Pareto Optimal Allocations for Probabilistic Sophisticated Variational Preferences on L^1 *

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Abstract

We prove the existence of comonotone Pareto optimal allocations within sets of acceptable allocations when decision makers have probabilistic sophisticated variational preferences on random endowments in L^1.

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1 Introduction

In this paper we prove the existence of Pareto optimal allocations of integrable random endowments when decision makers have probabilistic sophisticated variational prefer-

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ences.

Variational preferences were recently introduced and axiomatically characterized by Maccheroni, Marinacci, and Rustichini (2006). This broad class of preferences allows to model ambiguity aversion and includes several subclasses of preferences that have been extensively studied in the economic literature. Our study focuses on variational preferences that are in addition assumed to be probabilistic sophisticated. Probabilistic sophisticated preferences were introduced by Machina and Schmeidler (1992), further studied by Marinacci (2002) and by Maccheroni, Marinacci, and Rustichini (2006) and Strzalecki (2011b) for the case of variational preferences. A decision maker with probabilistic sophisticated preferences sees any two random endowments that have the same distribution under a reference probability measure as equivalent. Note that when dropping probabilistic sophistication there are simple examples in which Pareto optimal allocations do not exist; see Section 4.3.

The class of choice criteria which we consider represent probabilistic sophisticated variational preferences and are of the following type:

$$\mathcal{U}(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \quad X \in L^1, \tag{1.1}$$

where $u : \mathbb{R} \to \mathbb{R}\cup\{-\infty\}$ is a (not necessarily strictly) concave (not necessarily strictly) increasing utility function, $\mathcal{Q}$ is a rearrangement invariant set of probability measures and $\alpha(\mathbb{Q})$ is a suited rearrangement invariant penalization on $\mathbb{Q} \in \mathcal{Q}$; see Definition 2.1 for the details, and Remark 2.2 for the relation to variational preferences on Markov kernels. This broad class nests many well-known choice criteria studied in the economic and finance literature, in particular, the von Neumann and Morgenstern (1947) *expected utility*, the probabilistic sophisticated *maxmin expected utility preferences* introduced by Gilboa and Schmeidler (1989), and the probabilistic sophisticated *multiplier preferences* introduced by Hansen and Sargent (2000, 2001).

We assume that the decision makers have preferences on a space of future random payoff profiles which we identify with $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$, i.e. the space of $\mathbb{P}$-integrable random variables on a fixed non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ modulo $\mathbb{P}$-almost sure equality, where $\mathbb{P}$ is the reference probability measure. So far, most of the existing literature makes the assumption that payoff profiles are bounded, i.e. in L^∞, or even that the state space is finite. These assumptions are justified in many settings such as economies with finitely many commodities. However, there are many situations in which these boundedness assumptions are not appropriate. For instance, nearly all financial models involve unbounded distributions. This suggests the model space L^1 and the probabilistic sophistication of the preferences allows for that; see Remark 2.2.

In this paper we consider $n \geq 2$ decision makers with probabilistic sophisticated variational preferences on L^1 represented by choice criteria in (1.1). Given the initial endowments $W_i \in L^1$, $i = 1, \ldots, n$, of the decision makers, we establish the existence of Pareto optimal allocations of the aggregate endowment $W = W_1 + \ldots + W_n$ within sets of acceptable reallocations of W. A reallocation of W is defined as acceptable if it satisfies the individual (rationality) constraints (see inequalities (3.3)) of all decision makers. Individual constraints specify which payoff profiles the decision maker i is willing to accept in a reallocation of W in terms of an accepted utility loss or a required utility gain. We allow for any kind of individual constraints. For instance, a decision maker may only accept allocations which allot her an endowment which is at least as good as her initial one, while another decision maker may be willing to accept a worsening as compared to her initial endowment, or requires an improvement in order to take part in the re-sharing of W. The interest in investigating the existence of Pareto optimal allocations under such a variety of individual constraints is that the notion of acceptability might be quite different from one decision maker to another. These differences may for instance be due to outer factors, such as being a member of a society with its own rules and regulations, or having specific interests which are related to but not directly part of the reallocating problem and which do impact on the individual's behavior in the reallocating process.

Our main result is that when the decision makers' choice criteria are in the class (1.1), there always exists a comonotone Pareto optimal allocation of W within the set of acceptable reallocations. Comonotonicity means that the endowments in the allocation are continuous increasing functions of the aggregate endowment W.[1] In particular we prove the following. We establish the existence of comonotone Pareto optimal allocations for any choice of individual constraints when among the $n \geq 2$ decision makers there are no, or there is at most one cash additive choice criterion where cash additivity means that the choice criterion is affine if restricted to $\mathbb{R}$. If more than one choice criterion is cash additive, then the lack of sensitivity to comonotone re-sharings of sure payoffs amongst the cash additive decision makers poses some problems. In that case we show the existence of comonotone Pareto optimal allocations requiring that the utility of $-W^-$ is finite for those cash additive decision makers with non-trivial constraints. Moreover, if the domains of all utilities in (1.1) are bounded from below, we establish the existence of comonotone Pareto optimal allocations for any choice of individual constraints, without imposing any further requirements on the cash additive choice criteria amongst the n considered.

The existence of Pareto optimal allocations has so far only been established for a few subclasses of choice criteria of type (1.1). In case that all decision makers have von Neumann–Morgenstern expected utilities, existence results were already proved in the sixties by Borch (1962), Arrow (1963) and Wilson (1968). More recent is the proof of the existence of Pareto optimal allocations when all decision makers apply probabilistic sophisticated cash additive choice criteria on L^∞ (also called law invariant convex risk measures); see Jouini, Schachermayer, and Touzi (2008) and references therein. Filipović and Svindland (2008) extend this result to integrable (not necessarily

[1] Since probabilistic sophistication implies that the decision makers' preferences satisfy second-order stochastic dominance (Lemma 2.3), it is well understood that if there exists a Pareto optimal allocation, then there is a comonotone Pareto optimal allocation and that under strict concavity every Pareto optimum is necessarily comonotone, see Landsberger and Meilijson (1994) for the finite state space case. This is also linked to the fact that in our setting all decision makers have the common prior $\mathbb{P}$; see e.g. Billot, Chateauneuf, Gilboa, and Tallon (2000), Rigotti, Shannon, and Strzalecki (2008), Strzalecki and Werner (2011) and the references therein.

bounded) aggregate endowments. Even more recently, Dana (2011) proves the existence of Pareto optimal allocations when the decision makers have choice criteria within a class of probabilistic sophisticated, finitely valued, continuous, concave utility functions on L^∞ not necessarily representing variational preferences. In Dana (2011) at least one utility function is required to be cash additive and the others are assumed to be strictly concave. Rigotti, Shannon, and Strzalecki (2008) prove the existence of Pareto optimal allocations for variational preferences on the positive cone of L^∞ under the assumption of mutual absolute continuity. See Section 3.4 for a discussion of the positioning of our results in the known literature.

The paper is organized as follows. In Section 2 we introduce probabilistic sophisticated variational preferences on L^1, we recall some well-known subclasses of these preferences and some useful properties. The Pareto optimal allocations problem is studied throughout Section 3. After introducing the problem we recall the well-know characterization of Pareto optimal allocations as solutions to a weighted sup-convolution optimization problem. Then we state our main result on the existence of Pareto optimal allocations and the underlying theorems which prove the existence of solutions to the weighted sup-convolution optimization problem. In particular, we give bounds which specify a non-empty set of possible choices of weights for which the associated optimization problem admits solutions. Knowing this set is useful to derive Pareto optimal allocations explicitly. In Section 4 we illustrate our results by means of some examples. The first set of examples (Section 4.1) illustrates that the given bounds on the weights cannot be dropped. We also present a case study (Section 4.2) in which we explicitly characterize the Pareto optimal allocations between two decision makers, one with a Yaari (1987) type choice criterion and the other one with a choice criterion in a specific class containing, for instance, the probabilistic sophisticated multiplier preferences mentioned above. Finally, we give an example (Section 4.3) of decision makers with expected utility choice criteria under different subjective probabilities in which Pareto optimal allocations may or may not exist depending on the deviation of the subjective probabilities from each other and on the concavity of the respective utility functions.

2 Setup

Throughout this paper $(\Omega, \mathcal{F}, \mathbb{P})$ is an atom-less probability space, i.e. a probability space supporting a random variable with continuous distribution. Here Ω represents the states of the world, $\mathcal{F}$ is the σ-algebra of events, and $\mathbb{P}$ is the reference probability measure which resembles the decision makers' (reference) estimates of the likeliness of events. The decision makers have preferences on payoff profiles. A payoff profile is a real number (payoff) on each state of the world which is supposed to be consistent with the possible events $\mathcal{F}$. Hence, the set of payoff profiles is given by the set of random variables, i.e. $\mathcal{F}$-measurable functions $X : \Omega \to \mathbb{R}$, where $X(\omega)$ is the payoff given the state of the world $\omega \in \Omega$. The payoff profiles are assumed to be integrable with respect to $\mathbb{P}$, and we identify those payoff profiles which only deviate on an event of zero probability under $\mathbb{P}$. Therefore, the set of payoff profiles considered by the decision makers can be identified with the space $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$ of $\mathbb{P}$-integrable random variables modulo $\mathbb{P}$-almost sure (a.s.) equality. For a justification of the model space L^1 see Remark 2.2.

In this paper all equalities and inequalities between random variables are understood in the $\mathbb{P}$-a.s. sense. Given two random variables X and Y we write $X \overset{d}{=} Y$ to indicate that both random variables have the same distribution under the reference probability measure $\mathbb{P}$. The expectation (if well-defined) of a random variable X under a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ will be denoted by $\mathbb{E}_{\mathbb{Q}}[X]$. In case $\mathbb{Q} = \mathbb{P}$ we also write $\mathbb{E}[X] := \mathbb{E}_{\mathbb{P}}[X]$.

2.1 Probabilistic Sophisticated Variational Preferences on L^1

In this section we define probabilistic sophisticated variational preferences on L^1. In Remark 2.2 we briefly discuss how our setting, in particular our definition of probabilistic sophisticated variational preferences on L^1, fits into the existing literature on variational preferences.

Variational preferences were introduced by Maccheroni, Marinacci, and Rustichini (2006). In the same paper the authors also study the subclass of probabilistic sophisticated variational preferences. Probabilistic sophistication means that $X \stackrel{d}{=} Y$ implies $X \sim Y$ in the preference order.

Definition 2.1. (i) *A function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a* utility function *(on $\mathbb{R}$) if it is concave, right-continuous, increasing,* $\text{dom } u := \{x \in \mathbb{R} \mid u(x) > -\infty\} \neq \emptyset$*, and not constant in the sense that there exist $x, y \in \text{dom } u$ such that $u(x) \neq u(y)$.*

(ii) *Let Δ denote the set of all probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ which are absolutely continuous and have bounded densities with respect to $\mathbb{P}$, i.e. $\forall A \in \mathcal{F}$, $\mathbb{P}(A) = 0 \;\Rightarrow\; \mathbb{Q}(A) = 0$, and there exists $K > 0$ such that $\mathbb{P}\left(\frac{d\mathbb{Q}}{d\mathbb{P}} < K\right) = 1$. A set of probability measures $\mathcal{Q} \subset \Delta$ is* rearrangement invariant *if $\mathbb{Q} \in \mathcal{Q}$ and $\widehat{\mathbb{Q}} \in \Delta$ with $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \stackrel{d}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}$ implies that $\widehat{\mathbb{Q}} \in \mathcal{Q}$.*

(iii) *A decision maker has* probabilistic sophisticated variational preferences $\succeq$ *if for all $X, Y \in L^1$:*

$$X \succeq Y \quad \Leftrightarrow \quad \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right) \geq \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(Y)] + \alpha(\mathbb{Q})\right)$$

where u is a utility function, $\emptyset \neq \mathcal{Q} \subset \Delta$ is convex and rearrangement invariant, and $\alpha : \mathcal{Q} \to \mathbb{R}$ is a convex and rearrangement invariant function on $\mathcal{Q}$ in the sense that $\mathbb{Q}, \widehat{\mathbb{Q}} \in \mathcal{Q}$ with $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} \stackrel{d}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}$ implies $\alpha(\widehat{\mathbb{Q}}) = \alpha(\mathbb{Q})$. In addition α satisfies $\inf_{\mathbb{Q} \in \mathcal{Q}} \alpha(\mathbb{Q}) > -\infty$*. The numerical representation*

$$\mathcal{U} :\, L^1 \to \mathbb{R} \cup \{-\infty\},\; X \mapsto \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \tag{2.1}$$

is the choice criterion *used by the decision maker to quantify the utility of a payoff profile* $X \in L^1$.

In the mathematical finance literature choice criteria of type (2.1) are also called *law invariant robust utilities*; see e.g. Föllmer, Schied, and Weber (2009) and the references therein.

