x\rangle \geq C^{\max}(w, y)\big\}. \tag{4.4}$$

Proof If $0 \in T^{-1}(y)$ then IS means that $T^{-1}(y) = \mathbb{R}^m_+$ and the result is immediate. If $0 \notin T^{-1}(y)$ then we immediately obtain the result from Lemma 2.3.1. □

The main result of this section, Proposition 4.2.2 below, shows that, under strong disposability of outputs, the Shephard distance function and the the min-max cost function are dual to each other.

Proposition 4.2.2 *Let $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ be a technology such that, for all $y \in \mathbb{R}^n_+$, $T^{-1}(y)$ is closed and upward, that is, (IS) hold. Then, for all input vector $x \in \mathbb{R}^m_+ \backslash \{0\}$, one has*

$$D_i(x,y) = \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{C^{\max}(w,y)} \tag{4.5}$$

and

$$C^{\max}(w,y) = \inf_{x \in \mathbb{K}_1(w)} \frac{\langle w, x \rangle}{D_i(x,y)} \tag{4.6}$$

We show below that the result established in Proposition 4.2.2 can be generalized to the case of $\mathbb{B}$-convex input set.

Intuitively, this means that the producer seeks to find a virtual price minimizing the ratio between his or her cost and the minimum cost that is the cost function. Notice that strong disposability assumption is not required in the next result. However, notice that the ray spanned by the input vector we consider must intersect the input set.

Proposition 4.2.3 *Let $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ be a technology such that, for all $y \in \mathbb{R}^n_+$, $T^{-1}(y)$ is closed and $\mathbb{B}$-convex. Then, for all input vector $x \in \mathbb{R}^m_+ \backslash \{0\}$, such that $\mathbb{R}_{++}(x) \cap T^{-1}(y) \neq \emptyset$, equations (4.5) and (4.6) hold true.*

Proof: Let us denote $T^{-1}_+(y) = T^{-1}(y) + \mathbb{R}^m_+$. Since $T^{-1}_+(y)$ is an upward set, we deduce from Proposition 2.3.4 that,

$$\nu_{T^{-1}_+(y)}(x) = \inf_{w \in \mathbb{K}_1(x)} \frac{\langle w, x \rangle}{\sigma^{\max}_{T^{-1}_+(y)}(w)} \quad \text{and} \quad \sigma^{\max}_{T^{-1}_+(y)}(w) = \inf_{x \in \mathbb{K}_1(w)} \frac{\langle w, x \rangle}{\nu_{T^{-1}_+(y)}(x)}.$$

Consequently, we need to prove that if $\mathbb{R}_{++}(x) \cap T^{-1}(y) \neq \emptyset$ then we have: (i) $\sigma^{\max}_{T^{-1}(y)} = \sigma^{\max}_{T^{-1}_+(y)}$ and (ii) $\nu_{T^{-1}(y)} = \nu_{T^{-1}_+(y)}$. Let us prove (i). Since $T^{-1}(y) \subset T^{-1}(y) + \mathbb{R}^m_+$, we deduce that $\sigma^{\max}_{T^{-1}(y)} \geq \sigma^{\max}_{T^{-1}_+(y)}$. Moreover, for all $w \in \mathbb{R}^n_+$, the map $x \mapsto \max_{i=1,\ldots,n} w_i x_i$ is nondecreasing. Hence, we obtain the converse inequality

$$\sigma^{\max}_{T^{-1}(y)}(w) = \inf_{x \in T^{-1}(y)} \max_{i=1,\ldots,n} w_i x_i \leq \inf_{\substack{x \in T^{-1}(y) \\ u \in \mathbb{R}^n_+}} \max_{i=1,\ldots,n} w_i (x_i + u_i) = \sigma^{\max}_{T^{-1}_+(y)}(w),$$

which proves (i).

Suppose that $0 \in T^{-1}(y)$. Since we have $0 \in T^{-1}_+(y)$, it follows that $\nu_{T^{-1}_+(y)}(x) = \nu_{T^{-1}(y)}(x) = +\infty$ and (ii) holds true. Suppose that $0 \notin T^{-1}(y)$. In such a case $0 \notin T^{-1}_+(y)$ which implies that $\nu_{T^{-1}_+(y)}(x) < +\infty$. Clearly, we have $\nu_{T^{-1}_+(y)} \geq \nu_{T^{-1}(y)}$. Hence, we need to establish that $\nu_{T^{-1}_+(y)}(x) \leq \nu_{T^{-1}(y)}(x)$. Since, $0 \notin T^{-1}(y)$, $x \neq 0$ and $\mathbb{R}_{++}(x) \cap T^{-1}(y) \neq \emptyset$ we have $0 < \nu_{T^{-1}(y)}(x) \leq \nu_{T^{-1}_+(y)}(x) < +\infty$. Fix $\bar{\rho} = \left[\frac{\nu_{T^{-1}(y)}(x)}{\nu_{T^{-1}_+(y)}(x)}\right]$. Let us define

