Exit Options and Dividend Policy under Liquidity Constraints*

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Abstract

We introduce a post-entry liquidity constraint to the classic real option model of a firm with serially correlated profitability and an irreversible exit decision. We assume that a firm with no cash holdings and negative cash flow is forced to exit regardless of its future prospects. This creates a precautionary motive for holding cash, which must be traded off against the liquidity cost of holding cash. We characterize the optimal exit and dividend policy and analyze numerically its comparative statics properties. The firm pays dividends when it is in a sufficiently strong position in terms of cash flow and cash holdings, and the firm almost surely exits before running out of cash. The direct effect of the liquidity constraint is to impose inefficient exit, but in industry equilibrium it also creates a price distortion that leads to inefficient survival. (D81, D92, G35)

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1 Introduction

We analyze how financial frictions affect the optimal policy and survival prospects of a firm that operates under persistent cash flow uncertainty. The persistence in profitability implies that a firm should exit if the current cash flow falls sufficiently low. Financial frictions imply that a firm may also exit due to insufficient liquidity even when continuation would be economically efficient. Our model captures the interaction of these two dimensions—profitability and liquidity—underlying firm exit, and shows how the firm should optimally manage its cash reserves to cope with the liquidity constraint. The solution is a policy for exit and payouts that depends on the current levels of both profitability and cash holdings. We also analyze the associated steady state distribution of firms in a competitive industry, and show how it can involve either too much or too little exit, the latter case being a type of "survival of the fattest."

Our starting point is a standard real option model of a firm with serially correlated profitability and an irreversible exit decision.[1] In this setup the potential for future profits and the irreversibility of exit make it optimal for a firm to continue even when facing expected losses. Cash holdings are irrelevant in the absence of financial constraints and the optimal policy is simply a (negative) threshold level of profitability below which the firm exits. The optimal exit policy thus requires the ability to sustain negative cash flows indefinitely. It seems realistic in many contexts that a firm with a long history of losses would find it difficult to keep raising more funds. But as soon as there is a limit to a firm's ability to sustain losses the firm's problem changes in a fundamental way.

In our basic case we model the liquidity constraint as the complete inability to raise new funds. The firm has an initial stock of cash that can only be augmented with retained earnings. A firm without cash and with a negative cash flow is forced to exit immediately regardless of its future prospects, so firms have an incentive to hoard cash in order to avoid inefficient exit in the future. This precautionary saving is costly due to the liquidity premium: cash holdings earn interest at a rate below the discount rate. Therefore, if the firm is sufficiently safe from forced exit—with a sufficiently benign combination of cash flow and cash holdings—it is strictly optimal to pay out some of the cash to the owners. Thus, besides affecting the optimal exit policy, the model

[1] See e.g. Chapter 7 in Dixit and Pindyck (1994).

also generates the optimal dividend policy. At the same time, if the firm is currently unprofitable and the remaining cash holdings are relatively small, it can be optimal to pay out the remaining cash and close down operations rather than run the risk of forced exit later on. We call this feature of the optimal policy "precautionary exit." We characterize the optimal policy and analyze its dependence on the properties of the cash flow process. Our model leads to a free boundary partial differential equation problem that does not have an analytical solution. Instead of attempting to solve the firm's problem directly we formulate it as a recursive dynamic programming problem and show how it can be easily solved by value function iteration. The solution has an intuitive interpretation and we illustrate its comparative statics properties graphically. Our numerical results show that even a small liquidity premium has a large impact on optimal firm behavior.

We do not explicitly model the causes behind the liquidity constraint. One natural cause is asymmetric information: it can be difficult for a firm or a manager to credibly convey to investors the potential for profits.[2] Aside from the liquidity constraint, our model has no other imperfections such as agency problems.

The literal interpretation of the decision-maker in our basic model is a risk neutral owner-entrepreneur who can increase cash holdings only through retained earnings. Nevertheless, we believe our findings have relevance in the wider context. In an extension we assume that part of the fixed operating cost of the firm is due to debt service. We show that the main insights continue to hold even if the exit decision is made by the debtholders. Our model implies an endogenous negative relation between economic performance and balance sheet liquidity for distressed firms, which is consistent with recent empirical findings by Davydenko (2010) and by Acharya, Davydenko and Strebulaev (2011). In another extension we show that our results are robust to allowing the owners to raise new funds at a transaction cost; in effect the basic model assumes that this cost is prohibitive.

