On Some Semilattice Structures for Production Technologies

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Abstract

Tracing back from Charnes, Cooper and Rhodes [9] many approaches have been proposed to extend the DEA production model to non-convex technologies. The FDH method were introduced by Deprins, Simar and Tulkens [13] and it only assumes a free disposal assumption of the technology. This paper, continues further an earlier work by Briec and Horvath [7]. Among other things, a new class of semilattice production technologies is introduced. Duality results as well as computational issues are proposed.

Keywords: Non-parametric production technology, Semilattice, $\mathbb{B}$-convex sets, inverse $\mathbb{B}$-convexity, DEA, FDH.

1 Introduction

The interest of a convexity assumption in microeconomic production theory is intimately linked to duality theory. If one defines an optimization problem with respect to quantities, then there is a corresponding problem defined with respect to prices that has the same value. This approach is of great interest for microeconomics both for understanding the mathematics and for clarifying the economics (see Shephard [27]).

Convexity is also important from an operational standpoint. In their seminal paper Charnes, Cooper and Rhodes [9] proposed to model production technologies using a non-parametric approach that does not assume a functional form of the production set. Among other things, they showed how to determine the efficient observed production units in a sample of firms operating on a specific sector of the economy. They termed their model *DEA* (Data Envelopment Analysis). Banker, Charnes, Cooper [2] extended this approach to the case of variable returns to scale. One can loosely say

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that these technologies are constructed from the convex hull of all the observed production vectors representing each firm respectively. Convexity has another advantage, it allows to compute the efficiency scores of firms and yields a procedure to rank them.

While convexity has been traditionally invoked in economic theory, its use in production theory is questionable. Since prices are often lacking in the public sector, performance gauging is necessarily limited to technical rather than allocative efficiency. Tone and Sahoo [29] have mentioned another reason to question the role of convexity in production models, it is the presence of indivisibleness in all multistage production process. Indivisibilities are among the main ways in which economies of scale may emerge. This notion was introduced in economic literature by Kaldor [17] and Samuelson [26], among others. It is generally claimed that indivisibility argument fit in with the notion of fixed factor proportions in the neoclassical definition of scale. Another form of indivisibility by which scale may emerge is to consider the use of equipment, which has the characteristics of incorporating a factor proportionately less than its contribution to capacity when output is expanded. In their paper Tone and Sahoo [29] analyzed in details the potential for reorganization of inputs,which can emerge due to indivisibility of specific inputs. Their conclusion is that the presence of indivisibleness makes the technology structure non-convex.

Hence, after the growing interest received by DEA in the literature on operation research and efficiency analysis, several researchers proposed to relax convexity assumptions. Among others, Deprins, Simar and Tulkens [13], Tulkens [30] and Kerstens and Vanden Eeckaut [18] introduced and extended the so called FDH model (Free Disposal Hull) which relaxes convexity and only postulates a free disposal assumption of the technology. Petersen [22] and Bogetoft [4] also proposed some models based on a relaxed convexity assumption and, more recently, Podinovski [23] suggested a selective convexity approach merging DEA and FDH. Soleimani-damaneh et al. [25] and Boussemart et al. [3] developed approaches focusing on the specification of returns to scale in DEA. More recently Podinovski and Kuosmanen [24] proposed to model weak disposability in data envelopment analysis under relaxed convexity assumptions.

This paper proposes new models based on a notion of $\mathbb{B}$-*convexity* recently introduced by Briec and Horvath [5]. A upper (lower) semilattice is a partially ordered set in which each pair of elements has a least upper (best lower) bound. In a recent paper Briec and Horvath [7] proposed to evaluate performance of firms using $\mathbb{B}$-convexity. A production technology satisfying a $\mathbb{B}$-convexity assumption has an upper semilattice structure. This means that the least upper bound of a pair of input vectors can produce the upper bound of the outputs they can individually produce. Furthermore, it is assumed that inputs and outputs can be radially shrunk. This formulation may have some advantages regarding to DEA. First, it has a computational interest because efficiency scores can be formulated in closed forms. Hence, computation of efficiency measures requires a small number of arithmetic operations and does not require linear programming. Another advantage comes from the fact that DEA postulates a priori a returns to scale assumption of the technology. Modelling a non-parametric technology under a $\mathbb{B}$-convexity assumption implies that returns to scale *are not characterized a priori.* For example they may be locally nonincreasing or locally nondecreasing. In particular $\mathbb{B}$-convexity has the advantage to allow the variations of marginal productivity to form an alternating sequence, either increasing or decreasing, according to the frontier points of the production technology one consider.

This semilattice model has also some merits over the FDH model. First, the number of efficient firms characterizing the frontier is smaller than that one involved by the FDH model. FDH is based upon a minimum extrapolation principle and, from a certain viewpoint, it underestimates the size of the production possibilities set because of the minimum extrapolation principle. Furthermore, one can notice that a free disposal input set is always $\mathbb{B}$-convex. Moreover, in the FDH case, the marginal productivity is either null of infinite which is not the case with the $\mathbb{B}$-convex production model.

However, the approach proposed in [7] has also some drawbacks. In their paper the authors showed that a $\mathbb{B}$-convex technology is a Kohli technology under a free disposal assumption (see [19]). This means that the output set has a cubic structure that is a severe limitation to the proposed approach. To overcome this problem, this paper pays attention to an alternative formulation of $\mathbb{B}$-convexity. A distinction between $\mathbb{B}$-*convex* sets and *inverse* $\mathbb{B}$-*convex sets* is proposed. The former corresponds to a notion introduced in [5] and involves an upper semilattice structure while the latter assumes that the technology has a lower semilattice structure. Hence, the production set is a lower semilattice. This means that the lower bound of a pair of input vectors can produce the lower bound of the outputs they can individually produce. Furthermore, inputs and outputs can be proportionally expanded. $\mathbb{B}$-convexity and inverse $\mathbb{B}$-convexity have similar advantages regarding to the usual methods they are compared to above.

However, there exists some important differences between these two models. First, the mathematical structure of inverse $\mathbb{B}$-convex technologies differs from those based upon a $\mathbb{B}$-convexity assumption. It follows that output sets satisfying a free disposal assumption are always inverse $\mathbb{B}$-convex. Moreover, for such technologies, the output sets are not cubic. Hence, the Kholi property $\mathbb{B}$-convex technologies obey can be relaxed. Second, the $\mathbb{B}$-convex production model combines both an upper semilattice structure and a divisibility assumption. However, this is not the case of inverse $\mathbb{B}$-convex production technologies which involve a lower semilattice structure and are not based upon a divisibility assumption.

Duality plays a crucial role in the use of convexity to model economic problems. Though it is generally difficult to develop a dual framework for non-convex sets, it has been shown in [7] that $\mathbb{B}$-convex technologies satisfy some special type of dual properties based upon a $\min - \max$ approach.

Considering the inverse $\mathbb{B}$-convex case, one can obtain some new properties involving Leontief functional forms. Under a free disposal assumption the input set has a Leontief structure. This means that the input requirement set naturally involves a dual representation in term of Leontief function. The output case is more interesting and one can show that the traditional Farrell measure is dually linked to the Leontief function. Analyzing the relationships between productivity scale and marginal productivity, one can find some important differences between these two production models we focuss on. It appears that some properties inherited from the inverse $\mathbb{B}$-convex model seem to have a more natural economic interpretation than those involved with the $\mathbb{B}$-convex model.

The remainder of this paper is organized as follows. In section 2 we lay down the groundwork and present the basic notions of production technology. In particular, we focuss on the standard DEA and FDH models under an assumption of variable returns to scale. In section 3, $\mathbb{B}$-convexity and inverse $\mathbb{B}$-convexity are introduced. Section 4 provides duality results. An inverse $\mathbb{B}$-convex production model is proposed in section 5. Finally, section 6 proposes an approach for solving systems of maximum and minimum equations. Hence, some closed forms are derived to measure technical efficiency and a numerical example is proposed.

2 Non-Parametric Production Models

The following subsections are devoted to present basic concepts of production theory as well as traditional methods for estimating the production frontier in a non-parametric context.

2.1 The Background of Production Models

We first define the notations used in this section. Let $\mathbb{R}^d_+$ be the non negative Euclidean d-orthant; for all $z, w \in \mathbb{R}^d_+$ let us denote $z \leq w \iff z_i \leq w_i$ $\forall i \in [d]$[1].

Now let $m, n \in \mathbb{N}$ be two positive natural numbers such that $d = m + n$. A production technology transforms inputs $x = (x_1, ..., x_m)$ into outputs $y = (y_1, ..., y_n)$. The set $T \subset \mathbb{R}^{m+n}_+$ of all input-output vectors that are feasible is called the production set. Namely, it is defined as follows:

$$T = \big\{(x, y) \in \mathbb{R}^{m+n}_+ : x \text{ can produce } y\big\}. \tag{2.1}$$

T can also be characterized by an input correspondence $L : \mathbb{R}^n_+ \longrightarrow 2^{\mathbb{R}^m_+}$ defined by $L(y) = \big\{x \in \mathbb{R}^m_+ : (x, y) \in T\big\}$. $L(y)$ is the set of all the input vectors required to produce y. The technology can also be characterized from an output correspondence $P : \mathbb{R}^m_+ \longrightarrow 2^{\mathbb{R}^n_+}$, defined by $P(x) =$

[1] $[d] = \{1, ..., d\}$.

$\{y \in \mathbb{R}^n_+ : (x, y) \in T\}$. $P(x)$ is the set of all the output vectors obtainable from x. Now, let us denote

$$K = \mathbb{R}^m_+ \times (-\mathbb{R}^n_+). \tag{2.2}$$

K is called **the free disposal cone**. There are some standard assumptions the production technology must obey (see Shephard [27]):

T1: T is a closed set
T2: For any $z \in T$, $(z - K) \cap T$ is bounded.
T3: $T = (T + K) \cap \mathbb{R}^d_+$
T4: T is a convex set.

T1 is a standard mathematical requirement. T2 means that an infinite output cannot be produced from a finite input. T3 imposes that T is strongly disposable. T4 is the convexity assumption we shall relax in the remainder of the paper.

Among the input-output vectors of the production set, one can identify those that are efficient or weakly efficient. The efficient subset of T is defined by $E(T) = \{z \in T : \nexists w \in T, w \in z - K\backslash\{0\}\}$. The subset $E^w(T) = \{z \in T : \nexists w \in T, w \in int(z - K)\}$ is called the weak efficient subset.

Note that the definitions of $E(T)$ and $E^w(T)$ are valid T being convex or not. The next subsection presents the classical non-parametric approach for estimating a production technology.

2.2 Non-Parametric Convex and Non-Convex Technologies

Following Koopmans [20], Farrell [15], Charnes, Cooper and Rhodes [9] and Banker, Charnes, Cooper [2], the production set is traditionally defined from the convex hull that contains all the observations under a free disposal assumption. Under an assumption of variable returns to scale (see [2]), the production set is defined by $T_{DEA} = \big(Co(A) + K\big) \cap \mathbb{R}^d_+$ or equivalently

$$T_{DEA} = \Big\{(x, y) \in \mathbb{R}^{m+n}_+ : (-x, y) \leq \sum_{k=1}^{l} t_k\,(-x_k, y_k), t \geq 0, \sum_{k=1}^{l} t_k = 1\Big\}, \tag{2.3}$$

where $A = \{(x_k, y_k) : k = 1...l\} \subset \mathbb{R}^{m+n}_+$ a collection of l observed firms that operate on a specific sector of the economy and $Co(A)$ denotes the convex hull of A. This subset is, loosely speaking, the convex hull of a finite number of observed production vectors. In their seminal work, Charnes, Cooper Rhodes [9] defined the production set as the smallest convex cone containing all the observed firms. This implies an assumption of constant returns to scale. This model is obtained from (2.3) by dropping the constraint $\sum_{k=1}^{l} t_k = 1$.