$$p(x) = [\nu_{T^{-1}(y)}(x)]^{-1} x \quad \text{and} \quad p_+(x) = [\nu_{T^{-1}_+(y)}(x)]^{-1} x,$$

respectively. We need to prove that $p_+(x) \in T^{-1}(y)$. Since $T^{-1}(y)$ is closed and $\nu_{T^{-1}(y)}(x,y) > 0$, we have $p(x) \in T^{-1}(y)$. Moreover, there is some $x^0 \in T^{-1}(y)$ such that $p_+(x) \in x_0 + \mathbb{R}^n_+$. This implies that $x^0 \leq p_+(x)$. Let us consider the input vector

$$\bar{x} = x_0 \vee \bar{\rho}\, p(x).$$

Notice that $\bar{\rho}\, p(x) = p_+(x)$ and $\bar{\rho} \leq 1$. Since x_0, and $p(x)$ belong to $T^{-1}(x)$, we deduce from the $\mathbb{B}$-convexity of $T^{-1}(y)$ that $\bar{x} \in T^{-1}(y)$. Hence, since $x_0 \leq p_+(x)$, we have

$$\bar{x} = x_0 \vee p_+(x) = p_+(x) \in T^{-1}(y).$$

Therefore, $\nu_{T_+^{-1}(y)}(x) \leq \nu_{T^{-1}(y)}$ which ends the proof. □

Notice that the type of input sets depicted in the next figure may have some importance because they allow the case of a possible congestion of the technology. This means that there is a lack of the disposability of certain inputs which the use does not necessarily increase the production. These configurations of the technology frequently appear in agricultural and environmental economics.

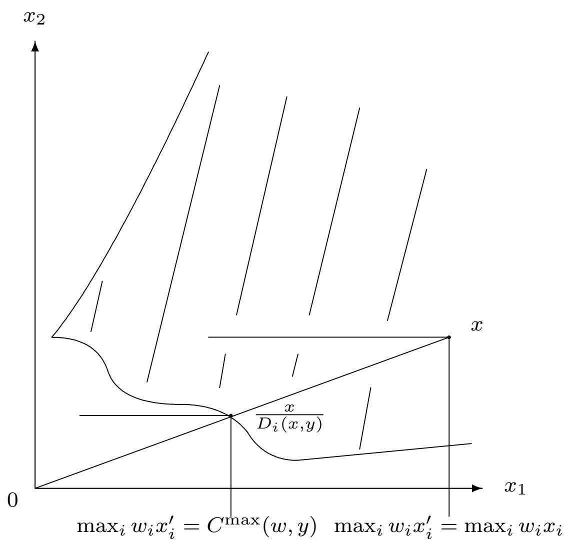

Figure 4.4.5 Duality and $\mathbb{B}$-convex input set.

4.3 A Particular class of technologies whose the output set is a semilattice

A technology $T \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$ is a **Kohli input price (KI) nonjoint technology** if there exist n single output technologies $T^j \subset \mathbb{R}^m_+ \times \mathbb{R}_+$ such that, for all input vectors $x \in \mathbb{R}^m_+$, $T(x) = T^1(x) \times \cdots \times T^n(x)$. More details can be found in [12] about this notion which is a generalization of the fixed-coefficient Leontief transformation.

Let $\{e_1, \cdots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. To a given technology $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ we associate n single output technologies defined as follows: $T^{[j]}(x) = \{y_j \in \mathbb{R}_+ : y_j e_j \in T(x)\}$.

Theorem 4.3.1 *Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology whose output sets are connected semilattices and assume that, for all x, $0 \in T(x)$. Then the following statements are equivalent.*

(1) *outputs are freely disposable, that is OS holds;*

(2) *for all $x \in \mathbb{R}_+^m$, $T(x) = T^{[1]}(x) \times \cdots \times T^{[n]}(x)$;*

(3) *for all $y \in \mathbb{R}_+^n$, $T^{-1}(y) = \bigcap_{j=1}^{n} T^{[j]-1}(y_j)$.*

Proof: If OS holds and if $y \in T(x)$ then, for all $j \in \{1, \cdots, n\}$, $y_j e_j$ belongs to $T(x)$; in other words, $y_j \in T^{[j]}(x)$. This shows that

$$T(x) \subset T^{[1]}(x) \times \cdots \times T^{[n]}(x).$$

The reverse inclusion is an obvious consequence of $y = \bigvee_{j=1}^{n} y_j e_j$ since $T(x)$ is a semilattice. We have shown that (1) implies (2).

