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The Free Rider Problem: a Dynamic Analysis*
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Abstract

We present a dynamic model of free riding in which n infinitely lived agents choose between private consumption and contributions to a durable public good g. We characterize the set of continuous Markov equilibria in economies with reversibility, where investments can be positive or negative; and in economies with irreversibility, where investments are non negative and g can only be reduced by depreciation. With reversibility, there is a continuum of equilibrium steady states: the highest equilibrium steady state of g is increasing in n, and the lowest is decreasing. With irreversibility, the set of equilibrium steady states converges to the highest steady state possible with reversibility, as depreciation converges to zero. We also show that in economies with reversibility there are always non-monotonic equilibria in which g converges to the steady state with damped oscillations; and there can be equilibria with persistent limit cycles.

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1 Introduction

The most significant economic examples of free rider problems (pollution, public goods and common pools, for example) are characterized by two key features. First, they concern a large number of agents who act independently and anonymously. Second, they have an important dynamic component since what matters to the agents is the stock of the individual contributions accumulated over time. These two features are particularly evident in environmental problems where individual contributions (the levels of pollution) are infinitesimal and mostly anonymous; and where the state of nature slowly evolves like a capital good.

There is a large literature studying Nash equilibria in static environments that has explored the first of these two features. This literature has formed most of the current understanding of free rider problems in economics and other social sciences. There is, however, a much more limited understanding of dynamic free rider problems. A number of important questions, therefore, still need to be fully answered. What determines the steady states of these problems and their welfare properties? How do we converge to the steady state? Do we always monotonically converge as in a planner's first best, or can we have endogenous cycles in the investments?

In this paper, we present a simple model of free riding to address the questions presented above. In the model, n infinitely lived agents allocate their income between private consumption and contributions to a public good g in every period. The public good is durable, so in period t we have $g_t = (1-d)g_{t-1} + \sum i_j^t$, where i_j^t is the contribution at time t of agent j, and d is the rate of depreciation. We consider two scenarios. First, we study economies with *reversibility*, in which in every period individual investments can either be positive or negative. Second, we study an economy where the investment is *irreversible*, so individual investments are non-negative and the public good can only be reduced by depreciation: $g_t \geq (1-d)g_{t-1}$.

We characterize the set of symmetric Markov equilibria in which strategies are continuous and depend only the payoff-relevant state variable g. The focus on continuous strategies is a robustness requirement: strategies that are not continuous are sensitive to small perturbations in the state, requiring an exact measurement of g that may seem unrealistic in most environments. The Markov restriction is natural in dynamic games like this with symmetric, anonymous (and possibly many) agents. We refer to these equilibria as *well behaved*.

To understand the results, it is useful to start from the static version of our free rider game. The game has a unique symmetric equilibrium in which g is independent of the number of the agents, and in which all agents contribute equally to its cost. The aggregate level of g, indeed, is equal to the level that each agent would choose alone in autarky. The agents' actions, moreover, are perfect strategic substitutes: the reaction function of each agent is decreasing in the others' contributions, so if we force an agent to invest more, all the other agents reduce their contribution by the same amount in the aggregate.

In the dynamic game, the results and the strategic interaction are very different. Consider first the economies with reversibility. We show that in these economies there is a continuum of well behaved equilibria, each characterized by a different stable steady state g_s. Most of the existing literature has focused (explicitly or implicitly) on monotonic equilibria, that is equilibria where the investment function is non decreasing in the state g. If we focus on this class, we can show that a point g_s is a stable steady state if and only if it is in the set:

$$\left[[u']^{-1}\left(1-\delta\left(1-d\right)/n\right),\ [u']^{-1}\left(1-\delta\left(1-d/n\right)\right)\right], \tag{1}$$

where $[u']^{-1}$ is the inverse of the first derivative of the utility from g, and δ is the discount factor. Non-monotonic equilibria, however, exist as well. When we extend the analysis to non monotonic equilibria, the upper bound of the set of feasible steady states remains the same, but the lower bound can be as low as $[u']^{-1}\left(1+\frac{\delta}{n}\left(n+d-2\right)\right)$. To interpret these thresholds, note that $[u']^{-1}\left(1-\delta\left(1-d\right)\right)$ is the level of g that would be accumulated by an agent alone in autarky. In the well-behaved equilibrium with the highest stable steady state, therefore, the steady state is increasing in the number of agents n, and higher than the steady state that would be reached in autarky by a single agent.[1] In the equilibrium with the lowest steady state, the steady state is decreasing in n and it is lower than the level that would be reached in autarky by a single agent. As the number of agents grows to infinity, moreover, (1) converges to $\left[[u']^{-1}\left(1\right), [u']^{-1}\left(1-\delta\right)\right]$: the multiplicity survives even in arbitrarily large economies.

Consider now the economies with irreversibility. Although there may be environments in which public investments can be described as being reversible (a museum or an art collection, for example), the most significant economic examples of dynamic free rider problems seem to be associated with irreversible investments: investments in physical capital like a bridge, a dam, or in less tangible assets like "social capital", or "state capacity" (see Besley and Persson [2010]) often cannot be easily reduced once they have been accumulated, and certainly cannot be transformed back into private consumption. In an economy with irreversibility the set of equilibrium steady states is much smaller than with reversibility. Indeed, we show that as the rate of depreciation converges to zero, this set converges to the upperbound of (1). This result may appear surprising. In a planner's solution the irreversibility constraint is irrelevant: it affects neither the steady state (that is unique), nor the convergence path.[2] All steady states in (1), moreover, are supported by

[1] This follows from (1) and the fact that, since we assume that $u(g)$ is concave, $[u']^{-1}(x)$ is decreasing in x.

[2] This of course is true if the initial state g_0 is smaller than the steady state, an assumption that we will maintain throughout this paper for simplicity of exposition.

equilibria with investment functions that are monotonically increasing in g. On the convergence path, therefore, investment is never reduced: it keeps increasing until the steady state is reached, and then it stops; the irreversibility constraint is, thus, never binding on the equilibrium path.

The reason why irreversibility is so important in a dynamic free rider game is precisely the fact that investments are inefficient. The intuition is as follows. In the equilibria with reversibility, the agents hold down their investments for fear that the other agents will appropriate part of it in the following periods. At some point the equilibrium investment function falls so low that the irreversibility constraint is binding. Even if this happens out of equilibrium, this affects the entire equilibrium investment function. In states just below the point in which the constraint is binding, the agents know that the constraint will not allow the other agents to reduce g when g passes the threshold. These incentives induce higher investments and a higher value function, with a ripple effect on the entire investment function on the equilibrium path. When depreciation is sufficiently small, this effect "forces" the agents to cooperate and induces a unique stable steady state as depreciation converges to zero. Despite the practical importance of the case with irreversibility, to our knowledge this paper presents the first clear characterization of equilibrium behavior in dynamic economies with a free riding problem and irreversibility.

For all the steady states described above and equilibria, we derive the associated convergence path and characterize a simple sufficient conditions that guarantees its uniqueness. When the equilibrium is monotonic, convergence is "standard:" the state gradually increases until the steady state is reached. As we have anticipated before, however, non-monotonic equilibria always exist. In non-monotonic equilibria convergence dynamics may be surprisingly complex. We show that there always are non-monotonic equilibria in which the states converges to the steady state with *damped oscillations*. We also show a sufficient condition that guarantees the existence of equilibria where the state converges to a *persistent cycle*. We construct an equilibrium in which the investment function can be solved in closed form in which the cycle has a 2-period orbit. To our knowledge this is the first paper to show the existence of endogenous cycles in a dynamic free rider game.

The reminder of the paper is organized as follows. Section 2 describes the model, and describes the benchmark case in which the public investment is chosen by a benevolent planner. Section 3 studies the well behaved equilibria in economies with reversibility. Section 4 studies the well behaved equilibria in economies with irreversibility. In Section 5 we discuss non-monotonic equilibria and cycles in g. In Section 6 we discuss preliminary evidence from a companion paper in which we examine some of the predictions of the theory in a laboratory experiment. Section 7 concludes.

1.1 Related literature

There is a long tradition of research on static free rider problems started by Samuelson (1954) and Olson [1965], and further developed in a large literature. Bergstrom, Blume and Varian [1986] present an elegant characterization of the problem, identifying general conditions for existence and uniqueness of the equilibrium, and characterizing its comparative statics.

Levhari and Mirman [1980] and Fershtman and Nitzan [1991] are early works studying Markov equilibria in dynamic free rider problems. Levhari and Mirman [1980] present a closed form characterization of a Markov equilibrium in a common pool problem in a discrete time setting. Fershtman and Nitzan [1991] characterize a Markov equilibrium in a voluntary public good contribution game similar to the game in our paper. They, however, focus on equilibria in linear strategies in a differential game with linear-quadratic preferences. These two papers are among the first to show the implications of free riding on the steady state of the economy (in terms of overexploitation of the pool in the first case, and of a lower public good accumulation in the second case). In addition, Fershtman and Nitzan [1991] is, to our knowledge, the first work to construct an equilibrium in which the steady state g is lower in a community with n agents than the level that would be chosen by an agent alone in autarky. None of these two seminal papers, however, present a full analysis of the set of Markov equilibria. Extending Fershtman and Nitzan's analysis, Wirl [1996] makes the important observation that multiple equilibria may exist, and that the linear equilibrium selected in Fershtman and Nitzan [1996] is indeed the worst possible in terms of welfare. Wirl [1996], however, who also restricts his analysis to a differential linear quadratic case, does not explicitly solve for equilibrium strategies, making it impossible to draw clear conclusion on the properties of the equilibria and the steady states. A number of papers following this literature have continued using the linear quadratic differential environment.[3] Little work has been done to extend the analysis beyond the differential case to environments with discrete time, and to characterize Markov equilibria with more general utility functions.[4]

Our paper contributes to this literature by studying the dynamic free riding problem in a discrete time setting with general utilities, and by presenting a sharp characterization of the equilibria. By doing this, we are able to clarify the conditions under which the *dynamic strategic substitutability effect* identified by Fershtman and Nitzan [1991] occurs, showing it is a general phenomenon that does not occur only in a linear equilibrium of a linear quadratic model. We are also able to identify conditions under which a novel *dynamic strategic complementarity effect*

[3] See, for example, Itaya and Shimomura [2001], Yanase [2006], Fujiwara and Matsueda [2009].

[4] Work has been done to study non-Markov equilibria. See, for example, Dutta and Sundaram [1993], and Gaitsgory an Nitzan [1994]. Gaitsgory an Nitzan [1994] present a folk theorem under the overtaking criterion that can be applied to the setting of Fershtman and Nitzan [1991].

arises, and provide a clean characterization of the extent to which it can alleviate the free rider problem.

None of the papers mentioned above have studied economies with irreversible investments. An analysis of a dynamic game of voluntary contributions with irreversibility is presented by Marx and Matthews [2000]. As in our paper, in this model n agents privately contribute to a public good and the good can be accumulated over time. They assume an environment with no depreciation and in which the agents are not allowed to reduce the stock of past investments (as in our irreversible economy case). There are however two substantial differences. First, the authors assume a different class of utilities: the public good provides a linear utility until a certain threshold is reached, and a premium if the threshold is reached. Second, the authors do not focus on Markov equilibria, but on the best sub-game perfect equilibrium as the discount factor converges to one. Because of these differences, Marx and Matthews [2000]'s paper has a different focus than our paper. Marx and Matthews [2000] are interested in characterizing conditions under which the public projects will be completed (i.e. they reach the threshold) for high levels of the discount factor. In our paper, we do not have a threshold to reach to complete the project, we are not interested in making assumptions on the discount factor, we assume a general non linear utility for g, and we focus on endogenous steady states. We see our paper as complementary to theirs.

A number of recent papers have studied models of dynamic public good contributions related to ours. In recent work, Dutta and Radner [2004] and Harstad [2011] present dynamic models of pollution in which agents can invest in pollution reducing technology. With respect to our work, these papers focus on environments and equilibria with simple dynamics, in which the agents' actions are constant after the first period. Harstad [2011] uses his model to study alternative contractual arrangements among the agents with different degrees of commitment.

Battaglini and Coate [2007] present a model in which a legislature chooses a public durable investment. As in this paper, the public investment is a capital good that can be accumulated and that depreciates slowly. Differently from this paper, the level of public investment is chosen through a process of non-cooperative bargaining. Related models are developed by Besley and Persson [2010] and Besley, Ilzetzki and Persson [2011] and applied to study the accumulation of what they call "Fiscal Capacity," i.e. economic institutions for tax compliance; and by Battaglini, Nunnari, and Palfrey [2011a] and [2011b] who also present experimental evidence on equilibrium behavior.[5]

[5] Other related papers are Battaglini and Coate [2008], Azzimonti, Battaglini and Coate [2011] and Barseghyan, Battaglini and Coate [2011], where the legislature can issue debt, and debt is the state variable.

All of these papers restrict their analysis to environments with reversibility.[6] Irreversibility seems a relevant and important feature for the type of dynamic public good studied in these papers, especially in less tangible assets like "fiscal capacity" or any other form of "social capital". We are confident that insights developed in our paper on irreversible economies can help understanding public investments even in these alternative models of public decision making in future research.

None of the papers cited above studies non-monotonic equilibria. To our knowledge this is the first paper to show the existence of non-monotonic equilibria in which convergence to the steady state occurs with damped oscillations, and of equilibria with persistent cycles in a dynamic game of free riding.

2 The model

Consider an economy with n agents. There are two goods: a private good x and a public good g. The level of consumption of the private good by agent i in period t is x_t^i, the level of the public good in period t is g_t. An allocation is an infinite nonnegative sequence $z = (x_\infty, g_\infty)$ where $x_\infty = (x_1^1, ..., x_1^n, ..., x_t^1, ..., x_t^n, ...)$ and $g_\infty = (g_1, ..., g_t, ...)$. We refer to $z_t = (x_t, g_t)$ as the allocation in period t. The utility U^j of agent j is a function of $z^j = (x_\infty^j, g_\infty)$, where $x_\infty^j = (x_1^j, ..., x_t^j, ...)$. We assume that U^j can be written as:

$$U^j(z^j) = \sum_{t=1}^{\infty} \delta^{t-1} \left[x_t^j + u(g_t) \right],$$

where $u(\cdot)$ is continuously twice differentiable, strictly increasing, and strictly concave on $[0, \infty)$, with $\lim_{g \to 0^+} u'(g) = \infty$ and $\lim_{g \to +\infty} u'(g) = 0$. The future is discounted at a rate δ.

There is a linear technology by which the private good can be used to produce public good, with a marginal rate of transformation $p = 1$. The private consumption good is nondurable, the public good is durable, and the stock of the public good depreciates at a rate $d \in [0, 1]$ between periods. Thus, if the level of public good at time $t-1$ is g_{t-1} and the total investment in the public good is I_t, then the level of public good at time t will be

$$g_t = (1-d)g_{t-1} + I_t.$$

We consider two alternative economic environments. In a *Reversible Investment Economy*

[6] An exception is Battaglini et al. [2011a] that presents an experimental study based on the theoretical model of this paper.

(RIE) the public policy in period t is required to satisfy three feasibility conditions:

$$\begin{aligned} x_t^j &\geq 0 \ \ \forall j, \ \forall t \\ I_t + (1-d)g_{t-1} &\geq 0 \ \ \forall t \\ I_t + \sum_{j=1}^{n} x_t^j &\leq W_t \ \ \forall t \end{aligned}$$

where W_t is the aggregate level of resources in the economy at time t. The first two conditions guarantee that allocations are nonnegative. The third condition requires that the current economy-wide budget is balanced. In an *Irreversible Investment Economy* (IIE), the second condition is substituted with:

$$I_t \geq 0 \ \ \forall t$$

The RIE corresponds to a situation in which the public investment can be scaled back in the future at no cost. An example can be an art collection, land for common use, etc. The IIE corresponds to situations in which once the investment is done it cannot be undone. This seems the appropriate case for investments in public infrastructure (say a bridge or a road), or less tangible investments like "social capital."

It is convenient to distinguish the state variable at t, g_{t-1}, from the policy choice g_t and to reformulate the budget condition. If we denote $y_t = (1-d)g_{t-1} + I_t$ as the new level of public good after an investment I_t when the last period's level of the public good is g_{t-1}, then the public policy in period t can be represented by a vector $(y_t, x_t^1, ..., x_t^n)$. Substituting y_t, the budget balance constraint $I_t + \sum_{j=1}^{n} x_t^j \leq W_t$ can be rewritten as:

$$\sum_{j=1}^{n} x_t^j + [y_t - (1-d)g_{t-1}] \leq W_t,$$

With this notation, we must have $x_t \geq 0, y_t \geq 0$ in a RIE, and $x_t \geq 0, y_t \geq (1-d)g_{t-1}$ in a IIE.

The initial stock of public good is $g_0 \geq 0$, exogenously given. Public policies are chosen as in the classic free rider problem. In period t, each agent j is endowed with $w_t^j = W/n$ units of private good, so $W_t = \sum_{i=1}^{n} w_t^j = W \ \forall t$. We assume that each agent has full property rights over a share of the endowment (W/n) and in each period chooses on its own how to allocate its endowment between an individual investment in the public good (which is shared by all agents) and private consumption, taking as given the strategies of the other agents. In a RIE, the level of individual investment can be negative, with the constraint that $i_t^j \in [-(1-d)g_t/n, W/n] \ \forall j$, where $i_t^j = W/n - x_t^j$ is the investment by agent j.[7] In a IIE, an agent's investment must satisfy

[7] This constraint guarantees that (out of equilibrium) the sum of reductions in g can not be larger than the stock

$\in [0, W/n]\ \forall j$. The total economy-wide investment in the public good in any period is then given by the sum of the agent investments.

The key difference with respect to the static free rider problem is that the public good can be accumulated over time. The level of the state variable g, therefore, creates a dynamic linkage across policy making periods.