The above approach summarizes the so-called DEA method (Data Envelopment Analysis). It is also possible to estimate a non-parametric technology which does not postulate a convexity assumption of the technology. The FDH approach was introduced in [13], [30] and [31] (FDH stands for "Free Disposal Hull"). The FDH hull of a data set A yields the non-parametric estimation production set defined by $T_{FDH} = (A + K) \cap \mathbb{R}_+^{m+n}$. One can give a more explicit form as follows:

$$T_{FDH} = \Big\{(x, y) \in \mathbb{R}_+^{m+n} : (-x, y) \leq \sum_{k=1}^{l} t_k\,(-x_k, y_k), t \in \{0,1\}^l, \sum_{k=1}^{l} t_k = 1\Big\}. \tag{2.4}$$

The main difference between the above convex and non-convex non-parametric models is that t is a real number in the former while it is valued in $\{0,1\}^l$ in the latter. FDH technologies are non-convex but postulates a free disposal assumption only.

Technical efficiency can then be measured using a notion of gauge function that looks, loosely speaking, for finding the closest point from any observed firms to the boundary of the production set. Along this line, the problem of measuring technical efficiency can be readily solved by linear programming. Let us define $\mathcal{T}$ as the class of all the production sets satisfying axioms $T1 - T3$. The Debreu-Farrell measure (Debreu [12], Farrell [15]) is the most usual measure of technical efficiency. It is essentially the inverse of the Shephard distance function (Shephard [27]). The input Debreu-Farrell efficiency measure is the map $E_i : \mathbb{R}_+^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by:

$$E_i(x, y, T) = \inf\big\{\lambda \geq 0 : (\lambda x, y) \in T\big\}. \tag{2.5}$$

In words, this measures the amount an input vector can be shrunk along a ray until it reaches the isoquant of the input set $L(y)$. If $T = T_{DEA}$, then it can be computed by linear programming. In the output case this technical efficiency measure is the map $E_o : \mathbb{R}_+^{m+n} \times \mathcal{T} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ defined by:

$$E_o(x, y, T) = \sup\big\{\theta \geq 1 : (x, \theta y) \in T\big\}. \tag{2.6}$$

The Debreu-Farrell measure can also be computed for FDH production technologies using an enumeration method (see Tulkens [30]).

3 On Some Class of Path-Connected Lower Semilattice

3.1 $\mathbb{B}$-convexity, Lattices and Semilattice Structures

This section lays down the groundwork for the use of $\mathbb{B}$-convexity in efficiency analysis. More details are given in Briec and Horvath [5] and Briec, Horvath and Rubinov [6]. A subset of C of $\mathbb{R}_+^d$ is $\mathbb{B}$-convex if for all $u, z \in C$ and all

$t \in [0,1]$ the join of u and tz lies in C. One can loosely say that $\mathbb{B}$-convexity is obtained from usual convexity making the formal substitution $+ \mapsto \max$. Semilattices structures play a crucial role to develop these notions. A subset $L \subset \mathbb{R}^d$ is said to be an **upper-semilattice** if $\forall z, t \in L$ then $z \vee t \in L$, where $z \vee t = (\max\{z_1, t_1\}, \ldots, \max\{z_d, t_d\})$. L is a **lower-semilattice** if $\forall z, t \in L$ then $z \wedge t \in L$, where $z \wedge t = (\min\{z_1, t_1\}, \ldots, \min\{z_d, t_d\})$. A **lattice** is a subset of $\mathbb{R}^d$ that is both an upper and a lower semilattice. In the remainder, we denote $\uparrow z = \{w \in \mathbb{R}^d : w \geq z\}$ and $\downarrow z = \{w \in \mathbb{R}^d : w \leq z\}$.

Let us consider $z_1, z_2, ..., z_l \in \mathbb{R}^d$. In the remainder of the paper we denote:

$$\bigvee_{k=1}^{l} z_k = \big(\max\{z_{1,1}, ..., z_{l,1}\}, \ldots, \max\{z_{1,d}, ..., z_{l,d}\}\big) \tag{3.1}$$

and

$$\bigwedge_{k=1}^{l} z_k = \big(\min\{z_{1,1}, ..., z_{l,1}\}, \ldots, \min\{z_{1,d}, ..., z_{l,d}\}\big). \tag{3.2}$$

The main objective of this contribution is the introduction of a new type of semilattice technologies. Hence, a connectedness property is important in our framework because it allows a production technique to be modified using a continuous transformation. Since in general a semilattice may not be a path-connected subset, $\mathbb{B}$-convex sets have some advantage in that aspect. A geometrical representation of the strings joining two points, in each case, is given in Appendix.

We come now to the introduction of $\mathbb{B}$-convexity. A subset $L \subset \mathbb{R}^d_+$ is said to be a $\mathbb{B}$-convex set, if $\forall u, z \in L$, and all $t \in [0,1]$ $u \vee tz \in L$. The basic properties of $\mathbb{B}$-convex sets are analyzed in [5]. From this definition a set C such that $\forall u, z \in C$, for all $s, t \geq 0$ $su \vee tz \in C$ is called a **$\mathbb{B}$-convex cone** .

Along this line, a notion of $\mathbb{B}$-convex hull can be provided. Let $A = \{z_1, ..., z_l\} \subset \mathbb{R}^d_+$ then the set

$$\mathbb{B}(A) = \Big\{ \bigvee_{k=1}^{l} t_k z_k, t \geq 0, \max_{k=1...l} \{t_k\} = 1 \Big\} \tag{3.3}$$

is called the $\mathbb{B}$-convex hull of A. A geometric representation is depicted in Figure 3.1.

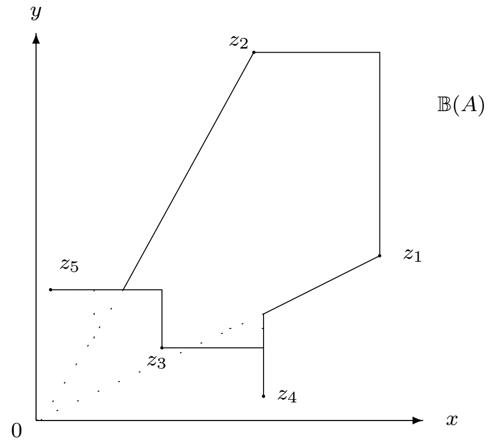

Figure 3.1 The $\mathbb{B}$-convex set $\mathbb{B}(A)$

Figure 3.1 is obtained using the geometrical construction of the strings connecting two points which is depicted in section 8.1.

3.2 Inverse $\mathbb{B}$-convex Sets

This section introduces a notion of **inverse $\mathbb{B}$-convex sets**. Paralleling our earlier definition of $\mathbb{B}$-convexity, inverse $\mathbb{B}$-convexity is obtained from usual convexity making the formal substitution $+ \mapsto \min$. It is shown in the remainder of this section that inverse $\mathbb{B}$-convex sets can be derived from $\mathbb{B}$-convex sets via a suitable homeomorphism. This mean that these notions are identical making a lexical change based on the formal substitution $\max \to \min$. Hence, all the results $\mathbb{B}$-convex sets satisfy can be transposed to inverse $\mathbb{B}$-convex sets via a suitable homeomorphism.

Definition 3.2.1 *Let $M \subset (\mathbb{R}_{++} \cup \{+\infty\})^d$. M is inverse $\mathbb{B}$-convex, if $\forall u, z \in M$, and $\forall t \in [1, +\infty]$ we have $u \wedge tz \in M$.*

Inverse $\mathbb{B}$-convex sets are isomorphically linked to $\mathbb{B}$-convex sets. To see that let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_{++} \cup \{+\infty\}$ be the inverse map defined by $\varphi(\alpha) \longrightarrow \frac{1}{\alpha}$.

A subset $M \subset \mathbb{R}^d_{++}$ is an inverse $\mathbb{B}$-convex set if and only if $L = \phi^{-1}(M)$ is a $\mathbb{B}$-convex set, where

$$\phi(z_1, ..., z_d) = (\varphi(z_1), ..., \varphi(z_d)) . \tag{3.4}$$

In words, a subset of $M \subset (\mathbb{R}_{++} \cup \{+\infty\})^d$ is inverse $\mathbb{B}$-convex if and only if its inverse is $\mathbb{B}$-convex. Though the respective geometric representation of $\mathbb{B}$-convex sets and inverse $\mathbb{B}$-convex sets are different they are both linked through an isomorphism over $(\mathbb{R}_{++} \cup \{+\infty\})^d$. These convexity concepts belong to a more general class of topological convexities described in [16]. All the properties stated in the earlier section do not apply for the case of vectors having null components. From [5] the next results are immediate and do not require a proof.

Lemma 3.2.2 *Suppose that M is an inverse $\mathbb{B}$-convex subset of $(\mathbb{R}_{++} \cup \{+\infty\})^d$. Then:*
(a) M is a lower-semilattice.
(b) If $\{z_1, ..., z_l\} \subset M$ and $(t_1, ..., t_l) \in [1, +\infty]^l$ then $\bigwedge_{k=1}^{l} t_k z_k \in M$.
(c) If $W \subset (\mathbb{R}_{++} \cup \{+\infty\})^d$ is inverse $\mathbb{B}$-convex, then $M \cap W$ is inverse $\mathbb{B}$-convex.
(d) M is a path connected set.

This last property holds regarding to the topology induced on $(\mathbb{R}_{++} \cup \{+\infty\})^d$ by the norm $x \mapsto \varphi^{-1}\big(\|\phi(x)\|\big)$, where $\|\cdot\|$ is a norm defined on $\mathbb{R}^n$. In [7], a $\mathbb{B}$-convex production model is constructed from the $\mathbb{B}$-convex hull of a finite number of points. A similar approach is proposed in the remainder of the paper. For this purpose, the inverse $\mathbb{B}$-convex hull of a finite set of points is a notion we need to introduce. We then provide the following definition.

Definition 3.2.3 *For all $A = \{z_1, ..., z_l\} \subset (\mathbb{R}_{++} \cup \{+\infty\})^d$, the set*

$$\mathbb{B}^{-1}(A) = \left\{ \bigwedge_{k=1}^{l} s_k z_k, \min_{k=1...l} s_k = 1 \right\}$$

is called the inverse $\mathbb{B}$-convex hull of A.

Now, we want to take into account the case where some components are null. Such a case cannot be directly derived from the isomorphism ϕ. Hence, the next definition provides a relaxed definition of inverse $\mathbb{B}$-convex sets to $\mathbb{R}^d_+$.

Definition 3.2.4 *A subset $M \subset \mathbb{R}^d_+$ is pseudo inverse $\mathbb{B}$-convex, if $\forall u, z \in M$, and $\forall t \geq 1$ we have $u \wedge tz \in M$.*

In general, it is easy to see that the formulation in 3.2.3 yields an inverse $\mathbb{B}$-convex set in the case where $A \subset \mathbb{R}^d_{++}$. However, if some components are null then it may not be path connected. This is the reason why we make a distinction between inverse $\mathbb{B}$-convex sets and **pseudo inverse $\mathbb{B}$-convex sets**. Clearly, an pseudo inverse $\mathbb{B}$-convex set having a nonempty interior is almost everywhere $\mathbb{B}$-convex with respect to the Lebesgue measure. For this reason, the definition above has some interest for the purpose of this paper.

Examples of the inverse $\mathbb{B}$-convex hull of two points are depicted in Figures 3.2.1.

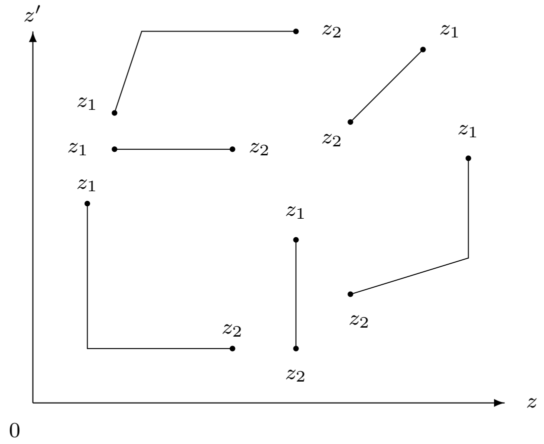

Figure 3.2.1 Inverse $\mathbb{B}$-convex hulls of 2 points.

More details about how to construct this inverse $\mathbb{B}$-convex hull can be found in Appendix 8.1. Figure 3.2.2 considers a more general case.