Assume that (2) holds. Then, for all $j \in \{1, \cdots, n\}$, $T^{[j]}(x)$ is connected, and it contains 0 by hypothesis. In conclusion $T^{[j]}(x)$ is an interval of $\mathbb{R}_+$ containing 0. If $y \in T(x)$ and if $y' \leq y$ then, for all $j \in \{1, \cdots, n\}$, $0 \leq y'_j \leq y_j \in T^{[j]}(x)$ and therefore $y'_j \in T^{[j]}(x)$. We have shown that $y' \in T^{[1]}(x) \times \cdots \times T^{[n]}(x)$ and also that (1) and (2) are equivalent. The equivalence of (2) and (3) is purely set theoretical. $\square$

We have seen in the course of the proof that $T^{[j]}(x)$ is an interval of $\mathbb{R}_+$ containing 0. If (2) holds then $T(x)$ is compact if and only if each $T^{[j]}(x)$ is a compact interval of $\mathbb{R}_+$ containing 0. From this one easily proves the following corollary.

Corollary 4.3.2 *Let $T \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a technology whose output sets are connected semilattices; assume that, for all x, $0 \in T(x)$ and that OS holds.*

(1) *T has compact values if and only if there exist functions $\varphi_j : \mathbb{R}_+^n \to \mathbb{R}_+$, the* **production functions of the technology***, such that, for all $x \in \mathbb{R}_+^m$, $T(x) = [0, \varphi_1(x)] \times \cdots \times [0, \varphi_n(x)]$, and therefore, for all $y \in \mathbb{R}_+^n$, $T^{-1}(y) = \bigcap_{j=1}^n \{x \in \mathbb{R}_+^m : y_j \leq \varphi_j(x)\}$;*

(2) *T has compact values and inputs are freely disposable, IS holds, if and only if all the production functions φ_j are increasing;*

(3) *T is a technology with compact values with constant return to scale if and only if, for all $j \in \{1, \cdots, n\}$, φ_j superhomogeneous, that is, for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}_+^m$, $\varphi(tx) \geq t\varphi(x)$.*

(4) *If T has compact values and all the value functions are upper semicontinuous then T has closed input sets.*

Proposition 4.3.3 *Let T be a technology with compact values for which OS, NFL and constant returns to scale hold. For all $j \in \{1, \cdots, n\}$, let $D_i^{[j]}$ be the Shephard's distance function of the single output technology $T^{[j]}$. Then, if all the φ_j are upper semicontinuous and positively homogenous maps, one has, for all $x, y \in \mathbb{R}_+^n \setminus \{0\}$ such that $T^{-1}(y) \neq \emptyset$,*

$$D_i(x, y) = \min_{j \in car(y)} \frac{\varphi_j(x)}{y_j} = \min_{j \in car(y)} D_i^{[j]}(x, y_j). \tag{4.7}$$

Proof: This is a reformulation of Lemma 2.3.6. □

A few remarks are in order. The results of this section specify the structure of a technology under some structural assumptions on the input and output sets. The production functions themselves are not explicitly given and very little can be said about their nature; as a matter of fact they could be rather arbitrary. Computing the Shepard's distance $D_i(x, y)$ from 4.7 could therefore be a hopeless task. Also one can see from 4.7 that for $D_i(x, y)$ to be equal to 1 it is sufficient to have $D_i^{[j]}(x, y_j) \geq 1$ for all j and $D_i^{[j_0]}(x, y_{j_0}) = 1$ for one index j_0. In other words, (x, y) is an efficient programm if, for all $j \in \{1, \cdots, n\}$, (x, y_j) is a feasible programm for the technology T^j and, for one j_0, (x, y_{j_0}) is an efficient programm for T^{j_0}. The KI nonjoint technologies involve a semilattice output set. The next section introduces a special class of technologies whose the input set is semilattice.

4.4 $\mathbb{B}$-convex estimation of a technology

To a given data set $A \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ we associate a technology whose values are compact connected semilattices. For these technologies the functions φ_j as well as the Shephard's distance functions can be explicitly computed. The construction is done as with the usual DEA model with the difference that the convex hull of a set A is replaced by what we call its $\mathbb{B}$-convex hull. Let $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ be a set of l observed production vectors. The subset of $\mathbb{R}_+^m \times \mathbb{R}_+^n$ defined by

$$T_v^{\vee} = \Big\{(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in \mathbb{B}(A) \text{ s.t. } x \geq x' \text{ and } y \leq y'\Big\} \tag{4.8}$$

is the $\mathbb{B}$**-convex estimation** of the production technology under a variable returns to scale assumption.
One can equivalently write $T_v^{\vee} = (\mathbb{B}(A) + K) \cap \left[\mathbb{R}_+^m \times \mathbb{R}_+^m\right]$, where
$K = \mathbb{R}_+^m \times (-\mathbb{R}_+^n)$.
This type of technology is depicted in Figure 4.4.6.