We also analyze the impact of the liquidity constraint at the level of an industry. Our concept of competitive industry equilibrium with entry and exit of firms is

[2]For evidence on the importance of liquidity constraints for firms, see, for example, Evans and Jovanovic (1989), Holtz-Eakin, Joulfaian and Rosen (1994), and Zingales (1998). There is also a literature on endogenous borrowing constraints, e.g., Albuquerque and Hopenhayn (2004), and DeMarzo and Sannikov (2006). Holmström and Tirole (2011, esp. Chapters 1-2) discuss why agency problems may cause a firm to face a liquidity constraint.

essentially that of Hopenhayn (1992), and we assume that the uncertainty faced by individual firms is due to idiosyncratic productivity shocks. In this setup the liquidity constraint causes an obvious overselectivity effect in terms of productivity: some marginally productive firms that should survive a temporary loss exit due to insufficient funds (or, more accurately, in order to preempt forced exit). This effect tends to make the remaining industry on average more productive by weeding out marginally productive firms that would need financing to survive. However, the liquidity constraint also induces some formerly productive firms with sufficient cash holdings to stay on even when their productivity falls below the socially efficient exit threshold. This is a type of "survival of the fattest" as coined by Zingales (1998). More specifically, we show that when the entry cost is sufficiently low the liquidity constraint in fact lowers the average productivity of firms in the industry.

Related literature

Our model builds on elements from the literature on the optimal exercise of options, where the seminal papers are by McDonald and Siegel (1986) who model the optimal timing of investment under uncertain cash flow, and by Dixit (1989) who analyzes the firm's optimal entry and exit decisions in the same framework. A large number of extensions to various directions is summarized by Dixit and Pindyck (1994). Our paper extends this line of research to another direction by adding a liquidity constraint that may prevent the firm from covering operating losses.

One paper that address the effects of liquidity constraints on the optimal exercise of real options is by Boyle and Guthrie (2003), who analyze the optimal timing of investment when uncertain wealth prior to the investment affects the firm's ability to finance the investment. Our paper, by contrast, focuses on post-investment uncertainty and its effects on optimal payouts and exit.

A special case of our model, where we assume away the liquidity premium, bears close resemblance to the problem of a financially constrained firm in Mello and Parsons (2000), who analyze the optimal hedging policy for a firm that faces persistent cash flow risk and cannot raise new funds. Gryglewicz (2011) presents a model of a financially constrained start-up firm, where the mean level of a stochastic cash flow is learned over time. Eventually, as firms mature, they either go bankrupt, or their confidence of being high type converges to certainty, in which case they face only i.i.d.

risk and their cash holdings increase without limit. In these models the firm has to choose the optimal exit policy, but it has no reason to ever pay out dividends.

It is important to make a clear distinction between our model and an ostensibly similar stream of literature that considers the problem of a liquidity constrained firm under non-persistent cash flow risk. This other literature models *cumulative* earnings as a Markovian stochastic process, which leads to independently distributed earnings across periods, whereas we model the *level* of earnings as the state variable which results in serially correlated earnings. Milne and Robertson (1996) is a representative model of a firm facing a memoryless profit stream under a financial constraint, where the firm faces exogenous liquidation if cash balance falls below a given threshold. The optimal policy is to accumulate a buffer stock of savings up to a point and pay out as dividends all income above that level. A number of other papers analyze various additional features in a similar framework: Radner and Shepp (1996) and Dutta and Radner (1999) add an operation policy that controls risk-return properties of the earnings process, Décamps and Villeneuve (2007) analyze the optimal exercise of a growth option, Peura and Keppo (2006) introduce a delay time to recapitalization, and Rochet and Villeneuve (2005) allow flexible allocation of reserves in risky and safe alternatives. Décamps, Mariotti, Rochet, and Villeneuve (2011) assume costly recapitalization, and analyze the implications of such financing frictions on the firm's cash management and stock price dynamics.