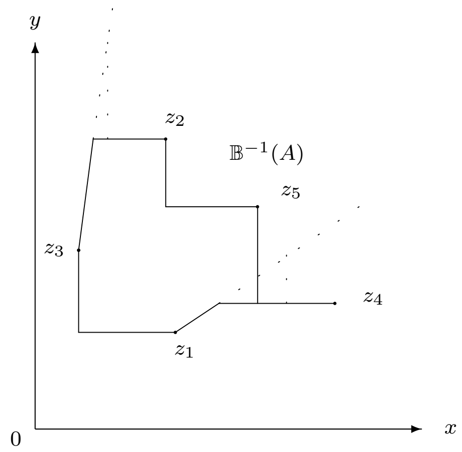

Figure 3.2.2 The inverse $\mathbb{B}$-convex set $\mathbb{B}^{-1}(A)$

In such a case, a geometrical representation is obtained by connecting each pair of points following the broken line described in Figure 3.2.1.

4 Duality

In this section some aspects of the economic meaning of $\mathbb{B}$-convexity are analyzed. A certain form of duality is proposed providing a price interpretation of technical efficiency measures.

4.1 Inputs and Output sets

We now analyze the input and output sets for each non-parametric technology. Clearly, if the subset T defined in 2.1 is a $\mathbb{B}$-convex (inverse $\mathbb{B}$-convex) technology then for all $(x, y) \in T$, the input set $L(y)$ and the output set $P(x)$ are $\mathbb{B}$-convex (inverse $\mathbb{B}$-convex). The next result shows that, in the case of FDH technology, the input set is $\mathbb{B}$-convex and the output set is inverse $\mathbb{B}$-convex.

Lemma 4.1.1 *Suppose $T \in \mathcal{T}$. Then:*
(a) *The input correspondence L has $\mathbb{B}$-convex values.*
(b) *The output correspondence P has pseudo inverse $\mathbb{B}$-convex values.*

The above property is immediate regarding to the definition of FDH technologies. However, it seems to have never been pointed in the literature on production frontier.

4.2 Some Duality Results in $\mathbb{B}$-convex Production Models

This section focusses on gauge functions and their dual relationships to maximum and minimum functions. For all $a \in \mathbb{R}^d$ maximum (resp. minimum) functions are defined by the map $z \mapsto \max_{i=1\ldots n} a_i z_i$ (resp. $z \mapsto \min_{i=1\ldots n} a_i z_i$). Some characterizations of inputs and output sets can be provided when input and output sets satisfy inverse $\mathbb{B}$-convexity and $\mathbb{B}$-convexity, respectively.

Proposition 4.2.1 *For all production technologies $T \in \mathcal{T}$, the following properties hold:*
(a) *For all $y \in \mathbb{R}^n_+$ if $L(y)$ is an inverse $\mathbb{B}$-convex set then there exists an uniqueness $\bar{x}_y \in \mathbb{R}^m_+$ such that $L(y) = \bar{x}_y + \mathbb{R}^m_+ = \uparrow \bar{x}_y$.*
(b) *For all $x \in \mathbb{R}^m_+$, if $P(x)$ is a $\mathbb{B}$-convex set then there exists $\bar{y}_x \in \mathbb{R}^n_+$ such that $P(x) = \big(\bar{y}_x - \mathbb{R}^n_+\big) \cap \mathbb{R}^n_+ = \downarrow \bar{y}_x \cap \mathbb{R}^n_+$.*

The result above has an immediate consequence on the structure of $\mathbb{B}$-convex technologies. It implies they have a functional representation.

Corollary 4.2.2 *Let $T \in \mathcal{T}$ be a production technology.*
(a) *If the input correspondence L has inverse $\mathbb{B}$-convex values then there exists a map $G : \mathbb{R}^n_+ \longrightarrow \mathbb{R}^m_+$ such that $T = \{(x, y) \in \mathbb{R}^{m+n}_+ : x \geq G(y)\}$.*
(b) *If the output correspondence P has $\mathbb{B}$-convex values then there exists a map $F : \mathbb{R}^m_+ \longrightarrow \mathbb{R}^n_+$ such that $T = \{(x, y) \in \mathbb{R}^{m+n}_+ : y \leq F(x)\}$.*

This property is immediate setting $G(x) = \bar{y}_x$ and $F(y) = \bar{x}_y$. Notice that Proposition 4.2.1.b means that the technology is output-cubic. In [7], it has been pointed out that such a technology is output cubic and can be related to Kohli technologies [19]. Similarly, Proposition 4.2.1.a means that the technology has a Leontief structure. Moreover, it is easy to see that both $L(y)$ and $P(x)$ also have a lattice structure. Notice that some results about the functional approach in $\mathbb{B}$-convexity can be found in [1].

Let $C_{\max} : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ be the function defined by:

$$C_{\max}(w, y) = \inf_{x \in L(y)} \max_{i=1\dots m} w_i x_i. \tag{4.1}$$

This function maps each input price $w \in \mathbb{R}^m_+$ and each output vector $y \in \mathbb{R}^n_+$ to the minimum of the maximal individual cost of each input. Thus, it is called the **max-cost function**. A symmetrical definition can be provided to maximize the revenue. The map $R_{\max} : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{-\infty\}$ defined by:

$$R_{\max}(p, x) = \sup_{y \in P(x)} \max_{j=1\dots n} p_j y_j \tag{4.2}$$

is called the **max-revenue function**. This function function maps each output price $p \in \mathbb{R}^n_+$ and each input vector $x \in \mathbb{R}^m_+$ to the maximum of the maximal individual revenue for each output.

Basically, these functions have some formal analogy with the cost and revenue functions making the formal substitution $+ \rightarrow \max$. Not surprisingly, their respective economic interpretations are different. For example, let us consider a firm that selects some input by minimizing this max-cost function. The map $x \mapsto \max_{i=1\dots m} w_i x_i$ can be interpreted as the maximum of the individual costs of each factor. By definition, $C_{\max}(w, y)$ gives the minimum amount of the maximal cost required to produce a production vector y. The corresponding optimization program implies that the optimal input is obtained imposing a particular proportion in the input consumption. Suppose that $a > 0$. It is easy to see that there is some $x^* \in L(y)$ minimizing the maximum cost such that $\frac{x_i^*}{x_k^*} = \frac{a_k^*}{a_i^*}$ for all $(i, k) \in [m]^2$. Notice that the form of the budget constraint can be related to the two-stage budgeting approach dating back to the creation of the separable utility model by Strotz [28]. It is further mentioned that Farrell technical efficiency measure is interpretable as a maximum ratio between the max-cost function and the max-cost of the observed input vector (Proposition 4.2.3). This duality property only requires a free disposal assumption. However, it has been shown in [7] that $\mathbb{B}$-convexity is sufficient.

Proposition 4.2.3 *For all $T \in \mathcal{T}$, if $(x, y) \in \mathbb{R}^{m+n}_+$ and $x \neq 0$, then:*

$$E_i(x, y, T) = \sup \left\{ C_{\max}(w, y) : \max_{i=1\dots m} w_i x_i = 1, w \geq 0 \right\}.$$

Moreover, if $0 \notin L(y)$, then:

$$C_{\max}(w, y) = \inf \left\{ \max_{i=1\dots m} w_i x_i : E_i(x, y, T) = 1, x \in \mathbb{R}^m_+ \right\}.$$

The proof of this property is an immediate consequence of the result established in [7] for the Shephard input distance function that is the inverse of the Farrell input measure.

The producer may also seek to maximize the quantity $\max_{j=1\dots n} p_j y_j$ that is the maximum of the maximal revenue he can expect from each outputs. It is further established that the Farrell output efficiency measure can be interpreted as a minimum ratio between this max-revenue function and the max-revenue of the observed output vector (Proposition 4.2.4).

Proposition 4.2.4 *Suppose that $T \in \mathcal{T}$. For all $(x, y) \in \mathbb{R}^{m+n}_+$, if $y \neq 0$ and $P(x)$ is a $\mathbb{B}$-convex set having a nonempty interior then:*

$$E_o(x, y, T) = \inf \left\{ R_{\max}(p, x) : \max_{j=1\dots n} p_j y_j = 1, p \geq 0 \right\}.$$

Moreover

$$R_{\max}(p, x) = \sup \left\{ \max_{j=1\dots n} p_j y_j : E_o(x, y, T) = 1, y \in \mathbb{R}^n_+ \right\}.$$

Let us introduce, the function $C_{\min} : \mathbb{R}^m_+ \times \mathbb{R}^n_+ \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$ defined by:

$$C_{\min}(w, y) = \inf_{x \in L(y)} \min_{i=1\dots m} w_i x_i. \tag{4.3}$$

This is called the **min-cost function**. Moreover, the function $R_{\min} : \mathbb{R}^n_+ \times \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+ \cup \{-\infty\}$ defined by

$$R_{\min}(p, x) = \sup_{y \in P(x)} \min_{j=1\dots n} p_j y_j \tag{4.4}$$

is called the **min-revenue function**.

These functions are constructed paralleling our earlier definitions of max-cost and max-revenue functions. In such a case, one assume that the producer seeks to minimize the function $x \mapsto \min_{i=1\dots m} w_i x_i$ that can be interpreted as the minimum cost that he (or she) can pay for each factor. Along this line, the function $C_{\min}(w, y)$ gives the minimum amount of the cost expected by the producer given the technology T. This implies that, on the input side, the technology has a Leontief structure. It is shown below that the Farrell input technical efficiency measure can then be interpreted as a maximum ratio between this min-cost function and the min-cost corresponding to the observed input vector (Proposition 4.2.5).

Proposition 4.2.5 *Suppose that $T \in \mathcal{T}$. For all $(x, y) \in \mathbb{R}^{m+n}_+$, if $x \neq 0$ and $L(y)$ is an inverse $\mathbb{B}$-convex set then:*

$$E_i(x, y, T) = \sup \big\{ C_{\min}(w, y) : \min_{i=1\dots m} w_i x_i = 1, w \geq 0 \big\}.$$

Moreover

$$C_{\min}(w, y) = \sup \big\{ \min_{i=1\dots m} w_i x_i : E_i(x, y, T) = 1, w \geq 0 \big\}.$$

Focusing on the output side, the producer seeks to maximize the quantity $\min_{j=1\dots n} p_j y_j$. Therefore the min-revenue function $R_{\min}(p, x)$ is interpreted as the minimum revenue he can expect from each outputs. It is shown below that the Farrell output measure can be interpreted as a minimum ratio between this min-revenue function and the minimum revenue corresponding to the observed output vector (Proposition 4.2.6).

To show this, it is useful to use a result established by Martínez, Rubinov and Singer [21]. Given some $y \in \mathbb{R}^n_+ \backslash P(x)$, it is always possible to find some $p \in \mathbb{R}^n_+$ such that $\min_{j\in[n]} p_j y_j > R_{\min}(p)$. Consequently, the output set $P(x)$ is the intersection of all its supporting half spaces. Namely, we have:

$$P(x) = \bigcap_{p\in\mathbb{R}^n_+} \{y \in \mathbb{R}^n_+ : \min_{i\in[n]} p_j y_j \leq R_{\min}(p, x)\}. \tag{4.5}$$

Proposition 4.2.6 *For all $T \in \mathcal{T}$, if $(x, y) \in \mathbb{R}^{m+n}_+$ and $y \neq 0$ then*

$$E_o(x, y, T) = \inf \big\{R_{\min}(p, x) : \min_{j=1\dots n} p_j y_j = 1, p \geq 0\big\}.$$

Moreover

$$R_{\min}(p, x) = \sup \big\{ \min_{j=1\dots n} p_j y_j : E_o(x, y, T) = 1, p \geq 0\big\}.$$

Duality of the output Farrell measure and min-revenue function is depicted in Figure 4.5.

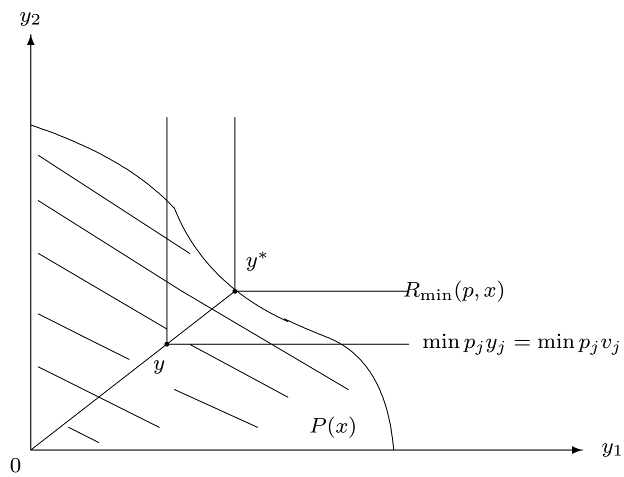

Figure 4.5 Duality with min-revenue function.