The choice criterion $\mathcal{U}$ in (2.1) is, clearly, rearrangement invariant (probabilistic sophisticated) and posses some other useful properties which are collected in Lemma 2.3 below. Due to Jensen's inequality for concave functions, the expectations in (2.1) are all well-defined, possibly taking the value $-\infty$. Note that $\mathcal{U}(X) = -\infty$ is possible for some $X \in L^1$. The interpretation is that the payoff profiles with utility $-\infty$ are totally unacceptable.

We remark that what we call a utility function in Definition 2.1 (i) satisfies relatively weak requirements and nests the vast majority of utilities proposed in the economic, finance, and insurance literature, including extreme cases such as increasing linear or affine functions. Moreover, by allowing u to take the value $-\infty$ we also incorporate the cases when the domain of the utility function u is bounded from below, as e.g. for the power utilities or the logarithmic utilities.

Remark 2.2. A standard approach to modeling preferences in presence of model ambiguity (Knightian uncertainty) is to consider preference orders on the set $\mathcal{M}$ of all Markov kernels $\mathcal{X}(\omega, dy)$ from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ (where $\mathcal{B}(\mathbb{R})$ denotes the Borel-σ-algebra) for which there exists a $k > 0$ such that $\mathcal{X}(\omega, [-k, k]) = 1$ for all $\omega \in \Omega$. It can then be shown under some mild additional assumptions that a preference order on $\mathcal{M}$ is in the class of variational preferences if and only if it admits a numerical representation of the form

$$\mathcal{U}(\mathcal{X}) = \inf_{Q \in C} \left(\int \int u(y)\mathcal{X}(\omega, dy)\, dQ(\omega) + \alpha(Q) \right), \quad \mathcal{X} \in \mathcal{M}. \tag{2.2}$$

Here u is a utility function on $\mathbb{R}$, and without loss of generality we may assume that C is the set of all *finitely additive* normalized measures, and $\alpha : C \to \mathbb{R} \cup \{\infty\}$ is a convex rearrangement invariant function with the additional property of being the minimal function for which $\mathcal{U}$ can be represented as in (2.2). For an axiomatic definition of variational preferences on Markov kernels and the details on their numerical representation (2.2) we refer to Föllmer, Schied, and Weber (2009). Notice that the space of all bounded payoff profiles L^∞ is naturally embedded into the space $\mathcal{M}$ by identifying each $X \in L^\infty$ with the associated kernel $\mathcal{X}(\omega, dy) = \delta_{X(\omega)}(dy)$ where δ_x denotes the Dirac measure given $x \in \mathbb{R}$. The restriction of $\mathcal{U}$ to L^∞ then takes the form

$$\mathcal{U}(X) = \inf_{Q \in C} \left(\int u(X)\, dQ + \alpha(Q) \right), \quad X \in L^\infty. \tag{2.3}$$

In case of probabilistic sophistication, using results in Svindland (2010a), it follows that $\mathcal{U}$ is $\sigma(L^\infty, L^\infty)$-upper semi continuous and thus we obtain a representation of $\mathcal{U}$ as an infimum over *σ-additive* probability measures in Δ:

$$\mathcal{U}(X) = \inf_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q}) \right), \quad X \in L^\infty, \tag{2.4}$$

where $\mathcal{Q} := \text{dom}\, \alpha \cap \Delta$.

Hence, these preferences on L^∞ are indeed consistent with our definition of probabilistic sophisticated variational preferences on L^1; see Definition 2.1. Moreover, the representation (2.4) shows that the choice criterion $\mathcal{U}$ and thus the corresponding preference order is canonically extended from L^∞ to L^1. $\diamondsuit$

2.2 Special Cases

In this section we list some well-known subclasses of choice criteria that are special cases of the class of choice criteria (2.1) considered in this paper.

(i) **Expected utility.** If in (2.1) $\mathcal{Q} = \{\mathbb{P}\}$ and $\alpha(\mathbb{P}) = 0$, then $\mathcal{U}$ is the classical expected utility criterion corresponding to *von Neumann-Morgenstern type preferences*: see von Neumann and Morgenstern (1947).

(ii) **Probabilistic sophisticated maxmin expected utility.** If $\alpha \equiv 0$ in (2.1), then the choice criterion $\mathcal{U}$ represents *probabilistic sophisticated maxmin expected* *utility preferences* (also called probabilistic sophisticated multiple prior preferences) axiomatized by Gilboa and Schmeidler (1989). This choice criterion has been largely used in the economic literature to model ambiguity aversion; see, e.g., Epstein and Wang (1994), Chen and Epstein (2002) and Yaari (1987). For the connection between maxmin expected utility and variational preferences see Maccheroni, Marinacci, and Rustichini (2006).

(iii) **Probabilistic sophisticated multiplier preferences criterion.** If the function α in (2.1) is the relative entropy of $\mathbb{Q} \in \mathcal{Q}$ with respect to $\mathbb{P}$, i.e. $\alpha(\mathbb{Q}) = \gamma H(\mathbb{Q} \mid \mathbb{P}) = \gamma \mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}} \log \frac{d\mathbb{Q}}{d\mathbb{P}}]$ for some $\gamma > 0$, then the choice criterion $\mathcal{U}$ represents the *probabilistic sophisticated multiplier preferences* introduced by Hansen and Sargent (2000, 2001). Like the maxmin expected utility, this choice criterion models the uncertainty of the decision maker on the reference model $\mathbb{P}$. For the connection between multiplier preferences and variational preferences see Strzalecki (2011a).

(iv) **Law invariant monetary utility.** If in (2.1) $u = id_{\mathbb{R}}$ (or more in general if $u = \alpha\, id_{\mathbb{R}} + \beta$ with $\alpha, \beta \in \mathbb{R}$, $\alpha \geq 0$), then the choice criterion $\mathcal{U}$ is known as law invariant *monetary utility function*. Monetary utilities (called convex risk measures when multiplied by -1) were introduced in order to assess the risk of future random endowments. In particular, monetary utilities are *cash additive* in the sense that $\mathcal{U}(X + a) = \mathcal{U}(X) + a$ for all $a \in \mathbb{R}$ (or more in general $\mathcal{U}(X + a) = \mathcal{U}(X) + \alpha a + \beta$). These functions have been largely investigated in the mathematical finance literature; see for instance Föllmer and Schied (2004) and the references therein.

2.3 Properties of the Choice Criterion (2.1)

In Lemma 2.3 we collect some properties of the choice criteria (2.1) which we will make frequently use of. Some of them are well-known, and proofs can be partly found for instance in Dana (2005), Föllmer and Schied (2004) and Maccheroni, Marinacci, and Rustichini (2006). References and proofs of the parts of Lemma 2.3 which we believe are not well-known are provided in Appendix A.

Lemma 2.3. *Consider a choice criterion* $\mathcal{U}$ *in* (2.1). *Then* $\mathcal{U}$ *has the following properties:*

(i) *properness:* $\mathcal{U} < \infty$ *and the domain* $\operatorname{dom}\mathcal{U} := \{X \in L^1 \mid \mathcal{U}(X) > -\infty\}$ *is not empty.*

(ii) *concavity:* $\mathcal{U}(\lambda X + (1-\lambda)Y) \geq \lambda\mathcal{U}(X) + (1-\lambda)\mathcal{U}(Y)$ *for all* $\lambda \in [0,1]$.

(iii) *monotonicity:* $X \geq Y$ *implies* $\mathcal{U}(X) \geq \mathcal{U}(Y)$.

(iv) *rearrangement invariance (probabilistic sophistication):* $X \overset{d}{=} Y$ *implies* $\mathcal{U}(X) = \mathcal{U}(Y)$.

(v) $\succeq_{ssd}$*-monotonicity:* $X \succeq_{ssd} Y$ *implies* $\mathcal{U}(X) \geq \mathcal{U}(Y)$, *where*

$$X \succeq_{ssd} Y \quad \Leftrightarrow \quad \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)], \text{ for all utility functions } u : \mathbb{R} \to \mathbb{R},$$

is the second order stochastic dominance order.

(vi) *upper semi-continuity: for all* $k \in \mathbb{R}$ *the level sets* $E_k := \{X \in L^1 \mid \mathcal{U}(X) \geq k\}$ *are closed in* L^1 *with respect to the topology induced by the norm* $\|\cdot\|_1 := \mathbb{E}[|\cdot|]$. *Equivalently, if* $(X_n) \subset L^1$ *converges to* $X \in L^1$ *(with respect to* $\|\cdot\|_1$*), then* $\mathcal{U}(X) \geq \limsup_{n\to\infty}\mathcal{U}(X_n)$.

Probabilistic sophisticated variational preferences do not only preserve second order stochastic dominance (Lemma 2.3 (v)) but consequently also the concave order, i.e. $X \succeq_c Y$ implies $\mathcal{U}(X) \geq \mathcal{U}(Y)$, where $\succeq_c$ denotes the concave order, that is

$$X \succeq_c Y \quad \Leftrightarrow \quad \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)] \text{ for all concave functions } u : \mathbb{R} \to \mathbb{R}. \tag{2.5}$$

Clearly, $X \succeq_c Y$ implies $X \succeq_{ssd} Y$. The property of preserving $\succeq_c$ is referred to as Schur concavity of $\mathcal{U}$. Indeed in case of monotone concave upper semi-continuous functions $\succeq_{ssd}$-monotonicity is equivalent to Schur concavity. See Dana (2005) for more details on concave order, second order stochastic dominance and functions preserving the concave order and the second order stochastic dominance; see also Rothschild and Stiglitz (1970), Föllmer and Schied (2004), and Maccheroni, Marinacci, and Rustichini (2006).

3 Pareto Optimal Allocations for Probabilistic Sophisticated Variational Preferences

Consider $n \geq 2$ decision makers with initial endowments $W_i \in L^1$. All decision makers are assumed to have probabilistic sophisticated variational preferences on L^1 and corresponding choice criteria

$$\mathcal{U}_i(X) = \inf_{\mathbb{Q}\in\mathcal{Q}_i} \left(\mathbb{E}_{\mathbb{Q}}[u_i(X)] + \alpha_i(\mathbb{Q})\right), \quad X \in L^1,\ i = 1, \ldots, n, \tag{3.1}$$

as defined in (2.1). We assume that $\mathcal{U}_i(W_i) > -\infty$ for all $i = 1, \ldots, n$, and let $W := W_1 + \ldots + W_n$ be the aggregate endowment. The aim of this section is to prove the existence of Pareto optimal allocations of W amongst the n agents within the set of reallocations of the aggregate endowment W that are accepted by all agents. The set $\mathbb{A}(W)$ of all acceptable allocations of W is

$$\begin{aligned}\mathbb{A}(W) \; := \; & \{(X_1, \ldots, X_n) \in (L^1)^n \mid \sum_{i=1}^n X_i = W, \\ & \mathcal{U}_i(X_i) > -\infty \text{ and } \mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - c_i \text{ for all } i \in \{1, \ldots, n\}\},\end{aligned} \tag{3.2}$$

where $c_i \in \mathbb{R} \cup \{\infty\}$, $i = 1, \ldots, n$. The constraint

$$\mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - c_i \tag{3.3}$$

expresses *the individual (rationality) constraint* of the decision maker i, specifying which payoff profiles X_i in a new re-allocation of W she is willing to accept. Indeed, the constant c_i measures to which extent the decision maker i is willing to accept a worsening of her situation as compared to her initial endowment or demands an improvement in order to take part in the re-sharing of the aggregate endowment W. Clearly, $c_i = 0$ represents the (classical) case when the decision maker will not accept any allocation which allots her an endowment which is not as least as good as her initial one. An extreme is $c_i = \infty$ which means that the decision maker is willing to accept any allocation with finite utility.[2] We also allow for situations in which the decision maker is to some bounded extent willing to accept a worsening as compared to her initial endowment ($c_i > 0$), or requires an improvement ($c_i < 0$).

Note that if $c_i \geq 0$ for all $i \in \{1, \ldots, n\}$, then the initial allocation $(W_1, \ldots, W_n)$ is acceptable, so in particular $\mathbb{A}(W) \neq \emptyset$. However, if some agents demand a strict improvement, it is in general not clear whether the set of acceptable allocations is non-empty. Hence, we make the following assumption.

Assumption 3.1. $$\mathbb{A}(W) \neq \emptyset.$$

We are interested in those reallocations of W that are Pareto optimal within $\mathbb{A}(W)$.

Definition 3.2. *An allocation* $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ *is* Pareto optimal *if* $(Y_1, \ldots, Y_n) \in \mathbb{A}(W)$ *and* $\mathcal{U}_i(Y_i) \geq \mathcal{U}_i(X_i)$ *for* $i = 1, \ldots, n$ *implies that* $\mathcal{U}_i(Y_i) = \mathcal{U}_i(X_i)$ *for all* $i = 1, \ldots, n$.