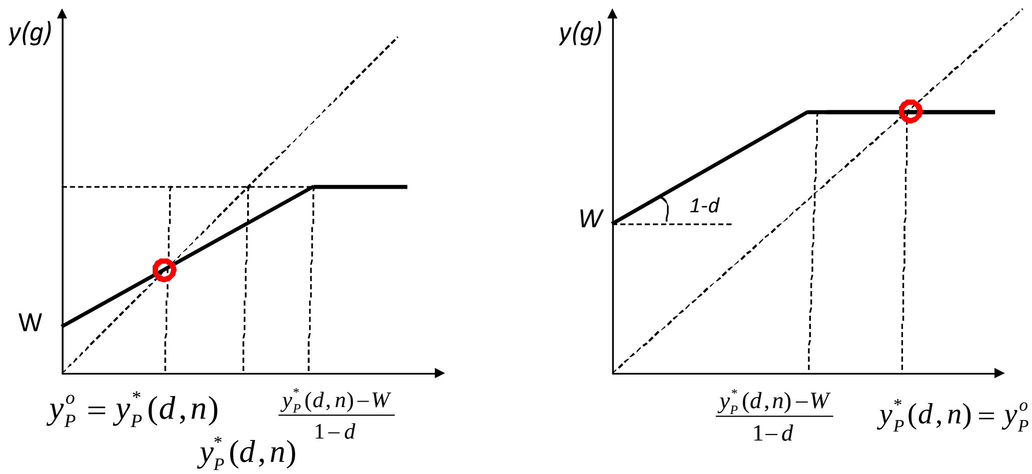
To study the properties of the dynamic free rider problem described above, we focus on symmetric Markov-perfect equilibria, where all agents use the same strategy, and these strategies are time-independent functions of the state, g. A strategy is a pair $(x(\cdot), i(\cdot))$: where $x(g)$ is an agent's level of consumption and $i(g)$ is an agent's level of investment in the public good in state g. Given these strategies, by symmetry, the public good in state g is $y(g) = (1-d)g + ni(g)$. Associated with any equilibrium is a value function, $v(g)$, which specifies the expected discounted future payoff to an agent when the state is g.

The focus on Markov equilibria seems particularly appropriate for this class of dynamic games. Free rider problems are typically intended to represent situations in which a large number of agents autonomously and independently contribute to a public good. In a large economy, it seems appropriate to require the equilibrium to be anonymous and independent from the action of any single agent. The Markov perfect equilibrium respects this property, by making strategies contingent only on the key economic state. An equilibrium is continuous if $y(g)$ and $v(g)$ are continuous in g; it is monotonic if $y(g)$ is non decreasing in g. We say that an equilibrium is *well-behaved* if it is continuous. In the reminder we will focus on well behaved equilibria. In Section 3 and 4 we first study monotonic well behaved equilibria; we then extend the analysis to non-monotonic equilibria in Section 5. We will therefore refer to a well behaved equilibrium simply as "equilibrium."

2.1 The planner's problem

As a benchmark with which to compare the equilibrium allocations, we first analyze the sequence of public policies that would be chosen by a benevolent planner who maximizes the sum of utilities of the agents. This is the welfare optimum because the private good enters linearly in each agent's utility function. Consider first an economy with reversible investment. The planner's problem has a recursive representation in which g is the state variable, and $v_P(g)$, the planner's value

of g. The analysis would be similar if we allow each agent to withdraw up to $(1-d)g$. In this case, however, we would have to assume a rationing rule in case the individuals withdraw more than $(1-d)g$. Withdrawing all g is never optimal at the individual level, since the marginal utility of the public good at 0 is infinite. When agents can withdraw without limits, however, we may have a spurious equilibrium in which agents adopt a stage dominated strategy of withdrawing all g: because the sum of the demands of all agents minus one is higher than $(1-d)g$, no agent's is pivotal in determining g and so it is weakly optimal to withdraw all. The constraint eliminates this uninteresting case.



Case 1: $W/d \leq y_P^*(d,n)$ **Case 2**: $W/d > y_P^*(d,n)$

Figure 1: The Planner's Problem

function can be represented recursively as:

$$v_P(g) = \max_{y,x} \left\{ \begin{array}{c} \sum_{j=1}^n x^i + nu(y) + \delta v_P(y) \\ s.t \ \ \sum_{j=1}^n x^i + y - (1-d)g \leq W, \ x^i \geq 0 \ \forall i, \ y \geq 0 \end{array} \right\} \tag{2}$$

By standard methods (see Stokey and Lucas [1989]), we can show that a continuous, strictly concave and differentiable $v_P(g)$ that satisfies (2) exists and is unique. The optimal policies have an intuitive characterization. When the accumulated level of public good is low, the marginal benefit of investing in g is high, and the planner finds it optimal to invest as much as possible: in this case $y_P(g) = W + (1-d)g$ and $\sum_{j=1}^n x^i = 0$. When g is high, the planner will be able to reach the level of public good $y_P^*(d,n)$ that solves the planner's unconstrained problem: i.e.

$$nu'(y_P^*(d,n)) + \delta v_P'(y_P^*(d,n)) = 1. \tag{3}$$

Applying the envelope theorem, we can show that at the interior solution $y_P^*(d,n)$ we have $v_P'(y_P^*(d,n)) = 1-d$. From (3), we therefore conclude that:

$$y_P^*(d,n) = [u']^{-1}\left(\frac{1-\delta(1-d)}{n}\right) \tag{4}$$

The investment function, therefore, has the following simple structure. For $g < \frac{y_P^*(d,n)-W}{1-d}$, $y_P^*(d,n)$ is not feasible: the planner invests everything and $y_P(g) = (1-d)g + W$. For $g \geq$

$\frac{y_P^*(d,n)-W}{1-d}$, instead, investment stops at $y_P(g) = y_P^*(d,n)$. In this case, without loss of generality, we can set $x^i(g) = (W + (1-d)g - y(g))/n \ \forall i$.[8] Summarizing, we have:

$$y_P(g) = \min\{W + (1-d)g, y_P^*(d,n)\}. \tag{5}$$

This investment function implies that the planner's economy converges to one of two possible steady states (see Figure 1). If $W/d \leq y_P^*(d,n)$, then the rate of depreciation is so high that the planner cannot reach $y_P^*(d,n)$, (except temporarily if the initial state is sufficiently large). In this case the steady state is $y_P^o = W/d$, and the planner invests all resources in all states on the equilibrium path (Figure 1, Case 1). If $W/d > y_P^*(d,n)$, $y_P^*(d,n)$ is sustainable as a steady state. In this case, in the steady state $y_P^o = y_P^*(d,n)$, and the (per agent) level of private consumption is positive: $x^* = (W + (1-d)g - y)/n > 0$ (Figure 1, Case 2).

An economy in which the planner's optimum can be feasibly sustained as a steady state is the most interesting case. With this in mind we define:

Definition 1. *An economy is said to be regular if* $W/d > y_P^*(d,n)$.

In the rest of the analysis we focus on regular economies.[9] This is done only for simplicity: extending the results presented below for economies with $W/d \leq y_P^*(d,n)$ can be done using the same techniques developed in this paper.

The planner's optimum for the IIE case is not very much different. The planner finds it optimal to invest all resources for $g \leq \frac{y_P^*(d,n)-W}{1-d}$. For $g \in \left(\frac{y_P^*(d,n)-W}{1-d}, \frac{y_P^*(d,n)}{1-d}\right)$, the planner finds it optimal to stop investing at $y_P^*(d,n)$, as before. For $g \geq \frac{y_P^*(d,n)}{1-d}$, $y_P^*(d,n)$ is not feasible, so it is optimal to invest 0, and to set $y_P(g) = (1-d)g$ This difference in the investment function for IIE, however, is essentially irrelevant for the optimal path and the steady state of the economy. Starting from any g_0 lower than the steady state y_P^*, levels of g larger or equal than $\frac{y_P^*(d,n)}{1-d}$ are impossible to reach, and the irreversibility constraint does not affect the optimal investment path.

3 Reversible investment economies

We first study equilibrium behavior when the investment in the public good is reversible. The optimization problem for agent j if the current level of public good is g, the agent's value function

[8] Indeed, the planner is indifferent regarding the distribution of private consumption.

[9] The case of $d = 0$ is also included as a regular economy.

is $v_R(g)$, and other agents' investment strategies are given by $x_R(g)$ can be represented as:

$$\max_{y,x} \left\{ \begin{array}{c} x + u(y) + \delta v_R(y) \\ \\ s.t \quad x + y - (1-d)g = W - (n-1)x_R(g) \\ \\ W - (n-1)x_R(g) + (1-d)g - y \geq 0 \\ \\ x \leq (1-d)g/n + W/n \end{array} \right\} \tag{6}$$

Agent j cannot choose y directly: it chooses only its level of private consumption and the level of its own contribution to the public investment. The agent, however, realizes that given the other agents' level of private consumption $(n-1)x_R(g)$, his/her investment ultimately determines y. It is therefore *as if* agent j chooses x and y, provided that he satisfies the feasibility constraints. The first constraint is the resource constraint: it requires that total resources, $W + (1-d)g$, are equal to the sum of private consumption, $(n-1)x_R(g) + x$, plus the public investment y. The second constraint requires that private consumption x is non negative. The third constraint requires that no agent can reduce y by more than his share $(1-d)g/n$.

In a symmetric equilibrium, all agents consume the same fraction of resources, so agent j can assume that in state g the other agents each consume:

$$x_R(g) = \frac{W + (1-d)g - y_R(g)}{n},$$

where $y_R(g)$ is the equilibrium level of investment in state g. Substituting the first constraint of (6) in the objective function, recognizing that agent j takes the strategies of the other agents as given, and ignoring irrelevant constants, the agent's problem can be written as:

$$\max_{y} \left\{ \begin{array}{c} u(y) - y + \delta v_R(y) \\ \\ y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y_R(g), \ y \geq \frac{n-1}{n} y_R(g) \end{array} \right\} \tag{7}$$

where it should be noted that agent j takes $y_R(g)$ as given.[10] The objective function shows that an agent has a clear trade off: a dollar in investment produces a marginal benefit $u'(y) + \delta v_R'(y)$, the marginal cost is -1, a dollar less in private consumption.[11] The first constraint shows that at the maximum the agent can increase the investment of the other players (i.e., $\frac{n-1}{n} y_R(g)$) by $\frac{W+(1-d)g}{n}$. The second constraint makes clear that at most the agent can eat his endowment W/n and his share of g, $(1-d)g/n$.

[10] Since $y_R(g)$ is the equilibrium level of investment, in a symmetric equilibrium $(n-1)y_R(g)/n$ is the level of investment that agent j expects from all the other agents, and that he/she takes as given in equilibrium.

[11] For simplicity of exposition we assume here that $v_R(g)$ is differentiable. Indeed, as we will see this is essentially without loss of generality since we will show that there is an equilibrium in which $v_R(g)$ is almost everywhere differentiable. We refer to the proofs in the appendix for the details.

A symmetric Markov equilibrium is therefore fully described in this environment by two functions: an aggregate investment function $y_R(g)$, and an associated value function $v_R(g)$. Two conditions must be satisfied. First, the level of investment must solve (7) given $v_R(g)$. The second condition for an equilibrium requires the value function $v_R(g)$ to be consistent with the agents strategies. Each agent receives the same benefit for the expected investment in the public good, and consumes the same share of the remaining resources, $(W + (1-d)g - y_R(g))/n$. This implies:

$$v_R(g) = \frac{W + (1-d)g - y_R(g)}{n} + u(y_R(g)) + \delta v_R(y_R(g)) \tag{8}$$

We can therefore define:

Definition 2. *An SME equilibrium in a dynamic free rider problem is a pair of functions, $y_R(\cdot)$ and $v_R(\cdot)$, such that for all $g \geq 0$, $y_R(g)$ solves (7) given the value function $v_R(\cdot)$; and for all $g \geq 0$, $v_R(g)$ solves (8) given $y_R(\cdot)$.*

For a given value function, if an equilibrium exists, the problem faced by an agent looks similar to the problem of the planner. There are two differences. First, in the objective function the agent does not internalize the effect of the public good on the other agents. This is the classic free rider problem, present in static models as well: it induces a suboptimal investment in g. The second difference with respect to the planner's problem is that the agent takes the contributions of the other agents as given. The incentives to invest depend on the agent's expectations about the other agents' behavior. This radically changes the nature of the equilibria. The more (or less) an agent (say agent i) expects the other agents to invest, the more (or less) i finds it optimal to invest. The relevant question is: Does this make the static free rider problem worse or better in a dynamic environment?

To answer this question, in this and in the next Sections, we first study the set of monotonic well behaved equilibria, that is equilibria in which $y(g)$ is non decreasing in g. This class of equilibria is quite important since all papers that have studied dynamic free rider problems have focused on particular monotonic equilibria (see for example Levhari and Mirman [1980] and Fershtman and Nitzan [1991], and more recently Harstad [2011]). Restricting the analysis on this class allows to focus on the key intuition without technical complications involved with non-monotonic equilibria. We extend the analysis to non-monotonic equilibria in Section 5.

As we show in Proposition 1, to study the set of possible steady states there is no loss of generality if we focus on concave equilibria. An equilibrium is said to be *concave* if $v(y;g)$ is weakly concave on y for any g, where $v(y;g)$ is the expected value of investing y in a state g:

$$v(y\,|g) = \frac{W + (1-d)g - y}{n} + u(y) + \delta v(y)$$

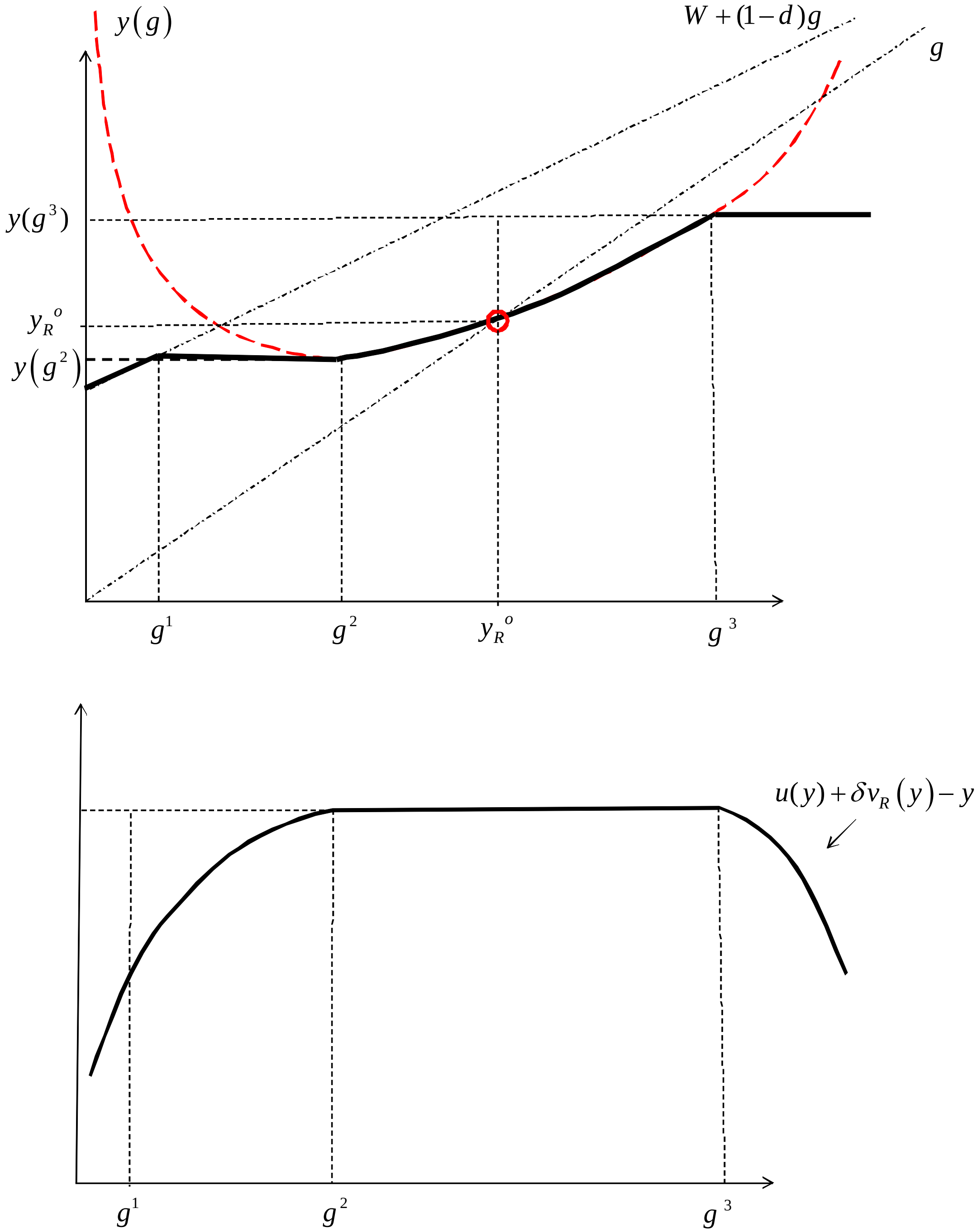


Figure 2: The equilibrium in an economy with reversibility

In a concave equilibrium, (7) is a standard concave programming problem similar to the planner's problem (2). The investment function, however, typically takes a more general form than the planner's solution (5). Figure 2 represents a typical equilibrium. The equilibrium investment function will generally take the following form:

$$y_R(g) = \begin{cases} \min\left\{W + (1-d)g, y\left(g^2\right)\right\} & g < g^2 \\ y\left(g\right) & g \in \left[g^2, g^3\right] \\ y\left(g^3\right) & g > g^3 \end{cases} \tag{9}$$

where g^2, g^3 are two critical levels g, and $y(g)$ is a function with values in $[g, W + (1-d)g]$. To see why $y_R(g)$ may take the form of (9), consider Figure 2. The top panel of the figure illustrates a canonical equilibrium investment function. The steady state is labeled y_R^o in the figure, the point at which the (bold) investment function intersects the (dotted) diagonal. The bottom panel of the figure graphs the corresponding objective function, $u(y) - y + \delta v_R(y)$. For $g < g^2$, the objective function of (7) is strictly increasing in y: either investing all resources is maximal (in Figure 2, $g \leq g^1$); or investment is at a level that maximizes the unconstrained objective function $u(y) + \delta v_R(y) - y$, i.e. corresponding to some $y\left(g^2\right) \in \left[g^2, g^3\right]$ (in Figure 2, $g^1 \leq g \leq g^2$). For $g > g^3$, the objective function is decreasing: the investment level is so high that the agents do not wish to increase g over $y(g^3)$. For intermediate levels of $g \in \left[g^2, g^3\right]$, an interior level of investment is chosen. This is possible because the objective function is flat in this region: an agent is indifferent between any state $g \in \left[g^2, g^3\right]$.