In Figure 4.5, y^* is the radial projection of y onto the frontier of the output set $P(x)$. Clearly, we have $R_{\min}(p, x) = \min_{j\in[n]} p_j y^*_j$ and $E_o(x, y, T) = \frac{\min_{j\in[n]} p_j y^*_j}{\min_{j\in[n]} p_j y_j}$. It is easy to check that if $p' \neq p$ and $\min_{j\in[n]} p'_j y_j = 1$ then $\frac{\min_{j\in[n]} p_j y^*_j}{\min_{j\in[n]} p_j y_j} \leq \frac{R_{\min}(p', x)}{\min_{j\in[n]} p_j y_j}$. Notice that this geometrical representation of our duality result has some formal analogy to the convex case by replacing the class of linear maps with that of the Leontief functions.

5 $\mathbb{B}$-convex and Inverse $\mathbb{B}$-convex Non-Parametric Technologies

5.1 The $\mathbb{B}$-convex Case: Some Known Results

This subsection presents the $\mathbb{B}$-convex non-parametric model introduced in [7]. We consider a collection $A = \big\{(x_k, y_k) : k = 1...l\big\}$ of l observed firms. The subset of $\mathbb{R}_+^{m+n}$ defined by

$$T_{\max} = \big(\mathbb{B}(A) + K\big) \cap \mathbb{R}_+^d \tag{5.1}$$

is called a **$\mathbb{B}$-convex non-parametric estimation** of the production technology. One can equivalently write:

$$T_{\max} = \Big\{(x, y) \in \mathbb{R}_+^{m+n} : x \geq \bigvee_{k=1}^{l} t_k x_k, y \leq \bigvee_{k=1}^{l} t_k y_k, \max_{k=1...l} t_k = 1, t \geq 0)\Big\}. \tag{5.2}$$

It has bee proved in [7] that $T_{\max}$ is a closed $\mathbb{B}$-convex set. Consequently, it also has an upper semilattice structure. The economic meaning of this model is discussed with more details in section 5.3. The basic idea is to replace usual convexity with $\mathbb{B}$-convexity. This implies a divisibility assumption and an upper semilattice structure of the technology. Consequently, the upper bound of two input production vectors can always produce the upper bound of the output vectors they are individually able to produce.[2] Comparing such an assumption to convexity, one can say that it has some advantages and drawbacks. Regarding the input side $\mathbb{B}$-convexity encompasses as a special case the situation where the technology assumes the inputs are freely disposable. Looking at the output side, $\mathbb{B}$-convexity implies, under a free disposal assumption, that the production set has an output cubic structure. This means that an assumption of output complementarity is implicitly made on the technology.

5.2 Inverse $\mathbb{B}$-convex Non-parametric Production Model

This section proposes an inverse $\mathbb{B}$-convex production model. This is constructed by analogy to the DEA model and the $\mathbb{B}$-convex structure proposed in the earlier section.

Definition 5.2.1 *Let $A = \big\{(x_k, y_k) : k = 1...l\big\} \subset \mathbb{R}_+^{m+n}$ a collection of l observed production vectors. The subset*

$$T_{\min} = \Big\{(x, y) \in \mathbb{R}_+^{m+n} : x \geq \bigwedge_{k=1}^{l} s_k\, x_k, y \leq \bigwedge_{k=1}^{l} s_k\, y_k, \min_{k=1...l} s_k = 1\Big\}$$

[2] There exists also DEA production models having a nonadditive algebraic structure, see for example[10, 11].

is called the inverse $\mathbb{B}$-convex non-parametric estimation of the production technology.

Note that if $A \subset \mathbb{R}^{m+n}_{++}$ then its inverse $\mathbb{B}$-convex hull $\mathbb{B}^{-1}(A)$ is well defined and one can equivalently write:

$$T_{\min} = \big(\mathbb{B}^{-1}(A) + K\big) \cap \mathbb{R}^d_+. \tag{5.3}$$

The next result establishes some basic results about the properties this inverse $\mathbb{B}$-convex model satisfies.

Proposition 5.2.2 *For all subset $A = \big\{(x_k, y_k) : k = 1...l\big\} \subset \mathbb{R}^{m+n}_+$ the production set $T_{\min}$ satisfies the following assumptions:*
(a) $T_{\min}$ is a closed set.
(b) $T_{\min}$ is an pseudo inverse $\mathbb{B}$-convex set.
(c) $T_{\min}$ is a lower semilattice, i.e $\forall z, w \in T_{\min}$, $z \wedge w \in T_{\min}$.

Paralleling the $\mathbb{B}$-convex case, this construction replaces usual convexity with pseudo inverse $\mathbb{B}$-convexity. This implies that production vectors are expandable. Moreover, the production set is endowed with a lower semilattice structure. Consequently, the lower bound of two input production vectors can always produce the lower bound of the output vectors they are individually able to produce. As in the $\mathbb{B}$-convex case, this approach also has both some merits and drawbacks. From the input side, inverse $\mathbb{B}$-convexity implies that, under a free disposal assumption, the input set has a Leontief structure. This means that the technology exhibits complementarity of inputs. Looking at the output side, inverse $\mathbb{B}$-convexity encompasses as a special case the situation where the technology only assumes that the outputs are freely disposable.

5.3 Comparison Between $\mathbb{B}$-convexity and Inverse $\mathbb{B}$-convexity

To make a comparison, we first depict the $\mathbb{B}$-convex and inverse $\mathbb{B}$-convex cases respectively.

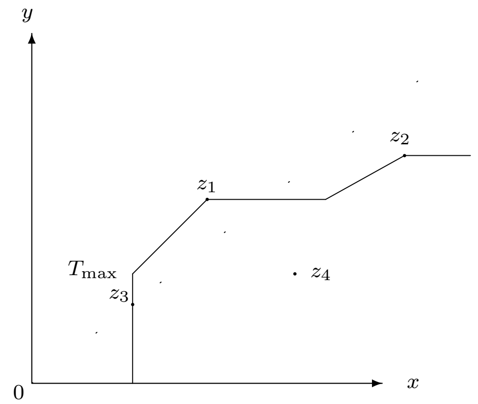

Figure 5.3 $\mathbb{B}$-convex estimation

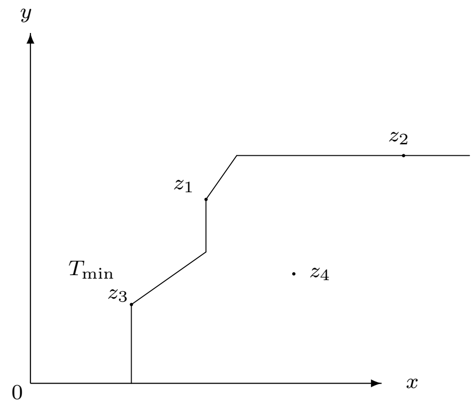

Figure 5.4 Inverse $\mathbb{B}$-convex estimation

The configuration of data set $A = \{z_1, z_2, z_3, z_4\}$ is identical in Figures 5.3. and 5.4. However, it is easy to see that the technologies resulting from this data sets are very different. Points z_1, z_2 and z_3 are weakly efficient both with respect to $T_{\max}$ and $T_{\min}$. An eyeball shows that the productivity average is greater in z_1 than in z_2 and z_3. Clearly, z_1 is a maximum productivity scale point. It is now of interest to compare the ways z_3 is connected to z_1 and z_1 is connected to z_2 respectively.

To do that, one should pay attention to the marginal productivities involved by the production frontier between z_3, z_1 and z_2. Since in both cases z_3 and z_2 are connected by a broken line, the marginal productivity is not defined everywhere. However, it is clearly defined almost everywhere and allows for a local comparison of the production frontiers of $T_{\max}$ and $T_{\min}$ respectively. Concerning $T_{\max}$, marginal productivity is nonincreasing from z_3 to z_1 and nondecreasing from z_1 to z_2. Hence, it follows that the technology is locally nonconvex between z_3 and z_1 and locally convex between z_1 and z_2. This implies that the marginal productivity at z_1 is infinite between z_3 and z_1 while it is finite and positive between z_1 and z_3. Clearly, the improvement of the average productivity required to reach the maximum productivity scale is achieved at the input level x_3.

Concerning $T_{\min}$ the above analysis is reversed. Marginal productivity is nondecreasing from z_3 to z_1 and nonincreasing from z_1 to z_2. Not surprisingly, it follows that the technology is locally convex between z_3 and z_1 and locally nonconvex between z_1 and z_2. In particular, this means that the marginal productivity at z_1 is positive and finite between z_3 and z_1 while it is null between z_1 and z_3. Consequently, a nondecreasing marginal productivity yields to an improvement of productivity scale. Conversely, a nonincreasing marginal productivity yields to a deterioration of productivity scale. Hence, some properties inherited from the inverse $\mathbb{B}$-convex model seem to have a more natural economic interpretation than those involved with the $\mathbb{B}$-convex model. In addition, the Leontief structure of the inverse $\mathbb{B}$-convex input sets seems to be preferable to the cubic form of the $\mathbb{B}$-convex outputs sets (see Proposition 4.2.1).

5.4 Mixed Estimation

In this section we also propose a mixed estimation that is defined as the intersection between the $\mathbb{B}$-convex and the inverse $\mathbb{B}$-convex estimations. Namely, the **mixed non-parametric estimation** is defined by

$$T_{\text{mix}} = T_{\max} \cap T_{\min}. \tag{5.4}$$

For all subset $A = \big\{(x_k, y_k) : k = 1...l\big\} \subset \mathbb{R}_+^{m+n}$ the mixed estimation T_{mix} is a closed set. Such an estimation is defined as the intersection of two technologies previously discussed.

The representation of is depicted in the following figure.

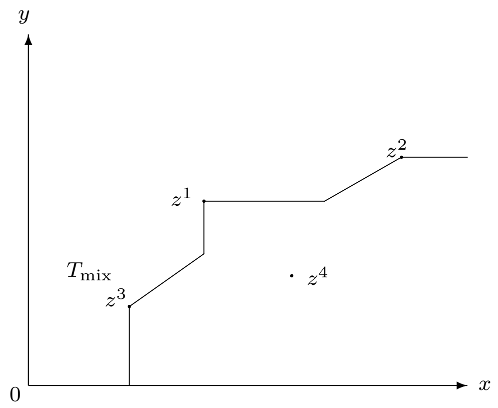

Figure 5.5 Mixed estimation

This approach has the advantage to provide a better extrapolation of the data. However, the technology is less "smooth" than in the $\mathbb{B}$-convex and inverse $\mathbb{B}$-convex cases.

6 Computation of Efficiency Measures

This section provides a general procedure to calculate Farrell efficiency measure in the input and output cases. This we do by focusing on the solutions of a system of maximum equations and minimum equations.