3.1 Characterization of Pareto Optimal Allocations

To prove the existence of Pareto optimal allocations in the set $\mathbb{A}(W)$ we use a well-known result that characterizes Pareto optimal allocations as solutions to the following weighted sup-convolution optimization problem

$$\text{Maximize} \sum_{i=1}^{n} \lambda_i \mathcal{U}_i(X_i) \quad \text{subject to } (X_1, \ldots, X_n) \in \mathbb{A}(W), \tag{3.4}$$

where $\lambda_i \geq 0$, $i = 1, \ldots, n$, are positive or zero weights. Note that under Assumption 3.1 the optimization problem (3.4) is well-posed.

[2] $c_i = \infty$ is understood as the restriction $\mathcal{U}_i(X_i) \geq \mathcal{U}_i(W_i) - \infty := -\infty$ being redundant.

Proposition 3.3. *If* $(X_1, \dots, X_n) \in \mathbb{A}(W)$ *is Pareto optimal, then there exist weights* $\lambda_i \geq 0$*,* $i = 1, \dots, n$*, not all equal to zero, such that the allocation* $(X_1, \dots, X_n)$ *solves* (3.4) *with these weights. Conversely, if* $(X_1, \dots, X_n)$ *solves* (3.4) *for some strictly positive weights* $\lambda_i > 0$*,* $i = 1, \dots, n$*, then* $(X_1, \dots, X_n)$ *is Pareto optimal.*

For the sake of completeness a proof of Proposition 3.3 is given in Appendix B. As we will see in Section 3.3, our techniques allow us to prove the existence of solutions to (3.4) only in case all weights are strictly positive, i.e. $\lambda_i > 0$ for all $i \in \{1, \dots, n\}$; see Theorems 3.9 and 3.10. Anyhow, we think that Pareto optimal allocations that are associated to strictly positive weights are the most interesting ones. Indeed $\lambda_j = 0$ implies that the decision maker j is not considered in the social welfare maximization problem (3.4), whereas $\lambda_j > 0$ in many cases will lead to a social welfare that is strictly higher than in case $\lambda_j = 0$. Moreover, in the following Lemma 3.4 we give conditions under which all Pareto optimal allocations correspond to solutions to (3.4) for strictly positive weights λ_i.

Lemma 3.4. *Suppose that* $c_i = \infty$ *for all* $i = 1, \dots, n$ *and that the* $\mathcal{U}_i$ *satisfy the following conditions for all* $i = 1, \dots, n$*:*

- $\operatorname{dom} \mathcal{U}_i = \operatorname{dom} \mathcal{U}_i + \mathbb{R}$*,*
- *non-satiation:* $\lim_{m \to \infty} \mathcal{U}_i(X + m) = \infty$ *for all* $X \in \operatorname{dom} \mathcal{U}_i$*.*

Then $(X_1, \dots, X_n) \in \mathbb{A}(W)$ *is Pareto optimal if and only if it solves* (3.4) *for some strictly positive weights* $\lambda_i > 0$*,* $i = 1 \dots, n$*.*

Proof. Let $(X_1, \dots, X_n) \in \mathbb{A}(W)$ be Pareto optimal and let $\lambda_i \geq 0$ be the corresponding weights from Proposition 3.3. The stated conditions ensure that $(X_1 + m_1, \dots, X_n + m_n) \in \mathbb{A}(W)$ for all $m_i \in \mathbb{R}$ such that $\sum_{i=1}^n m_i = 0$. Hence, if $\lambda_j = 0$ for some $j \in \{1, \dots, n\}$, then for any $k \in \{1, \dots, n\}$ such that $\lambda_k > 0$ we obtain

$$\begin{aligned}\sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i) &= \sup_{(Y_1, \dots, Y_n) \in \mathbb{A}(W)} \sum_{i=1}^n \lambda_i \mathcal{U}_i(Y_i) \\ &\geq \sup_{m \geq 0} \sum_{i \neq j,k} \lambda_i \mathcal{U}_i(X_i) + \lambda_k \mathcal{U}_k(X_k + m) + 0 \cdot \mathcal{U}_j(X_j - m) = \infty\end{aligned}$$

which is a contradiction to Pareto optimality of $(X_1, \dots, X_n)$. The rest follows from Proposition 3.3. $\square$

3.2 Comonotone Pareto Optimal Allocations

In this section we introduce the class of comonotone allocations which play a major role in the search for Pareto optima in case of probabilistic sophistication. It is known that if there exists a Pareto optimal allocation, then in particular there is also a comonotone one. This follows from Proposition 3.6 below.

Definition 3.5. *We denote by* CF *the set of all n-tuples $(f_1, \dots, f_n)$ of increasing functions $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$, such that $\sum_{i=1}^n f_i = \mathrm{Id}_{\mathbb{R}}$. These functions f_i are necessarily 1-Lipschitz-continuous. An allocation $(Y_1, \dots, Y_n) \in (L^1)^n$ of W, i.e. $\sum_{i=1}^n Y_i = W$, is* comonotone *if there exists $(f_i)_{i=1}^n \in$* CF *such that $Y_i = f_i(W)$ for all $i = 1, \dots, n$.*

Proposition 3.6. *For any $(X_1, \dots, X_n) \in \mathbb{A}(W)$ there exists a comonotone allocation $(Y_1, \dots, Y_n) \in \mathbb{A}(W)$ such that $Y_i \succeq_c X_i$ (and thus $Y_i \succeq_{ssd} X_i$) for all $i = 1, \dots, n$.*

Proof. First of all we recall a result that is often referred to as comonotone improvement: for any allocation $(X_1, \dots, X_n) \in (L^1)^n$ of W there exists a comonotone allocation $(Y_1, \dots, Y_n) \in (L^1)^n$ of W such that $Y_i \succeq_c X_i$ for all $i = 1, \dots, n$. The proof for the case when W is supported by a finite set goes back to Landsberger and Meilijson (1994). This has been further extended to aggregate endowments $W \in L^1$ by Filipović and Svindland (2008), Dana and Meilijson (2011), and Ludkovski and Rüschendorf (2008). Finally, by $\succeq_{ssd}$-monotonicity of the $\mathcal{U}_i$ (see Lemma 2.3 (v)) it follows that if $(X_1, \dots, X_n) \in \mathbb{A}(W)$, then any comonotone improvement $(Y_1, \dots, Y_n)$ of $(X_1, \dots, X_n)$ is acceptable as well, i.e. $(Y_1, \dots, Y_n) \in \mathbb{A}(W)$. $\square$

Hence, if there exists a solution to (3.4) for some non-negative weights $\lambda_i \geq 0$, then by $\succeq_{ssd}$-monotonicity of the $\mathcal{U}_i$ (see Lemma 2.3 (v)) and by virtue of Proposition 3.6

there must also exists a comonotone one. Indeed, when proving the existence of solutions to (3.4), and thus of Pareto optimal allocations, we will profit from the fact that we may restrict our attention to the set of comonotone acceptable allocations; see proofs of Theorems 3.9 and 3.10 in Appendix C.

Remark 3.7. The comonotone allocations have another desirable property. Suppose that $W \in L^p \subset L^1$ for some $p \in [1, \infty]$, and let $(f_i(W))_{i=1}^n$ be a comonotone allocation of W, i.e. $(f_i)_{i=1}^n \in \mathrm{CF}$. Then, by the 1-Lipschitz continuity of the f_i, it is easily verified that $(f_i(W))_{i=1}^n \in (L^p)^n$. Hence, any comonotone Pareto optimal allocation will posses the same integrability/boundedness properties as the aggregate endowment W. In that sense, further restricting the set of acceptable allocations by imposing additional integrability or even boundedness constraints in the formulation of problem (3.4) will yield the same comonotone solutions as solving the unrestricted problem. $\diamondsuit$

3.3 Main Results

Let $s_i := \inf \mathrm{dom}\, u_i \in \mathbb{R} \cup \{-\infty\}$, $i = 1, \ldots, n$, and $d_H^i := \lim_{x \to s_i} u_i'(x)$ (which may be ∞) and $d_L^i := \lim_{x \to \infty} u_i'(x) (\geq 0)$ where u' denotes the right-hand-derivative of u. Finally, let $N \subset \{1, \ldots, n\}$ be the set of all indices such that $d_H^i = d_L^i$, and $M := \{1, \ldots, n\} \setminus N$ the set of all indices such that $d_L^i < d_H^i$. Note that $N = \emptyset$ or $M = \emptyset$ is possible.

When $d_L^i = d_H^i$ and $s_i = -\infty$ (i.e. dom $u_i = \mathbb{R}$), then the corresponding choice criterion $\mathcal{U}_i$ is cash additive in the sense that $\mathcal{U}_i(X + m) = \mathcal{U}_i(X) + d_H^i m$, for all $m \in \mathbb{R}$ and $X \in L^1$; see Section 2.2 item (iii). In the case $d_L^i = d_H^i$ but $s_i > \infty$ (i.e. when the dom $u_i \subsetneq \mathbb{R}$ is bounded from below), we say that the choice criterion $\mathcal{U}_i$ is *quasi-cash additive.*

Theorem 3.8.

(i) *If $s_i > -\infty$ for all $i = 1, \ldots, n$, then for any individual constraints $(c_1, c_2, \ldots, c_n) \in (\mathbb{R} \cup \{\infty\})^n$ in $\mathbb{A}(W)$ there exists a comonotone Pareto optimal allocation of W.*

(ii) *If* $|N| \leq 1$*, then for any* $(c_1, c_2, \ldots, c_n) \in (\mathbb{R} \cup \{\infty\})^n$ *in* $\mathbb{A}(W)$ *there exists a comonotone Pareto optimal allocation of* W*.*

(iii) *Suppose that* $|N| \geq 2$*. Given any choice of individual constraints* $(c_1, c_2, \ldots, c_n)$ *in* $\mathbb{A}(W)$*, if* $s_i = -\infty$ *for all* $i \in N$*, and* $\mathcal{U}_j(-W^-) > -\infty$ *for all* $j \in N$ *such that* $c_j \in \mathbb{R}$ *then there exists a comonotone Pareto optimal allocation of* W*.*

Moreover, in the situation of (iii)*, if* $|N| = n$ *(i.e. all choice criteria are cash additive) the Pareto optimal allocations have the property of being up to a reallocation of cash, that is if* $(X_1, \ldots, X_n)$ *is a Pareto optimal allocation of* W*, then also* $(X_1 + m_1, \ldots, X_n + m_n)$ *is a Pareto optimal allocation whenever the numbers* $m_i \in \mathbb{R}$ *satisfy* $\sum_{i=1}^n m_i = 0$ *and* $(X_1 + m_1, \ldots, X_n + m_n) \in \mathbb{A}(W)$*.*

Proof. The proof follows immediately from Proposition 3.3, Theorem 3.9 and Theorem 3.10. □

Note that in Theorem 3.8 (i) the bounds $s_i > -\infty$ can be different among the n decision makers, and in Theorem 3.8 (ii) the domains of the u_i can be either bounded from below with arbitrary bounds or equal $\mathbb{R}$.

In the following we give the two theorems that jointly with Proposition 3.3 prove the existence of Pareto optimal allocations as stated in Theorem 3.8. In particular, these theorems specify a non-empty set of possible choices of weights $\lambda_1, \ldots, \lambda_n$ for which the associated optimization problem (3.4) has a solution. Notice that knowing this set is useful when Pareto optimal allocations need to be derived explicitly.

The first theorem treats the case when the domain of u_i is bounded from below for all $i = 1, \ldots, n$.

Theorem 3.9. *Suppose that* $s_i > -\infty$*,* $i = 1, \ldots, n$*. Then for any set of individual constraints* $c_i \in \mathbb{R} \cup \{\infty\}$ *and every set of strictly positive weights* $\lambda_i > 0$*,* $i = 1, \ldots, n$*,* (3.4) *admits a comonotone solution.*

The next theorem treats the general case in which the domains of the utilities u_i can be either unbounded or bounded from below. To ensure the existence of a solution to (3.4) in this setting, the strictly positive weights need to be chosen within certain bounds that depend on the extreme slopes d_L^j and d_H^j and thereby on the risk aversion of the decision makers.