Of course, were $u(y) - y + \delta v_R(y)$ strictly concave (and hence $v\left(y|g\right)$ strictly concave), one could have an investment function that appears as a special case of the general form (9), but where $g^2 = g^3$. In this case, we would have an investment function looking similar to $y_R(g)$ in (5), but with a different (lower) steady state and the upward sloping curve, $y\left(g\right)$, would collapse down to a single point at g^2. Thus, the possibility of a "flat" part in the objective function (i.e., $g^2 \neq g^3$), is not a mere intellectual curiosity: without it, $y_R(g)$ could not increase in $\left[g^2, g^3\right]$ as illustrated in Figure 2 and a higher steady state $y_R^o > y\left(g^2\right)$ would not be feasible. As in the planner's problem, with a strictly concave objective function in (7) the agents would either find it optimal to invest everything, or just enough to maintain the steady state. As we show in the remaining of this section, considering weakly concave equilibria is indeed essential to characterize all the equilibrium steady states of the game.[12]

For an investment curve as in Figure 2 to be an equilibrium we need to satisfy two necessary

[12] The fact that focusing on strictly concave equilibria implies a loss of generality has been also noted by Azzimonti, Battaglini and Coate [2009] who show that a strictly concave equilibrium does not exists in a dynamic model of public debt with a balanced budget rule; a weakly concave equilibrium, instead, always exists.

conditions. First, agents must be indifferent between investing and consuming for all states in $\left[g^2, g^3\right]$. If this condition does not hold, the agents do not find it optimal to choose an interior level $y(g)$. Given a value function v, the marginal utility of investments is $u'(g) + \delta v'(g) - 1$ so, we require:

$$\delta v'(g) = 1 - u'(g) \ \forall g \in \left[g^2, g^3\right] \tag{10}$$

Since the expected value function is: $v(g) = \frac{W+(1-d)g-y(g)}{n} + u(y(g)) + \delta v(y(g))$. We have:

$$v'(g) = \frac{1-d-y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g) \tag{11}$$

The investment function $y(g)$ must therefore solve the differential equation:

$$\frac{1-u'(g)}{\delta} = \frac{1-d-y'(g)}{n} + u'(y(g))y'(g) + \delta v'(y(g))y'(g) \tag{12}$$

This condition is useful only if we eliminate the last (endogenous) term: $\delta v'(y(g))y'(g)$. In the example of Figure 2 $y(g)$ is in $\left[g^2, g^3\right]$ for any $g \in \left[g^2, g^3\right]$. In this case, (10) implies $\delta v'(y(g)) = 1 - u'(y(g))$, and therefore:

$$\begin{aligned} \frac{1-u'(g)}{\delta} &= \frac{1-d-y'(g)}{n} + u'(y(g))y'(g) + \left[1-u'(y(g))\right]y'(g) \\ &= \frac{1-d+(n-1)y'(g)}{n} \end{aligned} \tag{13}$$

or

$$y'(g) = \frac{1-d-\frac{n\left(1-u'(g)\right)}{\delta}}{1-n} \tag{14}$$

The previous expression defines a simple differential equation with a solution $y(g)$, unique up to a constant.

To have a steady state at y_R^o, moreover, we need a second condition: $y(y_R^o) = y_R^o$. This equality provides the initial condition for (14), and so uniquely defines $y(g|y_R^o)$ in $\left[g^2, g^3\right]$ (see the dotted line in Figure 2). The investment function in Figure 2 is then fully characterized once we specify the levels g^2 and g^3.

Proposition 1, presented below, shows that we can indeed choose y_R^o and g^2, g^3 so that the investment function described above is a well behaved equilibrium. The proposition, moreover, fully characterizes the set of stable steady states that can be achieved in a monotonic well-behaved equilibrium. A steady state y_R^o is said to be stable if there is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o such that for any $N_{\varepsilon'}(y_R^o) \subseteq N_\varepsilon(y_R^o)$, $g \in N_{\varepsilon'}(y_R^o)$ implies $y_R(g) \in N_{\varepsilon'}(y_R^o)$. Intuitively, starting in a neighborhood of a stable steady state, g remains in a neighborhood of a stable steady state for all following periods.[13] Define the two thresholds:

$$y_R^*(d,n) = [u']^{-1}\left(1-\delta\frac{(1-d)}{n}\right), \text{ and } y_R^{**}(d,n) = [u']^{-1}\left(1-\delta\left(1-\frac{d}{n}\right)\right) \tag{15}$$

We say that an equilibrium steady state y_R^o is supported by a concave equilibrium if there is a concave equilibrium $y_R(g), v_R(g)$ such that $y_R(y_R^o) = y_R^o$. We have:

Proposition 1. *In a regular economy, an investment level y_R^o is a stable steady state of a monotonic, well-behaved equilibrium if and only if $y_R^o \in [y_R^*(d,n), y_R^{**}(d,n)]$. Each y_R^o is supported by a concave equilibrium with investment function $y_R(g\,|y_R^o)$ described by (9), where*

$$g^2 = \max\left\{\min_{g\geq 0}\left\{y\left(g\,|y^o\right) \leq W+(1-d)g\right\}, y_R^*(d,n)\right\},$$

g_3 is defined by $y\left(g^3\,|y_R^o\right) = y_R^{**}(d,n)$, *and $y(g) = y\left(g\,|y_R^o\right)$ is the the unique solution of (14) with initial condition $y\left(y_R^o\,|y_R^o\right) = y_R^o$.*

The reason why the steady state must be in $[y_R^*(d,n), y_R^{**}(d,n)]$ follows from three simple observations. First, an equilibrium steady state must be in the interior of the feasibility region, that is $y_R^o \in (0, W/d)$.[14] Intuitively, $y_R^o > 0$, since at 0 the marginal of the public good is infinite; and $y_R^o \leq W/d$ since even in the planner's solution we have this property. Second, in a stable steady state we must have $y_R'(y_R^o) \in (0,1)$. The highest and the lowest steady states, moreover, correspond to the equilibria with the highest and, respectively, the lowest $y_R'(g)$: so $y_R'(g) = 1$ and, respectively , $y'(g) = 0$. Third, since the solution is interior and the agents can choose the investment they like in a neighborhood of y_R^o, $y_R(g)$ can have positive slope at y_R^o only if the agents's objective function is flat in its neighborhood (otherwise the agents would choose the same optimum point irrespective of g). By the argument presented above, this implies (14). Using (14) and $y_R'(y_R^o) \leq 1$, we obtain the upper bound, $y_R^{**}(d,n)$; similarly, using (14) and $y_R'(y_R^o) \geq 0$, we obtain the lower bound, $y_R^{**}(d,n)$. Proposition 1 formalizes this argument, and it uses the construction described above to prove that $y_R^o \in [y_R^*(d,n), y_R^{**}(d,n)]$ is sufficient as well for y_R^o to be a stable steady state.

In the following three subsections we discuss three issues related to the dynamic properties of the equilibria. In Section 3.1 we discuss the efficiency of the steady states. In Section 3.2, we reinterpret Proposition 1 in the light of the type of strategic interaction (substitutability vs.

[13] Note that this condition is weaker than requiring that the state converges to the steady state from any point in its proximity, a property that is true if the slope of $y(g)$ is less than one at the steady state. In practice, however, all the steady states characterized below will have this stronger property. An intuitive way to think of stability, therefore, is that the slope of the investment function is less than one at the steady state.

[14] The feasibility set is given by $y \geq 0$ and $y \leq W + (1-d)g$, so a steady state must satisfy $y_R^o \geq 0$ and $y_R^o \leq W + (1-d)y_R^o$. The second inequality implies $y_R^o \leq W/d$.

complementarity) of the agents' actions. In Section 3.3 we discuss how the equilibrium converges to the steady state. We defer the discussion of non-monotonic equilibria to Section 5, after we have presented the case of economies with irreversibility.

3.1 Steady states and efficiency

Proposition 1 shows that, as in the static model, an equilibrium allocation is always inefficient: if $n > 1$ the steady state is always lower than the level that would be reached by a planner, $y_P^*(d,n) > y_R^{**}(d,n)$. We, however, have:

$$y_R^*(d,n) < [u']^{-1}(1-\delta(1-d)) < y_R^{**}(d,n).$$

Depending on the equilibrium choice, therefore, the accumulated level of g in a community with n players may be *either* higher or lower than the level that an agent alone in autarky would accumulate, which we denote by $\overline{y}(d)$:

$$\overline{y}(d) \equiv y_P^*(d,1) = [u']^{-1}(1-\delta(1-d)) \,. \tag{16}$$

This is in contrast to the static case (when $\delta = 0$), where the level of accumulation is independent of n.

In general, the relationship between the level of the steady state and the number of agents may depend on the equilibrium: the highest equilibrium steady state increases in n; the smallest steady state decreases in n. For large n, $y_R^*(d,n) \to [u']^{-1}(1)$ and $y_R^{**}(d,n) \to [u']^{-1}(1-\delta)$, while $\overline{y}(d) = [u']^{-1}(1-\delta(1-d))$ is independent of n. Of course, in the limit, all these equilibria are highly inefficient, since $y_P(d,n) = [u']^{-1}\left(\frac{1-\delta(1-d)}{n}\right) \to \infty$.

3.2 Substitutability and complementarity in the free rider problem

To interpret Proposition 1 it is useful to start from the special case in which $\delta = 0$ and so the free rider problem is static. In this case, we have $y_R^*(d,n) = y_R^{**}(d,n)$: there is a unique equilibrium "steady state" in which the agents invest $y_R^o = [u']^{-1}(1)$, independent of n. In addition, the agents' actions are pure strategic substitutes. In a symmetric equilibrium, each agent invests y_R^o/n. If agent j is forced to invest $1/n + \Delta$, then all the other agents find it optimal to reduce their investment exactly by $\Delta/(n-1)$.

In a dynamic game the strategic interaction is richer. Let us say that an investment function $y(g)$ displays *strategic substitutability* at a point of differentiability g, if $y'(g) < 1-d$. We have substitutability when $y'(g)$ is less than $1-d$ because in this case a marginal increase in investment at t by Δ is followed by a marginal reduction in investment (otherwise the stock would increase by at least $(1-d)$). Similarly, an investment function displays *strategic complementarity* (respectively, *neutrality*) at a point of differentiability g if $y'(g) > 1-d$ (respectively, $y'(g) = (1-d)$). The following result shows that for any $\delta > 0$, in a dynamic free rider problem we may have strategic substitutability, or strategic complementarity, or both, depending on whether the equilibrium steady state is less than or greater than $\overline{y}(d)$, respectively.[15]

For a given initial state g^0, an equilibrium investment function $y\left(g\,|y_R^o\right)$ with steady state y_R^o defines a *convergence path* $g^m \to y_R^o$, with $g^m = y\left(g^{m-1}\,|y^o\right)$.

Proposition 2. *The equilibrium investment strategy may display both strategic substitutability and strategic complementarity:*

- *If the steady state is $y_R^o \in (y_R^*(d,n), \overline{y}(d)]$, then there is an equilibrium in which starting from $y_o = 0$, the investment function displays strategic complementarity at no point on the equilibrium path. In addition, given the equilibrium path of this equilibrium $g^m \to y_R^o$ starting from any $y_o < y_R^o$, there is a m^* such that for $m > m^*$ we have strategic substitutability.*

- *If the steady state is $y_R^o \in (\overline{y}(d), y_R^{**}(d,n)]$* and $d > 0$*, then starting from $y_o < y_R^o$, for any equilibrium path $g^m \to y_R^o$, there is a m^* such that for $m > m^*$ we have strategic complementarity.*

The first point of Proposition 2 shows that when $y_R^o < \overline{y}(d)$ we can have equilibria exclusively characterized by strategic substitutability. These equilibria lead to steady state levels of g that are even below the level that would be reached by a single agent in perfect autarky, $\overline{y}(d)$. Proposition 2, therefore confirms and extends the main result of Fershtman and Nitzan [1991]. They showed that in the linear equilibrium of a linear-quadratic differential game, an agent could be worse in a dynamic setting than in a static setting because of the strategic substitutability. Proposition 2 shows that this phenomenon is a feature not only of this particular equilibrium selection or this environment, but is typical of the free rider problem and it is associated to a continuum of (non-linear) equilibria.

The second point of Proposition 2, however, shows that the equilibrium of the game is not necessarily characterized by strategic substitutability. Indeed, not only it is true that strategic complementarity is possible in equilibrium, but Proposition 2 shows that is a *necessary* feature of equilibria with sufficiently large steady states. An agent is willing to keep investing until $y_R^o > \overline{y}(d)$ only if he expects the other agents to react to his investment by increasing their own investments. This complementarity allows the agents to mitigate the free rider problem and

[15] In the boundary case of $d = 0$, there is no region of strategic complementarity because $\overline{y}(d) = y_R^{**}(d,n) = [u']^{-1}(1-\delta)$.

partially "internalize" the public good externality. In these equilibria, the agents accumulate more than what would be reached by an agent in perfect autarky.

3.3 Convergence paths in monotone equilibria

Even if we fix a steady state y_R^o, there could in principle be more than one equilibrium investment path consistent with it. In the equilibrium represented in Figure 2 the investment function stops being strictly increasing at g^2. We can however have equilibria in which the investment function stops being strictly increasing at a point g' in $\left[y_R^o, g^2\right]$. It can be proven that if an agent believes that the investment function stops increasing at this point, then it would indeed be optimal to stop investing at this point. This opens up a number of questions. Can we have multiple convergence paths to a given steady state? For a given steady state y_R^o, moreover, how fast can we converge to it starting from an initial state $g_0 < y_R^o$? We say that an equilibrium investment path $g^m \to y_R^o$ is *monotonically increasing* if $g^m \geq g^{m-1}$ $\forall m$. The following result shows that when the equilibrium is concave, then the equilibrium convergence path from below is uniquely defined[16] :

Proposition 3. *In a monotonic and concave equilibrium, the convergence path $g^m \to y_R^o$ starting from any $g^0 < y_R^o$ is uniquely defined and monotonically increasing. If $y_R^o > y_R^*(d,n)$ convergence is achieved only in the limit.*

The fact that, given a steady state, there is a unique monotonically increasing converging investment path is a feature that the equilibria of concave well behaved equilibria have in common with the planner's solution. There are however three notable differences.

First, the planner's solution is unique and it admits a unique steady state; in equilibrium, we may have a continuum of steady states.

Second, in the planner solution, convergence to the steady state is in finite time: the planner invests as much as possible until he/she reaches the steady state. In the equilibrium, instead, Proposition 3 shows that the state never actually reaches the steady state, convergence is only in the limit with the only exception being $y_R^o = y_R^*(d,n)$.

Third, for any of these steady states there may be non monotonic equilibria. These equilibria may be characterized by a cyclic investment path in which the state g alternates overshooting and undershooting the steady state, gradually spiraling into the steady state; or by persistent loops in which the steady state never converges. We will discuss these possibilities in detail in Section 5.

[16] The equilibrium investment function corresponding to a steady state $y_R^o > y_R^*$ is not generally uniquely defined for values of g above the steady state. For example, one could replace $y\left(g|y_R^o\right)$ with y_R^o for all values of $g > y_R^o$, and still have an equilbrium investment function that supports y_R^o as a steady state.

4 Irreversible economies

Most durable public investments are irreversible: once the investments are made, resources cannot be "withdrawn" and used for consumption or for other projects. Prominent examples are investments in public infrastructure (bridges, dams, etc.), basic research, and also less tangible investments such as "social capital". In this section, we study equilibrium behavior with irreversibility.

When the agents cannot directly reduce the stock of the public good, the optimization problem of an agent can be written like (6), but with an additional constraint: the individual level of investment cannot be negative; the only way to reduce the stock of g, is to wait for the work of depreciation. Following similar steps as before, we can write the maximization problem faced by an agent as:

$$\max_{y}\left\{\begin{array}{c} u(y)-y+\delta v_{IR}(y) \\ y\leq \frac{W+(1-d)g}{n}+\frac{n-1}{n}y_{IR}(g),\ y\geq \frac{(1-d)g}{n}+\frac{n-1}{n}y_{IR}(g) \end{array}\right\} \tag{17}$$

where the only difference with respect to (7) is the second constraint. To interpret it, note that it can be written as $y\geq(1-d)g+\frac{n-1}{n}\left[y_{IR}(g)-(1-d)g\right]$: the new level of public good cannot be lower than $(1-d)g$ plus the investments from all the other agents.