6.1 Systems of Maximum and Minimum Equations

Systems of maximum (resp. minimum) equations are obtained making the formal substitutions $+ \to \max$ (resp. $+ \to \min$). Maximum equation systems where studied in [5] to find the extreme points of a polytope. This paper also looks at the case of a minimum system of equations. As for maximum equation system, it is shown that some solutions can be obtained in closed form. We further establish that these results yield two general formula allowing to compute Farrell technical efficiency measures. For $k = 1...l$ and $i = 1...m$, let us denote

$$a^k = \left(a_1^k, \ldots, a_m^k\right) \quad \text{and} \quad a_i = \left(a_i^1, \ldots, a_i^l\right). \tag{6.1}$$

Maximum equations systems where studied in [5]. We briefly summarize their basic properties. The system of equations:

$$\begin{cases} \max\{a_1^1 x_1, \ldots, a_1^l x_l\} & = b_1 \\ \quad\vdots & \quad\vdots \\ \max\{a_m^1 x_1, \ldots, a_m^l x_l\} & = b_m \end{cases} \tag{6.2}$$

where $a^k \in \mathbb{R}_+^n$, $a^k \neq 0$ for $k = 1...l$ and $b \in \mathbb{R}_{++}^m$ is called a maximum equation system. If some b_i is null, then for all j, $a_i^j > 0$ implies $x_k = 0$. Therefore equation (i) can be suppressed. Consequently, such a system whose some components of b are null can be reduced to a system with $b \in \mathbb{R}^m_{++}$. Let $\mathcal{S}$ denote the set of solutions of system (6.2). Notice that the solutions can be characterized by the equivalence:

$$x \in \mathcal{S} \iff \bigvee_{k=1...l} x_j a^j = b \tag{6.3}$$

where $x \in \mathbb{R}^l_+$. Briec and Horvath [5] have established a necessary and sufficient conditions for the existence of some solution to system (6.2):

$$\mathcal{S} \neq \emptyset \iff \bigwedge_{i=1...n} b_i \phi(a_i) \in \mathcal{S} \tag{6.4}$$

where ϕ is the inverse function defined in (3.4).

This section pay attention to the case of a minimum equation system.

The system of equations:

$$\begin{cases} \min\{a_1^1 x_1, \dots, a_1^l x_l\} & = b_1 \\ \vdots & \vdots \\ \min\{a_m^1 x_1, \dots, a_m^l x_l\} & = b_m \end{cases} \tag{6.5}$$

where for $k = 1...l$, $a^k \in \mathbb{R}^l_{++}$, and $b \in \mathbb{R}^m_{++}$ is called a **minimum equation system**. If some b_i is infinite, the problem is symmetrical to that arising for a maximum equation system. For all k, $a_i^k < \infty$ implies $x_k = \infty$. Therefore equation (i) can be suppressed. Consequently, such a system whose some components of b are infinite can be converted to a system with $b \in \mathbb{R}^m_{++}$. Let us denote $\mathcal{Q}$ as the set of solutions satisfying (6.5). Notice that the solutions can be characterized by the equivalence:

$$x \in \mathcal{Q} \iff \bigwedge_{k=1...l} x_k a^k = b, \tag{6.6}$$

for all $x \in \mathbb{R}^l_{++}$. The next result establish a necessary and sufficient condition for the existence of some solution to the system (6.5)

Proposition 6.1.1 *If $a^k \in \mathbb{R}_{++}$ and $b_i \in \mathbb{R}_{++}$ for all $(i, k) \in [m] \times [l]$, one has*

$$\mathcal{Q} \neq \emptyset \iff \bigvee_{i=1...n} b_i \phi(a_i) \in \mathcal{Q}$$

To illustrate the result above a simple numerical example is provided. Let us consider the following system:

$$\begin{cases} \min\{2x, 3y\} & = 1 \\ \min\{4x, y\} & = 2 \end{cases} \tag{6.7}$$

We have $a_1 = (2, 3)$ and $a_2 = (4, 1)$. Now, $\phi(a_1) = \phi(2, 3) = (\frac{1}{2}, \frac{1}{3})$. Moreover $\phi(a_2) = \phi(4, 1) = (\frac{1}{4}, 1)$. Consequently since $b_1 = 1$ and $b_2 = 2$, we deduce $b_1\phi(a_1) \vee b_2\phi(a_2) = (\frac{1}{2}, \frac{1}{3}) \vee 2(\frac{1}{4}, 1) = (\frac{1}{2}, 2)$. It is now immediate to verify that $(\frac{1}{2}, 2)$ is the solution of S. We now study the case of an inequality system.

6.2 Systems of Maximum and Minimum Inequations

A **maximum-inequations system** is defined from a finite number of inequalities. The next result provides necessary and sufficient conditions for the existence of some solution.

Proposition 6.2.1 *Let us consider the two maximum-inequations systems:*

$$\left\{\begin{array}{ll} \max\{a_1^1 x_1, \ldots, a_1^l x_l\} & \leq b_1 \\ \vdots & \vdots \\ \max\{a_m^1 x_1, \ldots, a_m^l x_l\} & \leq b_m \end{array}\right. \tag{6.8}$$

where for $k = 1...l$, $a^k \in \mathbb{R}^m_+$, $a^k \neq 0$ *and* $b \in \mathbb{R}^m_{++}$, *and*

$$\left\{\begin{array}{ll} \max\{c_1^1 x_1, \ldots, c_1^l x_l\} & \geq d_1 \\ \vdots & \vdots \\ \max\{c_n^1 x_n, \ldots, c_n^l x_l\} & \geq d_n \end{array}\right. \tag{6.9}$$

where for $k = 1...l$, $c^k \in \mathbb{R}^n_+$, $c^k \neq 0$ *and* $d \in \mathbb{R}^n_{++}$. *Let* $\mathcal{IS}$ *and* $\mathcal{IS}'$ *be the solution sets of systems* (6.8) *and* (6.9) *respectively. Then*

$$\mathcal{IS} \cap \mathcal{IS}' \neq \emptyset \iff \bigwedge_{i=1...n} b_i \phi(a_i) \in \mathcal{IS} \cap \mathcal{IS}'.$$

A similar result can be stated in the context of a minimum-inequality system.

Proposition 6.2.2 *Let us consider the two minimum-inequations systems:*

$$\left\{\begin{array}{ll} \min\{a_1^1 x_1, \ldots, a_1^l x_l\} & \geq b_1 \\ \vdots & \vdots \\ \min\{a_m^1 x_1, \ldots, a_m^l x_l\} & \geq b_m \end{array}\right. \tag{6.10}$$

where for $k = 1...l$, $a^k \in \mathbb{R}^m_{++}$, *and* $b \in \mathbb{R}^m_{++}$, *and*

$$\left\{\begin{array}{ll} \min\{c_1^1 x_1, \ldots, c_1^l x_l\} & \leq d_1 \\ \vdots & \vdots \\ \min\{c_n^1 x_1, \ldots, c_n^l x_l\} & \leq d_n \end{array}\right. \tag{6.11}$$

where for $k = 1...l$, $c^k \in \mathbb{R}^n_{++}$ *and* $d \in \mathbb{R}^n_{++}$. *Let* $\mathcal{IQ}$ *and* $\mathcal{IQ}'$ *be the solution sets of* (6.10) *and* (6.11) *respectively. Then*

$$\mathcal{IQ} \cap \mathcal{IQ}' \neq \emptyset \iff \bigvee_{i=1...m} b_i \phi(a_i) \in \mathcal{IQ} \cap \mathcal{IQ}'.$$

6.3 Measurement of Technical Efficiency and $\mathbb{B}$-convex Non-Parametric Technologies

A general method to calculate the Farrell measures is proposed in this section. To simplify the notations, we assume that $A = \{(x_k, y_k) : k = 1...l\} \subset \mathbb{R}^{m+n}_{++}$. This means that we suppose the observed input and output vectors are positive. In the $\mathbb{B}$-convex case, a closed form calculating the Shephard input distance function have not yet been obtained. However, this was based upon another approach using the cubic structure of the technology (see Briec and Horvath [7]). The method we use in this section is based on max-programming. Moreover, since the Shephard distance function is essentially the inverse of the Farrell measure, the next results are equivalent to those obtained in [7]. In addition, a formula is given in the output case.

Proposition 6.3.1 *For all $A = \{(x_k, y_k) : k = 1...l\} \subset \mathbb{R}^{m+n}_{++}$, let us denote*

$$\alpha_{\bar{k},k} = \min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}.$$

(a) *For each $\bar{k} \in [l]$, the input Farrell technical efficiency measure is:*

$$E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\max}) = \max\Big\{ \max_{j=1...n} \min_{\substack{k \\ y_{\bar{k},j} \leq y_{k,j}}} \{\frac{y_{\bar{k},j}}{y_{\bar{k},k}\alpha_{\bar{k},k}}\}, \min_{k} \frac{1}{\alpha_{\bar{k},k}}\Big\}.$$

(b) *The Farrell output measure is:*

$$E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\max}) = \min_{j=1...n} \max_{k} \Big\{\frac{y_{k,j} \min\{\alpha_{\bar{k},k}, 1\}}{y_{\bar{k},j}}\Big\}.$$

6.4 Measurement of Technical Efficiency and Inverse $\mathbb{B}$-convex Non-Parametric Technologies

We now provide a method for calculating the Farrell measure over an inverse $\mathbb{B}$-convex set non-parametric technology.

Proposition 6.4.1 *Let $A = \{(x_k, y_k) : k = 1...l\} \subset \mathbb{R}^{m+n}_{++}$. Assume that $\forall k = 1...l$, $x_k \neq 0$. Let us denote*

$$\beta_{\bar{k},k} = \min_{j=1...n} \frac{y_{k,j}}{y_{\bar{k},j}}.$$

(a) *For all $\bar{k} \in [l]$, the input Farrell efficiency measure is:*

$$E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\min}) = \max_{i=1...m} \min_{k} \Big\{\frac{x_{k,i}}{x_{\bar{k},i} \min\{\beta_{\bar{k},k}, 1\}}\Big\}.$$

(b) *The output Farrell efficiency measure is:*

$$E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\min}) = \min\Big\{ \min_{i=1...m} \max_{\substack{k \\ x_{k,i} \leq x_{\bar{k},i}}} \{\frac{x_{\bar{k},i}\beta_{\bar{k},k}}{x_{k,i}}\}, \max_{k} \beta_{\bar{k},k}\Big\}.$$

Results hereinbefore enables us to formulate a direct method for calculating Farrell measures in a case of the above defined mixed technology.

Lemma 6.4.2 *Let* $A = \{(x_k, y_k) : k = 1...l\} \subset \mathbb{R}^{m+n}_{++}$. *Assume that for* $k = 1...l$, $x_k \neq 0$. *Then, for all* $\bar{k} \in [l]$ *the Farrell input measure is:*

$$E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\text{mix}}) = \max\{E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\max}), E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\min})\}$$

and the Farrell output measure is:

$$E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\text{mix}}) = \min\{E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\max}), E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\min})\}.$$

In the following, a numerical example is proposed. Farrell input and output measures are computed and compared in several cases.

Example 6.4.3 *The following data sample that can be found is Färe, Grosskopf and Lovell [14].*

Table 1. Data Sample

Firms	*Input*	*Output 1*	*Output 2*
1	*2*	*3/2*	*1*
2	*2*	*2*	*1*
3	*4*	*3*	*2*
4	*6*	*6*	*6*
5	*7*	*6*	*6*
6	*8*	*7*	*4*
7	*9*	*7*	*4*

Using the formula established in the sections above we obtain the following results.

Table 2. Farrell Measures.

Firms	*Input* $T_{\max}$	*Input* $T_{\min}$	*Input* T_{mix}	*Input* T_{FDH}	*Output* $T_{\max}$	*Output* $T_{\min}$	*Output* T_{mix}	*Output* T_{FDH}
1	*1*	*1*	*1*	*1*	*4/3*	*1*	*1*	*1*
2	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*
3	*3/4*	*1*	*1*	*1*	*4/3*	*1*	*1*	*1*
4	*1*	*1*	*1*	*1*	*1*	*1*	*1*	*1*
5	*6/7*	*6/7*	*6/7*	*6/7*	*1*	*1*	*1*	*1*
6	*1*	*7/8*	*1*	*1*	*1*	*1*	*1*	*1*
7	*8/9*	*7/9*	*8/9*	*8/9*	*1*	*1*	*1*	*1*

Not surprisingly, when firms are efficient regarding to the FDH technology, they may not be efficient with respect to the other models. The FDH production set is the smallest one satisfying a free disposal assumption. One can see that the efficiency scores are strongly dependant on the choice of the technology. If Farrell efficiency measures are output oriented, then all the firms are efficient in both the DEA, inverse $\mathbb{B}$*-convex and mixed cases. However, if the production technology is* $\mathbb{B}$*-convex, then firm 1 and 3 are not efficient. This is due to the fact that the output set of such a technology is cubic and multidimensional* ($n = 2$)*. If Farrell efficiency measures are input oriented, then the efficiency of production units depends on the type of* $\mathbb{B}$*-convex structure the technology satisfies. Firm 3 is efficient in the inverse* $\mathbb{B}$*-convex case, and inefficient in the* $\mathbb{B}$*-convex one. One can check that the situation is reversed for firm 6. Only firm 4 is efficient in the input and output cases for all the technologies presented in our example.*

7 Conclusion

We have introduced in this paper a production model based on $\mathbb{B}$-convexity and inverse $\mathbb{B}$-convexity. In particular, the inverse $\mathbb{B}$-convex model has been analyzed in depth and some earlier results established in [7] have been extended. Output measures of technical efficiency have also been computed and some additional methods of min-programming have been provided. We have shown that these $\mathbb{B}$-convex models can be used to overcome some limitations inherited from convex production technologies which do not take into account the presence of indivisibleness and impose a particular structure of returns to scale. Along this line further investigations should be made on the local structure of returns to scale $\mathbb{B}$-sets are based upon.