Theorem 3.10. *Consider the following bounds on the weights λ_i, $i = 1, \dots, n$, and some $\delta > 0$:*

$$\begin{aligned} \lambda_i &= \tfrac{\delta}{d_H^i} && \text{for all } i \in N, \\ \lambda_i d_L^i &< \delta < \lambda_i d_H^i && \text{for all } i \in M, \\ \tfrac{\lambda_i}{\lambda_j} &< \tfrac{d_H^j}{d_L^i} && \text{for all } i, j \in M, \end{aligned} \tag{3.5}$$

where for every $b > 0$ we set $\frac{b}{0} := \infty$, $\frac{b}{\infty} := 0$, and $\frac{\infty}{0} := \infty$ whereas $\frac{0}{\infty} := 0$. We consider two cases:

(i) *Suppose that $N = \emptyset$ or $|N| = 1$. Then (3.4) admits a comonotone solution for every set of individual constraints $c_i \in \mathbb{R} \cup \{\infty\}$, $i = 1, \dots, n$, and any set of weights $\lambda_i > 0$, $i = 1, \dots, n$, satisfying the constraints (3.5).*

(ii) *Suppose that $|N| \geq 2$. If $s_i = -\infty$ for all $i \in N$, and $\mathcal{U}_j(-W^-) > -\infty$ for all $j \in N$ such that $c_j \in \mathbb{R}$, then (3.4) admits a comonotone solution for every set of weights $\lambda_i > 0$, $i = 1, \dots, n$, satisfying the constraints (3.5). In particular, if $|N| = n$, the solutions have the property of being up to a reallocation of cash in the sense that if $(X_1, \dots, X_n)$ is a solution to (3.4) for some given weights, then also $(X_1 + m_1, \dots, X_n + m_n)$ is a solution to (3.4) with that weights whenever the numbers $m_i \in \mathbb{R}$ satisfy $\sum_{i=1}^n m_i = 0$ and $(X_1 + m_1, \dots, X_n + m_n) \in \mathbb{A}(W)$.*

We remark that when $N = \emptyset$ the three conditions in (3.5) reduce to the last one.

The proofs of Theorems 3.9 and 3.10 are provided in Appendix C. Theorem 3.10 is discussed in Examples 4.2, 4.3 and 4.4 where we illustrate that if we drop one of the conditions on the weights stated in (3.5) we cannot in general expect the existence of solutions to (3.4) any longer.

Remark 3.11. If $d_H^i = \infty$ and $d_L^i = 0$ for all $i = 1, \ldots, n$, as for instance when the decision makers' utilities u_i are chosen amongst the exponential, logarithmic or power utilities, then the bounds in (3.5) are void. Hence, Theorem 3.10 (i) implies the existence of a solution to (3.4) for any set of strictly positive weights $\lambda_i > 0$, $i = 1, \ldots, n$. $\Diamond$

As regards the uniqueness of Pareto optimal allocations, we have the following result. To this end we recall that a function $\mathcal{U} : L^1 \to \mathbb{R} \cup \{-\infty\}$ is strictly concave if $\mathcal{U}(\lambda X + (1-\lambda)Y) > \lambda\mathcal{U}(X) + (1-\lambda)\mathcal{U}(Y)$ whenever $\lambda \in (0,1)$ and $X \neq Y$.

Corollary 3.12. *Suppose that under the conditions stated in Theorem 3.9 (and Theorem 3.10, respectively) $(n-1)$ among the n choice criteria $\mathcal{U}_i$ are strictly concave. Then, for any given set of weights $\lambda_i > 0$, $i = 1, \ldots, n$, (and satisfying the bounds (3.5), respectively), the Pareto optimal allocation which solves the optimization problem* (3.4) *associated to* $(\lambda_1, \ldots, \lambda_n)$ *is unique and comonotone.*

Proof. For any given vector of positive weights $(\lambda_1, \ldots, \lambda_n)$ the set of solutions to the associated optimization problem (3.4) is convex because the $\mathcal{U}_i$ are concave. This together with the strict concavity of $(n-1)$ choice criteria implies that the solution to (3.4), if it exists, is unique. The comonotonicity of the solution follows from Theorem 3.9 (and from Theorem 3.10, respectively). $\square$

3.4 Positioning of the Main Results in the Known Literature

There is an extensive literature covering the existence of Pareto optimal allocations for (subclasses of) variational preferences when the state space is assumed to be finite; see e.g. Rigotti and Shannon (2005) and the references therein. However, it is well-know that in an infinite dimensional setting the standard analytical tools used in the finite dimensional framework do not work any longer.

The version of Theorem 3.10 when all the choice criteria are cash additive (i.e. when $|N| = n$) is known and proved in Jouini, Schachermayer, and Touzi (2008) for bounded aggregate endowments and in Filipović and Svindland (2008) for integrable (not necessarily bounded) aggregate endowments. Dana (2011) proves a version of Theorem 3.10 for a class of finitely valued, continuous, concave, rearrangement invariant utilities $\mathcal{U}_i : L^\infty \to \mathbb{R}$ on bounded endowments of which at least one utility is required to be cash additive, i.e. to be a monetary utility function, and the others are assumed to be strictly concave. In her setting, the $\mathcal{U}_i$ do not necessarily correspond to variational preferences. Dana (2011) derives similar bounds on the weights λ_i given in (3.5) in her setting. Theorem 3.10 for $|N| < n$ and unbounded merely integrable endowments is to our knowledge new.

Theorem 3.9 is well-known in case all choice criteria are expected utilities; see e.g. Lemma 3.57 in Föllmer and Schied (2004). Rigotti, Shannon, and Strzalecki (2008) prove the existence of Pareto optimal allocations for variational preferences on L^∞_+, which corresponds to the case $s_i = 0$ for all $i = 1, \ldots, n$, without requiring probabilistic sophistication but under the additional assumption of mutual absolute continuity. Apart from these results, we think Theorem 3.9 is new.

4 Examples

This section collects our examples. In Subsection 4.1 we show that dropping the conditions stated in (3.5) we cannot expect the existence of solutions to (3.4) any longer. In Subsection 4.2 we characterize the Pareto optimal allocations in the case of two decision makers, one with Yaari type preferences and the other one with preferences in a class of probabilistic sophisticated preferences that contains the probabilistic sophisticated multiplier preferences as an example. In Subsection 4.3 we show that if the decision makers' preferences are not probabilistic sophisticated with respect to the same reference probability measure, then there are cases in which Pareto optimal allocations do not exist. Since our examples will only involve two decision makers, in the following we give a version of Theorem 3.10 for two decision makers.

Theorem 4.1. *Suppose $n = 2$. We consider two cases:*

(i) *Suppose that $d_L^i < d_H^i$ for at least one $i \in \{1, 2\}$. If the weights $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfy*

$$\frac{\lambda_2}{\lambda_1} \in \left(\frac{d_L^1}{d_H^2}, \frac{d_H^1}{d_L^2} \right), \tag{4.1}$$

then for any $c_1, c_2 \in \mathbb{R} \cup \{\infty\}$ there exists a comonotone solution to (3.4).

(ii) *Suppose that $d_H^1 = d_L^1$, $d_H^2 = d_L^2$, $s_1 = s_2 = -\infty$ and $\mathcal{U}_i(-W^-) > -\infty$ for any $i = 1, 2$ such that $c_i \in \mathbb{R}$. If $\lambda_i = \frac{\delta}{d_H^i}$, $i = 1, 2$, for some $\delta > 0$, then there exists a comonotone solution to (3.4).*

4.1 Examples Illustrating the Bounds (3.5)

Example 4.2. Illustration of Theorem 4.1 (i): Let $\mathcal{U}_1(X) = \mathbb{E}[d_L X^+ - d_H X^-]$ and $\mathcal{U}_2(X) := \mathbb{E}[X]$, $X \in L^1$, where $0 < d_L < 1 < d_H$. Moreover, suppose that $c_1 = c_2 = \infty$ and that $W \geq 0$. If $\frac{\lambda_2}{\lambda_1} < d_L$, consider the allocations $(W + k, -k) \in \mathbb{A}(W)$, $k \in \mathbb{R}_+$. Then

$$\begin{aligned} \lambda_1 \mathcal{U}_1(W + k) + \lambda_2 \mathcal{U}_2(-k) &= \lambda_1 \mathbb{E}[d_L(W + k)] - \lambda_2 k \\ &= \lambda_1 \mathbb{E}[d_L W] + (\lambda_1 d_L - \lambda_2)k \to \infty \quad \text{for } k \to \infty \end{aligned}$$

because $\lambda_1 d_L - \lambda_2 > 0$. Hence, (3.4) admits no solution. Analogously, (3.4) admits no solution in case $\frac{\lambda_2}{\lambda_1} > d_H$. However for $\frac{\lambda_2}{\lambda_1} \in [d_L, d_H]$ we have that (3.4) admits a solution which is the comonotone allocation $(0, W)$. $\diamondsuit$

Notice that in Example 4.2 there exists a solution to (3.4) even if $\frac{\lambda_1}{\lambda_2}$ equals one of the bounds given in (4.1). However, if the choice criterion of one of the decision makers is strictly concave, then often (3.4) admits no solution at the interval bounds of (4.1) either. This is illustrated by the following example.

Example 4.3. Illustration of Theorem 4.1 (i): Let $W = 0$, $c_1 = c_2 = \infty$ and consider two utility functions u_1 and u_2 with $d_H^i < \infty$ and $d_L^i > 0$, $i = 1, 2$. Let $\mathcal{U}_i(X) :=$

$\mathbb{E}[u_i(X)]$, $X \in L^1$. Suppose that $u_2'(a) = d_L^2 + \frac{1}{2\sqrt{a}}$ for large $a > 0$ (in particular u_2' does not attain d_L^2) and that $u_1'(-a) = d_H^1$ for $a > 0$. Then for $\frac{\lambda_2}{\lambda_1} = \frac{d_H^1}{d_L^2}$ and some constant $k > 0$ we have

$$\sup_{a\in\mathbb{R}} \lambda_1 \mathcal{U}_1(-a) + \lambda_2 \mathcal{U}_2(a) = \sup_{a\in\mathbb{R}} \lambda_1 u_1(-a) + \lambda_2 u_2(a) \geq \sup_{a\geq 0} \lambda_2 \sqrt{a} + k = \infty.$$

Similar arguments show that in general we cannot expect the existence solutions to (3.4) in case $\frac{\lambda_2}{\lambda_1}$ equals the lower bound $\frac{d_L^1}{d_H^2}$ either. $\Diamond$

Example 4.4. Illustration of Theorem 4.1 (ii): Suppose that $u_1(x) = d^1 x$ and $u_2(x) = d^2 x$ for some $d^1, d^2 > 0$ and suppose that $c_1 = c_2 = \infty$. If $\frac{\lambda_2}{\lambda_1} \neq \frac{d^1}{d^2}$, then $(\lambda_1 d^1 - \lambda_2 d^2) \neq 0$ and considering the allocations of type $(W + k, -k) \in \mathbb{A}(W)$ for some constant k yields

$$\sup_{k\in\mathbb{R}} \lambda_1 \mathcal{U}_1(W + k) + \lambda_2 \mathcal{U}_2(-k) = \lambda_1 \mathcal{U}_1(W) + \lambda_2 \mathcal{U}_2(0) + \sup_{k\in\mathbb{R}} (\lambda_1 d^1 - \lambda_2 d^2) k = \infty.$$

Hence, (3.4) admits no solution. $\Diamond$

4.2 Yaari Preferences Versus Multiplier Preferences

In the following we characterize the Pareto optimal allocations between two decision makers endowed with two different probabilistic sophisticated variational preferences when $c_1 = c_2 = \infty$. In particular we assume that decision maker 1 has Yaari (1987) type preferences represented by the choice criterion

$$\mathcal{U}_1(X) = \frac{1}{\alpha} \int_0^\alpha q_X(s)\, ds, \quad X \in L^1, \tag{4.2}$$

where $\alpha \in (0, 1)$ and $q_X(s) := \inf\{x : \mathbb{P}(X \leq x) \geq s\}$, $s \in (0, 1)$, is the quantile function of X. Note that the choice criterion $-\mathcal{U}_1$ is the well-known Average Value at Risk (AVaR) coherent risk measure, that is

$$\mathcal{U}_1(X) = -\mathrm{AVaR}_\alpha(X) = \min_{\mathbb{Q}\in\mathcal{Q}_1} \mathbb{E}_\mathbb{Q}[X] = \min_{\mathbb{Q}\in\mathcal{Q}_1} \int_0^1 q_X(s) q_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(1 - s)\, ds$$

where $\mathcal{Q}_1 := \{\mathbb{Q} \ll \mathbb{P} \mid \frac{d\mathbb{Q}}{d\mathbb{P}} \leq \frac{1}{\alpha}\}$; see e.g. Föllmer and Schied (2004) Theorems 4.47 and 4.54. The latter representation of $\mathcal{U}_1$ in particular reveals that, as a support function over the $\sigma(L^\infty, L^1)$-compact set $\mathcal{Q}_1$, $\mathcal{U}_1$ is concave, upper semi-continuous, and finite-valued and hence continuous on the Banach space $(L^1, \|\cdot\|_1)$; see Ekeland and Téman (1999), Part I, Corollary 2.5. As for decision maker 2, her probabilistic sophisticated variational preferences are represented by a choice criterion $\mathcal{U}_2$ as in (2.1) satisfying some additional properties which are listed in Proposition 4.6. Examples of choice criteria for decision maker 2 are the probabilistic sophisticated multiplier preferences (see Section 2.2) with any strictly increasing utility $u_2 : \mathbb{R} \to \mathbb{R}$. Other examples are obtained choosing $\mathcal{U}_2(\cdot) = U_2(u_2(\cdot))$, where

$$U_2(X) := \mathbb{E}[X] - \beta \mathbb{E}\left[(X - \mathbb{E}[X])_-^p\right]^{\frac{1}{p}}, \quad X \in L^1,$$

for some $\beta \in [0,1]$ and $p \in [1, \infty]$, is a semi-deviation utility and u_2 is the same as above.