As in the reversible case, a continuous symmetric Markov equilibrium is fully described in this environment by two functions: an aggregate investment function $y_{IR}(g)$, and an associated value function $v_{IR}(g)$. The aggregate investment function $y_{IR}(g)$ must solve (17) given $v_{IR}(g)$. The value function $v_{IR}(g)$ must be consistent with the agents' strategies. Similarly, as in the reversible case, we must have:

$$v_{IR}(g)=\frac{W+(1-d)g-y_{IR}(g)}{n}+u\left(y_{IR}(g)\right)+\delta v_{IR}(y_{IR}(g)) \tag{18}$$

We can therefore define:

Definition 2. *An SME equilibrium in a dynamic free rider problem with irreversibility is a pair of functions, $y_{IR}(\cdot)$ and $v_{IR}(\cdot)$, such that for all $g\geq 0$, $y_{IR}(g)$ solves (17) given the value function $v_{IR}(\cdot)$, and for all $g\geq 0$, $v_{IR}(g)$ solves (18) given $y_{IR}(g)$.*

As pointed out in Section 2.1, when public investments are efficient, irreversibility is irrelevant for the equilibrium allocation. The investment path chosen by the planner is unaffected because the planner's choice is *time consistent*: he never finds it optimal to increase g if he plans to reduce it later. In the monotone equilibria characterized in the previous section, the investment function may be inefficient, but it is weakly increasing in the state. Agents invest until they reach a steady

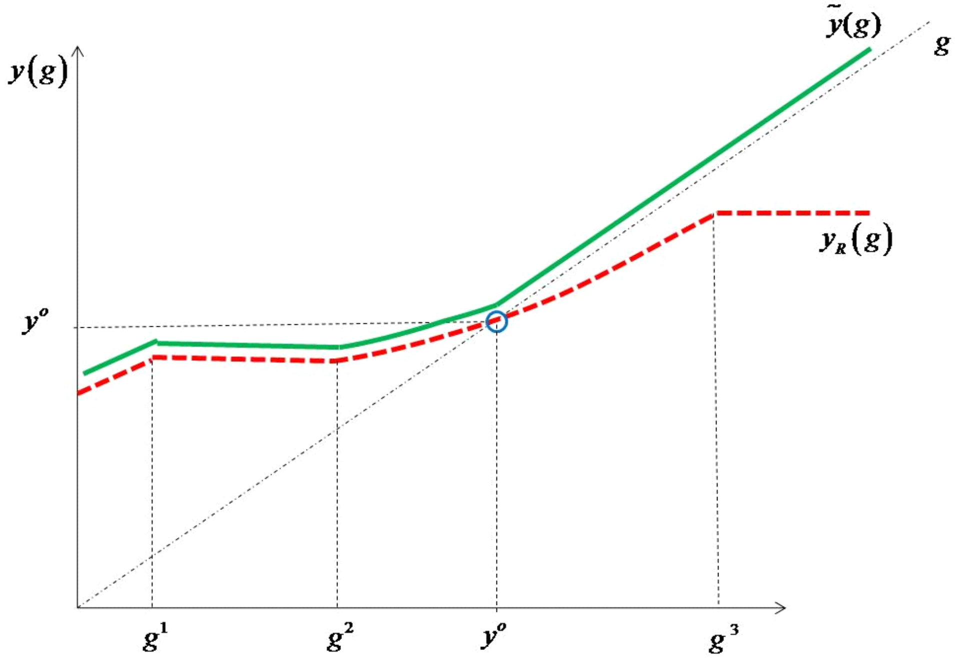


Figure 3: The irreversibility constraint and the reversible equilibrium.

state, and then they stop. It may seem intuitive, therefore, that irreversibility is irrelevant in this case too. In this section we show that, to the contrary, irreversibility changes the equilibrium set: it induces the agents to significantly increase their investment and it leads to significantly higher steady states when depreciation is small.

To illustrate the impact of irreversibility on equilibrium behavior, suppose for simplicity that $d = 0$ and consider Figure 3, where the red dashed line represents some arbitrary monotone equilibrium with steady state y^o in the model with reversibility. Next, suppose we ignore the irreversibility constraint where it is not binding, so we keep the same investment function for $g \leq y^o$ where $y_R(g) \geq g$ and then set the investment function equal to g when $y_R(g) < g$. This gives us the modified investment function $\widetilde{y}_R(g)$, represented by the green solid line. This investment function induces essentially the same allocation: the same steady state and the same convergence path for any initial $g_0 \leq y^o$. Unfortunately, $\widetilde{y}_R(g)$ is no longer an equilibrium. On the left of y^o the objective function, $u(y) - y + \delta v_{IR}(y)$, is flat. On the right of y^o, the objective function would remain flat if the investment were the red dashed line as with reversibility; with irreversibility, however, the constraint $y \geq g$ forces the investment to increase at a faster rate than $y_R(g)$. Because $y_R(g)$ is ex ante suboptimal, the "forced" increase in investment makes the objective function increase on the right of y^o. But then choosing y^o would no longer be optimal in state y^o, so it cannot be a steady state.[17]

Does a well-behaved equilibrium exist? Yes. What does it look like? It can be verified that there is an equilibrium with a steady state that corresponds to the point at which the solution of the differential equation (14) has slope $(1-d)$. In particular, let $\widehat{y}(g)$ be the solution of (14) with the initial condition $\widehat{y}(\overline{y}(d)) = (1-d)\overline{y}(d)$, which is the unique solution of (14) that is tangent to $(1-d)g$. Define y^o_{IR} as the fixed point of this function:

$$\widehat{y}(y^o_{IR}(d,n)) = y^o_{IR}(d,n). \tag{19}$$

where $y^o_{IR}(d,n)$ may depend on n since $\widehat{y}(g)$ is a function of n. We have:

Proposition 4. *In a regular economy with irreversibility, there is a concave and monotonic well-behaved equilibrium with investment function:*

$$y_{IR}(g) = \begin{cases} \min\left\{W + (1-d)g, \widehat{y}\left(g^2_{IR}\right)\right\} & g \leq g^2_{IR} \\ \widehat{y}\,(g) & g^2_{IR} < g \leq \overline{y}(d) \\ (1-d)g & g \geq \overline{y}(d) \end{cases} \tag{20}$$

where $g^2_{IR} = \max\left\{\min_{g\geq 0}\left\{\widehat{y}\,(g) \leq W + (1-d)g\right\}, y^*_R(d,n)\right\}$*, and* $\widehat{y}\,(g)$ *is the the unique solution of (14) with initial condition* $\widehat{y}\,(\overline{y}(d)) = (1-d)\overline{y}(d)$*. The associated steady state is* $y^o_{IR}(d,n)$ *as defined in (19).*

Proposition 4 establishes that the dynamic free rider game with reversibility admits an equilibrium with standard concavity properties. Figure 4 represents the associated investment function (20). Note that the investment function merges smoothly with the irreversibility constraint: indeed, at the point of the merger $\widehat{y}(d)$ (where the constraint becomes binding), the investment function has slope $1-d$. This feature is essential to avoid the problem illustrated in Figure 3, and to obtain a concave equilibrium.

Contrary to what happens in a reversible economy, however, now there is a loss of generality in assuming that the equilibrium is concave. Indeed, it can be proven that there are (non concave) equilibria in which the steady state is higher than $y^o_{IR}(d,n)$.[18] Can we have steady states lower

[17] This problem does not arise with the planner's solution because the planner's solution is time consistent. After the planner's steady state y^*_P is reached the planner would keep g at y^*_P. If the planner's is forced to increase y on the right of y^*_P, we would have a kink at y^*_P, but it would be a "downward" kink. Such a kink makes the objective function fall at a faster rate on the right of the steady state, so it preserves concavity and it does not disturb the optimal solution. The kink is "upward" in the equilibrium with irreversibility because the steady state is not optimal, so the irreversibility constraint, $y \geq g$, increases expected welfare. This creates a sort of "commitment device" for the future; the agents know that g can not be reduced by the others (or their future selves).

[18] For brevity we omit the proof of this result. The proof is a variant of the proofs of Proposition 1 and 4.

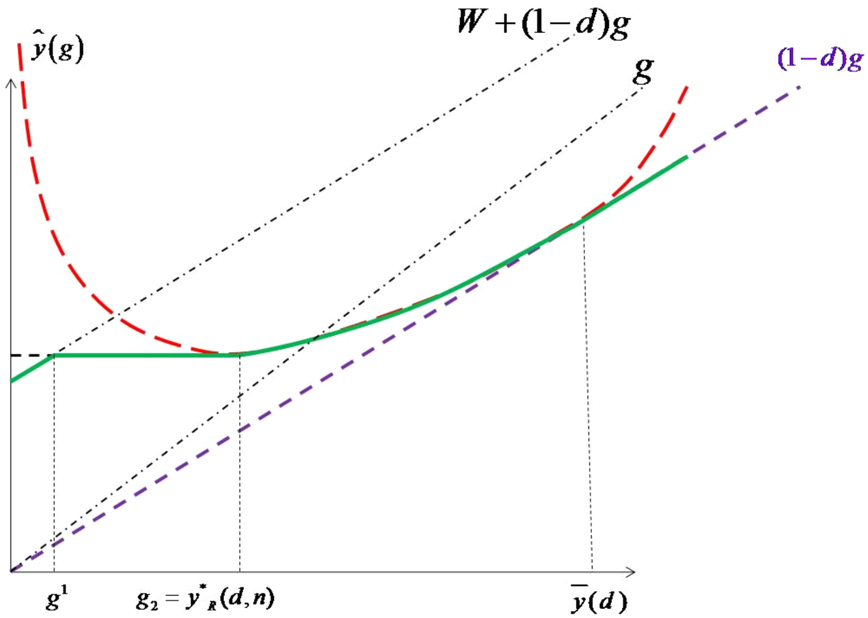


Figure 4: The irreversible equilibrium as $d \to 0$.

than $y^o_{IR}(d,n)$? How high can an equilibrium steady state be? The following result shows that when depreciation is not too high, all stable steady states of a well-behaved equilibrium must be close to $y^o_{IR}(d,n)$, and they all converge to $\overline{y}(0)$ as $d \to 0$:

Proposition 5. *There is lower bound $y^*_{IR}(d,n) \geq y^*_R(d,n)$ such that y_{IR} is a stable steady state of a monotonic well-behaved equilibrium only if $y_{IR} \in [y^*_{IR}(d,n), y^{**}_R(d,n)]$. Moreover, as $d \to 0$ both $y^*_{IR}(d,n) \to \overline{y}(0)$, and $y^{**}_R(d,n) \to \overline{y}(0)$.*

There is an intuitive explanation for Proposition 5. Because of decreasing returns, the investment in g declines over time, and so the constraint $y \geq (1-d)g$ is binding in any equilibrium when g is high enough. When this happens the agents are forced to keep the investment higher than what they would like. Since the equilibrium is inefficiently low (because the agents do not fully internalize the social benefit of g), the constraint $y \geq (1-d)g$ increases expected welfare in these states. The states where the constraint $y \geq (1-d)g$ is binding are typically out of equilibrium, that is on the right of the steady state: in the equilibrium illustrated in Figure 4, for example, the constraint is binding for $g > y^o_{IR}(d)$. The irreversibility constraint, however, has

Intuitively this may happen if the point at which the irreversibility constraint is binding is sufficiently high in the out of equilibrium region (as, for example, when d is very high). In these cases the point where the irreversibility constraint is binding is still a point where the right derivative is higher than the left derivative: still both derivatives are lower than one, so incentives in lower states are unaffected.

a ripple effect on the entire investment function. In a left neighborhood of $\overline{y}(d)$, the constraint is not binding; still, the agents expect that the other agents will preserve their investment, so the strategic substitutability will not be too strong. Steady states lower than $y^o_{IR}(d)$ can occur with reversibility because the agents expect high levels of "strategic substitutability." However, the construction of such equilibria relies on the existence of states where agents can invest negative amounts. Proposition 5 shows that when d is sufficiently low, the irreversibility constraint makes these expectations impossible in equilibrium, inducing an equilibrium steady state close to the maximal steady state of the reversible case, $y^{**}_R(d, n)$. Thus, as $d \to 0$, there is a unique well-behaved Markov equilibrium steady state in the irreversible case, i.e., $y^o_{IR} = y^{**}_R(d, n)$.

5 Non-monotonic equilibria and cycles

Following the previous literature on dynamic free rider problems, to this point we have focused on monotonic equilibria: that is on equilibria in which $y(g)$ is non-decreasing. Can there be non-monotonic equilibria? Non-monotonic equilibria are of particular interest: when $y(g)$ is decreasing, the dynamics of the equilibrium can be much more complicated, and perhaps surprising; in principle, there could be spiraling-in convergence to a steady state, or limit cycles. This raises a number of important questions: in particular, can the steady state be higher than $y^{**}_R(d, n)$? or lower than $y^*_R(d, n)$? Can we have equilibria where the investment path does not converge to a steady state, but instead have persistent cycles? In the reminder of this section we will focus on the case of non-monotonic Markov equilibria in economies with reversibility.[19]

The first result we present establishes the existence of stable steady states supported by non-monotonic equilibria, and it bounds the set of steady states that can be supported in these equilibria. Define the lower bound threshold:

$$y^{***}_R(d, n) = [u']^{-1}\left(1 + \frac{\delta}{n}(n + d - 2)\right)$$

Note that $y^{***}_R(d, n) < y^*_R(d, n)$. We have:

Proposition 6. *In any regular economy there exists a $\Delta > 0$ such that any point in*

$$[y^*_R(d, n) - \Delta,\ y^*_R(d, n)]$$

is a stable steady state of a concave non-monotonic equilibrium. In these equilibria, the investment path converges to the steady state with damped oscillations as in Figure 5. A *stable steady state supported by a non-monotonic equilibrium must be in $[y^{***}_R(d, n), y^{**}_R(d, n)]$.*

[19] The analysis for economies with irreversibility is similar when d is sufficiently large. Obviously, however, the equilibrium must be monotonic in the limit as $d \to 0$ in the irreversible case since, in this case, we must have $y_t = y(y_{t-1}) \geq y_{t-1}$.

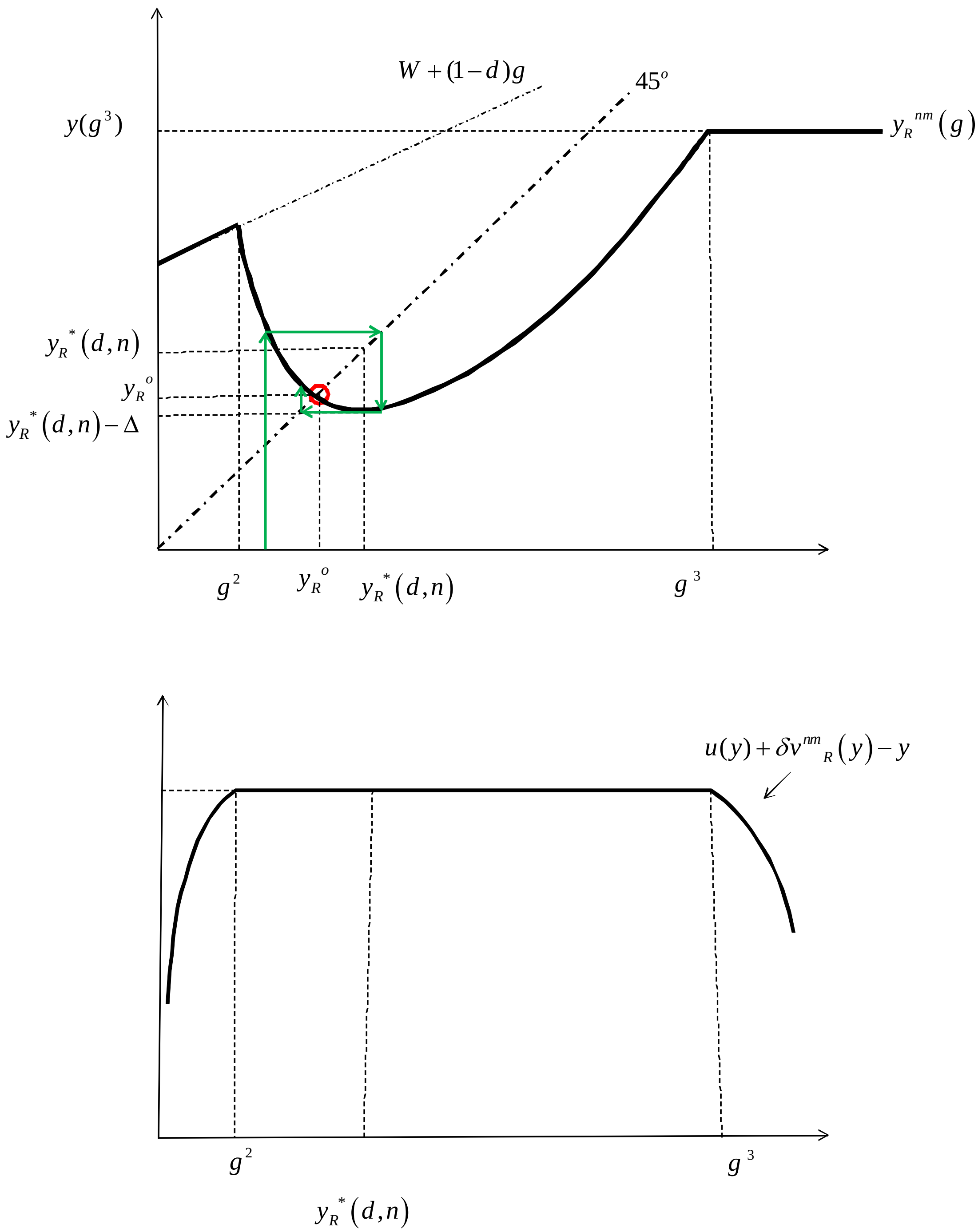


Figure 5: Non-monotone equilibrium with steady state y_R^o.

Figure 5 represents a non-monotone equilibrium with a stable steady state. In this equilibrium the investment function $y_R^{nm}(g)$ intersects the 45^0 line at a stable steady state y^o from above, with $[y_R^{nm}]'(y^o) \in (-1, 0)$. The convergence path, therefore, is characterized by damped oscillations (as represented by the green arrows in Figure 5). We can draw two lessons from this proposition. First, the result presented in Section 3 that the stable steady states are bounded above by $y_R^{**}(d,n)$ still holds true in the larger class of equilibria that may be non-monotonic. Second, however, the lowest steady state now can be strictly lower than $y_R^*(d,n)$. This result extends and reinforces the finding that the steady state of a dynamic free rider game may be lower in a community with n agents than when there is only one agent in autarky. Proposition 6, therefore, confirms the main conclusions reached in the previous sections studying monotonic equilibria.

The intuition for the existence of non-monotonic equilibria and why steady states below $y_R^*(d,n)$ can be supported in these equilibria is relatively straightforward. As in the equilibria constructed in Section 3, an agent's objective function $u(y) + \delta v_R^{nm}(y) - y$ in these equilibria is weakly concave, with a "flat" top in the interval (g_2, g_3), as illustrated in the lower panel in Figure 5. The flat region allows the investment function to be in (g_2, g_3), since the agents are indifferent in this region. As described in Section 3, however, in order to have this property in the region (g_2, g_3), the investment function must satisfy the differential equation (14). When g is sufficiently large, this equation implies that $y'(g) > 0$. In these cases, g is such that the objective function remains flat only if the agents expect the other agents to continue investing. When g is very large (i.e. $g > \overline{y}(d)$), then $y'(g) > 1 - d$, so we have strategic complemetarity. When g is small, however, the private value of investment $u'(g)$ is very large: indeed $u'(g) \to \infty$ as $g \to 0$. In these cases the agents are willing to choose an interior level of g only if they expect that the other agents will reduce the stock of investment: this reduction compensates for the high level of private value of the investment. This is the reason why for $g < y_R^*(d,n)$, the equilibrium investment function must be negatively sloped. If we are constructing a steady state larger than $y_R^*(d,n)$, it is not necessary to support it by a non-monotonic investment path; but a steady state lower than $y_R^*(d,n)$ can be supported *only* by a non-monotonic investment path.