The attraction of modeling the level of profits as a memoryless process is that it results in one-dimensional state-space, which yields analytical solutions. The drawback is that the liquidity constraint is then the *only* reason why the firm would ever exit, because the future always looks equally profitable. This is reasonable for a firm that consists of financial assets whose prices react to news in an efficient market but is less suited as a model of a firm facing uncertainty over real (non-financial) operations. In our setup, the firm's profitability (the level of expected profit flow) fluctuates, making entry and exit natural features of the economy irrespective of whether there are liquidity constraints or not. Having a first-best benchmark that involves firm exit allows us to analyze how the liquidity constraint affects firm survival, and how, at industry level, it impacts firm selection.

There are also a few papers on the macroeconomic effects of financial frictions that are related to ours. Cooley and Quadrini (2001), Gomes (2001), and Jones (2003) use as building blocks models of firm dynamics with serially correlated productivity.

In Gomes's and Jones's papers firms also face an exit decision, and in the latter paper the financial constraint may force the firm to exit in states where it would be socially efficient to continue. However, due to different focus, none of these papers characterize the joint exit-payout policy of the firm.

Our setup is also to some extent related to the models of precautionary saving. The seminal papers on precautionary saving by Zeldes (1989) and Deaton (1991) analyze the problem of optimal lifetime consumption. Under serially correlated income shocks the state space is two-dimensional (savings and expected income) as in our model; the key difference is that consumers do not face an exit decision. For consumers, precautionary saving results from the convexity of marginal utility, whereas in our model it results from the threat of forced exit.

In the next section we characterize the problem of the firm, and then in section 3 we solve the firm's optimal policy under the liquidity constraint and analyze its comparative statics. Extensions to debt and recapitalization are analyzed in sections 4 and 5 respectively. The implications of the liquidity constraint for a competitive industry are analyzed in section 6.

2 The Problem of the Firm

The firm faces a stochastic revenue flow x_t that follows geometric Brownian motion

$$dx_t = \mu x_t dt + \sigma x_t dw_t, \tag{1}$$

where dw_t is the increment of a standardized Wiener process (i.e., with mean zero and variance dt). The firm earns a profit flow $\pi_t = x_t - c$ where the fixed cost c is a positive constant. Exit is irreversible and without an additional exit cost or scrap value. (The entry decision will only show up in industry equilibrium.) The objective is to maximize the expected present value of the income to the owners, discounted at rate $\rho > \mu$.

There are two fundamentally different cases. An unconstrained firm can accumulate negative profits indefinitely if needed. The problem of an unconstrained firm is described by the standard real option model of optimal exit. The sole decision is to choose the exit threshold for x_t, so there is no meaningful decision for when (if at all) to retain cash or pay dividends.

A constrained firm has to worry about its ability to cover negative profits, because it is forced to exit if it has no cash while it faces a negative cash flow. The optimal exit policy depends both on revenue x_t and cash holdings s_t. The firm's cash holdings are augmented by the profit flow and by the interest earned on the cash holdings at an exogenous rate $r \leq \rho$. The difference $\rho - r$ is the liquidity premium. If $r < \rho$ then the cash held inside the firm incurs a cost to the owners, so they face a meaningful decision of how to pay dividends.[3] The downside of payouts is that reduced cash holdings lower the capability to cover any future losses. We start by assuming that the liquidity constraint is very stark in the sense that it is not possible to inject more cash into the firm. We later extend the model to the case where new funds may be raised at some transaction cost; the basic version can be thought of as a special case in which such transaction costs are prohibitive.

2.1 Unconstrained Firm

The unconstrained firm will exit if the cash flow becomes too negative. The value function $V^*(x)$ gives the expected discounted future cash flows for a firm with current revenue level $x_t = x$, and it is defined by the familiar differential equation:

$$\rho V^*(x) = x - c + \mu x V_x^*(x) + \frac{\sigma^2}{2} x^2 V_{xx}^*(x) \tag{2}$$

(see e.g. Dixit and Pindyck 1994, Chapter 7) with the constraints that V_x^* be continuous ("smooth pasting") and have a finite limit. This ODE has a well-known closed-form solution. The firm exits when x_t falls to x^* given by

$$x^* = \frac{\beta(\rho - \mu)}{\beta - 1} \frac{c}{\rho}, \tag{3}$$

$$\text{where } \beta = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\rho}{\sigma^2}} < 0. \tag{4}$$