Our case study is inspired by and extends an example in Jouini, Schachermayer, and Touzi (2008), Proposition 3.2. In Jouini, Schachermayer, and Touzi (2008) the choice criterion $\mathcal{U}_2$ is required to be cash additive, that is $u_2 \equiv \mathrm{Id}_{\mathbb{R}}$. Proposition 4.6 below shows that the functional form of the Pareto optimal allocations obtained in Jouini, Schachermayer, and Touzi (2008) stays the same also when, dropping the requirement $u_2 \equiv \mathrm{Id}_{\mathbb{R}}$, we allow for a larger class of preferences for the second agent.

Before giving the result, we recall the definition of strict risk aversion conditional on lower-tail events.

Definition 4.5. (i) *Let $X \in L^1$ and $A \in \mathcal{F}$ with $\mathbb{P}(A) > 0$. The set A is a* lower tail-event for X if $\operatorname{ess\,inf}_A X < \operatorname{ess\,sup}_A X \leq \operatorname{ess\,inf}_{A^c} X$ where $\operatorname{ess\,inf}_A X := \sup\{m \in \mathbb{R} \mid \mathbb{P}(X > m \mid A) = 1\}$ ($\sup \emptyset := -\infty$) and $\operatorname{ess\,sup}_A X := \inf\{m \in \mathbb{R} \mid \mathbb{P}(X \leq m \mid A) = 1\}$ ($\inf \emptyset := \infty$).

(ii) A function $\mathcal{U} : L^1 \to \mathbb{R} \cup \{-\infty\}$ is *strictly risk averse conditional on lower tail-events if $\mathcal{U}(X) < \mathcal{U}(X 1_{A^c} + \mathbb{E}[X \mid A] 1_A)$ for every $X \in \operatorname{dom} \mathcal{U}$ and any set A*

which is a lower tail-event for X.

Proposition 4.6. *Suppose that $c_1 = c_2 = \infty$. Let decision maker 1 be represented by* (4.2) *and decision maker 2 by a choice criterion $\mathcal{U}_2 = \inf_{\mathbb{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[u_2(X)] + \alpha_2(\mathbb{Q}))$ as in* (2.1) *with the following additional properties*

- $\operatorname{dom} \mathcal{U}_2 = \operatorname{dom} \mathcal{U}_2 + \mathbb{R}$,
- *non-satiation:* $\lim_{m\to\infty} \mathcal{U}_2(X + m) = \infty$,
- *$\mathcal{U}_2$ is strictly monotone, i.e. $X \geq Y$ and $\mathbb{P}(X > Y) > 0$ implies $\mathcal{U}_2(X) > \mathcal{U}_2(Y)$,*
- *$\mathcal{U}_2$ is strictly risk averse conditionally on lower-tail events.*

Given any initial endowments $W_i \in \operatorname{dom} \mathcal{U}_i$, $i = 1, 2$, the comonotone Pareto optimal allocations of the aggregate endowment $W = W_1 + W_2$ exist and take the following form

(4.3) $(X_1, X_2) = (-(W - l)_- + k, W \vee l - k)$ *for some $l \in \mathbb{R}\cup\{-\infty\}$ and $k = k(l) \in \mathbb{R}$.*

If $\mathcal{U}_2$ is in addition strictly concave then according to Corollary 3.12 all Pareto optimal allocations are comonotone and take the form in (4.3). The proof of Proposition 4.6 is essentially the same as in Jouini, Schachermayer, and Touzi (2008), the differences come from the facts that u_2 is not necessarily the identity on $\mathbb{R}$ and W is not necessarily bounded. For the sake of readability we provide the main ideas of the proof in Appendix D.

4.3 (Non-)Existence of Pareto Optima Outside the Subclass of Probabilistic Sophisticated Variational Preferences

Consider two decision makers with expected utility choice criteria $\mathcal{U}_1(X) = \mathbb{E}_{\mathbb{P}}[u_1(X)]$ and $\mathcal{U}_2(X) = \mathbb{E}_{\widetilde{\mathbb{P}}}[u_2(X)]$, $X \in L^\infty$, where the probability measures $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent but not equal, and $u_i : \mathbb{R} \to \mathbb{R}$ are utility functions with $u_i(0) = 0$, $i = 1, 2$. Notice that the decision makers are probabilistic sophisticated in different worlds, i.e.

with respect to different reference probabilities. Hence there is $\epsilon > 0$ such that the sets $A := \{\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \geq 1+\epsilon\}$ and $B := \{\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \leq 1-\epsilon\}$ have positive probability (under $\mathbb{P}$). Suppose that $c_1 = c_2 = \infty$ and that $(Y_1, Y_2) \in \mathcal{A}(0)$ is a Pareto optimal allocation. According to Lemma 3.4 - which does not rely on probabilistic sophistication - if the u_i are 'nice', then (Y_1, Y_2) is the solution to

$$\lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2) = \sup_{Y \in L^1} \lambda_1 \mathcal{U}_1(-Y) + \lambda_2 \mathcal{U}_2(Y)$$

for some weights $\lambda_i > 0$, $i = 1, 2$. However,

$$\begin{aligned} \lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2) &\geq \sup_{t>0} \lambda_1 \mathbb{E}_{\mathbb{P}}[u_1(-t1_A)] + \lambda_2 \mathbb{E}_{\mathbb{P}}\left[u_2(t1_A)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right] \\ &\geq \sup_{t>0}(\lambda_1 u_1(-t) + \lambda_2(1+\epsilon)u_2(t))\mathbb{P}(A) \end{aligned} \tag{4.4}$$

and similarly

$$\begin{aligned} \lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2) &\geq \sup_{t>0} \lambda_1 \mathbb{E}_{\mathbb{P}}[u_1(t1_B)] + \lambda_2 \mathbb{E}_{\mathbb{P}}\left[u_2(-t1_B)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right] \\ &\geq \sup_{t>0}(\lambda_1 u_1(t) + \lambda_2(1-\epsilon)u_2(-t))\mathbb{P}(B). \end{aligned} \tag{4.5}$$

Now it is easy to construct situations in which (4.4) or (4.5) explode and thus contradict the Pareto optimality of (Y_1, Y_2). If for instance $d_L^i > 0$ and $d_H^i < \infty$ for $i = 1, 2$, then

$$(4.4) \geq \sup_{t>0}(\lambda_2(1+\epsilon)d_L^2 - \lambda_1 d_H^1)\mathbb{P}(A)t \tag{4.6}$$

and

$$(4.5) \geq \sup_{t>0}(\lambda_1 d_L^1 - \lambda_2(1-\epsilon)d_H^2)\mathbb{P}(B)t. \tag{4.7}$$

(4.6) or (4.7) explode apart from the case[3]

$$\frac{(1+\epsilon)d_L^2}{d_H^1} \leq \frac{\lambda_1}{\lambda_2} \leq \frac{(1-\epsilon)d_H^2}{d_L^1}. \tag{4.8}$$

So in particular we must have that

$$\frac{(1+\epsilon)d_L^2}{d_H^1} \leq \frac{(1-\epsilon)d_H^2}{d_L^1}. \tag{4.9}$$

[3]Note the similarity between the bounds in (4.8) and the bounds in (4.1).

However, if e.g. $1-\epsilon/2 < d_L^i \leq d_H^i < 1+\epsilon/2$, $i=1,2$, then (4.9) is not satisfied which in the end contradicts the Pareto optimality of (Y_1, Y_2).

But there are also cases in which Pareto optimal allocations exists. Suppose that there are constants $K > 1 > k > 0$ such that $k \leq d\widetilde{\mathbb{P}}/d\mathbb{P} \leq K$ and suppose that u_2 is such that $\frac{d_H^2}{d_L^2} \geq \frac{K}{k}$. The latter condition implies that u_2 is concave enough in the sense that there is a utility function $\widetilde{u}_2$ which dominates $v(x) := k u_2(x) 1_{\{x<0\}} + K u_2(x) 1_{\{x \geq 0\}}$, $x \in \mathbb{R}$. Indeed, as v is concave on the half axises $x < 0$ and $x \geq 0$ respectively, and by the requirement on the concavity of u_2, there are $x_0 < 0$ and $x_1 > 0$ and a joint constant $L > 0$ such that $k u_2'(x_0) > K u_2'(x_1)$ and $x \mapsto k u_2'(x_0) x + L$ dominates v on $x < 0$ and $x \mapsto K u_2'(x_1) x + L$ dominates v on $x \geq 0$, so $\widetilde{u}_2(x) := K u_2'(x_1) x^+ - k u_2'(x_0) x^- + L$ does the job. Consequently for all $W \in L^1$ and for any $\lambda_1, \lambda_2 > 0$ we have

$$
\begin{aligned}
(4.10)\quad \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1 \mathcal{U}_1(X_1) + \lambda_2 \mathcal{U}_2(X_2) &= \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1 \mathcal{U}_1(X_1) + \lambda_2 \mathbb{E}_{\mathbb{P}}\left[u_2(X_2) \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \right] \\
&\leq \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1 \mathcal{U}_1(X_1) + \lambda_2 \mathbb{E}_{\mathbb{P}}\left[v(X_2) \right] \\
(4.11)\quad &\leq \sup_{(X_1,X_2)\in\mathcal{A}(W)} \lambda_1 \mathcal{U}_1(X_1) + \lambda_2 \mathbb{E}_{\mathbb{P}}\left[\widetilde{u}_2(X_2) \right].
\end{aligned}
$$

Since $\widetilde{\mathcal{U}}_2(\cdot) := \mathbb{E}_{\mathbb{P}}[\widetilde{u}_2(\cdot)]$ is of type (2.1) we know that (4.11) is bounded and admits a comonotone solution if λ_1, λ_2 satisfy the conditions stated in Theorem 4.1. Now it is easy to construct situations in which the above inequalities are indeed equalities, and solutions to (4.11) thus coincide with solutions to the left hand side of (4.10). Suppose for instance that $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = k 1_B + K 1_{B^c}$ for some set $B \in \mathcal{F}$ with $\mathbb{P}(B) = \frac{K-1}{K-k}$, and that $u_2(x) = d_L^2 x^+ - d_H^2 x^-$. Then we may choose $\widetilde{u}_2(x) = v(x) = K d_L^2 x^+ - k d_H^2 x^-$. Depending on u_1, any situation in which the extreme allocation $(W, 0)$ is a solution to (4.11) (like in Example 4.2), this allocation obviously also solves the left hand side of (4.10). Furthermore, whenever there is a solution (Y_1, Y_2) to (4.11) such that $B = \{Y_2 \leq 0\}$, the allocation (Y_1, Y_2) solves (4.10) too, because then $u_2(Y_2)\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} = \widetilde{u}_2(Y_2)$. Apparently, if $\widetilde{\mathbb{P}} \neq \mathbb{P}$, the existence of Pareto optima depends on parameters such as the deviation of the measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ from each other relative to the concavity of the utilities u_i. Hence, if the decision makers are not probabilistic sophisticated with respect to the same reference probability measure, then existence results like Theorem 3.8 do not hold in general any longer.

A Proof of Lemma 2.3

The following lemma is crucial for the proofs of Lemma 2.3 and Theorems 3.9 and 3.10.