References

[1] Adilov, G. and A.M. Rubinov (2006), $\mathbb{B}$-convex sets and functions, *Numerical Functional Analysis and Optimization*, 27, pp. 237-257.

[2] Banker, R.D., A. Charnes, and W.W. Cooper (1984), Some models for estimating technical and scale inefficiencies in data envelopment analysis, *Management Science*, 30, pp. 1078-92.

[3] Boussemart, J-P., W. Briec, N. Peypoch and C. Tavéra (2009), α-return to scale in multi-ouput technologies, *European Journal of Operational Research*, 197, pp. 332-339.

[4] Bogetoft, P. (1996), DEA on relaxed convexity assumptions, *Management Science*, 42, pp. 457-465.

[5] Briec, W. and C.D. Horvath (2004), $\mathbb{B}$-convexity, *Optimization*, 53, pp. 103-127.

[6] Briec, W., C.D. Horvath and A. Rubinov (2005), Separation in $\mathbb{B}$-convexity, *Pacific Journal of Optimization*, 1, pp. 1-27.

[7] Briec, W. and C.D. Horvath (2009), A $\mathbb{B}$-convex Production Model for Evaluating Performance of Firms, *Journal of Mathematical Analysis and Applications*, 355, pp. 131-144.

[8] Briec, W. and C.D. Horvath (2010), On the separation of convex sets in some idempotent semimodules, *Linear Algebra and its Applications*, forthcomming.

[9] Charnes, A., W.W. Cooper, and E. Rhodes (1978), Measuring the efficiency of decision making units, *European Journal of Operational Research* 26, pp. 429-444.

[10] Charnes, W, W.W. Cooper, L. Seiford, and J. Stutz (1982), A multiplicative model for efficiency analysis, *Socio-Economic Planning Sciences*, 16, pp. 223-224.

[11] Charnes, A, W.W.Cooper, L. Seiford, and J. Stutz (1983), Invariant multiplicative efficiency and piecewise Cobb-Douglas envelopments, *Operations Research Letters*, 2 , pp. 1010-1013.

[12] Debreu, G. (1951), The coefficient of resource utilization, *Econometrica*, 19 , pp. 273-292.

[13] Deprins, D., L. Simar and H. Tulkens (1984), Measuring labour efficiency in post offices. In: Marchand, M., Pestieau, P. and Tulkens, H., Editors, 1984. *The Performance of Public Enterprises*, North-Holland, Amsterdam, pp. 243-267.

[14] Färe, R, S. Grosskopf and C.A.K. Lovell (1985), *The Measurement of Efficiency of Production*, Klüwer- Kluwer-Nijhoff Publ., Boston.

[15] Farrell, M.J. (1957), The measurement of productive efficiency, *Journal of the Royal Statistical Society*, 120, pp. 253-281.

[16] Horvath, C.D. (2008), Topological convexities, selections and fixed points, *Topology and its Applications*, 155, pp. 830-850.

[17] Kaldor, N. (1934), The equilibrium of the firm, *Economic Journal*, 34, pp. 60-76.

[18] Kerstens, K. and P. Vanden Eeckaut (1999), Estimating returns to scale using non-parametric deterministic technologies: A new method based on goodness-of-fit, *European Journal of Operational Research*, 9, pp. 206-214.

[19] Kohli, U. (1983), Nonjoint Technologies, *Review of Economic Studies*, 50, pp. 209-19.

[20] Koopmans, T. (1951), Analysis of Production as an Efficient Combination of Activities, in: T. Koopmans (ed) *Activity Analysis of Production and Allocation*, New haven, Yale Univ. Press, pp. 33-97.

[21] Martínez-Legaz, J-E., Rubinov, A.M. and I. Singer (2002), Downward Sets and their Separations and Approximation Properties, *Journal of Global Optimization*, 2, pp. 111-137.

[22] Petersen, N.C. (1990), DEA on a relaxed set of assumptions, *Management Science*, 36, pp. 305-314

[23] Podinovski, V. (2005), Selective convexity in DEA models, *European Journal Of Operational Research*, 161, pp. 552-563.

[24] Podinovski, V. and T. Kuosmanen (2011), Modelling weak disposability in data envelopment analysis under relaxed convexity assumptions, *European Journal of Operational Research*,211 (3), pp. 577-585.

[25] Soleimani-damaneh, M., G.R. Jahanshahlooa and M. Reshadia (2001), On the estimation of returns-to-scale in FDH models , *European Journal of Operational Research*, 174, pp. 1055-1059.

[26] Samuelson, P.A. (1947/65), *Foundations of Economic Analysis*, Atheneum, New York.

[27] Shephard, R.W. (1970), *Theory of cost and production functions*, Princeton University Press, Princeton, New Jersey, 1970.

[28] Strotz, R.H. (1957), The empirical implications of a utility tree, *Econometrica*, 25, pp. 269-280.

[29] Tone, K. and B. K. Sahoo (2003), Scale, Indivisibilities and Production Function in Data Envelopment Analysis, *International Journal of Production Economics*, 84(2), pp. 165-192.

[30] Tulkens, H. (1993), On FDH analysis: Some Methodological Issue and an Application to Retail Banking, Court and Urban Transit, *Journal of Productivity Analysis*, 4 (1-2) 183-210

[31] Tulkens, H and P. Vanden-Eeckaut (1995), Non-parametric Efficiency, Progress and Regress Measures for Panel Data: Methodological Aspects, *European Journal of Operational Research*, 80 (3), pp. 474-499.

8 Appendix

8.1 The Geometrical Representation of $\mathbb{B}$-convex and Inverse $\mathbb{B}$-convex Sets

The following figures depict the geometric form of the broken lines joining two points with respect to the convexity type that is considered. Figure 8.1.1 depicts the case x_1 and x_2 are not ordered.

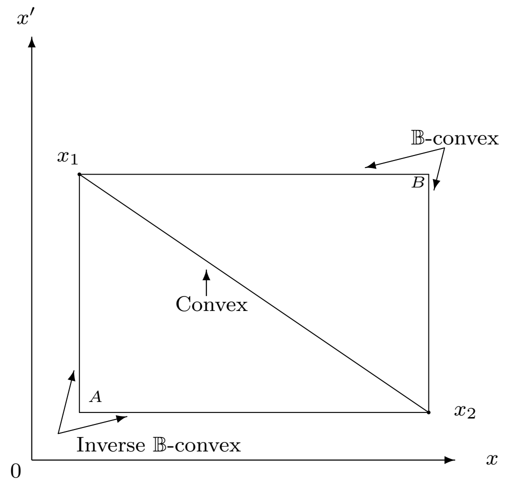

Figure 8.1.1 The segment lines joining x_1 and x_2 when they are not ordered

The $\mathbb{B}$-convex set $\mathbb{B}(x_1, x_2)$ is the broken line connecting points x_1, B and x_2. The inverse $\mathbb{B}$-convex set $\mathbb{B}^{-1}(x_1, x_2)$ is the broken line joining points x_1, A and x_2. Figure 8.1.2 depicts a case x_1 and x_2 are ordered.

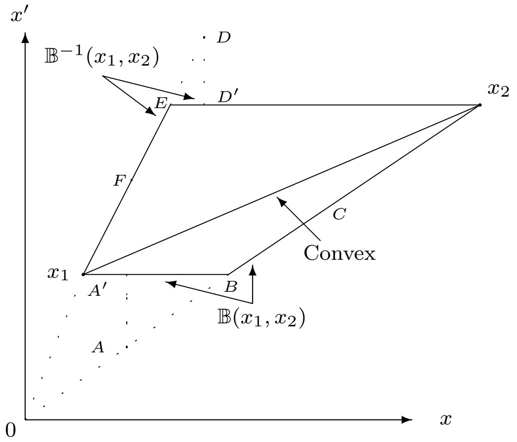

Figure 8.1.2 x_2 under the halfline spanned by x_1

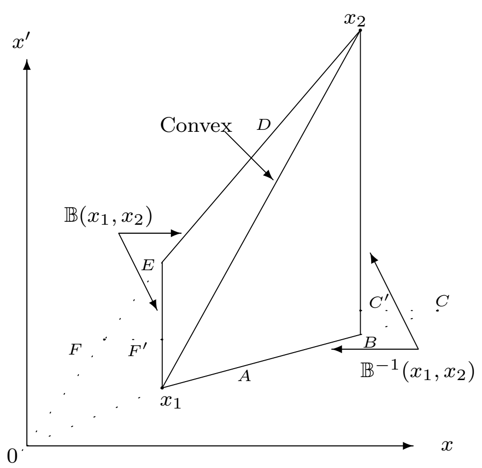

Figure 8.1.3 x_2 upper the half line spanned by x_1.

In Figure 8.1.2 we consider the case $x_1 \leq x_2$ and x_2 under the half line spanned by x_1 (the half line $[x_1, D]$ in Figure 8.1.2). We first analyze the case $\mathbb{B}$-convex case. It is clear that point A can be written as $A = t_A x_2$ with $t_A \leq 1$. Taking the maximum vector between A and x_1 yields the point A'.

Moreover, any $C \in [B, x_2]$ can be written $C = t_C x_2$ the maximum between C and x_1 belongs to the segment line $[B, x_2]$. Hence the $\mathbb{B}$-convex set $\mathbb{B}(x_1, x_2)$ is the broken line joining points x_1, B and x_2. A symmetrical analysis can be provided in the inverse $\mathbb{B}$-convex case. In Figure 8.1.2 point D can be written as $D = s_D D x_1$ with $s_D \geq 1$. Taking the minimum vector between D and x_2 yields the point D'. Moreover, any $F \in [x_1, E]$ can be written $F = s_F x_1$ the minimum between F and x_2 is F and therefore belongs to the segment line $[x_1, E]$. Consequently the inverse $\mathbb{B}$-convex set $\mathbb{B}^{-1}(x_1, x_2)$ is the broken line joining points x_1, E and x_2.
Figure 8.1.2. depicts the case $x_1 \leq x_2$ and x_2 is under the half line spanned by x_1. Using a similar approach the case x_2 is upper the half line spanned by x_1 is illustrated in Figure 8.1.3.