The unconstrained value function is

$$V^*(x) = \begin{cases} \left(\frac{c}{\rho} - \frac{x^*}{\rho - \mu}\right)\left(\frac{x}{x^*}\right)^\beta + \frac{x}{\rho - \mu} - \frac{c}{\rho} & \text{for } x \geq x^*, \\ 0 & \text{for } x < x^*. \end{cases} \tag{5}$$

[3] Another literal interpretation is that the owner-manager is risk neutral but discounts consumption by more than the return on cash. Decamps et al (2008) interpret $\rho - r$ as a reduced form of an agency cost, caused by the manager engaging in wasteful activities with the firm's liquid assets.

2.2 Constrained Firm

The constrained firm has an initial cash balance s_0 that is exogenous to the problem. Cash earns interest at rate $r \leq \rho$. At any moment t, the firm can run down its cash balance by paying dividends. Paying dividends is costless and instantaneous. The objective of the firm is to maximize the expected discounted stream of dividend payments. We denote by $D := \{D_t\}_{t \geq 0}$ the cumulative dividend process. The restrictions that we impose on this process are the following. First, since we allow only positive dividend payments, the process must be increasing. Second, the dividend payment dD_t at time t can only be conditioned on past history of cash-flows. Stated in technical terms, D must be adapted to the filtration generated by the Brownian motion $\{w_t\}_{t \geq 0}$. Third, we assume that D is right-continuous (upward jumps in D represent lumpy dividend payments). Finally, D must satisfy the liquidity constraint, which requires that $s_t \geq 0$ for all t, where the dynamics of the cash balance s_t are given by:

$$ds_t = (x_t - c + rs_t)\, dt - dD_t. \tag{6}$$

The firm is forced to exit if $x_t \leq c$ and $s_t = 0$, so the exit time τ is given by

$$\tau := \inf \{t \geq 0 : x_t \leq c \text{ and } s_t = 0\}. \tag{7}$$

The objective of the firm is to choose a dividend process to maximize:

$$\sup_D \mathbb{E} \int_{t=0}^{\tau} e^{-\rho t} dD_t \tag{8}$$

subject to (6), (7), and $s_t \geq 0$ for all $t \in [0, \tau]$. Note that this formulation allows voluntary exit when $x_t < c$ and $s_t > 0$ by paying out the remaining cash as the liquidation value: $dD_t = s_t$.[4]

The firm's problem becomes much more intuitive once recast as a Markovian control problem with suitably chosen state variables. Note that the history at time t consists of past cash flows $\{x_{t'}\}_{0 \leq t' \leq t}$, past dividends $\{D_{t'}\}_{0 \leq t' \leq t}$, and the initial cash balance s_0. Since the cash flow process is Markovian, the part of the history that defines the probability distribution for future incomes is summarized in the current cash flow level x_t. Similarly, the part of the history that defines the firm's capacity to satisfy the liquidity constraint is summarized as the current cash holdings

[4] We allow voluntary exit when $x_t > c$, but this would never be optimal.

s_t, as derived from past cash flows and dividend payments through equation (6). Therefore, the pair (x_t, s_t) summarizes the history part that is payoff relevant for the future, and is sufficient for deciding the optimal policy at t by the Bellman's Principle of Optimality. Consequently, we may denote by $V(x, s)$ the value of the firm that solves (8) starting from an arbitrary state point $(x_0, s_0) = (x, s)$.

In effect, the problem of the firm is to choose between three policy options at each point of the state space. First, the firm may exit, which is irreversible, and results in the exit value s_t. Second, the firm may pay a positive dividend dD_t to the owners, which shifts the firm in the state space to cash balance level $s_t - dD_t$. Third, the firm can continue without paying dividends, in which case the cash balance evolves according to

$$\frac{ds_t}{dt} = x_t - c + rs_t. \tag{9}$$

The solution to the firm's problem is a division of the (x, s) −space into regions in each of which one of the three policy options is optimal. The following Proposition characterizes the solution in the case where $r < \rho$ (the special case $r = \rho$ will be discussed later). For illustration, see Figure 1.