Lemma A.1. *Let*

$$\mathcal{U}(X) = \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})\right), \quad X \in L^1,$$

be a choice criterion as (2.1)*. Define*

$$U(X) := \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}\left[X\right] + \alpha(\mathbb{Q})\right), \quad X \in L^1, \tag{A.1}$$

so that $\mathcal{U}(\cdot) = U(u(\cdot))$*. Then,* U *is a proper, rearrangement invariant,* $\succeq_{ssd}$*-monotone, upper semi-continuous, monotone, cash additive (*$\mathcal{U}(X+m) = \mathcal{U}(X)+m$ *for all* $m \in \mathbb{R}$*), and concave function. Moreover, we have that*

$$U(X) \leq \mathbb{E}[X] + U(0) \quad \textit{for all } X \in L^1. \tag{A.2}$$

Proof. We give a brief version of the proof since many of the presented arguments are standard and can for instance be found in Föllmer and Schied (2004). Cash additivity and monotonicity are obvious by definition of U and properness follows from $\inf_{\mathbb{Q}\in\mathcal{Q}} \alpha(\mathbb{Q}) > -\infty$. Concavity and upper semi-continuity follow from the fact that U is a point-wise infimum over continuous affine functions. To see that U is rearrangement invariant we note that for $X \in L^1$ and $Z \in L^\infty$ we have

$$\sup_{\widetilde{Z}\overset{d}{=}Z} \mathbb{E}[X\widetilde{Z}] = \int_0^1 q_X(s)q_Z(s)\,ds \tag{A.3}$$

where $q_Y(s) := \inf\{x \mid \mathbb{P}(Y \leq x) \geq s\}$ denotes the (left-continuous) quantile function of a random variable Y. The relation (A.3) is a consequence of the (upper)

Hardy-Littlewood inequality and some analysis. A proof can be found in Föllmer and Schied (2004), Lemma 4.55, or in a slightly more general version in Svindland (2010b), Lemma C.2. As e.g. in the proof of Föllmer and Schied (2004), Theorem 4.54, by (A.3) and rearrangement invariance of $\mathcal{Q}$ and α we obtain that

$$\begin{aligned}
U(X) &= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} X\right] + \alpha(\mathbb{Q}) \right) \quad = \quad \inf_{\mathbb{Q}\in\mathcal{Q}} \inf_{\widetilde{\mathbb{Q}} \stackrel{d}{=} \mathbb{Q}} \left(\mathbb{E}\left[\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} X\right] + \alpha(\widetilde{\mathbb{Q}}) \right) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(- \left(\sup_{\widetilde{\mathbb{Q}} \stackrel{d}{=} \mathbb{Q}} \mathbb{E}\left[-\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} X \right] \right) + \alpha(\mathbb{Q}) \right) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\int_0^1 -q_{-X}(s) q_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(s)\, ds + \alpha(\mathbb{Q}) \right) \\
&= \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\int_0^1 q_X(1-s) q_{\frac{d\mathbb{Q}}{d\mathbb{P}}}(s)\, ds + \alpha(\mathbb{Q}) \right)
\end{aligned}$$

in which, with some abuse of notation, we write $\widetilde{\mathbb{Q}} \stackrel{d}{=} \mathbb{Q}$ instead of $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{P}} \stackrel{d}{=} \frac{d\mathbb{Q}}{d\mathbb{P}}$ and use the fact that $q_X(1-s) = -q_{-X}(s)$ for almost all $s \in (0,1)$. Clearly, the last term in the equations only depends on the distribution of X under $\mathbb{P}$. Hence, the rearrangement invariance follows. According to Dana (2005), Theorem 4.1, an upper semi-continuous monotone concave function is rearrangement invariant if and only if it is $\succeq_{ssd}$-monotone as defined in Lemma 2.3 (v). The final statement follows from the fact that $\mathbb{E}[X] \succeq_{ssd} X$ by Jensen's inequality for concave functions. Thus $\succeq_{ssd}$-monotonicity and cash additivity imply that $U(X) \leq U(\mathbb{E}[X]) = \mathbb{E}[X] + U(0)$. $\square$

Proof of Lemma 2.3. Let

$$\mathcal{U}(X) := \inf_{\mathbb{Q}\in\mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q}) \right), \quad X \in L^1,$$

as in (2.1) and let U be defined as in Lemma A.1 such that $\mathcal{U}(\cdot) = U(u(\cdot))$.

(i): This follows from Jensen's inequality for concave functions and the fact that by definition dom $u \neq \emptyset$ and $\inf_{\mathbb{Q}\in\mathcal{Q}} \alpha(\mathbb{Q}) > -\infty$.

(ii): is obvious.

(iii): This follows from the concavity of u and the monotonicity and concavity of U.

(iv): Since U is rearrangement invariant (Lemma A.1), $\mathcal{U}(\cdot) = U(u(\cdot))$ is rearrangement invariant too.

(v): According Dana (2005), Theorem 4.1, an upper semi-continuous monotone concave function is rearrangement invariant if and only if it is $\succeq_{ssd}$-monotone. The upper semi-continuity of $\mathcal{U}$ is proved in the next item.

(vi): Let $k \in \mathbb{R}$ and $(X_n)_{n\in\mathbb{N}} \subset E_k := \{X \in L^1 \mid \mathcal{U}(X) \geq k\}$ be a sequence converging in $(L^1, \|\cdot\|_1)$ to some X. Then we may choose a subsequence which we also denote by $(X_n)_{n\in\mathbb{N}}$ which converges $\mathbb{P}$-a.s. to X too. We consider the following two cases: either the right-hand derivative u' of u is bounded on the domain of u or it is unbounded. In the first case, if the right-hand derivative u' of u is bounded on the domain of u, let $C > 0$ such that $u'(x) \leq C$ for all $x \in \text{dom } u$. The right continuity of u implies that in this case dom u is closed in $\mathbb{R}$. Since $X_n \in \text{dom } U$ for all $n \in \mathbb{N}$, we must have that $X_n \in \text{dom } u$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$ (see (A.2)) and therefore $X \in \text{dom } u$ $\mathbb{P}$-a.s. Monotonicity and concavity of u imply that

$$|u(X_n) - u(X)| \leq (u'(X) \vee u'(X_n))|X_n - X| \leq C|X_n - X|.$$

Hence, we conclude that the sequence $u(X_n)$ converges to $u(X)$ in L^1, and by upper semi-continuity of U we infer that

$$\mathcal{U}(X) = U(u(X)) \geq \limsup_{n\to\infty} U(u(X_n)) \geq k,$$

so E_k is closed. Now suppose that u' is unbounded on the domain of u. Then there exists a strictly decreasing sequence $(a_r)_{r\in\mathbb{N}} \subset \text{dom } u$ such that $u'(a_1) > 0$, $u'(a_r) < \infty$, and $\lim_{r\to\infty} u'(a_r) = \infty$. By the same arguments as presented in the first case the sequence $u(X_n \vee a_r)$ converges in L^1 to $u(X \vee a_r)$, because u' is bounded on $[a_r, \infty]$ and $(X_n \vee a_r)_{n\in\mathbb{N}}$ converges $\mathbb{P}$-a.s. and in $(L^1, \|\cdot\|_1)$ to $X \vee a_r$. Hence, by upper semi-continuity and monotonicity of U (Lemma A.1) as well as monotonicity of u we obtain

$$\mathcal{U}(X \vee a_r) = U(u(X \vee a_r)) \geq \limsup_{n\to\infty} U(u(X_n \vee a_r)) \geq \limsup_{n\to\infty} U(u(X_n)) \geq k.$$

Now let $a := \lim_{r\to\infty} a_r \geq -\infty$. Then dom $u \subset [a, \infty)$, and $X_n \geq a$ $\mathbb{P}$-a.s., because $X_n \in$ dom $\mathcal{U}$. Hence, $X = \lim_{n\to\infty} X_n \geq a$ $\mathbb{P}$-a.s. too and thus $\lim_{r\to\infty} X \vee a_r = X$. Moreover, by right-continuity and monotonicity of u we have $\lim_{r\to\infty} u(X \vee a_r) = u(X)$ monotonously. Therefore, we infer from applying the monotone convergence theorem that

$$\begin{aligned}\mathcal{U}(X) &= \inf_{\mathbb{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[u(X)] + \alpha(\mathbb{Q})) \quad = \quad \inf_{\mathbb{Q}\in\mathcal{Q}} \lim_{r\to\infty} (\mathbb{E}_{\mathbb{Q}}[u(X \vee a_r)] + \alpha(\mathbb{Q})) \\ &\geq \limsup_{r\to\infty} \inf_{\mathbb{Q}\in\mathcal{Q}} (\mathbb{E}_{\mathbb{Q}}[u(X \vee a_r)] + \alpha(\mathbb{Q})) \quad = \quad \limsup_{r\to\infty} \mathcal{U}(X \vee a_r) \\ &\geq k.\end{aligned}$$

Hence, also in this case E_k is closed, so $\mathcal{U}$ is upper semi-continuous.

□

B Proof of Proposition 3.3

Let $(X_1, \ldots, X_n) \in \mathbb{A}(W)$ be Pareto Optimal. Then the non-empty convex sets $C := \{(\mathcal{U}_1(X_1), \ldots, \mathcal{U}_n(X_n))\}$ and $V = \{(\mathcal{U}_1(Y_1), \ldots, \mathcal{U}_n(Y_n)) \mid (Y_1, \ldots, Y_n) \in \mathbb{A}(W)\} - \mathbb{R}^n_{++}$ in $\mathbb{R}^n$, where $\mathbb{R}^n_{++} := \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_i > 0,\ i = 1, \ldots, n\}$, have empty intersection due to the Pareto optimality of $(X_1, \ldots, X_n)$. Hence, there exists a non-trivial linear functional $(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ such that

$$\sum_{i=1}^{n} \lambda_i \mathcal{U}_i(X_i) \geq \sum_{i=1}^{n} \lambda_i (\mathcal{U}_i(Y_1) - y_i) \tag{B.1}$$

for all $(Y_1, \ldots, Y_n) \in \mathbb{A}(W)$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n_{++}$; see Rockafellar (1974), Theorem 11.2. We infer that $\lambda_i \geq 0$ for all i because otherwise choosing $y_i >> 0$ would yield a contradiction. The last assertion of Proposition 3.3 is obvious.

C Proofs of Theorems 3.9 and 3.10

The following Lemma C.1 is an Arzela-Ascoli type argument which will be crucial in the proof of Theorem 3.10. It can be derived from Tychonoff's compactness theorem or by a diagonal sequence argument. For a proof see e.g. Filipović and Svindland (2008).

Lemma C.1. *Let $f_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, be a sequence of increasing 1-Lipschitz-continuous functions such that $f_n(0) \in [-K, K]$ for all $n \in \mathbb{N}$ where $K \geq 0$ is a constant. Then there is a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ and an increasing 1-Lipschitz-continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ for all $x \in \mathbb{R}$.*

Let $\mathrm{CFN} := \{(f_i)_{i=1}^n \in \mathrm{CF} \mid f_1(0) = \ldots = f_n(0) = 0\}$. Note that

$$\mathrm{CF} = \{(f_i + a_i)_{i=1}^n \mid (f_i)_{i=1}^n \in \mathrm{CFN}, a_i \in \mathbb{R}, \sum_{i=1}^n a_i = 0\}.$$

According to Proposition 3.6 there exists a solution to (3.4) for some given weights $(\lambda_1, \ldots, \lambda_n)$ if and only if there is a solution to

$$\text{(C.1)} \quad \text{Maximize } \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W) + a_i) \qquad \text{subject to } (f_i)_{i=1}^n \in \mathrm{CFN},\ a_i \in \mathbb{R},$$
$$\sum_{i=1}^n a_i = 0, (f_i(W) + a_i)_{i=1}^n \in \mathbb{A}(W).$$

Proof of Theorem 3.10. We will prove the existence of a solution to (C.1). Fix a set of weights $(\lambda_1, \ldots, \lambda_n)$ satisfying the conditions (3.5). First of all, we observe that if for some $j \in M$ the right-hand-derivative u_j' does not attain the values d_H^j and/or d_L^j, we can always find non-negative numbers $\widetilde{d}_H^j$ and/or $\widetilde{d}_L^j$ in the image of u_j' such that, for the already given set of weights $(\lambda_1, \ldots, \lambda_n)$, the conditions (3.5) still hold true if we replace the d_H^j and/or d_L^j by $\widetilde{d}_H^j$ and/or $\widetilde{d}_H^j$. We assume that all d_H^j and/or d_L^j which are not attained by the corresponding u_j' are replaced as in the described manner, and for the sake of simplicity we keep the notation d_H^j and d_L^j. By concavity of the u_i there is a constant k such that for all $i = 1, \ldots, n$ the affine functions $\mathbb{R} \ni x \mapsto d_L^i x + k$ and $\mathbb{R} \ni x \mapsto d_H^i x + k$ both dominate u_i. Using this, we will show that

$$\text{(C.2)} \qquad P \ := \ \sup\Big\{ \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W) + a_i) \mid (f_i)_{i=1}^n \in \mathrm{CFN},\ a_i \in \mathbb{R},$$
$$\sum_{i=1}^n a_i = 0, (f_i(W) + a_i)_{i=1}^n \in \mathbb{A}(W)\Big\} < \infty,$$

and that this supremum is realized over a bounded set of comonotone allocations where the bound is given by W. More precisely, we will prove that there exists some constant $K > 0$ depending on W such that

$$\text{(C.3)} \qquad P \;=\; \sup\Big\{ \sum_{i=1}^{n} \lambda_i \mathcal{U}_i(f_i(W)+a_i) \mid (f_i)_{i=1}^n \in \mathrm{CFN},\; a_i \in [-K,K], \\ \sum_{i=1}^{n} a_i = 0, (f_i(W)+a_i)_{i=1}^n \in \mathbb{A}(W) \Big\}.$$