It is natural to ask whether the lower bound $y_R^{***}(d,n)$ of Proposition 6 is tight. The answer to this question depends on the exact shape of the utility function $u(g)$. Define $y_\Delta(g)$ to be the solution to (14) with initial condition $y_\Delta(y_R^*(d,n)) = y_R^*(d,n) - \Delta$. The function $y_\Delta(g)$ is differentiable, with a minimum at $y_R^*(d,n)$, and it intersects the 45^o at two points: y_Δ^o and y_Δ^{oo} with $y_\Delta^o < y_R^*(d,n) < y_\Delta^{oo}$.[20] The next assumption on the underlying fundamentals yields a

[20] Since $y_\Delta(g)$ is convex and continuous, with a minimum at $y_R^*(d,n)$, and such that $\lim_{g\to 0} y_\Delta(g) > 0$ and $y_R^*(d,n) > y_\Delta(y_R^*(d,n))$ by construction, we have $y_\Delta^o \in (0, y_R^*(d,n))$. Since, in addition, $\lim_{g\to\infty} y_\Delta(g) = \infty$, we have $y_\Delta^{oo} > y_R^*(d,n)$.

sufficient condition for $y_R^{***}(d,n)$ to be a tight lower bound:

Assumption 2. There is a $\Delta \in (0, y_R^*(d,n))$ such that $y'_\Delta(y_\Delta^o) < -1$ and $y_\Delta(y_R^{**}(d,n) - \Delta) < y_\Delta(y_\Delta^{oo})$.

This condition can be easily verified in specific examples. Let the utility function be $u(g) = g^\alpha/\alpha$, parametrized by α. In this case we can solve for $y_\Delta(g)$ in closed form:

$$y_\Delta(g) = A(\alpha, n, \delta, d, \Delta) + \frac{n}{\alpha\delta(1-n)} g^\alpha - \frac{(n - \delta(1-d))}{\delta(1-n)} g \tag{21}$$

where:

$$A(\alpha, n, \delta, d, \Delta) = \left(\frac{n}{n-\delta(1-d)}\right)^{\frac{1}{1-\alpha}} \left[1 - \left(\frac{n}{n-\delta(1-d)}\right)^{\frac{2\alpha-1}{1-\alpha}}\right] \frac{n}{\alpha\delta(1-n)} + \frac{b(n-\delta(1-d))}{\delta(1-n)} - \Delta$$

If, for instance, we assume $n = 3$, $\delta = .75, d = 0.2, \alpha = 0.1$, then we have:

$$y_\Delta(g) = 19.733 - \Delta - 20.0 g^{0.1} + 1.6g$$

and $y_R^*(d,n) = 1.2814$. When $\Delta \geq 1$, $y_\Delta(g)$ admits a fixed-point y_Δ^o lower than 0.63581 and such that $y'_\Delta(y_\Delta^o) \leq -1.4063$, so the first part of Assumption 2 is verified. When $\Delta \leq 1.2$, moreover, it admits a fixed-point y_Δ^{oo} such that $y_\Delta(y_R^{**}(d,n) - \Delta) < y_{\underline{\Delta}}(y_\Delta^{oo})$. In this example, therefore, Assumption 2 is verified at least for any $\Delta \in (1, 1.2)$.[21] We have:

Proposition 7. *Given Assumption 2, in any regular economy, there is a W^* such that for any $W > W^*$ the dynamic free rider game has an equilibrium in which y_o is a stable steady state for any $y_o \in [y_R^{***}(d,n), y_R^{**}(d,n)]$.*

Note that the level W^* that guarantees the result in Proposition 7 does not need to be very large. For the class of equilibria that we construct we must have $W > y(y_R^*(d,n) - \Delta)$. In the example presented above this implies $W \geq 1.6$.

The last question we would like to address is the possibility of cycles. In general we have a *cycle* when there is a time path $(g_t^*)_{t=0}^\infty$ with $g_0^* = g_0$ and $g_t^* = y(g_{t-1}^*)$ such that g_t^* does not converge to any point. We have:

Proposition 8. *In any regular economy, if Assumption 2 holds there is a W^{**} such that for any $W > W^{**}$ the dynamic free rider game has an equilibrium with a cycle.*

A cycle may be periodic in the sense that it generates an *orbit* of a given finite length. Let $y^1(g) = y(g)$ and define the p_{th}-iterate recursively as $y^p(g) = y(y^{p-1}(g))$, $p = 1, 2, ...$ We say that

[21] Note, moreover, that since (21) is continuous in all parameters, Assumption 2 is satisfied in an open set of *all* fundamental parameters.

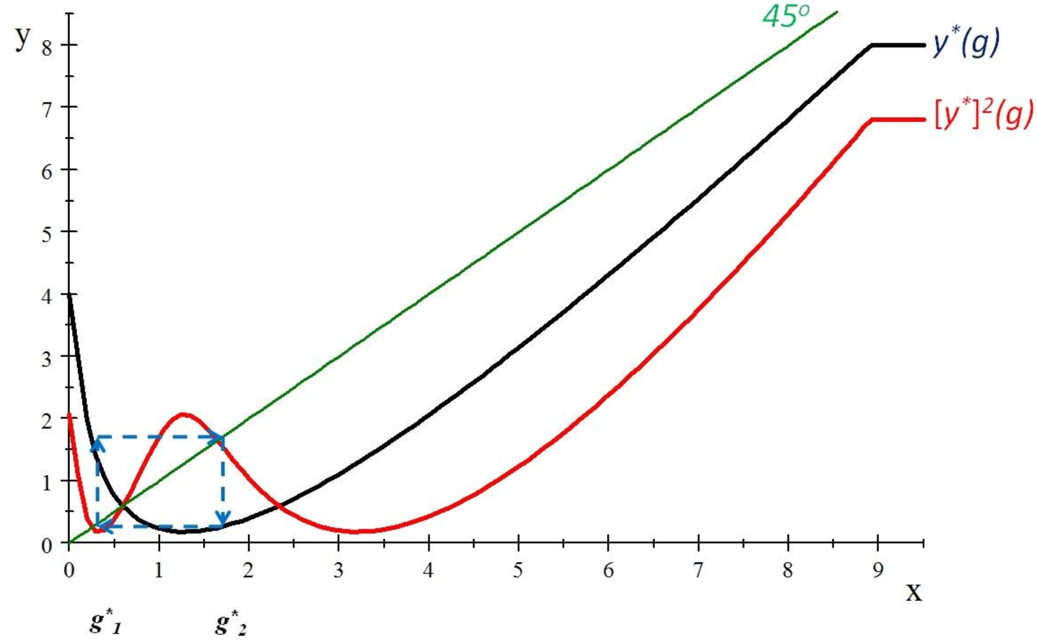


Figure 6: An example of an equilibrium converging to a cycle with a 2-state orbit.

we have a *p-cycle* if there is a point y_p such that $y_p = y^p(y_p)$. When we have a p-cycle the state rotates in an orbit of p points defined by the set $\{(y_1, ..., y_p)|y_j = y^p(y_j) \text{ for } j = 1, ..., p\}$. Under the specific parametric assumptions of the example above, the orbit of a 2-cycle can be explicitly solved for. Letting $\Delta = 1.1$, the equilibrium investment function can be solved in closed form:

$$y^*(g) = \min\left(\min\left(W + (1-d)g, 1.6g - 20.0g^{0.1} + 18.633\right), 11.5\right)$$

Figure (6) represents $y^*(g)$ for this example and its 2-iteration $[y^*]^2(g) = y^*(y^*(g))$. As it can be seen $[y^*]^2(g)$ (the red line) has two stable fix points $g_1 = 0.59535$ and $g_2 = 1.6478$ such that $g_1 = y^*(g_2)$, and $g_2 = y^*(g_1)$, so $\{g_1, g_2\}$ constitute a 2-period orbit. In the example of Figure 6, therefore, the state oscillates between g_1 and g_2. This cycle, moreover, is stable: if we start from any initial state g_0 there is a limit *2-cycle*, in the sense that the investment path converges to this two-period orbit.

6 Laboratory Evidence about the dynamic public good provision

In this paper, we develop a theory of the accumulation of a durable public good under what amounts to a "voluntary contributions" mechanism. While there is an abundance of evidence about the free rider problem in such games for static environments[22] (literally hundreds of experimental papers), there have been no studies on the provision of durable public goods.[23] In a companion paper (Battaglini, Nunnari, and Palfrey [2011b]), we report the results from a laboratory experiment designed explicitly to provide empirical evidence about behavior in the dynamic environments studied in this paper. These experiments shed light on a number of questions.

First, the theory makes a clear prediction about the effect of reversibility of the public investments. With no depreciation, the unique equilibrium steady state when investments cannot be reversed supports public good levels that are strictly higher than almost all stable steady states in the reversible case (even the ones that are not strictly concave or non-monotonic), and weakly higher than *all* the stable equilibria in the reversible case. This prediction can be sharpened by focusing on strictly concave equilibria when they exist. In Battaglini, Nunnari, and Palfrey [2011b] we show that in a reversible economy there is a unique steady state that can be supported by a strictly concave equilibrium: $y^*(d, n)$ that is always strictly lower than the unique equilibrium steady state of an irreversible economy. [24]

Second, in the present paper, we restrict attention to a subset of equilibria, namely continuous Markov perfect equilibria with stable steady states. There are of course many other equilibria, some using folk theorem types of discontinuous strategies or non-Markov strategies to support very efficient paths, and others with cycles and orbits. These assumptions (continuous, Markov, and subgame perfect) are both strong assumptions and testable assumptions. Furthermore, they are commonly employed equilibrium restrictions in the analysis of dynamic games such as the ones we study. An important empirical question is whether such strong restrictions are valid. While a laboratory experiment cannot answer this question for general games, we are able to answer it specifically for the class of games considered in this paper. A laboratory experiment allows us to see whether in the most favorable conditions - small numbers of players and no confounding

[22] Ledyard (1996) provides a comprehensive survey of static public goods experiments. Since the publication of that survey, there have been dozens more papers published on the subject.

[23] Several experimental papers consider multistage mechanisms for provision of public goods, but in all cases, the environments are static in the sense of the mechanisms being one-shot multistage games and the public goods are nondurable. See for example Smith (1980), Ferejohn et al. (1982), Harrison and Hirschleifer (1989), Choi et al. (2008), Duffy et al. (2007).

[24] The theoretical model also makes predictions about comparative statics with respect to the model parameters (utility functions, discount factor, depreciation rate, and the number of agents).

factors or uncertainty - players may be able to support highly efficient investment paths that are inconsistent with these kinds of equilibrium restrictions. Experiments on the repeated prisoners dilemma suggest the likelihood that the infinite horizon dynamics will allow efficient or nearly efficient public goods provision, far more efficient than the continuous Markov perfect equilibria identified in this paper.[25]

The experimental design in Battaglini, Nunnari, and Palfrey [2011b] closely mirrors the theoretical model, and addresses both sets of questions.[26] The results largely confirm the theory. For the first set of questions, we conduct the games with and without reversibility, holding constant the other parameters of the game. We also vary the group size between $n = 3$ and $n = 5$. The main result is that we observe much higher public good investments when investments are irreversible, consistent with our theoretical characterization of stable steady states. We address the second set of questions in two different ways. First, we analyze investment decisions from the experiment described above, to see whether they are consistent with continuous Markov perfect equilibria with stable steady states. We find strong evidence for forward-looking Markov behavior in the reversible case. The stock of the public good levels out in most cases at a point above the predicted range of equilibrium steady states, but far below the efficient solution.[27]

The second way we address the Markov equilibrium question is by conducting a one-period version of the reversible investment game, where the payoffs are given by the equilibrium value functions corresponding to the unique strictly concave equilibrium steady state: $y_R^* = 1.78$ for $n = 3$, and $y_R^* = 1.38$ for $n = 5$. That is, for each possible y, the payoffs the subjects receive are equal to the theoretical equilibrium continuation value functions of the infinite horizon game. We vary the initial states across many different plays of the game which allows us to measure the empirical investment functions. If the theory is correct, we would observe the same investment patterns in this one-period version of the game as we do in the baseline treatment that closely mirrored the infinite horizon game. If the pattern of investment is much different, this suggests that the players are not optimizing relative to the equilibrium value functions. We observe similar patterns of investment in the one period version of the game and the infinite horizon version.

[25] For example Dal Bo (2005) and other papers report highly efficient non-Markov equilibrium outcomes in repeated prisoner's dilemma games.

[26] The infinite horizon is implemented in the laboratory by the now-standard method of a stationary random termination probability. In all treatments of the experiment, induced utility functions are quadratic, the discount factor is $\delta = .75$, and there is no depreciation ($d = 0$).

[27] With reversible investment, the median period 10 public good level is 4.00 for $n = 3$ and 7.21 for $n = 5$. This compares with equilibrium steady state ranges of $1.78 - 16$ and $1.38 - 16$, respectively, and efficient planner's solution levels equal to 144 and 400, respectively. In the irreversible case we observe levels much higher than the unique equilibrium steady state of 16: the median period 10 public good level is 73.38 for $n = 3$ and 91.75 for $n = 5$.

7 Conclusions

In this paper we have studied a simple model of free riding in which n infinitely lived agents choose between private consumption and contributions to a durable public good g. We have considered two possible cases: economies with reversible investments, in which in every period individual investments can either be positive or negative; and economies with irreversible investments, in which the public good can only be reduced by depreciation. For both cases we have characterized the set of steady states that can be supported by Markov equilibria in continuous strategies. As we have argued in the introduction, this seems the most appropriate equilibrium concept for this class of problems.

We have highlighted three main results. First, we have shown that economies with reversible investments have typically a continuum of equilibria. In the best equilibrium the steady state is higher in a community with n agents than in autarky, and it is increasing in n; in the worst equilibrium, the steady state is lower in autarky, and it decreases in n. We have argued that, while in a static free rider's problem the players' contributions are strategic substitutes, in a dynamic model they may be strategic complements. Second, we have shown that in economies with irreversible investments, the set of steady states that can be supported by well behaved equilibria is much smaller: indeed, as depreciation converges to zero, the set of equilibrium steady states converges to the best equilibrium that can be reached in economies with reversible investments. Irreversibility, therefore, helps the agents removing the coordination problem that plagues most of the equilibria in the reversible case, and so it necessarily induces higher investment. Third, we have shown that there are non-monotonic equilibria with complex and (perhaps) surprising dynamic properties, in which the state converges to the steady with damped oscillations, and in which there are persistently cycles. Indeed, we have constructed a closed form example in which the state perpetually rotates in a 2-period orbit.

Although in this paper we have focused on a free rider problem in which agents act independently and there is no institution to coordinate their actions, the idea we have developed have a wider applicability and can be used to study dynamic games in other environments as well. In future work, it would be interesting to investigate economies with irreversible investments or the existence of endogenous cycles when public decisions are taken by legislative bargaining or other types of centralized political processes.

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Appendix

7.1 Proof of Proposition 1

Let $y_R^*(d,n)$ and $y_R^{**}(d,n)$ be defined by (15). Since we are in a regular economy, we have $W/d > y_R^{**}(d,n)$. We first prove here that for any $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$, there is Markov equilibrium with steady state equal to y^o (sufficiency). Then we prove that the steady state must be in $[y_R^*(d,n), y_R^{**}(d,n)]$ (necessity).

7.1.1 Sufficiency

To construct the equilibrium we proceed in 3 steps.

Step 1. We first construct the strategies and prove their key properties. Let $y(g\,|y^o)$ be the solution of the differential equation when we require the initial condition: $y(y^o\,|y^o) = y^o$, for $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$. Let $g^2(y)$ be:

$$g^2(y) = \max\left\{\min_{g\geq 0}\{y(g\,|y^o) \leq W + (1-d)g\}, y_R^*(d,n)\right\}. \tag{22}$$

This, essentially, is the largest point between the point at which $y(g\,|y^o)$ crosses from below $W + (1-d)g$, and $y_R^*(d,n)$. Let $g^3(y^o)$ be defined by $y\left(g^3(y^o)\,|y^o\right) = y_R^{**}(d,n)$.

Lemma A.1. *$y'(g\,|y^o) \in (0,1)$ in $\left[g^2(y^o), y_R^{**}(d,n)\right]$ and $y''(g\,|y^o) \geq 0$.*

Proof. From (14), $y'(g\,|y^o) \geq 0$ for $g \geq y_R^*(d,n)$, *and* $y'(g\,|y^o) \leq 1$ for $g \leq y_R^{**}(d,n)$. Since $y''(g\,|y^o) = \frac{n}{1-n}\left[\frac{u''(g)}{\delta}\right]$, $y''(g) > 0$. ■

Lemma A.2. *For any $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$, $g^3(y^o) \geq y_R^{**}(d,n)$.*

Proof. Note that $y(y_R^{**}(d,n)\,|y^o)$ is increasing in y^o. Moreover $y(y_R^{**}(d,n)\,|y_R^{**}(d,n)) = y_R^{**}(d,n)$. So $y(y_R^{**}(d,n)\,|y^o) < y_R^{**}(d,n)$ for $y^o < y_R^{**}(d,n)$. It follows that $g^3(y^o) \geq y_R^{**}(d,n)$ for any $y^o \leq y_R^{**}(d,n)$. ■

We have:

Lemma A.3. *$y(g\,|y^o) \in (0, W + (1-d)g)$ in $\left(g^2(y^o), g^3(y^o)\right)$.*

Proof. First note that $y\left(g^2(y^o)|y^o\right) \leq W + (1-d)g^2(y^o)$. Since $y'(g\,|y^o) < 1$ for $g < y_R^{**}(d,n)$ we must have $y(g\,|y^o) < W + (1-d)g$ for $g \in \left(g^2(y^o), y_R^{**}(d,n)\right)$. For $g > y_R^{**}(d,n)$, we have $W + (1-d)g > W + (1-d)y_R^{**}(d,n)$. Since $y(g\,|y^o) < y_R^{**}(d,n)$ in $\left(g^2(y^o), g^3(y^o)\right)$, We have $y(g\,|y^o) < y_R^{**}(d,n) < W + (1-d)y_R^{**}(d,n) < W + (1-d)g$ in $\left[y_R^{**}(d,n), g^3(y^o)\right)$ as well. Similarly, since $y'(g\,|y^o) \geq 0$ for $g > g^2(y^o)$ and $y\left(g^2(y^o)|y^o\right) \geq 0$, we must have $y(g\,|y^o) > 0$ for $g > g^2(y^o)$. Note that $y\left(g^2(y^o)\,|y^o\right) \geq 0$ since $y'(g\,|y^o) \in (0, 1-d)$ in $[y_R^*(d,n), y^o]$ implies that $y(g\,|y^o) > g$ for all $g \in [y_R^*(d,n), y^o]$. ■

For any $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$, we now define the investment function:

$$y_R(g\,|y^o\,) = \begin{cases} \min\left\{W+(1-d)g, y\left(g^2(y^o)\,|y^o\,\right)\right\} & g \leq g^2(y^o) \\ y\,(g\,|y^o\,) & g^2(y^o) < g \leq g^3(y^o) \\ y_R^{**}(d,n) & g \geq g^3(y^o) \end{cases}$$

For future reference, define $g^1(y^o) = \max\left\{0, \left(y\left(g^2(y)\,|y^o\,\right) - W\right)/(1-d)\right\}$. This is the point at which $W + (1-d)g^2(y^o) = y\left(g^2(y^o)\,|y^o\,\right)$, if positive. Clearly, we have $g^1(y^o) \in \left[0, g^2(y^o)\right]$. We have:

Lemma A.4. *For any* $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$, $y\,(g\,|y^o\,) \in \left[g^2(y^o), g^3(y^o)\right]$ *for* $g \in \left[g^2(y^o), g^3(y^o)\right]$.