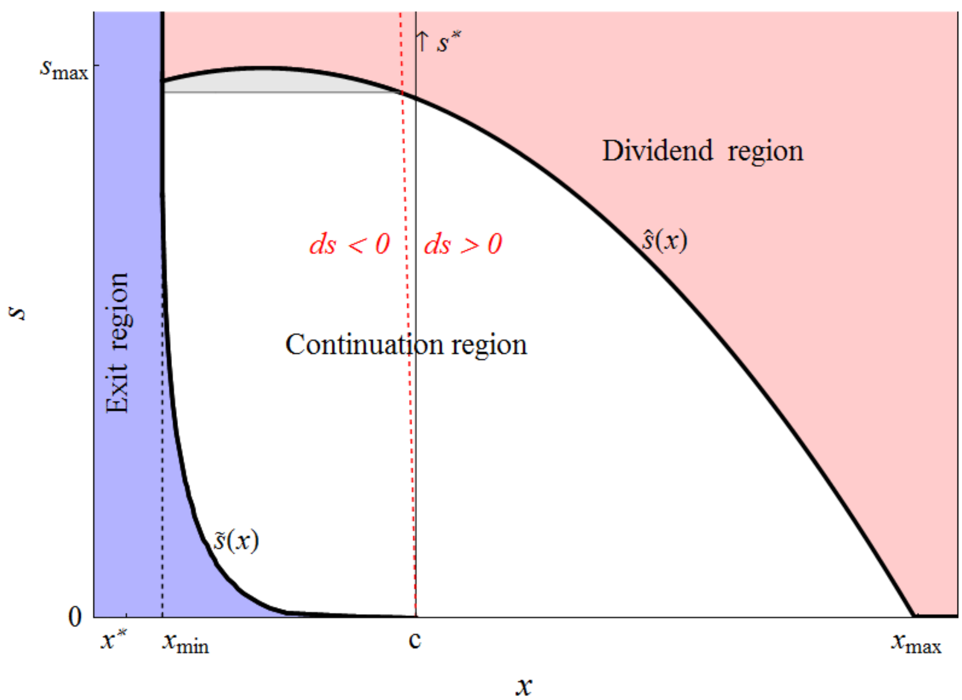


Figure 1: Optimal policy regions of a liquidity constrained firm.

Proposition 1 (Optimal policy when $0 \leq r < \rho$**)** *There are constants* $x_{\min} \in (x^*, c)$*,* $x_{\max} > c$*, and* $s_{\max} > 0$ *such that the optimal policy has the following features:*

1. *If* $x_t \leq x_{\min}$*, it is optimal to exit immediately irrespective of* s_t*.*

2. *If* $x_{\min} < x_t < c$*, then there is a cut-off value* $\widetilde{s}(x_t) > 0$ *such that it is optimal to exit if and only if* $s_t \leq \widetilde{s}(x_t)$*.* $\widetilde{s}(x_t)$ *is decreasing in* x_t *and* $\lim_{x_t \to c} \widetilde{s}(x_t) = 0$*.*

3. *If* $x_t \geq c$*, it is optimal not to exit, irrespective of* s_t*.*

4. *If* $x_t > x_{\max}$ *or* $s_t > s_{\max}$*, it is strictly optimal to pay out some dividends (and continue thereafter, if* $x_t > x_{\min}$*).*

The proof is in Appendix A. Figure 1 illustrates the optimal policy. This is the key figure of our paper. The life span of a firm is a stochastic path in the $(x, s)-$space. While the firm stays inside the *continuation region* its law of motion is given by equations (1) and (9). The gray area inside the continuation region is a transitory region: after leaving it, a firm that follows the optimal policy cannot return there. The firm never ventures inside the *dividend region*, because payouts (which move the firm down along s-axis) keep it from crossing the boundary of that region. When revenue x is sufficiently high, the dividend region reaches all the way to the $s = 0$ line, where the firm operates with zero cash holdings and continually pays out all of the profit flow as dividends. The firm's life span ends when it hits the boundary of the *exit region* for the first time.

We will next explain the intuition for why the optimal policy takes the form depicted in Figure 1.