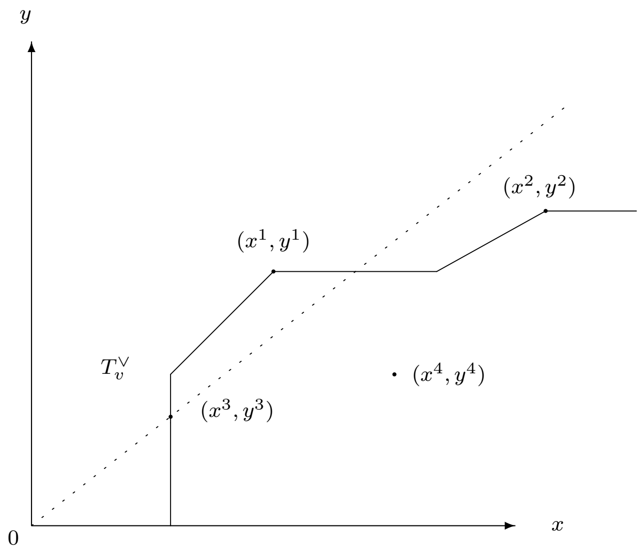

Figure 4.4.6 $\mathbb{B}$-convex estimation

Proposition 4.4.1 *For all subsets* $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ *of* l *observed production vectors, the nonparametric technology* $T_v^\vee$ *has the following properties:*

(1) $T_v^\vee$ *is a closed* $\mathbb{B}$*-convex subset of* $\mathbb{R}_+^m \times \mathbb{R}_+^n$*;*

(2) TS *(therefore OS and IS) holds;*

(3) *for all* $x \in \mathbb{R}_+^m$, $T_v^\vee(x)$ *is a compact* $\mathbb{B}$*-convex subset of* $\mathbb{R}_+^n$*;*

(4) *for all* $y \in \mathbb{R}_+^n$, $T_v^{\vee -1}(y)$ *is a closed* $\mathbb{B}$*-convex subset of* $\mathbb{R}_+^m$.

Proof: (1) From the definition of $T_v^\vee$ and from 4.1 the set $T_v^\vee$ is a $\mathbb{B}$-convex subset of $\mathbb{R}_+^m \times \mathbb{R}_+^n$. From Remark 4.1.2, the values of $T_v^\vee$ are $\mathbb{B}$-convex.

The set A is finite, which implies that $\mathbb{B}(A)$ is compact. Since the sum of a compact set with a closed set is closed we deduce that $\mathbb{B}(A) + K$ is closed. Consequently, $(\mathbb{B}(A) + K) \cap \left[\mathbb{R}_+^m \times \mathbb{R}_+^m\right]$ is a closed set and therefore $T_v^\vee$ is closed.

(2) If $x \leq \hat{x}$ and $y \in T_v^\vee(x)$ then there exists $(x', y') \in \mathbb{B}(A)$ such that $x \geq x'$ and $y \leq y'$; we trivially also have $\hat{x} \geq x'$ which shows that $y \in T_v^\vee(\hat{x})$. In other words $T_v^\vee(x) \subset T_v^\vee(\hat{x})$. Similarly, if $y \leq \hat{y}$ then $T_v^{\vee -1}(\hat{y}) \subset T_v^{\vee -1}(y)$.

(3) $T_v^\vee(x)$ is the projection on $\mathbb{R}_+^m$ of $\left[\{x\} \times \mathbb{R}_+^n\right] \cap T_v^\vee$, which is $\mathbb{B}$-convex in $\mathbb{R}_+^m \times \mathbb{R}_+^n$; by Remark 4.1.2, $T_v^\vee(x)$ is a $\mathbb{B}$-convex set.

Let us see that $T_v^\vee(x)$ is closed. If $(y^k)_{k\in\mathbb{N}}$ is a sequence of elements of $T_v^\vee(x)$ which converges to $y^\star$ then take a sequence $\left((x^k, y'^k)\right)_{k\in\mathbb{N}}$ in $\mathbb{B}(A)$ such that, for all $k \in \mathbb{N}$, $x \geq x^k$ and $y'^k \geq y^k$. From the compactness of $\mathbb{B}(A)$ we deduce, as in (1), that $y^\star \in T_v^\vee(x)$. To complete the proof of this part we verify that $T_v^\vee(x)$ is bounded. If $y \in T_v^\vee(x)$ there exists $y' \in \mathbb{B}(A)$ such that $y \leq y'$. By 4.2 there exists $(t_1, \cdots, t_l) \in [0, 1]$ such that $y' = \bigvee_{i=1}^l t_l y_l$ and therefore $y' \leq \bigvee_{i=1}^l y_l$.