To this end, we define the functions

$$U_i(X) := \inf_{\mathbb{Q}\in\mathcal{Q}_i} \left(\mathbb{E}_{\mathbb{Q}}[X] + \alpha_i(\mathbb{Q})\right), \quad X \in L^1,\; i=1,\dots,n,$$

as in Lemma A.1. Consider $(f_i)_{i=1}^n \in \mathrm{CFN}$ and $a_i \in \mathbb{R}$ such that $\sum_{i=1}^n a_i = 0$ and $(f_i(W)+a_i)_{i=1}^n \in \mathbb{A}(W)$. Let $I := \{i \in \{1,\dots,n\} \mid a_i < 0\}$ and $J := \{1,\dots,n\} \setminus I$. By applying monotonicity, cash additivity and finally property (A.2) of the U_i (see Lemma A.1) we obtain:

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i \mathcal{U}_i(f_i(W)+a_i) &= \sum_{i=1}^{n} \lambda_i U_i(u_i(f_i(W)+a_i)) \\
&\le \sum_{i\in I\cap M} \lambda_i U_i(d_H^i(f_i(W)+a_i)+k) + \\
&\qquad \sum_{j\in J\cap M} \lambda_j U_j(d_L^j(f_j(W)+a_j)+k) + \\
&\qquad\qquad \sum_{l\in N} \lambda_l U_l(d_H^l(f_l(W)+a_l)+k) \\
&\le k\sum_{i=1}^{n} \lambda_i + \sum_{i\in I\cap M} \lambda_i U_i(d_H^i f_i(W)) + \\
&\quad \sum_{j\in J\cap M} \lambda_j U_j(d_L^j(f_j(W))) + \sum_{l\in N} \lambda_l U_l(d_H^l(f_l(W))) + \\
&\quad \left(\min_{i\in I\cap M} \lambda_i d_H^i\right) \sum_{i\in I\cap M} a_i + \left(\max_{j\in J\cap M} \lambda_j d_L^j\right) \sum_{j\in J\cap M} a_j + \delta \sum_{l\in N} a_l \\
&\le k\sum_{i=1}^{n} \lambda_i + \mathbb{E}[W^+] \sum_{i=1}^{n} \lambda_i d_H^i + \sum_{i=1}^{n} \lambda_i U_i(0) + \\
&\quad \left(\min_{i\in I\cap M} \lambda_i d_H^i\right) \sum_{i\in I\cap M} a_i + \left(\max_{j\in J\cap M} \lambda_j d_L^j\right) \sum_{j\in J\cap M} a_j + \delta \sum_{l\in N} a_l. \qquad \text{(C.4)}
\end{aligned}$$

Suppose that $N = \emptyset$, then we further estimate

$$\text{(C.5)} \quad \text{(C.4)} \leq \mathbb{E}[W^+] \sum_{i=1}^{n} \lambda_i d_H^i + k \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \lambda_i U_i(0) - \left(\min_{i \in I} \lambda_i d_H^i - \max_{j \in J} \lambda_j d_L^j \right) a$$

where $a := \sum_{i \in J} a_i \, (\geq 0)$. If $N \neq \emptyset$ and $\sum_{l \in N} a_l < 0$, then we estimate

$$\text{(C.6)} \qquad \text{(C.4)} \leq \mathbb{E}[W^+] \sum_{i=1}^{n} \lambda_i d_H^i + k \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \lambda_i U_i(0) - \left(\delta - \max_{j \in J \cap M} \lambda_j d_L^j \right) \tilde{a}$$

for $\tilde{a} := \sum_{i \in J \cap M} a_i \, (\geq 0)$ using (3.5). And similarly, if $N \neq \emptyset$ and $\sum_{l \in N} a_l \geq 0$, then we estimate

$$\text{(C.7)} \qquad \text{(C.4)} \leq \mathbb{E}[W^+] \sum_{i=1}^{n} \lambda_i d_H^i + k \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \lambda_i U_i(0) - \left(\min_{i \in I \cap M} \lambda_i d_H^i - \delta \right) \hat{a}$$

for $\hat{a} := \sum_{i \in J \cup N} a_i \, (\geq 0)$. Consequently we infer that

$$P \leq \mathbb{E}[W^+] \sum_{i=1}^{n} \lambda_i d_H^i + k \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \lambda_i U_i(0) < \infty.$$

Choose any allocation $(X_1, \ldots, X_n) \in \mathbb{A}(W)$. Then we have that $P \geq \sum_{i=1}^{n} \lambda_i \mathcal{U}_i(X_i) =: \widetilde{k}$. Letting

$$A := \min_{i=1,\ldots,n} \lambda_i d_H^i - \max_{j=1,\ldots,n} \lambda_j d_L^j$$

if $N = \emptyset$, or

$$A := \min \left\{ \left(\delta - \max_{j \in M} \lambda_j d_L^j \right), \left(\min_{i \in M} \lambda_i d_H^i - \delta \right) \right\}$$

if $N \neq \emptyset$, we infer from (C.5), (C.6), and (C.7) that the supremum in (C.2) is realized over allocations such that

$$|a_i| \leq \frac{|\widetilde{k}| + \mathbb{E}[W^+] \sum_{i=1}^{n} \lambda_i d_H^i + k \sum_{i=1}^{n} \lambda_i + |\sum_{i=1}^{n} \lambda_i U_i(0)|}{A} =: \overline{K}$$

for all $i \in M$, and $|\sum_{i \in N} a_i| \leq \overline{K}$ too. Note that $A > 0$ due to the conditions (3.5) on the weights λ_i. In the following we argue that in case $|N| > 1$ we may also assume that the a_i belonging to $i \in N$ are bounded due to the insensitivity of the cash additive

$\mathcal{U}_i$, $i \in N$, to constant re-sharings of 0 amongst themselves. To this end note that the choice of the λ_i and the requirement $s_i = -\infty$ for $i \in N$ implies

(C.8)
$$\sum_{i\in N} \lambda_i \mathcal{U}_i(f_i(W) + a_i + m_i) = \sum_{i\in N} \lambda_i \mathcal{U}_i(f_i(W) + a_i) + \delta \sum_{i\in N} m_i = \sum_{i\in N} \lambda_i \mathcal{U}_i(f_i(W) + a_i),$$

whenever $m_i \in \mathbb{R}$ such that $\sum_{i\in N} m_i = 0$. Hence, adding constants m_i such that $\sum_{i\in N} m_i = 0$ to the endowments of the decision makers in N does not affect the contribution of the allocation to P which immediately implies that we may assume $|a_i| \leq \overline{K}$ for all $i \in N$ if $c_i = \infty$ for all $i \in N$. If the latter condition does not hold, that is, if the set $N_b \subset N$ of indices $i \in N$ such that $c_i \in \mathbb{R}$ is not empty, we also need to consider cash amounts that might be needed to make the endowment $f_i(W) + a_i$ acceptable. This is the point where the assumption $\mathcal{U}_i(-W^-) > -\infty$ for all $i \in N_b$ enters (whenever $|N| > 1$). Using the linearity and monotonicity of $\mathcal{U}_i$ we obtain that $f_i(W) + z$ is acceptable for decision maker $i \in N_b$ whenever

$$z \geq \frac{\mathcal{U}_i(W_i) - \mathcal{U}_i(-W^-) - c_i}{d_H^i}.$$

Moreover, acceptability of $f_i(W) + a_i$ implies (again using linearity and monotonicity of $\mathcal{U}_i$) for all $i \in N_b$:

$$a_i \geq \frac{\mathcal{U}_i(W_i) - c_i - \mathcal{U}_i(0)}{d_H^i} - \mathbb{E}[W^+].$$

Therefore we may assume that there exists a constant $\widehat{K} > 0$ such that $|a_i| \leq \widehat{K}$ for all $i \in N_b$ and thus $|\sum_{i\in N_u} a_i| \leq \overline{K} + |N_b|\widehat{K} =: K$ where $N_u := N \setminus N_b$. Since $c_i = \infty$ for all $i \in N_u$ we are back to the already treated situation and may by cash invariance assume that indeed $|a_i| \leq K$ for all $i \in N_u$, and finally that $|a_i| \leq K$ for all $i = 1, \ldots, n$. Hence, (C.2) and (C.3) are proved. By virtue of (C.3) we may choose a sequence $((f_i^p)_{i=1}^n)_{p\in\mathbb{N}} \subset \mathrm{CF}$ with $f_i^p(0) \in [-K, K]$ for all $i = 1, \ldots, n$ and $p \in \mathbb{N}$ such that $(f_i^p(W))_{i=1}^n \in \mathbb{A}(W)$ for all $p \in \mathbb{N}$ and

$$P = \lim_{p\to\infty} \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i^p(W)).$$

According to Lemma C.1 there exists a subsequence, which we for the sake of simplicity also denote by $(f_i^p)_{i=1}^n$, which converges pointwise to some $(f_i)_{i=1}^n \in \mathrm{CF}$. As $|f_i^p(W)| \leq |W| + K$ for all $i = 1, \dots, n$ and $p \in \mathbb{N}$, we may apply the dominated convergence theorem which yields $f_i(W) \in L^1$, and $\lim_{p\to\infty} \mathbb{E}[|f_i(W) - f_i^p(W)|] = 0$ for all $i = 1, \dots, n$. By upper semi-continuity of the $\mathcal{U}_i$ (Lemma 2.3) we have

$$\mathcal{U}_i(W_i) - c_i \leq \limsup_{p\to\infty} \mathcal{U}_i(f_i^p(W)) \leq \mathcal{U}_i(f_i(W)),$$

and

$$\begin{aligned} P &= \lim_{p\to\infty} \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i^p(W)) \quad \leq \quad \sum_{i=1}^n \lambda_i \limsup_{p\to\infty} \mathcal{U}_i(f_i^p(W)) \\ &\leq \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W)). \end{aligned}$$

Hence, we infer that $(f_i(W))_{i=1}^n \in \mathbb{A}(W)$ (since $P > -\infty$) and

$$P = \sum_{i=1}^n \lambda_i \mathcal{U}_i(f_i(W)).$$

For the last part of (ii) suppose that $|N| = n$, and let $(X_1, \dots, X_n)$ be a solution to (3.4) for the given weights. If $m_i \in \mathbb{R}$ such that $\sum_{i=1}^n m_i = 0$, then the same computation as in (C.8) yields $\sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i + m_i) = \sum_{i=1}^n \lambda_i \mathcal{U}_i(X_i)$. □

Proof of Theorem 3.9. Recall (C.1) and let $(f_i)_{i=1}^n \in \mathrm{CFN}$, $a_i \in \mathbb{R}$ with $\sum_{i=1}^n a_i = 0$ such that $(f_i(W) + a_i)_{i=1}^n \in \mathbb{A}(W)$. Since in particuar $\mathcal{U}_i(f_i(W) + a_i) > -\infty$, we must have that $f_i(W) + a_i \geq s_i$ for all $i = 1, \dots, n$. Let $\widetilde{K} := \sum_{i=1}^n |s_i|$. Then

$$-(|W| + \widetilde{K}) \leq f_i(W) + a_i = W - (\sum_{j\neq i} f_j(W) + a_j) \leq |W| + \widetilde{K}.$$

Hence, we deduce that (C.3) holds with $K := 2 \operatorname{essinf} |W| + \widetilde{K}$. The rest of the proof now follows the lines of the proof of Theorem 3.10. □

D Proof of Proposition 4.6

For the proof of Proposition 4.6 we will need some (additional) tools from convex duality theory which we briefly introduce in the following. The details and proofs of the statements can e.g. be found in Ekeland and Téman (1999). Let $\mathcal{U}$ be a choice criterion as in (2.1). The dual function of $\mathcal{U}$ is

$$\mathcal{U}^*(Z) := \sup_{Y \in L^1} \mathcal{U}(Y) - E[YZ], \quad Z \in L^\infty,$$

which is convex and $\sigma(L^\infty, L^1)$-lower semi-continuous, i.e. the level sets $E_k := \{Z \in L^\infty \mid \mathcal{U}^*(Z) \leq k\}$ are closed in the $\sigma(L^\infty, L^1)$-topology for all $k \in \mathbb{R}$. Moreover, $\mathcal{U}^*$ is rearrangement invariant by the same arguments as applied in the proof of Lemma A.1 ($-\mathcal{U}^*$ is concave and upper semi-continuous) and therefore $\mathcal{U}^*$ is $\succeq_c$-antitone according to Dana (2005), Theorem 4.1. Since $\mathcal{U}$ is concave and upper semi-continuous (Lemma 2.3), it follows from the Fenchel-Moreau theorem that

$$\text{(D.1)} \qquad \mathcal{U}(X) = \mathcal{U}^{**}(X) := \inf_{Z \in L^\infty} E[ZX] + \mathcal{U}^*(Z), \quad X \in L^1.$$

Again the very same techniques as in the proof of Lemma A.1 show that $\mathcal{U}$ is rearrangement invariant if and only if $\mathcal{U}^*$ is rearrangement invariant. The superdifferential of $\mathcal{U}$ at some $X \in L^1$ is

$$\partial \mathcal{U}(X) := \{Z \in L^\infty \mid \mathcal{U}(Y) \leq \mathcal{U}(X) + \mathbb{E}[Z(Y - X)] \, \forall Y \in L^1\}.$$

Notice that

$$\text{(D.2)} \qquad Z \in \partial \mathcal{U}(X) \quad \Leftrightarrow \quad \mathcal{U}(X) = E[ZX] + \mathcal{U}^*(Z)$$

and that monotonicity of $\mathcal{U}$ implies $\partial \mathcal{U}(X) \subset \operatorname{dom} \mathcal{U}^* \subset L^\infty_+$.