Continuation Region

The point of accumulating cash is to use it as a buffer that prevents inefficient exit. To see this, consider a situation where the firm's current cash holding s_t is small but strictly positive, and where the profit flow is exactly zero, i.e. $x_t = c$. The firm is not currently making losses and there is a positive option value associated with future profits, so it cannot be optimal to exit. Neither can it be optimal to pay out s_t as dividends, because this would cause the firm to immediately move down to the point $(x = c, s = 0)$, which means that the firm is forced to exit within the "next instant" thus losing the option value. Therefore, there must be a non-empty continuation region, where it is optimal to retain cash inside the firm despite the difference between the discount rate and the rate of return on cash holdings.

Now let's consider the properties of the value function in the continuation region. Define the value of the constrained firm $V(x,s)$ as gross of the cash holdings, so the value at the time of exit is $V(x,s) = s$. Using Ito's lemma, we can write the differential dV as:

$$dV(x,s) = V_s(x,s)\,ds + V_x(x,s)\,dx + \frac{1}{2}V_{xx}(x,s)\,(dx)^2 . \tag{10}$$

Taking the expectation and letting dt be small yields:

$$E(dV) = V_s(x,s)\,ds + V_x(x,s)\,\mu x dt + \frac{1}{2}V_{xx}(x,s)\,\sigma^2 x^2 dt,$$

where ds is from (9). The Bellman equation is $V(x,s) = E(V + dV)/(1+\rho dt)$, which can be solved for $\rho V dt = E(dV)$, leading to the following PDE:

$$\rho V(x,s) = (x - c + rs)\,V_s(x,s) + \mu x V_x(x,s) + \frac{\sigma^2}{2}x^2 V_{xx}(x,s) . \tag{11}$$

Note that this PDE does not contain a cash flow term. The reason is that, in the continuation region, the cash flow between the firm and its owners is zero: Positive cash flow adds to the cash balance and negative flow subtracts from it.

The PDE (11) does not have a closed-form solution. Further, it is valid only in the continuation region, the boundaries of which must be optimally chosen as part of the solution. We will next discuss the properties of these boundaries, which constitute the optimal exit and dividend policies. The numerical solution of the problem is discussed in Section 3.1.

Exit Policy

The liquidity constraint can only reduce the continuation value of the firm, so the constrained firm should certainly exit whenever the unconstrained would, i.e., when $x_t \le x^*$. In addition, the firm is forced to exit when it has no cash to cover the current loss, i.e., when $(x_t \le c, s_t = 0)$. This gives a fixed boundary for the value of the firm:

$$V(x,0) = 0 \text{ for } x \le c. \tag{12}$$

The firm should clearly never exit while current profits are positive $(x_t > c)$. Now consider a firm with a very small s_t and with $x_t < c$. This firm is depleting its cash but could in principle still continue. However, it is very likely to be forced to exit in the near future. For any $x_t < c$, and for sufficiently small s_t, the firm is so unlikely to bounce back to a positive cash flow before s hits zero that the owners are better off exiting immediately and just taking the remaining s_t.[5] Thus, there must be a boundary between exit and continuation regions that lies strictly above $s = 0$ for $x < c$. We call exiting when $x_t > x^*$ and $s_t > 0$ *precautionary exit*.

We denote the *exit threshold* by $\tilde{s}(x)$, defined in $x \in [x_{\min}, c]$ where $x_{\min}$ is, in practical terms, the lowest revenue at which the firm ever operates. The lower is x_t, the less valuable the continuation value of the firm, and thus the higher the s required for continuation to be optimal, so $\tilde{s}'(x) < 0$ in $x \in (x_{\min}, c)$.

Inside the continuation region the value of the firm must exceed the exit value s. At the exit boundary the firm is indifferent between taking the exit value and the continuation value, so

$$V(x,\ \tilde{s}(x)) = s. \tag{13}$$

Inside the continuation region the marginal value of cash must be at least unity, else the owners would be better off by paying out cash. Smooth pasting at the exit boundary requires

$$V_s(x,\ \tilde{s}(x)) = 1, \tag{14}$$

$$V_x(x,\ \tilde{s}(x)) = 0. \tag{15}$$

It may seem unintuitive that the exit boundary is strictly above zero for all $x < c$. To see this point more formally, suppose, by contrast, that the continuation region in fact reached all the way down to $s = 0$ for some interval $[x', c]$, where $x' < c$. Since a cashless firm is forced to exit at $s = 0$, the boundary value $V(x, 0) = 0$ is fixed for all $x \leq c$. This implies that the first and second derivatives with respect to x must also be zero within this interval: $V_x(x, 0) = V_{xx}(x, 0) = 0$ for all $x \in (x', c)$. Substituting these into the PDE (11) that holds in the continuation region yields $V_s(x, 0) = 0$ within this interval. But this leads to a contradiction, because $V_s \geq 1$ must hold in the continuation region, or else cash would be more valuable outside than inside the firm. It follows that the continuation region cannot reach down to $s = 0$ for $x < c$.