The proof of (4) is entirely similar to the proof of (3). □

Theorem 4.3.1 and Proposition 4.4.1 can not be directly used for the technology $T_v^{\vee}$ since the condition $0 \in T_v^{\vee}(x)$ might not hold. As one can see, $0 \in T_v^{\vee}(x)$ if and only if $T_v^{\vee}(x) \neq \emptyset$.

As with the standard DEA model we define the $\mathbb{B}$-convex estimation of the technology under a constant return to scale assumption by replacing in the definition of $T_c^{\vee}$ the $\mathbb{B}$-convex hull of a subset A of $\mathbb{R}_+^d$, that is $\mathbb{B}(A)$, by its **$\mathbb{B}$-convex conic hull of** A, that is

$$\mathbb{B}_c(A) = \Big\{ \bigvee_{k=1}^{l} t_k z^k : (t_1, \cdots, t_l) \in \mathbb{R}_+^d \Big\}. \tag{4.9}$$

This gives the following technology, which is clearly $\mathbb{B}$-convex and satisfies a constant return to scale assumption.

$$T_c^{\vee} = \Big\{ (x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n : \exists (x', y') \in \mathbb{B}_c(A) \text{ s.t. } x \geq x' \text{ and } y \leq y' \Big\}. \tag{4.10}$$

Lemma 4.4.2 *Let T be a closed production technology under a constant return to scale assumption. There is no free lunch (NFL) if and only if for all $x \in \mathbb{R}_+^m$ $T(x)$ is compact.*

Proof: Assume that there is no free lunch. $T(x)$ is the projection on $\mathbb{R}_+^m$ of $\big[\{x\} \times \mathbb{R}_+^n\big] \cap T$, which is closed in $\mathbb{R}_+^m \times \mathbb{R}_+^n$, thus $T(x)$ is a closed set. Hence, all we need to prove is that $T(x)$ is bounded. Suppose that there is some $x \in \mathbb{R}_+^m$ such that $T(x)$ is not bounded and let us show a contradiction. If $T(x)$ is not bounded then there is a sequence $(y^k)_{k\in\mathbb{N}}$ in $T(x)$ such that $\lim_{k\to\infty} \|y^k\| = +\infty$, where $\|\cdot\|$ is an arbitrary norm defined on $\mathbb{R}^n$. Consequently, there is some k_0 such that for all $k \geq k_0$ $\|y^k\| \geq 1$. By hypothesis, $(x, y^k) \in T$ for all natural number k. Since T satisfies a constant returns to scale assumption, we have for all $k \geq k_0$, $\big(\frac{x}{\|y^k\|}, \frac{y^k}{\|y^k\|}\big) \in T$. Moreover

$$\Big(\frac{x}{\|y^k\|}, \frac{y^k}{\|y^k\|}\Big) \in [0, x] \times S(0, 1)$$

where $S(0, 1) = \{y \in \mathbb{R}^n : \|y\| = 1\}$ is the unit sphere of $\mathbb{R}^n$, and $[0, x] = \{v \in \mathbb{R}^n : 0 \leq v \leq x\}$. It follows that

$$\Big(\frac{x}{\|y^k\|}, \frac{y^k}{\|y^k\|}\Big) \in T \cap \Big([0, x] \times S(0, 1)\Big).$$

Since $T \cap \Big([0, x] \times S(0, 1)\Big)$ is closed and bounded, it is a compact set and one can find a subsequence $(k_l)_{l\in\mathbb{N}}$ such that $\Big(\big(\frac{x}{\|y^{k_l}\|}, \frac{y^{k_l}}{\|y^{k_l}\|}\big)\Big)_{l\in\mathbb{N}}$ converges to some $(x^\star, y^\star) \in T \cap \Big([0, x] \times S(0, 1)\Big)$. Obviously, since $\lim_{l\to\infty} \|y^{k_l}\| = +\infty$, one has $x^\star = 0$. Moreover, $\|y^\star\| = 1$ and, therefore $y^\star \neq 0$. Since there is no free lunch, this is a contradiction and the first part of the proof is established. The reciprocal is immediate. If NFL does not hold then there is some $y \neq 0$ such that $(0, y) \in T$. Under a constant returns to scale assumption, $\mathbb{R}_{++}(0, y) \subset T$ and consequently $T(0)$ is not bounded. □

Most of the following properties are immediate. The compactness of $T_c^\vee(x)$ for all input vector x is an immediate consequence of the lemma above.