8.2 Proof of results

Proof of Lemma 4.1.1: (a) By hypothesis T satisfies a free disposal assumption. Consequently, for all $y \in \mathbb{R}^n_+$ the subset $L(y)$ also satisfies a free disposal assumption. Hence, from [7] $L(y)$ is $\mathbb{B}$-convex. (b) Fix some $x \in \mathbb{R}^m_+$ and suppose that $P(x)$ is a nonempty subset of $\mathbb{R}^n_+$. Assume that $y, v \in P(x)$. We need to prove that for all $s \geq 1$ one has $y \wedge sv \in P(x)$. By hypothesis, since $u, v \geq 0$ we have $0 \leq y \wedge sv \leq y$. Since T satisfies a free disposal assumption, it follows that $y \wedge sv \in P(x)$ which ends the proof. $\square$

Proof of Proposition 4.2.1: a) The first step of the proof is to establish that $L(y)$ has a minimal element. Let us consider the map $f : \mathbb{R}^m_+ \longrightarrow \mathbb{R}_+$ defined by $f(x) = x_1 + \cdots + x_m$. This function is continuous and nondecreasing on $\mathbb{R}^m_+$. Fix some $u \in L(y)$ and define $L^u(y) = L(y) \cap \downarrow u$. Since $L(y)$ is inverse $\mathbb{B}$-convex it is a lower semilattice. Moreover $\downarrow u$ is also a lower semilattice. Thus $L^u(y)$ is a lower semilattice and since $\mathbb{R}^m_+ \cap \downarrow u$ is closed and bounded it follows that $L^u(y)$ is a compact lower semilattice of $\mathbb{R}^m_+$. Consequently there is some $\bar{x}_y \in L^u(y)$ that achieves the infimum of f on $L^u(y)$. We first prove that $\bar{x}_y$ is a minimal element of $L^u(y)$. Suppose this is not the case. Then there is some $x \in L^u(y)$ such that $x \notin \uparrow \bar{x}_y$. Hence $x \wedge \bar{x}_y \neq \bar{x}_y$ which implies that $f(x \wedge \bar{x}_y) < f(\bar{x}_y)$. However, since $L^u(y)$ is a lower semilattice $x \wedge \bar{x}_y \in L^u(y)$ and this is a contradiction. Thus $\bar{x}_y$ is a minimal element of $L^u(y)$. To complete the proof, remark that for all $v \in L(y)$ one has $\bar{x}_y \wedge v \leq \bar{x}_y \leq u$. Hence $\bar{x}_y \wedge v \in L^u(y)$ that implies $\bar{x}_y \leq \bar{x}_y \wedge v$. Consequently $\bar{x}_y = \bar{x}_y \wedge v$ and we deduce that for all $v \in L(y)$ one has $v \geq \bar{x}_y$. Thus $\bar{x}_y$ is a minimal element of $L(y)$ and it follows that $L(y) \subset \bar{x}_y + \mathbb{R}^m_+$.

Furthermore, since the strong disposal assumption hods $\bar{x}_y + \mathbb{R}^m_+ \subset L(y)$ which proves (a). (b) From Axiom T3, $P(x)$ is bounded for all $x \in \mathbb{R}^m_+$. Since $P(x)$ is $\mathbb{B}$-convex and closed, one can deduce from [6] that it has a maximal element. Hence $P(x) \subset \left(\bar{y}_x - \mathbb{R}^n_+\right) \cap \mathbb{R}^n_+$. However since the free disposal assumption holds the converse inclusion is also true which ends the proof. $\square$

Proof of Proposition 4.2.4: We first prove that there is some $\bar{p}$ such that $E_o(x,y,T) = \frac{R_{\max}(\bar{p},x)}{\max_{j\in[n]} \bar{p}_j y_j}$. Since $P(x)$ is compact and $\mathbb{B}$-convex set it has a maximal element $\bar{y}$. Moreover $P(x)$ has a nonempty interior. Thus its maximal element lies in the interior of $\mathbb{R}^n_+$ and this implies that $\bar{y}_x > 0$. From Proposition 4.2.1, $P(x) = \big(\bar{y}_x - \mathbb{R}^n_+\big) \cap \mathbb{R}^n_+$. It follows that $P(x) = \{v \in \mathbb{R}^n_+ : \max_{j\in[n]} \frac{v_j}{\bar{y}_{x,j}}\} \leq 1$. It follows that $E_o(x,y,T) = \min_{j\in[n]} \frac{\bar{y}_{x,j}}{y_j} = \big[\max_{j\in[n]} \frac{y_j}{\bar{y}_{x,j}}\big]^{-1}$. Fix $\bar{p} = (\frac{1}{\bar{y}_{x,1}}, ..., \frac{1}{\bar{y}_{x,n}})$. By construction we have $R_{\max}(\bar{p},x) = 1$. It follows that, for all $p \in \mathbb{R}^n_+$ we have

$$E_o(x,y,T) = \frac{R_{\max}(\bar{p},x)}{\max_{j\in[n]} \bar{p}_j y_j} \leq \sup\big\{\theta : \max_{j\in[n]} p_j \theta y_j \leq R_{\max}(p,x)\big\}.$$

We then deduce that $E_o(x,y,T) \leq \frac{R_{\max}(p,x)}{\max_{j\in[n]} p_j y_j}$ and normalizing the price vectors this proves the first part of the statement. The second statement is immediate from the fact that the maximum function is nondecreasing and positively semi-homogenous. Hence, its maximum is achieved by a frontier point such that $E_o(x,y,T) = 1$. $\square$

Proof of Proposition 4.2.5: We first prove that there is some $\bar{w}$ such that $E_i(x,y,T) = \frac{C_{\min}(\bar{w},y)}{\min_{j\in[n]} \bar{w}_i x_i}$. Since $L(y)$ is an inverse $\mathbb{B}$-convex this means that $L(y) \subset \mathbb{R}^m_{++}$. Moreover, from Proposition 4.2.1 $L(y)$ has a minimal element $\bar{x}_y$. Thus its minimal element lies in the interior of $\mathbb{R}^m_+$ and this implies that $\bar{x}_y > 0$. From Proposition 4.2.1, it follows that $L(y) = \{u \in \mathbb{R}^m_+ : \min_{i\in[m]} \frac{u_i}{\bar{x}_{y,i}}\} \geq 1$. Hence $E_i(x,y,T) = \max_{i\in[m]} \frac{\bar{x}_{y,i}}{x_i} = \big[\min_{i\in[m]} \frac{x_i}{\bar{x}_{y,i}}\big]^{-1}$. Fix $\bar{w} = (\frac{1}{\bar{x}_{y,1}}, ..., \frac{1}{\bar{x}_{y,m}})$. By construction we have $C_{\min}(\bar{w},y) = 1$. It follows that, for all $w \in \mathbb{R}^m_+$ we have

$$E_i(x,y,T) = \frac{C_{\min}(\bar{w},y)}{\min_{i\in[m]} \bar{w}_i x_i} \geq \sup\big\{\lambda : \min_{i\in[m]} w_i \lambda x_i \geq C_{\min}(w,y)\big\}.$$

We then deduce that $E_i(x,y,T) \geq \frac{C_{\min}(w,y)}{\min_{i\in[m]} w_i x_i}$ and normalizing the price vectors this proves the first part of the statement. The second statement is immediate from the fact that the minimum function is nondecreasing and positively semi-homogeneous. Hence, its minimum is achieved by a frontier point such that $E_i(x,y,T) = 1$. $\square$

Proof of Proposition 4.2.6: (a) Since $P(x)$ is a subset of $\mathbb{R}^n_+$, we have for all $y \in \mathbb{R}^n_+$

$$E_o(x,y,T) = \sup\{\theta : \theta y \in P(x)\}.$$

One equivalently has

$$E_o(x,y,T) = \inf\{\theta : \theta y \notin P(x)\}.$$

Since $P(x) = \bigcap_{p\in\mathbb{R}^n_+} \{v \in \mathbb{R}^n_+ : \min_{j\in[n]} p_j v_j \le R_{\min}(p,x)\}$, it follows that

$$\begin{aligned}\mathbb{R}^n_+ \backslash P(x) &= \bigcup_{p\in\mathbb{R}^n_+} \mathbb{R}^n_+ \backslash \{v \in \mathbb{R}^n_+ : \min_{j\in[n]} p_j v_j \le R_{\min}(p,x)\} \\ &= \bigcup_{p\in\mathbb{R}^n_+} \{v \in \mathbb{R}^n_+ : \min_{j\in[n]} p_j y v_j > R_{\min}(p,x)\}.\end{aligned}$$

Consequently,

$$\begin{aligned}E_o(x,y,T) &= \inf\{\theta : \theta y \in \bigcup_{p\in\mathbb{R}^n_+} \{v \in \mathbb{R}^n_+ : \min_{j\in[n]} p_j v_j > R_{\min}(p,x)\}\} \\ &= \inf_{p\in\mathbb{R}^n_+} \inf\{\theta : \theta y \in \{v \in \mathbb{R}^n_+ : \min_{j\in[n]} p_j v_j > R_{\min}(p,x)\}\} \\ &= \inf_{p\in\mathbb{R}^n_+} \inf\{\theta : \min_{j\in[n]} p_j \theta y > R_{\min}(p,x)\} \\ &= \inf_p \frac{R_{\min}(p,x)}{p_j y_j}.\end{aligned}$$

Making an immediate normalization yields the result. (b) The second statement is immediate from the fact that the minimum function is nondecreasing and positively semi-homogenous. Hence, its maximum is achieved by a frontier point such that $E_o(x,y,T) = 1$.□

Proof of Proposition 5.2.2: (a) The map $s \mapsto \min_{k=1,\ldots,l} s_k$ is continuous, consequently $B = \{\bigwedge_{k=1}^{l} s_k(x_k,y_k) : \min s_k = 1\}$ is a closed subset of $\mathbb{R}^{m+n}_+$. Since by definition $T_{\min} = (B+K)\cap\mathbb{R}^{m+n}_+$ it follows that $T_{\min}$ is a closed. (b) Assume that $(x,y),(u,v) \in T_{\min}$. In such a case, there exists $s_1,\ldots,s_l \ge 0$, such that $\min\{s_1,\ldots,s_l\} = 1$, $x \ge \bigwedge_{k=1}^{l} s_k\, x_k$ and $y \le \bigwedge_{k=1}^{l} s_k\, y_k$. Moreover, there exists $t_1,\ldots,t_k \ge 0$ with $\min\{t_1,\ldots,t_l\} = 1$ such that $u \ge \bigwedge_{k=1}^{l} t_k\, x_k$ and $v \le \bigwedge_{k=1}^{l} t_k\, y_k$. Now, let $q,r \ge 0$ such that $\min\{q,r\} = 1$. We have $qx \wedge ru \ge q\left(\bigvee_{k=1}^{l} s_k\, x_k\right) \wedge r\left(\bigwedge_{k=1}^{l} t_k\, x_k\right) = \bigwedge_{k=1}^{l} \min\{qs_k, rt_k\}\, x_k$. Similarly $qy \wedge sv \le \bigwedge_{k=1}^{l} \min\{qs_k, rt_k\}\, y_k$. But it is easy to see that $\min_{k=1\ldots l} \min\{qs_k, rt_k\} = 1$. Consequently $q(x,y) \vee s(u,v) \in T_{\min}$ and $T_{\min}$ is an pseudo inverse $\mathbb{B}$-convex set. (c) is an immediate consequence of (b) since an pseudo inverse $\mathbb{B}$ convex set is a lower semilattice setting $q = r = 1$. □

Proof of Proposition 6.1.1: By definition the inverse map ϕ is defined on $\mathbb{R}^l_{++}$ by $\phi(x) = (\varphi(x_1),\ldots,\varphi(x_l))$ where $\varphi(x_k) = \frac{1}{x_k}$ for each k. However, for all $i \in [m]$, we have

$$\min_{k\in[l]} a_i^k x_k = b_i \iff \varphi\big(\max_{k\in[l]} \phi(a_i^k)\,\phi(x_k)\big) = b_i.$$

The reciprocal of φ is φ itself. Hence, the problem of finding an optimal solution can the be converted to solve a system of maximum equations defined by

$$\max_{k\in[l]} \left\{\phi(a_i^k)\,\phi(x_k)\right\} = \varphi(b_i) \quad i \in [m].$$

Consequently, since the reciprocal of ϕ is also ϕ itself, we deduce from (6.4)

$$\mathcal{Q} \neq \emptyset \iff \bigwedge_{i=1...m} \varphi(b_i)a_i \in \phi\Big(\mathcal{Q}\Big),$$

which yields the result. □

Proof of Proposition 6.2.1: Assume that $\mathcal{IS} \cap \mathcal{IS}' \neq \emptyset$. If $\bar{x} \in \mathcal{IS} \cap \mathcal{IS}'$, then $\bar{x} \in \mathcal{IS}$. But from equation (6.4) the vector $\bigwedge_{i=1...m} b_i\phi(a_i)$ satisfies conditions for being in $\mathcal{IS}$. Moreover, it is maximal in $\mathcal{IS}$. Consequently, we have $\bar{x} \leq \bigwedge_{i=1...m} b_i\phi(a_i)$. Since the maximum function is non decreasing and $\bar{x} \in \mathcal{IS}'$, we deduce that $\bigwedge_{i=1...m} b_i\phi(a_i) \in \mathcal{IS}'$. Therefore $\bigwedge_{i=1...m} b_i\phi(a_i) \in \mathcal{IS} \cap \mathcal{IS}'$, and the first part of the equivalence is proven. Since the converse is immediate the result is stated. □