Proof. Since $y\,(g\,|y^o\,)$ it is monotonic non-decreasing in $g \in \left[g^2(y^o), g^3(y^o)\right]$,

$$y\,(g\,|y^o\,) \in \left[y\left(g^2(y^o)\,|y^o\,\right), y\left(g^3(y^o)\,|y^o\,\right)\right] \quad \forall g \in \left[g^2(y^o), g^3(y^o)\right].$$

Since $y\,(g\,|y^o\,)$ has slope lower than one in $\left[g^2(y^o), g^3(y^o)\right]$ and $y\,(y^o\,|y^o\,) = y^o$ for $y^o \geq g^2(y^o)$, we must have $y\left(g^2(y^o)\,|y^o\,\right) \geq g^2(y^o)$, so $y\,(g\,|y^o\,) \geq g^2(y^o)$ *for* $g \in \left[g^2(y^o), g^3(y^o)\right]$. Similarly, $y\left(g^3(y^o)\,|y^o\,\right) \leq g^3(y^o)$, so $y\,(g\,|y^o\,) \leq g^3(y^o)$ *for* $g \in \left[g^2(y^o), g^3(y^o)\right]$. ■

Step 2. We now construct the value functions corresponding to each steady state $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$. For $g \in \left[g^2(y^o), g^3(y^o)\right]$ define the value function recursively as

$$v(g|y^o) = \frac{W+(1-d)g - y\,(g\,|y^o\,)}{n} + u(y\,(g\,|y^o\,)) + \delta v(y\,(g\,|y^o\,)). \tag{23}$$

By Theorem 3.3 in Stokey and Lucas (1989), the right han side of (23) is a contraction: it defines a unique, continuous and differentiable value function $v_0(g\,|y^o\,)$ for this interval of g. (differentiability follows from the differentiability of $y\,(g\,|y^o\,)$). We have

Lemma A.5. For any $y^o \in [y_R^*(d,n), y_R^{**}(d,n)]$ and any $g \in \left[g^2(y^o), g^3(y^o)\right]$, $u'(g) + \delta v_0'(g; y^o) = 1$.

Proof. Note that by Lemma A.4, for $g \in \left[g^2(y^o), g^3(y^o)\right]$, we have $y\,(g\,|y^o\,) \in \left[g^2(y^o), g^3(y^o)\right]$. From (14) we can write (for simplicity we write $y'(g\,|y^o\,) = y'(g)$):

$$\frac{1-u'(g)}{\delta} = \frac{1-y'(g)}{n} + u'(y(g))y'(g) + \left[1-u'(y(g))\right]y'(g)$$

for any $g \in \left[g^2(y^o), g^3(y^o)\right]$. But then using (14) again allows to substitute $1-u'(y(g))$ to obtain:

$$\begin{aligned} \frac{1-u'(g)}{\delta} &= \frac{1-y'(g)}{n} + u'(y(g))y'(g) \\ &\quad + \delta\left[\frac{1-y'(y(g))}{n} + u'(y^2(g))y'((y(g)) + \left[1-u'(y^2(g))\right]y'(y(g))\right]y'(g) \end{aligned}$$

where $y^0(g) = g$, $y^1(g) = y(g)$, $y^m(g) = y(y^{m-1}(g))$, and $[y']^0(g) = 1$, $[y']^1(g) = y'(g)$, and $[y']^m(g) = y'([y']^{m-1}(g))$. Iterating we have:

$$\begin{aligned}
\frac{1-u'(g)}{\delta} &= \lim_{n\to\infty}\sum\nolimits_{j=0}^{n}\delta^j\left[\frac{1-y'(y^j(g\,|y^o\,)\,|y^o\,)}{n}+u'(y^{j+1}(g))y'(y^j(g\,|y^o\,)\,|y^o\,)\right]\prod\nolimits_{i=0}^{j}[y']^i(y^{i-1}(g)) \\
&= v'(g\,|y^o\,)
\end{aligned}$$

This implies $u'(g)+\delta v_0'(g;y^o)=1$. ■

In the rest of the state space we define the value function recursively. In $\left[g^1(y^o), g^2(y^o)\right]$, if $g^1(y^o) < g^2(y^o)$, the value function is defined as:

$$v_0(g\,|y^o\,) = \frac{W+(1-d)g - y\left(g^2(y^o)\,|y^o\right)}{n} + u(y\left(g^2(y^o)\,|y^o\,\right)) + \delta v_0(y\left(g^2(y^o)\,|y^o\,\right)) \tag{24}$$

for $y\left(g^2(y^o)\,|y^o\right) \in \left[g^2(y^o), g^3(y^o)\right]$.

Lemma A.6. *For any $g \in \left[g^1(y^o), g^3(y^o)\right]$, $u(g)+\delta v(g\,|y^o\,)$ is concave and has slope larger or equal than 1.*

Proof. If $g^1(y^o) = g^2(y^o)$, the result follows from the previous lemma. Assume therefore, $g^1(y^o) < g^2(y^o)$. In this case $g^2(y^o) = y_R^*(d,n)$. For any $g \in \left[g^1(y^o), g^2(y^o)\right]$, $y(g;y^o) = y\left(y_R^*(d,n)\,|y^o\,\right)$. So we have $v_0'(g\,|y^o\,) = (1-d)/n$ implying: $u'(g)+\delta v_0'(g\,|y^o\,) = u'(g)+\delta(1-d)/n > 1$ since $g \le g^2(y^o) = y_R^*(d,n)$. The statement then follows from this fact and Lemma A.5. ■

Consider $g < g^1(y^o)$. In $\left[g_{-1}, g^1(y^o)\right]$ the value function is defined as:

$$v_{-1}(g\,|y^o\,) = u(W+(1-d)g) + \delta v_0(W+(1-d)g\,|y^o\,)$$

where $g_{-1} = \max\left\{0, \frac{g^1(y^o)-W}{1-d}\right\}$. Assume that we have defined the value function in $g \in \left[g_{-t}, g_{-(t-1)}\right]$ as v_{-t}, for all t such that $g_{-(t-1)} > 0$. Then we can define $v_{-(t+1)}$ as:

$$v_{-(t+1)}(g\,|y^o\,) = u(W+(1-d)g) + \delta v_{-t}(W+(1-d)g\,|y^o\,),$$

in $\left[g_{-(t+1)}, g_{-t}\right]$ with $g_{-(t+1)} = \frac{g_{-t}-W}{1-d}$.

Lemma A.7. *For any $g \in [0, g^3(y^o)]$, $u(g)+\delta v(g\,|y^o\,)$ is concave and it has slope greater than or equal than 1.*

Proof. We prove this by induction on t. Consider now the interval $\left[\frac{g^1(y^o)-W}{1-d}, g^1(y^o)\right]$. In this range we have

$$v_{-1}'(g\,|y^o\,) = [u'(W+(1-d)g) + \delta v_0'(W+(1-d)g\,|y^o\,)]\,(1-d) \ge 1-d$$

since $W+(1-d)g \in \left[g^1(y^o), g^3(y^o)\right]$. It follows that for $g \in \left[\frac{g^1(y^o)-W}{1-d}, g^1(y^o)\right]$:

$$u'(g)+\delta v'_{-1}(g\,|y^o\,) \geq u'(g)+\delta(1-d) > 1 \tag{25}$$

Where the last inequality follows from the fact that $g \leq g^2(y^o) < y_R^{**}(d,n)$. Note, moreover, that the right and left derivative of $v(g\,|y^o\,)$ at $g^1(y^o)$ are the same. To see this note that by the argument above, the left derivative is $(1-d)/n$; by Lemma A.5, however, the right derivative is $\left(1-u'(y_R^*(d,n))\right)/\delta = (1-d)/n$ as well. We conclude that $u'(g)+\delta v'_{-1}(g\,|y^o\,)$ is concave, it has derivative larger than 1. Assume that we have shown that for $g \in \left[g_{-t}, g^3(y^o)\right]$, $u(g)+\delta v_{-t}(g\,|y^o\,)$ is concave and $u'(g)+\delta v'_{-t}(g\,|y^o\,) > 1$. Consider in $g \in \left[g_{-(t+1)}, g_{-t}\right]$. We have:

$$v'_{-(t+1)}(g\,|y^o\,) = \left[u'(W+(1-d)g)+\delta v'_{-t}(W+(1-d)g\,|y^o\,)\right](1-d) \geq 1-d$$

since $W+(1-d)g \geq \left[g_{-t}, g^3(y^o)\right]$. So $u'(g)+\delta v'_{-(t+1)}(g\,|y^o\,) \geq u'(g)+\delta(1-d) \geq 1$. By the same argument as above, moreover, v is concave at g_{-t}. We conclude that for any $g \leq g^1$, $u(g)+\delta v(g\,|y^o\,)$ is concave and it has slope larger than 1. ■

We can define the value function for $g \geq g^3(y^o)$ as:

$$v_1(g\,|y^o\,) = \frac{W+(1-d)g-y_R^{**}(d,n)}{n} + u(y_R^{**}(d,n)) + \delta v_0(y_R^{**}(d,n)\,|y^o\,)$$

since, by Lemma A.2, $g^3(y^o) \geq y_R^{**}(d,n)$.

Lemma A.8. *For any $g \geq 0$, $u(g)+\delta v(g\,|y^o\,)$ is concave and it has slope less than or equal than 1.*

Proof. For $g > g^3(y^o)$, $v'(g\,|y^o\,) = (1-d)/n$. Since, by Lemma A.2, $g \geq y_R^{**}(d,n) \geq y_R^*(d,n)$, we have $u'(g)+\delta v'(g\,|y^o\,) < 1$. Previous lemmas imply $u(g)+\delta v(g|y^o)$ is concave and has slope greater than or equal than 1 for $g \leq g^3(y^o)$. This establishes the result. ■

Step 3. Define

$$x(g\,|y^o\,) = \frac{W+(1-d)g-y(g\,|y^o\,)}{n}, \text{ and } i(g\,|y^o\,) = \frac{y(g\,|y^o\,)-(1-d)g}{n}$$

as the levels of per capita private consumption and investment, respectively. Note that by construction, $x(g\,|y^o\,) \in [0, W/n]$. We now establish that $y(g\,|y^o\,)$, $x(g\,|y^o\,)$ and the associated value function $v\,(g\,|y^o\,)$ defined in the previous steps constitute an equilibrium. We first show that given $y(g\,|y^o\,)$, $v\,(g\,|y^o\,)$ describes the expected continuation value to an agent, starting at state g. Since $y(g\,|y^o\,) \in \left[g^2(y^o), g^3(y^o)\right]$ for $g \in \left[g^2(y^o), g^3(y^o)\right]$, $v\,(g\,|y^o\,)$ must be described by (23) for $g \in \left[g^2(y^o), g^3(y^o)\right]$. By construction, moreover, $v(g\,|y^o\,)$ is the expected continuation value to an agent in all states $g \geq g^3(y^o)$, and $g \leq g^2(y^o)$. We now show that $y(g\,|y^o\,)$ is an optimal reaction function given $v(g\,|y^o\,)$. An agent solves the problem (7), where $y_R(g) = y(g\,|y^o\,)$. Note that $y(g\,|y^o\,)$ satisfies the constraints of this problem if $y(g\,|y^o\,) \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y(g\,|y^o\,)$, so if $y(g\,|y^o\,) \leq W + (1-d)g$; and if $y(g\,|y^o\,) \geq \frac{n-1}{n} y(g\,|y^o\,)$, so if $y(g\,|y^o\,) \geq 0$. Both conditions are automatically satisfied by construction. If $g < g^1(y^o)$, we have $u'(y) + \delta v'(y) \geq 1$ for all $y \in [0, W + (1-d)g]$, so $y(g\,|y^o\,) = W + (1-d)g$ is optimal. If $g \geq g^1(y^o)$, then $y(g\,|y^o\,)$ is an unconstrained optimum, so again it is an optimal reaction function.

7.1.2 Necessity

We now prove that any stable steady state of an equilibrium must be in $[y_R^*(d,n), y_R^{**}(d,n)]$. We proceed in two steps.

Step 1. We first prove that $y_R^o \leq y_R^{**}(d,n)$. Suppose to the contrary that there is stable steady state at $y_R^o > y_R^{**}(d,n)$. We must have $y_R^o \in (y_R^{**}(d,n), W/d]$, since it is not feasible for a steady state to be larger than W/d. Consider a left neighborhood of y_R^o, $N_\varepsilon(y_R^o) = (y_R^o - \varepsilon, y_R^o)$. The value function can be written in $g \in N_\varepsilon(y_R^o)$ as:

$$\begin{aligned} v_R(g) &= \frac{W + (1-d)g - y_R(g)}{n} + u(y_R(g)) + \delta v_R(y_R(g)) \\ &= u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g) + \frac{W + (1-d)g}{n} + (1 - 1/n)\, y_R(g) \end{aligned}$$

In $N_\varepsilon(y_R^o)$ the constraint $y \geq \frac{n-1}{n} y_R(g)$ cannot be binding, else we would have $y_R(g) = (1 - 1/n) y_R(g)$, so $y_R(g) = 0$: but this is not possible in a neighborhood of $y_R^o > 0$. We consider two cases.

Case 1. Suppose first that $y_R^o < W/d$. We must therefore have that $y_R(g) < W + (1-d)g$ in $N_\varepsilon(y_R^o)$, so the constraint $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y$ is not binding. The solution is in the interior of the constraint set of (7), and the objective function $u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g)$ is constant for $g \in N_\varepsilon(y_R^o)$.

Lemma A.9. *There is a neighborhood $N_\varepsilon(y_R^o)$ in which $y_R(g)$ is strictly increasing.*

Proof. Suppose to the contrary that, for any $N_\varepsilon(y_R^o)$, there is an interval in $N_\varepsilon(y_R^o)$ in which $y_R(g)$ is constant. Using the expression for $v_R(g)$ presented above, we must have $v_R'(g) = (1-d)/n$ for any g in this interval. Since $N_\varepsilon(y_R^o)$ is arbitrary, then we must have a sequence $g^m \to y_R^o$ such that $v_R'(g^m) = (1-d)/n\ \forall m$. We can therefore write:

$$\begin{aligned} v_R^-(y_R^o) &= \lim_{\Delta \to 0} \frac{v_R(y_R^o) - v_R(y_R^o - \Delta)}{\Delta} = \lim_{\Delta \to 0} \lim_{m \to \infty} \frac{v_R(g^m) - v_R(g^m - \Delta)}{\Delta} \\ &= \lim_{m \to \infty} \lim_{\Delta \to 0} \frac{v_R(g^m) - v_R(g^m - \Delta)}{\Delta} = \frac{1-d}{n} \end{aligned}$$

where $v_R^-(y_R^o)$ is the left derivative of $v_R(g)$ at y_R^o, and the second equality follows from the continuity of $v_R(g)$. Consider now a marginal reduction of g at y_R^o. The change in utility is (as

$\Delta \to 0$):

$$\begin{aligned} \Delta U\left(y_R^o\right) &= u(y_R^o - \Delta) - u(y_R^o) + \delta\left[v_R(y_R^o - \Delta) - v_R(y_R^o)\right] + \Delta \\ &= \left[1 - \left(u'(y_R^o) + \delta\frac{1-d}{n}\right)\right]\Delta \end{aligned}$$

In order to have $\Delta U\left(y_R^o\right) \leq 0$, we must have $u'(y_R^o)+\delta(1-d)/n \geq 1$. This implies $y_R^o \leq y_R^*(d,n) < y_R^{**}(d,n)$, a contradiction. Therefore, if there is stable steady state at $y_R^o > y_R^{**}(d,n)$, then $y_R(g)$ is strictly increasing in a neighborhood $N_\varepsilon(y_R^o)$. ■

Lemma A.9 implies that there is a neighbrohood $N_\varepsilon(y_R^o)$ in which $u(g)+\delta v_R(g)-g$ is constant. Since y_R^o is a stable steady state and $y_R(g)$ is strictly increasing. Moreover, for any open left neighborhood $N_{\varepsilon'}(y_R^o) = (y_R^o - \varepsilon', y_R^o) \subset N_\varepsilon(y_R^o)$, $g \in N_{\varepsilon'}(y_R^o)$ implies $y_R(g) \in N_{\varepsilon'}(y_R^o)$). These observations imply:

Lemma A.10. *There is a neighborhood $N_\varepsilon(y_R^o)$ in which*

$$y_R'(g) = \frac{n}{n-1}\left(\frac{1-u'(g)}{\delta} - \frac{1-d}{n}\right) \tag{26}$$

Proof. There is a $N_\varepsilon(y_R^o)$ and a constant K such that $\delta v_R\left(g\right) = K + g - u(g)$ for $g \in N_\varepsilon(y_R^o)$). Hence $v_R\left(g\right)$ is differentiable in $N_\varepsilon(y_R^o)$. Moreover, $y_R(g) \in N_\varepsilon(y_R^o)$ for all $g \in N_\varepsilon(y_R^o)$. Hence $u(y_R(g)) + \delta v\left(y_R(g)\right) - y_R(g)$ is constant in $g \in N_\varepsilon(y_R^o)$ as well. These observations and the definition of $v_R(g)$ imply that $v_R'(g) = \frac{1-d}{n} + \left(1 - \frac{1}{n}\right) y_R'(g)$ in $N_\varepsilon(y_R^o)$ (where $y_R(g)$ must be differentiable otherwise $v_R(g)$ would not be differentiable). Given that $u'(g) + \delta v_R'(g) = 1$ in $g \in N_\varepsilon(y_R^o)$, we must have:

$$u'(g) + \delta v_R'(g) = u'(g) + \delta\left[\frac{1-d}{n} + \left(1 - \frac{1}{n}\right) y_R'(g)\right] = 1$$

which implies (26) for any $g \in N_\varepsilon(y_R^o)$. ■

Let g^m be a sequence in $N_\varepsilon(y_R^o)$ such that $g^m \to y_R^o$. We must have

$$\begin{aligned} y_R^-(y_R^o) &= \lim_{\Delta\to 0}\frac{y_R(y_R^o) - y_R(y_R^o - \Delta)}{\Delta} = \lim_{\Delta\to 0}\lim_{m\to\infty}\frac{y_R(g^m) - y_R(g^m - \Delta)}{\Delta} \\ &= \lim_{m\to\infty}\lim_{\Delta\to 0}\frac{y_R(g^m) - y_R(g^m - \Delta)}{\Delta} = \frac{n}{n-1}\left(\frac{1-u'(y_R^o)}{\delta} - \frac{1-d}{n}\right) \end{aligned}$$

where $y_R^-(y_R^o)$ is the left derivative of $y_R(y_R^o)$, and the second equality follows from continuity. Consider a state $(y_R^o - \Delta)$. For y_R^o to be stable we need that for any small Δ:

$$y_R(y_R^o - \Delta) \geq y_R^o - \Delta = y_R(y_R^o) + (y_R^o - \Delta) - y_R^o$$

where the equality follows from the fact that $y_R(y_R^o) = y_R^o$. As $\Delta \to 0$, this implies $y_R^-(y_R^o) \leq 1$ in $N_\varepsilon(y_R^o)$. By (27), we must therefore have:

$$\frac{n}{n-1}\left(\frac{1-u'(y_R^o)}{\delta} - \frac{1-d}{n}\right) \leq 1$$

This implies: $y_R^o \leq y_R^{**}(d,n)$, a contradiction.