Lemma D.1. *Let $\mathcal{U}$ be a choice criterion as in* (2.1) *and let $X \in L^1$ such that $\partial \mathcal{U}(X) \neq \emptyset$. Then there exists a decreasing function $h : \mathbb{R} \to [0, \infty)$ such that $h(X) \in \partial \mathcal{U}(X)$.*

Proof. Let $Z \in \partial\mathcal{U}(X)$ and $h : \mathbb{R} \to \mathbb{R}_+$ be a measurable function such that $h(X) = E[Z \mid X]$. By (D.2), $E[Z \mid X] \succeq_c Z$ (Jensen's inequality), $\succeq_c$-antitonicity of $\mathcal{U}^*$ and finally (D.1) it follows that

$$\mathcal{U}(X) = E[ZX] + \mathcal{U}^*(Z) \geq E[E[Z \mid X]X] + \mathcal{U}^*(E[Z \mid X]) \geq \mathcal{U}(X).$$

Thus $h(X) \in \partial\mathcal{U}(X)$ too. Note that (A.3) and rearrangement invariance of $\mathcal{U}^*$ imply

$$\mathcal{U}(X) \leq \int_0^1 q_{h(X)}(1-t)q_X(t)\,dt + \mathcal{U}^*(h(X))$$

in the same way as the similar argument presented in the proof of Lemma A.1. Hence we obtain that

$$\begin{aligned}\mathcal{U}(X) &\leq \int_0^1 q_{h(X)}(1-t)q_X(t)\,dt + \mathcal{U}^*(h(X)) \\ &\leq E[h(X)X] + \mathcal{U}^*(h(X)) \quad = \quad \mathcal{U}(X)\end{aligned}$$

where we applied the Hardy-Littlewood inequalities in the second step; see Föllmer and Schied (2004) Theorem A.24. Consequently $E[h(X)X] = \int_0^1 q_{h(X)}(1-t)q_X(t)\,dt$ which guarantees that h might be chosen as to be decreasing; see again Föllmer and Schied (2004) Theorem A.24. □

Lemma D.2. *Let $\mathcal{U}$ be a choice criterion as in* (2.1) *which is strictly risk averse conditional on lower-tail events. Let $(X, Z) \in L^1 \times L^\infty$ be such that $Z \in \partial\mathcal{U}(X)$ and $X = f(W)$, $Z = h(W)$ for some $W \in L^1$ and an increasing function $f : \mathbb{R} \to \mathbb{R}$ and a decreasing function $h : \mathbb{R} \to \mathbb{R}_+$. Consider the set $A := \{Z = \operatorname{ess\,sup} Z\}$. If $\mathbb{P}(A) > 0$, then X is constant on the set A.*

Proof. Assume that $\mathbb{P}(A) > 0$ and, by contradiction, that X is not constant on A. Since f is increasing and h is decreasing A is a lower tail-event of X. As $\mathcal{U}$ is strictly risk averse conditional on lower-tail events it follows that

$$\mathcal{U}(X) < \mathcal{U}(\overline{X}) \tag{D.3}$$

where $\overline{X} = X1_{A^c} + \mathbb{E}[X \mid A]1_A$. But $\mathbb{E}[ZX] = \mathbb{E}[Z\overline{X}]$ and $Z \in \partial\mathcal{U}(X)$ imply $\mathcal{U}(\overline{X}) \leq \mathcal{U}(X) + \mathbb{E}[Z(\overline{X} - X)] = \mathcal{U}(X)$ which contradicts (D.3). □

Lemma D.3. *Let $\mathcal{U}$ be a choice criterion as in* (2.1) *which is in addition strictly monotone and let $(X, Z) \in L^1 \times L^\infty$ such that $Z \in \partial\mathcal{U}(X)$. Then $Z > 0$ a.s.*

Proof. Let $A := \{Z = 0\}$. As $\mathcal{U}(X+1_A) \le \mathcal{U}(X)+\mathbb{E}[Z1_A] = \mathcal{U}(X)$, strict monotonicity of $\mathcal{U}$ implies $\mathbb{P}(A) = 0$. $\square$

Proof of Proposition 4.6. Let $(X_1, X_2) \in \mathbb{A}(W)$ be a comonotone Pareto optimal allocation of W, and let $f, g : \mathbb{R} \to \mathbb{R}$ be increasing functions such that $f + g = \mathrm{Id}_{\mathbb{R}}$ and $(X_1, X_2) = (f(W), g(W))$. According to Lemma 3.4 there exists $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\lambda_1\mathcal{U}_1(X_1) + \lambda_2\mathcal{U}_2(X_2) = \max_{(Y_1,Y_2)\in\mathbb{A}(W)} \lambda_1\mathcal{U}_1(Y_1) + \lambda_2\mathcal{U}_2(Y_2). \tag{D.4}$$

Note that the function

$$\mathcal{U}_{\lambda_1,\lambda_2}(Y) := \sup_{(Y_1,Y_2)\in\mathbb{A}(Y)} \lambda_1\mathcal{U}_1(Y_1) + \lambda_2\mathcal{U}_2(Y_2), \quad Y \in L^1,$$

is concave, increasing and

$$\mathrm{dom}\,\mathcal{U}_{\lambda_1,\lambda_2} = \mathrm{dom}\,\mathcal{U}_1 + \mathrm{dom}\,\mathcal{U}_2 = L^1 + \mathrm{dom}\,\mathcal{U}_2 = L^1.$$

Moreover $\mathcal{U}_{\lambda_1,\lambda_2}(\cdot) \ge \lambda_1\mathcal{U}_1(\cdot) + \lambda_2\mathcal{U}_2(0)$ implies that there exists an open set in L^1 on which $\mathcal{U}_{\lambda_1,\lambda_2}$ is bounded from below, because $\lambda_1\mathcal{U}_1$ as a continuous concave function has this property; see Ekeland and Téman (1999) Proposition 2.5. Hence $\mathcal{U}_{\lambda_1,\lambda_2}$ is continuous on L^1 and therefore everywhere superdifferentiable; see e.g. Ekeland and Téman (1999) Proposition 2.5 and Proposition 5.2. $\mathcal{U}_{\lambda_1,\lambda_2}$ is also rearrangement invariant. This can be deduced by verifying that the dual

$$\mathcal{U}^*_{\lambda_1,\lambda_2} = (\lambda_1\mathcal{U}_1)^* + (\lambda_2\mathcal{U}_2)^*$$

is rearrangement invariant as a sum of rearrangement invariant functions; see also the introductory remarks of this section. According to Lemma D.1 there exists a decreasing function $h : \mathbb{R} \to \mathbb{R}_+$ such that

$$Z := h(W) \in \partial\mathcal{U}_{\lambda_1,\lambda_2}(W) = \partial\lambda_1\mathcal{U}_1(X_1) \cap \partial\lambda_2\mathcal{U}_2(X_2). \tag{D.5}$$

The inclusion $\subset$ in (D.5) is due to the fact that for all $Z \in \partial \mathcal{U}_{\lambda_1,\lambda_2}(W)$ we have that

$$\mathcal{U}_{\lambda_1,\lambda_2}(Y) \leq \mathcal{U}_{\lambda_1,\lambda_2}(W) + \mathbb{E}[Z(Y-W)] \quad \text{for all } Y \in L^1$$

and thus by (D.4) and definition of $\mathcal{U}_{\lambda_1,\lambda_2}$ that

$$\lambda_1 \mathcal{U}_1(Y_1) + \lambda_2 \mathcal{U}_2(Y_2) \leq \lambda_1 \mathcal{U}_1(X_1) + \lambda_2 \mathcal{U}_2(X_2) + \mathbb{E}[\tilde{Z}(Y_1 + Y_2 - (X_1 + X_2))]$$

for all $Y_1, Y_2 \in L^1$. Now $Z \in \partial \lambda_i \mathcal{U}_i(X_i)$, $i = 1, 2$, follows. The converse inclusion in (D.5) follows similarly. By definition of the supergradient, this implies that $\frac{Z}{\lambda_1} \in \partial \mathcal{U}_1(X_1)$ and $\frac{Z}{\lambda_2} \in \partial \mathcal{U}_2(X_2)$. From $\frac{Z}{\lambda_2} \in \partial \mathcal{U}_2(X_2)$ and the strict monotonicity of $\mathcal{U}_2$ it follows that $\mathbb{P}(Z = 0) = 0$; see Lemma D.3. Note that

$$\mathcal{U}_1(Y) = \int_0^1 q_Y(t)\, d\varphi(t), \quad Y \in L^1,$$

where $\varphi(t) := \frac{t}{\alpha} \wedge 1$ for $t \in [0,1]$ is an increasing continuous (on $(0,1)$) function. Since $\frac{Z}{\lambda_1} \in \partial \mathcal{U}_1(X_1)$ we have also that

$$\mathcal{U}_1(X_1) = \int_0^1 q_{X_1}(t) q_{\frac{Z}{\lambda_1}}(1-t)\, dt = \int_0^1 q_{X_1}(t)\, d\psi(t)$$

where $\psi(t) := \frac{1}{\lambda_1} \int_0^t q_Z(1-s)\, ds$, $t \in [0,1]$, is another increasing continuous function. Hence, we obtain that

$$\int_0^1 q_{X_1}(t)\, d\varphi(t) - \int_0^1 q_{X_1}(t)\, d\psi(t) = \mathcal{U}_1(X_1) - \mathcal{U}_1(X_1) = 0$$

and integration by parts (Dunford and Schwartz 1976, III.6.21, Theorem 22) of each integral in combination with a limiting argument by means of $X_1^n := -n \vee X_1 \wedge n$, $n \in \mathbb{N}$, yields

$$\int_0^1 (\psi(t) - \varphi(t))\, dq_{X_1}(t) = 0. \tag{D.6}$$

As $\frac{Z}{\lambda_1} \in \mathcal{Q}_1$, we observe that $\psi \leq \varphi$. Hence (D.6) can only be satisfied if q_{X_1} is constant on $\{\psi < \varphi\}$.

Since $\mathbb{P}(Z = 0) = 0$, we have $q_Z(1-s) > 0$ for any $s \in (0,1)$, so in particular $\psi(t) < 1$ for all $t < 1$. Moreover, $\psi(0) = 0$ and the slope of ψ is at most $\frac{1}{\alpha}$. Therefore we have $\beta := \inf\{t \mid \psi(t) < \varphi(t)\} \in [0,1)$ and

$$q_{X_1} \text{ is constant on } (\beta, 1) \subset \{\psi < \varphi\}. \tag{D.7}$$

If $\beta > 0$, it follows for any $t \in [0, \beta]$ that $\frac{t}{\alpha} = \psi(t) = \int_{1-t}^{1} \frac{1}{\lambda_1} q_Z(s) ds$. As $q_{\frac{Z}{\lambda_1}} \leq \frac{1}{\alpha}$, we deduce that

$$q_Z(s) = \frac{\lambda_1}{\alpha} = \operatorname{ess\,sup} Z \text{ for all } s \in (1-\beta, 1]. \tag{D.8}$$

Since $\frac{Z}{\lambda_2} \in \partial \mathcal{U}_2(X_2)$ and as $\mathcal{U}_2$ is strictly risk averse conditionally on lower tail-events, X_2 is constant on $\{Z = \operatorname{ess\,sup} Z\}$; see Lemma D.2. Recall that $X_1 = f(W)$, $X_2 = g(W)$ and $Z = h(W)$ for increasing functions g and f and a decreasing function h. Consequently (D.7) implies that $f(W)$ is constant on $W^{-1}(l, \infty)$ whereas (D.8) implies that $h(W)$ and thus $g(W)$ are constant on $W^{-1}(-\infty, l)$ for $l := q_W(\beta)$ ($:= -\infty$ if $\beta = 0$). In conjunction with the fact that f, g are continuous we deduce that X_1 and X_2 ought to be of the following form

$$X_1 = (W - l)1_{\{W \leq l\}} + k, \quad X_2 = l1_{\{W \leq l\}} + W1_{\{l \leq W\}} - k, \quad \text{where } k \in \mathbb{R}.$$

□

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