Corollary 4.4.3 *Let $T_c^\vee$ be the $\mathbb{B}$-convex estimation of a technology associated to the data $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ under a constant return to scale assumption. Then, if $T_c^\vee$ satisfies OS and IS , for all $x \in \mathbb{R}_+^m$, $0 \in T_c^\vee(x)$ and $T_c^\vee(x)$ is $\mathbb{B}$-convex. Moreover, if for $k = 1 \cdots l$ one has $y_k \neq 0$ then $T_c^\vee$ satisfies NFL and $T_c^\vee(x)$ is compact for all $x \in \mathbb{R}_+^m$.*

We close this section with a lemma which reduces the study of $T_c^\vee$ to that of $T_v^\vee$. To do that we fix some notations. Let $1_l \in \mathbb{R}^l$ be the vector whose coordinates are all equal to 1. Let $\{e^1, \cdots, e^l\}$ be the elements of the canonical basis of $\mathbb{R}^l$. Given $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ let

$$\tilde{A} = \big\{\big((x^1, e^1), (y^1, e^1)\big), \cdots, \big((x^l, e^l), (y^l, e^l)\big)\big\} \subset \mathbb{R}_+^{m+l} \times \mathbb{R}_+^{n+l}. \tag{4.11}$$

Since we will have to consider different data sets we will write $\widetilde{T_c^\vee}$, the production set constructed from the data set $\tilde{A}$.

Lemma 4.4.4 *Given $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$, for all $(x, y) \in \mathbb{R}_+^m \times \mathbb{R}_+^n$, $(x, y) \in T_v^\vee$ if and only if there exists $k_0 \in \{1, \cdots, l\}$ such that $\big((x, 1_l), (y, e^{k_0})\big) \in \widetilde{T_c^\vee}$.*

Proof: By definition of $T_v^\vee$, $(x, y) \in T_v^\vee$ if and only if there exists $(\rho_1, \cdots, \rho_l) \in [0, 1]^l$ such that

$$\max_{i \leq k \leq l} \rho_k = 1, \quad x \geq \bigvee_{k=1}^{l} \rho_k x^k \quad \text{and} \quad y \leq \bigvee_{k=1}^{l} \rho_k y^k \tag{4.12}$$

and, by definition of $T_c^\vee$, $\big((x, 1_l), (y, e^{k_0})\big) \in \widetilde{T_c^\vee}$ if and only if there exists $(\rho_1, \cdots, \rho_l) \in \mathbb{R}_+^l$ such that

$$(x, 1_l) \geq \bigvee_{k=1}^{l} \rho_k (x^k, e^k) \quad \text{and} \quad (y, e^{k_0}) \leq \bigvee_{k=1}^{l} \rho_k (y^k, e^k). \tag{4.13}$$

Notice that (4.13) reduces to $x \geq \bigvee_{k=1}^{l} \rho_k x^k$, $y \leq \bigvee_{k=1}^{l} \rho_k y^k$, $0 \leq \rho_k \leq 1$ for all k and $\rho_{k_0} = 1$. □

4.5 Computing the Shephard's distance

Proposition 4.5.1 *Let $T_c^{\vee}$ be the $\mathbb{B}$-convex estimation of a technology associated to the data $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}_+^m \times \mathbb{R}_+^n$ under a constant return to scale assumption and such that there is no free lunch. Let, for all $k \in \{1, \cdots, l\}$,*

$$\alpha_k(x) = \min_{i \in car(x^k)} \frac{x_i}{x_i^k}. \tag{4.14}$$

Then, for all $y \in \mathbb{R}_+^n \backslash \{0\}$ such that $T_c^{\vee -1}(y) \neq \emptyset$,

$$D_i(x, y) = \min_{j \in car(y)} \frac{\max_{1 \leq k \leq l} \left\{\alpha_k y_j^k(x)\right\}}{y_j}. \tag{4.15}$$

Furthermore for all $x \in \mathbb{R}_+^m \backslash \{0\}$,

$$T_c^{\vee}(x) = [0, \varphi_1(x)] \times \cdots \times [0, \varphi_n(x)]. \tag{4.16}$$

where

$$\varphi_j(x) = \max_{1 \leq k \leq l} \left\{\alpha_k(x) y_j^k\right\}. \tag{4.17}$$

Proof: Let for all, $k \in \{1, \cdots, l\}$,

$$\alpha_k(x) = \min_{i \in \mathrm{car}(x^k)} \frac{x_i}{x_i^k}. \tag{4.18}$$

and, for all $j \in \{1, \cdots, n\}$,

$$\varphi_j(x) = \max_{1 \leq k \leq l} \left\{\alpha_k y_j^k(x)\right\}. \tag{4.19}$$

Since there is no free lunch $\mathrm{car}(x^k) \neq \emptyset$ and $\alpha_k(x)$ is well defined for all $k \in \{1, \cdots, l\}$.