Case 2. Assume now that $y_R^o = W/d$ and it is a strict local maximum of the objective function $u(y) + \delta v_R(y) - y$. In this case in a left neighborhood $N_\varepsilon(y_R^o)$, we have that the upperbound $y \leq \frac{W+(1-d)g}{n} + \frac{n-1}{n} y_R(g)$ is binding: implying $y_R(g) = W + (1-d)g$ in $N_\varepsilon(y_R^o)$. We must therefore have a sequence of points $g^m \to y_R^o$ such that $g^m = y_R(g^{m-1})$ and $y_R(g^m) = W + (1-d)g^m$ $\forall m$. Given this, we can write:

$$\begin{aligned} v_R(g^m) &= u(g^{m+1}) + \delta v_R(g^{m+1}) = u(g^{m+1}) + \delta \left[u(g^{m+2}) + \delta v_R(g^{m+2}) \right] \\ &= \sum\nolimits_{j=0}^{\infty} \delta^j u(W + (1-d) g^{m+j}) \end{aligned}$$

note that since $g^{m+1} = W + (1-d)g^m$, the derivative of g^{m+1} with respect to g^m is $\left[g^{m+1}\right]' = (1-d)$. By an inductive argument, it is easy to see that $\left[g^{m+j}\right]' = (1-d)^j$. So $v_R(g^m)$ is differentiable and:

$$\delta v_R'(g^m) = \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(W + (1-d)g^{m+j}).$$

Since $u'(g^m) + \delta v_R'(g^m) \geq 1$, we have:

$$u'(g^m) + \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(W + (1-d)g^{m+j}) \geq 1$$

for all m. Consider the limit as $m \to \infty$. Since $u'(g)$ is continuous and $g^m \to y_R^o$, we have:

$$\begin{aligned} 1 &\leq \lim_{m\to\infty} \left[u'(g^m) + \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(W + (1-d)g^{m+j}) \right] \\ &= u'(y_R^o) + \sum\nolimits_{j=0}^{\infty} \left[\delta(1-d)\right]^{j+1} u'(y_R^o) = \frac{u'(y_R^o)}{1 - \delta(1-d)} \end{aligned}$$

This implies $y_R^o \leq [u']^{-1}(1 - \delta(1-d)) < y_R^{**}(d,n)$, a contradiction.

Case 3. Assume now that $y_R^o = W/d$, but it is not a strict maximum of $u(y) + \delta v_R(y) - y$ in any left neighborhood. It must be that $u(y) + \delta v_R(y) - y$ is constant in some left neighborhood $N_\varepsilon(y_R^o)$. If this were not the case, then in any left neighborhood we would have an interval in which $y_R(g)$ is constant, but this is impossible by Lemma A.9. But then if $u(y) + \delta v_R(y) - y$ is constant in some $N_\varepsilon(y_R^o)$, the same argument as in Case 1 of Step 1 implies a contradiction.

Step 2. We now prove that $y_R^o \geq y_R^*(d,n)$. Assume there is stable steady state at $y_R^o < y_R^*(d,n)$. Since $\lim_{g\to 0} u'(g) = \infty$, $y_R^o > 0$. There is therefore a neighborhood $N_\varepsilon(y_R^o) = (y_R^o, y_R^o + \varepsilon)$ in which $y_R(g)$ satisfies all the constraints of (7) and it maximizes $u(y) + \delta v_R(y) - y$ We conclude that the objective function $u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g)$ is constant in $N_\varepsilon(y_R^o)$. By the same argument as in Lemma A.9 it follows that there is a neighborhood $N_\varepsilon(y_R^o)$ in which $y_R(g)$ is strictly increasing. Since y_R^o is a stable steady state and $y_R(g)$ is strictly increasing in $N_\varepsilon(y_R^o)$, there is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o such that for any open right neighborhood $N_{\varepsilon'}(y_R^o) = (y_R^o, y_R^o + \varepsilon') \subset N_\varepsilon(y_R^o)$, $g \in N_{\varepsilon'}(y_R^o))$ implies $y_R(g) \in N_{\varepsilon'}(y_R^o)$. By the same argument as in Lemma A.10, it follows that there is a $N_{\varepsilon'}(y_R^o)$ in which $y_R'(g)$ is given by (26). Equation (26), however, implies that $y_R'(g) \geq 0$ only for states $g \geq y_R^*(d,n)$. This implies that $y_R(g)$ is non-monotonic, a contradiction. ■

7.2 Proof of Proposition 2

We start from the first point of Proposition 2. Consider the equilibrium described in Proposition 1 with steady state y_R^o. If $g \in \left[g^2(y^o), g^3(y^o)\right]$, then $y_R(g) = y\left(g \left| y_R^o\right.\right)$. From (14) we can see that $y'\left(g \left| y_R^o\right.\right) \leq 1-d$ if and only if $g \leq \overline{y}(d)$. If $y_R^o \leq \overline{y}(d)$, then $y_R'(g) \leq 1-d$ at any point of differentiability less or equal than y_R^o (i.e. almost everywhere, since $y_R(g)$ in this equilibrium is almost everywhere differentiable). In addition, since $y_R(g)$ is strictly increasing in $\left[g^2(y^o), \overline{y}(d)\right]$, there is a m^* such that for $m > m^*$, $g^{m+1} = y(g^m \left| y_R^o\right.)$, and hence $y'(g^m \left| y_R^o\right.) < 1-d$.

We now show that if the steady state $y_R^o \in (\overline{y}(d), y_R^{**}(d,n)]$, then starting from $y_o \leq y_R^o$, for any equilibrium path $g^m \to y_R^o$, there is a m^* such that for $m > m^*$ we have strategic complementarity. First note that $d > 0$ implies $\overline{y}(d) < y_R^{**}(d,n)$, so $y_R^o \in (\overline{y}(d), y_R^{**}(d,n)]$ exists. Since $y_R^o \leq y_R^{**}(d,n)$, it must be that $y_R^o \in (\overline{y}(d), W + (1-d)y_R^o)$. By continuity, therefore, there is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o in which $y_R(g)$ is interior of the constraint set of problem (7), and so $u(y_R(g)) + \delta v_R(y_R(g)) - y_R(g)$ is constant. Using the same argument as in Step 1 in Section 7.1.2, we can establish that $y_R(g)$ is equal to $y(g \left| y_R^o\right.)$ in $N_\varepsilon(y_R^o)$, i.e. the function that solves (26) with the initial condition $y_R^o = y(y_R^o \left| y_R^o\right.)$ in a left neighborhood of y_R^o. It follows that starting from $y_o < y_R^o$, for any equilibrium path $g^m \to y_R^o$, there is a m^* such that for $m > m^*$, $g^{m+1} = y(g^m \left| y_R^o\right.)$, and $y'(g^m \left| y_R^o\right.)$, is given by (26). Since by assumption $y_R^o > \overline{y}(d)$, for m large enough we must have $g^m > \overline{y}(d)$, and so $y'(g^m \left| y_R^o\right.) > (1-d)$. ■

7.3 Proof of Proposition 3

Let $y_R^o \in [y_R^*(d,n), y_R^{**}(d,n)]$ be the steady state of $y_R(g)$. Define:

$$g^2\left(y_R^o\right) = \max\left\{\min_{g \geq 0}\left\{y\left(g \left| y_R^o\right.\right) \leq W + (1-d)g\right\}, y_R^*(d,n)\right\}$$

We now show that, as in (9), $y_R(g)$ is equal to $y(g \left| y_R^o\right.)$ (i.e., a function that solves the differential equation (14) with the initial condition $y(y_R^o \left| y_R^o\right.) = y_R^o$) for all $g \in [g_2(y_R^o), y_R^o]$.

Consider first the states $g \in [y_R^*(d,n), y_R^o]$. Assume that there is an interval $(g_1, g_2) \subset [y_R^*(d,n), y_R^o]$ in which $y_R(g)$ is constant. For any point in (g_1, g_2) we would have $v_R'(g) = (1-d)/n$. Since $g_1 \geq y_R^*(d,n)$, we have $u(g) + \delta v_R'(g) = u(g) + \delta(1-d)/n < 1$ for $g \in (g_1, g_2)$. Since $u(g) + \delta v_R(g)$ is concave, we must have that the left derivative of $u(g) + \delta v_R(g)$ at y_R^o exists and it strictly smaller than 1. But this is in contradiction with the optimality of y_R^o at y_R^o. We conclude that $y_R(g)$ is strictly increasing in $(y_R^*(d,n), y_R^o]$.

Because $y_R^o \leq y_R^{**}(d,n) < W/d$, we must have $y_R^o < W + (1-d)y_R^o$. There is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o in which $y_R(g) \in (0, W + (1-d)g)$. As it can be immediately be verified, this implies that there is a neighborhood $N_\varepsilon(y_R^o)$ of y_R^o in which $y_R(g)$ is interior of the constraint set of problem (7). Using the same argument used in Step 1 of Section 7.1, we conclude that that $y_R(g)$ must be equal is equal to $y(g\,|y_R^o)$ in $N_\varepsilon(y_R^o)$. Since $y(g\,|y_R^o)$ is strictly increasing in $[y_R^*(d,n), y_R^o]$, we must have that $y_R(g) = y(g\,|y_R^o)$ until $y(g\,|y_R^o)$ becomes flat, or it intersects from below the curve $W + (1-d)g$: that is for $g \geq g_2(y_R^o)$. We conclude that a monotonic investment function below steady state y_R^o, can differ from $y\left(g^2(y)\,|y_R^o\right)$ only for $g < g_2(y_R^o)$.

To prove the rest of the proposition, we now have two cases:

Case 1. Assume first $g_2(y_R^o) > y_R^*(d,n)$. In this case we know that the right derivative of $u(g) + \delta v_R(g)$ at $g_2(y_R^o)$ is 1. The left derivative of $u(g) + \delta v_R(g)$ at $g_2(y_R^o)$ can be equal to one only if $y_R(g)$ is equal to $y(g\,|y_R^o)$. For $g \in [y_R^*(d,n), g_2(y_R^o)]$, $y'(g\,|y_R^o) < (1-d)$. Since we have $y(g_2(y_R^o)\,|y_R^o) = W + (1-d)y(g_2(y_R^o)\,|y_R^o)$, we would have $y(g\,|y_R^o) > W + (1-d)y(g\,|y_R^o)$, which is unfeasible. So the left derivative of $u(g) + \delta v_R(g)$ at $g_2(y_R^o)$ must be strictly greater than one. This implies $y_R(g) = W + (1-d)g$, as in (9). We conclude that if $g_2(y_R^o) > y_R^*(d,n)$, any investment function with steady state y_R^o must be equal to $y_R(g\,|y_R^o)$ for $g \leq y_R^o$ (i.e., in this case, $W + (1-d)g$). This implies that the convergence path to the steady state from any $g_0 \leq y_R^o$ is unique.

Case 2. Assume now $g_2(y_R^o) = y_R^*(d,n)$. In this case we know that the right derivative of $u(g) + \delta v_R(g)$ at $g_2(y_R^o)$ is 1. The left derivative of $u(g) + \delta v_R(g)$ at $g_2(y_R^o)$ can be equal to one only if $y_R(g)$ is equal to $y(g\,|y_R^o)$. For $g \leq y_R^*(d,n)$, $y'(g\,|y_R^o) < 0$, so the investment function would not be monotonic. So the left derivative of $u(g) + \delta v_R(g)$, for any $g < g_2(y_R^o)$, must be strictly larger than one. Define $g_1(y_R^o)$ as in the proof of Proposition 1, $g^1(y^o) = \max\left\{0, \left(y\left(g^2(y)\,|y^o\right) - W\right)/(1-d)\right\}$: this is the point at which $W + (1-d)g$ is equal to $y\left(g^2(y)\,|y^o\right)$, if positive. For $g \leq g^1(y^o)$, since the objective function is strictly increasing, $W + (1-d)g < g^2(y)$, so $y_R(g) = W + (1-d)g$. By monotonicity, in $\left[g^1(y^o), g^2(y^o)\right]$, we must have $y(g) = y\left(g^2(y)\,|y^o\right)$. This, again, implies that the convergence path to the steady state from any $g_0 \leq y_R^o$ is unique.

The result that the convergence path is monotonically increasing follows from the fact that $y_R(g)$ is increasing and $y_R(g) > g$ for $g < y_R^o$. The result that convergence is only in the limit if $y_R^o > y_R^*(d,n)$, follows from the fact that $y_R(g)$ is strictly increasing in $(y_R^*(d,n), y_R^o)$. ■

7.4 Proof of Proposition 4

Since we are in a regular economy, we have $W/d > y_R^{**}(d,n)$. To construct the equilibrium we proceed in two steps.

Step 1. We first construct the strategies. Remember that $\overline{y}(d)$ is defined by (16). This is the point at which the solution of the differential equation (14) has slope $(1-d)$. Given this, let $\widehat{y}(g)$ be the solution of (14) with the initial condition $\widehat{y}(\overline{y}(d)) = (1-d)\overline{y}(d)$. Define g_{IR}^2 as:

$$g_{IR}^2 = \max\left\{\min_{g\geq 0}\left\{\widehat{y}(g) \leq W + (1-d)g\right\}, y_R^*(d,n)\right\}. \tag{27}$$

The investment function is defined as:

$$y_{IR}(g) = \begin{cases} \min\left\{W + (1-d)g, \widehat{y}\left(g_{IR}^2\right)\right\} & g \leq g_{IR}^2 \\ \widehat{y}\,(g) & g_{IR}^2 < g \leq \overline{y}(d) \\ (1-d)g & g \geq \overline{y}(d) \end{cases}$$

Using the same argument as in the proof of Proposition 1, we can prove that $y_{IR}(g)$ is continuous and almost everywhere differentiable with right and left derivative at any point, and $y_{IR}(g) \in [(1-d)g, W + (1-d)g]$ for any g. Finally, it is easy to see that $y_{IR}(g)$ has a unique fixed-point y_{IR}^o such that $y_{IR}(y_{IR}^o) = y_{IR}^o \in \left[g_{IR}^2, \overline{y}(d)\right]$.

Step 2. We now construct the value function $v_{IR}(g)$ associated to $y_{IR}(g)$, and prove that $y_{IR}(g), v_{IR}(g)$ is an equilibrium. For $g \leq \overline{y}(d)$, we define the value function exactly as in Step 2 of Section 7.1.1. For $g \geq \overline{y}(d)$, note that $y_{IR}(g) < g$, so we can define the value function recursively as:

$$v_{IR}(g) = \frac{W}{n} + u((1-d)g) + \delta v_{IR}((1-d)g). \tag{28}$$

The value function defined above is continuous in g. Using the same argument as in Step 2 of Section 7.1.1 we can show that $u(g) + \delta v(g; y_{IR}^o) - y$ is weakly concave in g for $g \leq \overline{y}(d)$; it is strictly increasing in $\left[0, g_{IR}^2\right]$, and flat in $\left[g_{IR}^2, \overline{y}(d)\right]$. Consider now states $g > \overline{y}(d)$. Let $g^4 = \frac{\overline{y}(d)}{1-d}$. In $\left[\overline{y}(d), g^4\right]$, we must have $(1-d)g \in \left[g_{IR}^2, \overline{y}(d)\right]$. Note that $u'(g) + \delta v_{IR}'(g) = 1$ for $g \in \left[g_{IR}^2, \overline{y}(d)\right]$, so by (28) we have

$$v_{IR}'(g) = (1-d)\left[u'((1-d)g) + \delta v_{IR}'((1-d)g)\right] = 1-d$$

for $g \in \left[\overline{y}(d), g^4\right]$. This fact implies that $u'(g) + \delta v_{IR}'(g) = u'(g) + \delta\,(1-d)$ for any $g \in \left[\overline{y}(d), g^4\right]$, and hence it is concave in this interval. It follows that $v_{IR}(g)$ is concave in $g \leq g^4$ because $u'(g) + \delta v_{IR}'(g) \leq 1$ for any $g \in \left[\overline{y}(d), g^4\right]$. Using a similar approach we can prove that $v_{IR}(g)$ is concave for all g, and we have $u'(g) + \delta v'_{IR}(g) \leq 1$ for $g \geq \overline{y}(d)$. To prove that $y_{IR}(g), v_{IR}(g)$ is an equilibrium, we proceed exactly as in Step 3 of Section 7.1.1 to establish that $y_{IR}(g)$ is optimal given $v_{IR}(g)$, and that $v_{IR}(g)$ satisfied (18) given $y_{IR}(g)$.■

7.5 Proof of Proposition 5

We proceed in 2 steps.