Proof of Proposition 6.2.2: The proof is obtained from Proposition 6.2.1 making an elementary change in the variables. □

Proof of Proposition 6.3.1: (a) Let us consider the inequations system:

$$\begin{cases} \bigvee_{k=1...l} t_k x_k & \leq & \lambda x_{\bar{k}} \\ \bigvee_{k=1...l} t_k y_k & \geq & y_{\bar{k}} \\ \max_{k=1...l} t_k & = & 1 \qquad t \geq 0 \end{cases} \tag{8.1}$$

It can be rewritten:

$$\begin{cases} \bigvee_{t=1...l} t_k x_k & \leq & \lambda x_{\bar{k}} \\ \max_{k=1...l} t_k & \leq & 1 \\ \bigvee_{k=1...l} t_k y_k & \geq & y_{\bar{k}} \\ \max_{k=1...l} t_k & \geq & 1 \qquad t \geq 0 \end{cases} \tag{8.2}$$

From Proposition 6.2.1 the solution set S set of the system (8.1) is nonempty if and only if $\bigwedge_{i=1...m} \lambda x_{\bar{k},i}\phi({}^t x_i) \wedge 1^m$ is solution, where the ${}^t x_i$'s denote the vectors ${}^t x_i = (x_{i,1}, ..., x_{i,l})$ and 1^l is a l-dimensional vector with all component equal to one. Therefore, since $A \subset \mathbb{R}^d_{++}$ and from the definition of $T_{\max}$ we need to solve the minimization program:

$$\begin{aligned} &\inf \lambda \\ &st \max_{k=1...l} \left\{ x_{k,i} \min\left\{ \min_{i=1...m} \lambda \frac{x_{\bar{k},i}}{x_{k,i}}, 1\right\}\right\} \leq \lambda x_{\bar{k},i} \ i = 1...m \\ &\max_{k=1...l} \left\{ y_{k,j} \min\left\{ \min_{i=1...m} \lambda \frac{x_{\bar{k},i}}{x_{k,i}}, 1\right\}\right\} \geq y_{\bar{k},j} \ j = 1...n \\ &\max_{k=1...l} \left\{ \min\left\{ \min_{i=1...m} \lambda \frac{x_{\bar{k},i}}{x_{k,i}}, 1\right\}\right\} = 1 \end{aligned} \tag{8.3}$$

This yields:

$$
\begin{aligned}
&\inf \lambda \\
&st. \max_{k=1...l} \left\{ x_{k,i} \min \left\{ \min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}, \frac{1}{\lambda} \right\} \right\} \leq x_{\bar{k},i} \ i = 1...m \\
&\max_{k=1...l} \left\{ y_{k,j} \min \left\{ \lambda \min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}, 1 \right\} \right\} \geq y_{\bar{k},j} \ j = 1...n \\
&\max_{k=1...l} \left\{ \min \left\{ \lambda \min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}, 1 \right\} \right\} = 1
\end{aligned}
\tag{8.4}
$$

From the notations above, this program can be rewritten:

$$
\begin{aligned}
&\inf \lambda \\
&st. \max_{k=1...l} \left\{ x_{k,i} \min \left\{ \alpha_{\bar{k},k}, \frac{1}{\lambda} \right\} \right\} \leq x_{\bar{k},i} \ i = 1...m \\
&\max_{k=1...l} \left\{ y_{k,j} \min \left\{ \lambda \alpha_{\bar{k},k}, 1 \right\} \right\} \geq y_{\bar{k},j} \ j = 1...n \\
&\max_{k=1...l} \left\{ \min \left\{ \lambda \alpha_{\bar{k},k}, 1 \right\} \right\} = 1
\end{aligned}
\tag{8.5}
$$

Equivalently:

$$
\begin{aligned}
&\inf \lambda \\
&st \max_{k=1...l} \left\{ \min \left\{ x_{k,i} \alpha_{\bar{k},k}, \frac{1}{\lambda} x_{k,i} \right\} \right\} \leq x_{\bar{k},i} \ i = 1...m \\
&\max_{k=1...l} \left\{ \min \left\{ \lambda y_{k,j} \alpha_{\bar{k},k}, y_{k,j} \right\} \right\} \geq y_{\bar{k},j} \ j = 1...n \\
&\max_{k=1...l} \left\{ \min \left\{ \lambda \alpha_{\bar{k},k}, 1 \right\} \right\} = 1
\end{aligned}
\tag{8.6}
$$

To solve this optimization program, we first mention the following intermediary properties. For all real numbers $\beta_1, \beta_2, b > 0$, we have:

(i) $\inf \left\{ \lambda \geq 0 : \min \left\{ \beta_1, \beta_2 \frac{1}{\lambda} \right\} \leq b \right\} = \begin{cases} 0 & b \geq \beta_1 \\ \frac{\beta_2}{b} & b < \beta_1 \end{cases}$

(ii) $\inf \left\{ \lambda \geq 0 : \min \left\{ \beta_1 \lambda, \beta_2 \right\} \geq b \right\} = \begin{cases} +\infty & \text{if} \quad b > \beta_2 \\ \frac{b}{\beta_1} & \text{if} \quad b \leq \beta_2 \end{cases}$

(iii) $\inf \left\{ \lambda \geq 0 : \min \left\{ \beta_1 \lambda, \beta_2 \right\} = b \right\} = \begin{cases} +\infty & \text{if} \quad b > \beta_2 \\ \frac{b}{\beta_1} & \text{if} \quad b \leq \beta_2 \end{cases}$

Since, by definition $x_{\bar{k},i} \geq \alpha_{\bar{k},k} x_{k,i}$, it follows that only the constraints $j = 1...n$ and the last constraint are active. Moreover, by definition, the condition $y_{\bar{k},j} \leq y_{k,j}$ implies that $\min \left\{ \lambda : \min \left\{ \lambda y_{k,j} \alpha_{\bar{k},k}, y_{k,j} \right\} \geq y_{\bar{k},j} \right\} = \frac{y_{\bar{k},j}}{\alpha_{\bar{k}<k} y_{k,j}} \ \forall k, j$. Moreover $\min \left\{ \lambda : \min \left\{ \alpha_{\bar{k},k} \lambda, 1 \right\} = 1 \right\} = \frac{1}{\alpha_{\bar{k},k}}$. We deduce:

$$E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\max}) = \max\Big\{ \max_{j=1...n} \min_{\substack{k \\ y_{\bar{k},j} \le y_{k,j}}} \{\frac{y_{\bar{k},j}}{y_{k,j}\alpha_{\bar{k},k}}\}, \min_k \frac{1}{\alpha_{\bar{k},k}}\Big\}.$$

(b) To compute the Farrell output measure we need to solve the maximization program:

$$\begin{aligned} &\sup \theta \\ &st. \max_{j=1...p} \Big\{x_{k,i} \min\{\min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}, 1\}\Big\} \le x_{\bar{k},i} \ i = 1...m \\ &\max_{k=1...l} \Big\{y_{k,j} \min\{\min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}, 1\}\Big\} \ge \theta y_{\bar{k},j} \ j = 1...n \\ &\max_{k=1...l} \Big\{ \min\{\min_{i=1...m} \frac{x_{\bar{k},i}}{x_{k,i}}, 1\}\Big\} = 1 \end{aligned} \tag{8.7}$$

This yields:

$$\begin{aligned} &\sup \theta \\ &st. \max_{k=1...l} \Big\{x_{k,i} \min\{\alpha_{\bar{k},k}, 1\}\Big\} \le x_{\bar{k},i} \ i = 1...m \\ &\max_{k=1...l} \Big\{y_{k,j} \frac{1}{\theta} \min\{\alpha_{\bar{k},k}, 1\}\Big\} \ge y_{\bar{k},j} \ j = 1...n \\ &\max_{k=1...l} \Big\{ \min\{\alpha_{\bar{k},k}, 1\}\Big\} = 1 \end{aligned} \tag{8.8}$$

Since, by definition $x_{\bar{k},i} \ge \alpha_{\bar{k},k} x_{k,i}$, it follows that the constraints $i = 1...m$ hold. Moreover, the last constraint holds also (just take $k = \bar{k}$). Now, the last constraint is active. Moreover, $\max\{\theta : \frac{1}{\theta} y_{k,j} \min\{\alpha_{\bar{k},k}, 1\} \ge y_{\bar{k},j}\} = \frac{y_{k,j} \min\{\alpha_{\bar{k},k}, 1\}}{y_{\bar{k},j}}$ for $j = 1...n$. We deduce that:

$$E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\max}) = \min_{j=1...n} \max_k \Big\{\frac{y_{k,j} \min\{\alpha_{\bar{k},k}, 1\}}{y_{\bar{k},j}}\Big\}. \square$$

Proof of Proposition 6.4.1: (a) The minimization program one should solve is:

$$\begin{aligned} &\inf \lambda \\ &st. \bigwedge_{k=1...l} s_k x_k \le \lambda x_{\bar{k}} \\ &\bigwedge_{k=1...l} s_k y_k \ge y_{\bar{k}} \\ &\min_{k=1...l} s_k = 1 \end{aligned} \tag{8.9}$$

This can be rewritten:

$$\begin{aligned} & \inf \lambda \\ & st.\, \phi\left(\bigvee_{k=1...l} \varphi(s_k)\phi(x_k)\right) \leq \lambda x_{\bar{k}} \\ & \phi\left(\bigvee_{k=1...l} \varphi(s_k)\phi(y_k)\right) \geq y_{\bar{k}} \\ & \varphi\left(\max_{k=1...l} \varphi(s_k)\right) = 1 \end{aligned} \tag{8.10}$$

or equivalently, setting $t_k = \varphi(s_k)$ for $k = 1...l$ and $\theta = \varphi(\lambda)$:

$$\begin{aligned} & \max \lambda \\ & st. \bigvee_{k=1...l} t_k \phi(x_k) \geq \theta \phi(x_{\bar{k}}) \\ & \bigvee_{k=1...l} t_k \phi(y_k) \leq \phi(y_{\bar{k}}) \\ & \max_{k=1...l} t_k = 1, t \geq 0 \end{aligned} \tag{8.11}$$

Now if $\theta*$ is solution of the program above, then we have $E^i(x_{\bar{k}}, y_{\bar{k}}, T_{\min}) = \varphi(\theta*)$. Hence, using the results in Proposition 6.3.1, we obtain:

$$E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\min}) = \left[\min_{i=1...m} \max_k \left\{\frac{x_{\bar{k},i} \min\{\beta_{\bar{k},k}, 1\}}{x_{k,i}}\right\}\right]^{-1}$$

(b) Using a similar approach, the Farrell output measure can be calculated using Proposition 6.3.1.b, and we obtain:

$$E_o(x_{\bar{k}}, y_{\bar{k}}, T_{\min}) = \left[\max\left\{\max_{i=1...m} \min_{\substack{k \\ x_{k,i} \leq x_{\bar{k},i}}} \{\frac{x_{k,i}}{x_{\bar{k},i}\beta_{\bar{k},k}}\}, \min_k \frac{1}{\beta_{\bar{k},k}}\right\}\right]^{-1} . \square$$

Proof of Lemma 6.4.2: Let us fix $\lambda_{\max} = E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\max})$ and $\lambda_{\min} = E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\min})$. From the strong disposability assumption $\{(\lambda x_{\bar{k}}, y_{\bar{k}}) : \lambda \geq \lambda_{\max}\} \subset T_{\max}$ and $\{(\lambda x_{\bar{k}}, y_{\bar{k}}) : \lambda \geq \lambda_{\min}\} \subset T_{\min}$. Consequently, their intersection is a subset of T_{mix}. It follows that $\{(\lambda x_{\bar{k}}, y_{\bar{k}}) : \lambda \geq \max\{\lambda_{\max}, \lambda_{\min}\}\} \subset T_{\text{mix}}$. Hence $E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\text{mix}}) \leq \max\{E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\max}), E_i(x_{\bar{k}}, y_{\bar{k}}, T_{\min})\}$. Moreover, since $T_{\text{mix}} \subset T_{\max}$ and $T_{\text{mix}} \subset T_{\min}$, the converse inequality is immediate, which yields the result. The proof of the second statement is similar. $\square$