Recall that $(x, y) \in T_c^{\vee}$ if and only if there exists $(\rho_1, \cdots, \rho_l) \in \mathbb{R}_+^l$ such that $x \geq \bigvee_{k=1}^l \rho_k x^k$ and $y \leq \bigvee_{k=1}^l \rho_k y^k$. The first inequality says that, for all $i \in \{1, \cdots, m\}$ and all $k \in \{1, \cdots, l\}$, $x_i \geq \rho_k x_i^k$ which is clearly equivalent to $\alpha_k(x) \geq \rho_k$. If the second inequality holds then $y \leq \displaystyle\bigvee_{k=1}^{l} \alpha_k(x) y^k$ also holds. The inequality $x \geq \displaystyle\bigvee_{k=1}^{l} \alpha_k(x) x^k$ always holds. In other words, $(x, y) \in T_c^{\vee}$ if and only if $y \leq \displaystyle\bigvee_{k=1}^{l} \alpha_k(x) y^k$ or, equivalently, for all $j \in \{1, \cdots, n\}$, $y_j \leq \max\limits_{1 \leq k \leq l} \alpha_k(x) y_j^k$. From Corollary 2.3.6 the result is nos immediate. □

Proposition 4.5.2 *Let* $A = \{(x^1, y^1), \cdots, (x^l, y^l)\} \subset \mathbb{R}^m_+ \times \mathbb{R}^n_+$. *Suppose that there is no free lunch. For all* $k \in \{1, \cdots, l\}$ *and all* $(x, y) \in \mathbb{R}^m_+ \times \mathbb{R}^n_+$ *let*

$$\alpha_k(x) = \min_{i \in car(x^k)} \frac{x_i}{x_i^k}, \quad \alpha(x) = \max_{1 \le k \le l} \alpha_k(x).$$

and

$$\beta_j(x, y) = \max_{\substack{1 \le k \le l \\ y_j \le y_j^k}} \frac{\alpha_k(x) y_j^k}{y_j}$$

For all $y \in \mathbb{R}^n_+ \backslash \{0\}$ *such that* $T_v^{\vee -1}(y) \neq \emptyset$ *the Shephard's distance function is given by:*

$$D_i(x, y) = \min\Big\{ \min_{j \in car(y)} \beta_j(x, y), \alpha(x) \Big\}. \tag{4.20}$$

Furthermore, for all $x \in \mathbb{R}^m_+$, $T_v^\vee(x) \neq \emptyset$ *if and only if* $\alpha(x) \ge 1$ *and, if* $\alpha(x) \ge 1$, *then*

$$T_v^\vee(x) = \prod_{j=1}^n [0, \varphi_j(x)] \tag{4.21}$$

where

$$\varphi_j(x) = \max_{1 \le k \le l} \min\{\alpha_k(x) y_j^k, y_j^k\}. \tag{4.22}$$

Proof: Let $\widetilde{D_i}$ be the Shephard's distance function computed on $\widetilde{T_v^\vee}$. Using Lemma 4.4.4 the computation of D_i is reduced to that of $\widetilde{D_i}$. Starting from $(x, y) \in T_v^\vee$ if and only if there exists $k_0 \in \{1, \cdots, l\}$ such that $\big(x + 1_l, y + e^{k_0}\big) \in \widetilde{T_c^\vee}$ we proceed with $\widetilde{T_c^\vee}$ as in the proof of Proposition 4.5.1 to find that $(x, y) \in T_v^\vee$ if and only there exists k_0 such that

$$\min_{j \in car(y)} \max_{1 \le k \le l} \min \left\{ \alpha_k(x) \frac{y_j^k}{y_j}, \frac{y_j^k}{y_j} \right\} \ge 1 \tag{4.23}$$

and

$$\min \big\{ \alpha_{k_0}(x), 1 \big\} \ge 1 \tag{4.24}$$

from which (4.20) easily follows. For the last part, that is (4.21) and (4.22) recall that $T_v^{\vee -1}(y)$ is closed and coradial and use part 2 of Proposition 2.2.3.□

Example 4.5.3 *The following data sample, with* $m = 1$, $n = 2$ *and* $l = 7$, *can be found is Färe, Grosskopf and Lovell [10].*

Table 1. Data Sample

Firms	*Input*	*Output 1*	*Output 2*
1	2	3/2	1
2	2	2	1
3	4	3	2
4	6	6	6
5	7	6	6
6	8	7	4
7	9	7	4

For $k, k' \in \{1, \cdots, l\}$ let $\alpha_k(x^{k'}) = a_{k',k'}$ and let A be the $l \times l$ matrix whose coefficients are the numbers $a_{k',k}$.