Step 1. The same argument used in Step 1 of Section 7.1.2 shows that no equilibrium stable steady state can be greater than $y_R^{**}(d,n) = [u']^{-1}(1-\delta(1-d/n))$. The same argument used in Step 2 in Section 7.1.2 we can show that no equilibrium can be less than $y_R^{*}(d,n)$, so $y_{IR}^{*}(d,n) \geq y_R^{*}(d,n)$.

Step 2. Consider a sequence $d^m \to 0$. For each d^m there is at least an associated equilibrium $y_m(g)$, $v_m(g)$ with steady state y_m^o. If follows trivially that $\lim_{m\to\infty} y_R^{**}(d^m,n) = [u']^{-1}(1-\delta) = \overline{y}(0)$.

What remains to be shown is that $\lim_{m\to\infty} y_{IR}^{*}(d^m,n) = [u']^{-1}(1-\delta) = \overline{y}(0)$. Let Γ_m be the set of equilibrium steady states when the rate of depreciation is d^m. We now show by contradiction that for any $\xi > 0$, there is a $\widetilde{m}$ such that for $m > \widetilde{m}$, $\inf_y \Gamma_m \geq \overline{y}(0) - \xi$. Since $\inf_y \Gamma_m \leq y_R^{**}(d,n)$, this will immediatly imply that $y_{IR}^{*}(d,n) \to \overline{y}(0)$. Suppose to the contrary there is a sequence of steady states y_m^0, with associated equilibrium investment and value functions $y_m(g)$, $v_m(g)$, and an $\xi > 0$ such that $y_m^0 < \overline{y}(0) - \xi$ for any arbitrarily large m. Define $y_m^0(g) = y_m(g)$, and $y_m^j(g) = y_m(y_m^{j-1}(g))$ and consider a marginal deviation from the steady state from y_m^0 to $y_m^0 + \Delta$. By the irreversibility constraint we have $y_m(g) \geq (1-d^m)\,g$. Using this property and the fact that y_m^0 is a steady state, so $y_m^j(y_m^0) = y_m^0$, we have:

$$y_m(y_m^0+\Delta) - y_m(y_m^0) \geq (1-d^m)\left(y_m^0+\Delta\right) - y_m^0 = (1-d^m)\,\Delta - d^m y_m^0$$

This implies that, as $m \to \infty$, for any given Δ:

$$\frac{y_m(y_m^0+\Delta) - y_m^0}{\Delta} \geq 1 + o_1(d^m)$$

where $o_1(d^m) \to 0$ as $m \to 0$. We now show with an inductive argument that a similar property holds for all iterations $y_m^j(y_m^0)$. Assume we have shown that:

$$\frac{y_m^{j-1}(y_m^0+\Delta) - y_m^0}{\Delta} \geq 1 + o_{j-1}(d^m)$$

where $o_{j-1}(d^m) \to 0$ as $m \to 0$. We must have:

$$y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^j(y_m^0) \geq (1-d^m)\, y_m^{j-1}(y_m^0+\Delta) - y_m^0$$

We therefore have:

$$y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^0 \geq y_m^{j-1}(y_m^0+\Delta) - y_m^0 - d^m y_m^{j-1}(y_m^0+\Delta)$$

so we have:

$$\begin{aligned} \frac{y_m(y_m^{j-1}(y_m^0+\Delta)) - y_m^0}{\Delta} &\geq \frac{y_m^{j-1}(y_m^0+\Delta) - y_m^0}{\Delta} - \frac{d^m y_m^{j-1}(y_m^0+\Delta)}{\Delta} \\ &\geq 1 + o_j\left(d^m\right) \end{aligned} \tag{29}$$

where $o_j\left(d^m\right) = o_{j-1}\left(d^m\right) - \frac{d^m y_m^{j-1}(y_m^0+\Delta)}{\Delta}$, so $o_j\left(d^m\right) \to 0$ as $m \to 0$.

We can write the value function after the deviation to $y_m^0 + \Delta$ as:

$$V(y_m^0+\Delta) = \sum\nolimits_{j=0}^{\infty} \delta^{j-1}\left[\frac{W + (1-d^m)\, y_m^{j-1}(y_m^0+\Delta) - y_m^j(y_m^0+\Delta)}{n} + u(y_m^j(y_m^0+\Delta))\right]$$

For any given function $f(x)$, define $\Delta f(x) = f(x+\Delta) - f(x)$. We can write:

$$\begin{aligned} \Delta V(y_m^0)/\Delta &= \sum\nolimits_{j=0}^{\infty} \delta^{j-1}\left[\begin{array}{c} \frac{(1-d^m)\Delta y_m^{j-1}(y_m^0)/\Delta - \Delta y_m^j(y_m^0)/\Delta}{n} \\ \\ +\left[u'(y_m^0) + o\left(\Delta\right)\right]\Delta y_m^j(y_m^0)/\Delta \end{array}\right] \\ &\geq \sum\nolimits_{j=0}^{\infty} \delta^{j-1}\left[\begin{array}{c} \frac{(1-d^m)(1+o_{j-1}(d^m)) - (1+o_j(d^m))}{n} \\ \\ +\left[u'(y_m^0) + o\left(\Delta\right)\right](1 + o_j\left(d^m\right)) \end{array}\right] \end{aligned} \tag{30}$$

where $o\left(\Delta\right) \to 0$ as $\Delta \to 0$. In the first equality we use the fact that if we choose Δ small, since $y_m(g)$ is continuous, $\Delta y_m^j(y_m^0)$ is small as well. This implies that

$$\left(u(y_m^j(y_m^0+\Delta)) - u(y_m^j(y_m^0))\right) / \left[y_m^j(y_m^0+\Delta) - y_m^j(y_m^0)\right]$$

converges to $u'(y_m^j(y_m^0))$ as $\Delta \to 0$. The inequality in 30 follows from (29). Given Δ, as $m \to \infty$, we therefore have $\lim_{m\to\infty} \Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)+o(\Delta)}{1-\delta}$. We conclude that for any $\varepsilon > 0$, there must be a Δ_ε such that for any $\Delta \in (0, \Delta_\varepsilon)$ there is a m_Δ guaranteeing that $\Delta V(y_m^0)/\Delta \geq \frac{u'(y_m^0)}{1-\delta} - \varepsilon$ for $m > m_\Delta$. After a marginal deviation to $y_m^0 + \Delta$, therefore, the change in agent's objective function is:

$$u'(y_m^0) + \delta\Delta V(y_m^0)/\Delta - 1 \geq \frac{u'(y_m^0)}{1-\delta} - \delta\varepsilon - 1$$

for m sufficiently large. A necessary condition for the un-profitability of a deviation from y_m^0 to $y_m^0 + \Delta$ is therefore:

$$y_m^0 \geq \left[u'\right]^{-1}\left(1 - \delta + \delta\varepsilon\left(1-\delta\right)\right). \tag{31}$$

Since ε can be taken to be arbitrarily small, for an arbitrarily large m, (31) implies $y_m^0 \geq \overline{y}(0) - \xi/2$, which contradicts $y_m^0 < \overline{y}(0) - \xi$. We conclude that $y_{IR}^*(d,n) \to \overline{y}(0)$ as $d \to 0$. ■

7.6 Proof of Proposition 6

Since we are in a regular economy, we have $W/d > y_R^{**}(d,n)$. To construct the equilibrium we proceed in two steps.

Step 1. We first construct the strategies. Let $y_\Delta(g)$ be the solution of (14) with the initial condition $y_\Delta(y_R^*(d,n)) = y_R^*(d,n) - \Delta$. Let $g^2(\Delta)$ be:

$$g^2(\Delta) = \max\left\{\min_{g\geq 0}\{y_\Delta(g) \leq W + (1-d)g\}, y_R^*(d,n) - \Delta\right\} \tag{32}$$

Let $g^3(\Delta)$ be defined by $y_\Delta\left(g^3(\Delta)\right) = y_R^{**}(d,n)$. The investment function is defined as:

$$y_R^{nm}(g) = \begin{cases} \min\left\{W + (1-d)g, y_\Delta\left(g^2(\Delta)\right)\right\} & g \leq g^2(\Delta) \\ y_\Delta(g) & g^2(\Delta) < g \leq g^3(\Delta) \\ y_\Delta(g^3(\Delta)) & g \geq g^3(\Delta) \end{cases} \tag{33}$$

It follows immediately from the proof of Proposition 1 that $y_\Delta(g)$ is continuous and almost everywhere differentiable with right and left derivative at any point. Since $\lim_{g\to\infty} y_\Delta(g) > 0$ as $g \to 0$, there is a fixed-point $y_\Delta^o \in \left(g^2(\Delta), y_R^*(d,n)\right)$ such that $y_\Delta(y_\Delta^o) = y_\Delta^o$. We have:

Lemma A.11. *There is a Δ^* such that for $\Delta \leq \Delta^*$: $y_\Delta(g) \in \left[g^2(\Delta), g^3(\Delta)\right]$ for any $g \in \left[g^2(\Delta), g^3(\Delta)\right]$.*

Proof. First note that $g^2(\Delta)$ is continuous and decreasing in Δ. We first show that there is a Δ' such that for $\Delta \leq \Delta'$ $y_\Delta(g) \leq W + (1-d)g$ *for any* $g \in \left[g^2(\Delta), g^3(\Delta)\right]$. Assume not. Then there must be a sequences $\Delta^m \to 0$ and $(g^m)_m$ with $g^m \in \left[g^2(\Delta^m), g^3(\Delta^m)\right]$, such that $y_{\Delta^m}(g^m) > W + (1-d)g^m$ for any m and $g^m \to y_R^*(d,n)$ (since $g^2(\Delta^m) \to y_R^*(d,n)$). This implies: $\lim_{m\to\infty} y_{\Delta^m}(g^m) \geq W + (1-d)\lim_{m\to\infty} g^m$. Note that

$$\begin{aligned} W + (1-d)\lim_{m\to\infty} g^m &= W + (1-d)y_R^*(d,n) \\ &\leq \lim_{m\to\infty} y_{\Delta^m}(g^m) = y_R^*(d,n) \end{aligned}$$

implying $W/d \leq y_R^{**}(d,n)$. However, in a regular economy $W/d > y_R^{**}(d,n)$, a contradiction. Since by construction $y_\Delta(g) \geq y_R^*(d,n) - \Delta$, we conclude that $y_\Delta(g) \geq g^2(\Delta)$ for $\Delta \leq \Delta'$. We now prove that there is a Δ'' such that, for $\Delta \leq \Delta''$, $y_\Delta(g) \leq g^3(\Delta)$. Note that for $g \geq y_\Delta^o$, we have $y_\Delta(g) \leq g^3(\Delta)$ by construction (since $y_\Delta(g) \leq g$). For $g < y_\Delta^o$, we have $g \leq y_\Delta\left(g^2(\Delta)\right)$. If the statement were not true, then we would have sequences $\Delta^m \to 0$ and $g^m \to y_R^*(d,n)$ such that $y_{\Delta^m}(g^m) > y_R^{**}(d,n)$, but this is impossible since $y_{\Delta^m}(g^m) \to y_R^*(d,n) < y_R^{**}(d,n)$. ■

Step 2. Using Lemma A.11, we can now construct the concave value function $v_R^{nm}(g)$ associated with the investment function in (33) exactly as in Proposition 1. The proof that $v_R^{nm}(g)$, $v_R^{nm}(g)$ is an equilibrium is identical to the corresponding proof in Proposition 1 and omitted. To prove that the steady state y^o_Δ is stable and that convergence to it is characterized by damped oscillations, we need to prove that $[y_R^{nm}]'(y^o_\Delta) \in (-1,0)$. Note that since $y^o_\Delta < y^*_R(d,n)$, $[y_R^{nm}]'(y^o_\Delta) < 0$; and since y^o_Δ is arbitrarily close to $y^*_R(d,n)$ for Δ small, then $[y_R^{nm}]'(y^o_\Delta) \in (-1,0)$.

Step 3. The fact that a steady state is bounded above by $y^{**}_R(d,n)$ is already proven as in Proposition 1. To show that a steady state is bounded below by $y^{***}_R(d,n)$ we proceed by contradiction. Suppose to the contrary that there is a steady state $y^o < y^{***}_R(d,n)$. Since it must be $y^o > 0$, in a right neighborhood of y^o we must have that the investment function $y(g)$ is interior in $(0, W + (1-d)g)$. Following an argument similar to the argument of Proposition 1 we can show that in this range the investment function cannot be constant (otherwise the agents would want to invest more), and that $y(g)$ must satisfy (26). Since y^o is a stable equilibrium we must have $|y'(y^o)| < 1$. This implies $y^o \geq y^{***}_R(d,n)$. ■

7.7 Proof of Proposition 7

Let $y_\Delta(g)$ be the solution of (14) with the initial condition $y_\Delta(y^*_R(d,n)) = y^*_R(d,n) - \Delta$. Let $g^2(\Delta)$ be:

$$g^2(\Delta) = \max\left\{\min_{g\geq 0}\{y_\Delta(g) \leq W + (1-d)g\}, y^*_R(d,n) - \Delta\right\} \tag{34}$$

and define $g^3(\Delta) = y^{oo}_\Delta$, where y^{oo}_Δ is the fixed point of $y_\Delta(g)$ to the right of $y^*_R(d,n)$. The investment function is defined as:

$$y_R^{nm}(g) = \begin{cases} \min\left\{W + (1-d)g, y_\Delta\left(g^2(\Delta)\right)\right\} & g \leq g^2(\Delta) \\ y_\Delta(g) & g^2(\Delta) < g \leq g^3(\Delta) \\ y_\Delta(g^3(\Delta)) & g \geq g^3(\Delta) \end{cases} \tag{35}$$

It follows immediately that $y_\Delta(g)$ is continuous and almost everywhere differentiable with right and left derivative at any point. Let y^o_Δ is the fixpoint of $y_\Delta(g)$ on the left of $y^*_R(d,n)$. Note that y^o_Δ is decreasing in Δ and continuous in Δ. Since by Assumption 2 there is a Δ' such that $y'_{\Delta'}(g) < -1$, for any $y^o \in [y^{***}_R(d,n), y^*_R(d,n)]$ we must have a Δ such that $y_\Delta(y^o) = y^o$. We only need to prove that, in correspondence to this Δ, $y_\Delta(g)$ is an equilibrium investment function for W sufficiently large. We have:

Lemma A.12. Given Assumption 2, for any $y^o \in [y^{***}_R(d,n), y^*_R(d,n)]$, there is a Δ and a W_Δ such that for $W > W_\Delta$, $y_\Delta(y^o) = y^o$ and $y_\Delta(g) \in \left[g^2(\Delta), g^3(\Delta)\right]$ *in* $g \in \left[g^2(\Delta), g^3(\Delta)\right]$.

Proof. The fact that there is a a Δ such that $y_\Delta(y^o) = y^o$ (for any W) follows from the previous argument. We now show that when W is large enough we also have $y_\Delta(g) \in \left[g^2(\Delta), g^3(\Delta)\right]$ for

$g \in \left[g^2(\Delta), g^3(\Delta)\right]$. It is immemdiate to see that for W large enough, $y_\Delta(g) \leq W + (1-d)g$ for $g \in \left[g^2(\Delta), g^3(\Delta)\right]$. Given this, $y_\Delta(g) \geq g^2(\Delta)$ follows directly. We next prove that $y_\Delta(g) \leq g^3(\Delta)$. For $g \geq y_\Delta^o$ we have $y_\Delta(g) \leq g^3(\Delta)$ by construction. Assume now $g \leq y_\Delta^o$. We have $y_\Delta(g) \leq y_\Delta(y_R^*(d,n) - \Delta) \leq y_\Delta(y_\Delta^{oo})$, where the second inequality follows from Assumption 2. Since by definition $y_\Delta(y_\Delta^{oo}) = y_\Delta^{oo} = g^3(\Delta)$, we conclude that $y_\Delta(g) \in \left[g^2(\Delta), g^3(\Delta)\right]$ for $g \in \left[g^2(\Delta), g^3(\Delta)\right]$. ■

Using Lemma A.12, one can construct a concave value function $v_R^{nm}(g)$ associated to $y_R^{nm}(g)$ in the same way as in Proposition 1. ■

7.8 Proof of Proposition 8

We can construct an equilibrium with investment function $y_\Delta(g)$ exactly as in Proposition 7. Let $y_\Delta^o < y_R^*(d,n)$ be the fixed point of $y_\Delta(g)$, and let $g^2(\Delta)$ and $g^3(\Delta)$ be defined as in Proposition 7. By Assumption 2 we can choose Δ so that $y_\Delta^o < y_R^{***}(d,n)$, and so in a open right neighborhood $N_\varepsilon(y_\Delta^o)$ of y_Δ^o we have $y'_\Delta(g) < -1$. Consider an initial state $g_0 \in \left(g^2(\Delta), g^3(\Delta)\right)$ with $g_0 \neq y_\Delta^o$. It is immediate to see that no path $\{g_m^*\}$ with $g_0^* = g_0$ and $g_m^* = y_\Delta(g_{m-1}^*)$ can converge. The path can converge only to y_Δ^o, since this is the unique fixed point of $y_\Delta(g)$ in $\left(g^2(\Delta), g^3(\Delta)\right)$. But since $y'_\Delta(g) < -1$ in $N_\varepsilon(y_\Delta^o)$ there is no $\{g_m^*\}$ converging to y_Δ^o starting from $g_0 \neq y_\Delta^o$ as well. We conclude that this equilibrium has a cycle. ■