With the data from Table 1 we obtain:

$$A = \begin{pmatrix} 1 & 1 & 2 & 3 & 7/2 & 4 & 9/2 \\ 1 & 1 & 2 & 3 & 7/2 & 4 & 9/2 \\ 1/2 & 1/3 & 1 & 3/2 & 7/4 & 2 & 9/4 \\ 1/3 & 1/3 & 2/3 & 1 & 7/6 & 4/3 & 3/2 \\ 2/7 & 2/7 & 4/7 & 6/7 & 1 & 8/7 & 9/7 \\ 1/4 & 1/4 & 4/8 & 3/4 & 7/8 & 1 & 9/8 \\ 2/9 & 2/9 & 4/9 & 2/3 & 7/9 & 8/9 & 1 \end{pmatrix}.$$

The values of the Shephard distance function for the DEA and FDH estimations are listed in Table 2 where $\mathbb{B}$ stands for $\mathbb{B}$-convex and C for convex.

Table 2. Distance Function and Technologies.

Firms	*$\mathbb{B}$-VRS*	*$\mathbb{B}$-CRS*	*C-VRS*	*C-CRS*	*FDH*
1	1	4/3	1	4/3	1
2	1	1	1	1	1
3	4/3	4/3	4/3	4/3	1
4	1	1	1	1	1
5	7/6	7/6	7/6	7/6	7/6
6	8/7	8/7	1	8/7	1
7	9/8	9/8	9/8	9/7	9/8

The results obtained under a $\mathbb{B}$-convexity assumption are no less than those obtained in the convex and FDH cases. One can easily check that the inefficiency score are greater in the CRS models than in the VRS models. This is especially true in $\mathbb{B}$-convex versions. Notice that firms 2 and 4 are efficient for all types of estimations. Moreover, the FDH estimation being the minimal extrapolation of the "observed" data-set yields the largest number of efficient firms. Finally, notice that firm 6 lies on the frontier of the convex VRS model though it is inefficient in the $\mathbb{B}$-convex model.

References

[1] Adilov, G. and A.M. Rubinov (2006), $\mathbb{B}$-convex sets and functions, *Numerical Functional Analysis and Optimization*, Vol. 27(3-4), pp. 237-257.

[2] Banker, R.D., A. Charnes and W.W. Cooper (1984), Some Model for Estimating Technical and Scale Innefficiency in Data Envelopement Analysis, *Management Science*, Vol. 30, pp. 1078-1092.

[3] Briec, W. and C.D. Horvath (2004), $\mathbb{B}$-convexity, *Optimization*, Vol. 53, pp. 103-127.

[4] Briec, W., C.D. Horvath and A. Rubinov (2005), Separation in $\mathbb{B}$-convexity, *Pacific Journal of Optimization*, Vol. 1, pp. 13-30.

[5] Briec, W. and C.D. Horvath (2008), Nash points, Ky Fan inequality and equilibria of abstract economies in Max-Plus and $\mathbb{B}$-convexity, *Journal of Mathematical Analysis and Applications* Vol. 341, pp. 188-199.

[6] Briec, W. and C.D. Horvath (2008), Halfspaces and Hahn-Banach like properties in $\mathbb{B}$-convexity and Max-Plus convexity, *Pacific Journal of Optimization* Vol. 4(2), pp. 293-317.

[7] Charnes, A., W.W. Cooper, and E. Rhodes (1978). Measuring the efficiency of decision making units *European Journal of Operational Research* Vol. 2(6), pp. 429-444.

[8] Debreu, G. (1951), The coefficient of resource utilization, *Econometrica*, Vol. 19, pp. 273-292.

[9] Deprins. D, L. Simar and H. Tulkens (1984), Measuring labour efficiency in post offices. In: Marchand, M., Pestieau, P. and Tulkens, H., Editors, 1984. *The Performance of Public Enterprises*, North-Holland, Amsterdam, pp. 243-267.

[10] Färe, R., S. Grosskopf and C.A.K. Lovell (1985), *The Measurement of Efficiency of Production*, Klüwer-Nijhoff Publ., Boston.

[11] Farrell, M.J. (1957), The measurement of productive efficiency, *Journal of the Royal Statistical Society*, Vol. 120 , pp. 253-281.

[12] Kohli, U. (1983), Nonjoint Technologies. *Review of Economic Studies* Vol. 50, pp. 209-19.

[13] Martínez-Legaz, J.-E., A. M. Rubinov and I. Singer (2002), Downward sets and their separation and approximation properties, *Journal of Global Optimization*, Vol. 23(2), pp. 111-137.

[14] Rubinov, A. (2000), *Abstract Convexity and Global Optimization*, Kluwer, Dordrecht.

[15] Shephard, R.W. (1970), *Theory of Cost and Production Functions*, Princeton University Press, Princeton NJ.

[16] Zaffaroni, A. (2006), Monotonicity along rays and consumer duality with nonconvex preferences, *Quaderno del Dipartimento di Scienze Economiche e Matematico-Statistiche*, Università di Lecce, 2006.