The only way in which a firm following the optimal policy can extinguish all funds is to hit exactly the zero-flow-zero-stock point for cash, $\{x_t, s_t\} = \{c, 0\}$. Thus the

[5] It can be shown that the probability with which the firm bounces up to positive profits before running out of cash goes to zero at a rate faster than s.

constrained firm will experience a forced exit with probability 0.[6] Practically all exit by liquidity constrained firms is precautionary.

We assume that the scrap value of the firm is zero, so the exit value of the firm is simply equal to its cash holdings. In the unconstrained case, adding a positive scrap value would be equivalent to adding the rental opportunity cost of the scrap value to the flow cost. However, in the presence of a liquidity constraint an opportunity cost is not equivalent to an operating cost as only the latter requires liquidity. In the extreme, having a scrap value so high that the optimal exit threshold of an unconstrained firm is positive, the firm cannot face negative cash flows during its lifetime so the liquidity constraint is redundant. Apart from this extreme case, the problem would not be qualitatively changed by a positive scrap value.

Dividend Policy

When $r < \rho$, holding cash is costly. The benefit of holding cash is that it may allow the firm to avoid a forced exit in the future when the option value of continuation would still be positive. This benefit is bounded above by $V^*(c)$, the unconstrained continuation value at the zero profit flow. Since the cost of holding cash increases without bound in s, there exists, for any x, some s high enough such that it is better to stop accumulating cash. This threshold value, denoted $\hat{s}(x)$, defines the boundary between the continuation region and the dividend region. It can be interpreted as a target level of cash holdings that depends on current profitability. The firm makes payouts to owners to make sure it doesn't hold more than the target level of cash; hence we call $\hat{s}$ the *dividend threshold*. The value of the firm above the dividend threshold must be:

$$V(x,s) = V(x,\hat{s}(x)) + (s - \hat{s}(x)), \text{ when } s > \hat{s}(x).$$

For sufficiently high x the possibility of forced exit is so remote that it is not worth holding on to any cash. We denote the threshold above which it is optimal to not hold any cash by $x_{\max}$. In the limit $x \to \infty$, the prospect of forced exit becomes irrelevant, and thus the value of the firm must converge to the value of the unconstrained:

$$\lim_{x\to\infty} V(x,s) = V^*(x) + s. \tag{16}$$

[6] The firm's position in (x,s)-space cannot evolve along the boundaries of the continuation region because, if $s_t = 0$ and $x_t > c$ then $ds > 0$, and if $x_t < c$ then the firm exits if it hits the boundary $\{x, \tilde{s}(x)\}$.

At the dividend threshold, cash is equally valuable inside as it is outside the firm, where one dollar is of course worth one dollar. Thus, the value matching condition

$$V_s(x, \hat{s}(x)) = 1 \tag{17}$$

must hold at the dividend threshold. The associated smooth-pasting condition requires[7]

$$\begin{aligned} V_{ss}(x, \hat{s}(x)) &= 0, && (18) \\ V_{xs}(x, \hat{s}(x)) &= 0. && (19) \end{aligned}$$

The firm is constrained at the margin only in the continuation region; there having a dollar more would increase the value of the firm by more than a dollar: $V_s(x,\ s) > 1$.

When the firm hits the dividend threshold from inside it pays out just enough cash to not cross the boundary. However, if the firm were to start at $s_0 > \hat{s}(x_0)$, then it would immediately pay out the excess $s_0 - \hat{s}(x_0)$ as a lump sum dividend. (For a new firm this means that the owners have more than enough funds to endow the firm with the optimal level of precautionary cash holdings; the "lump dividend" at the start is then the cash that owners retain for themselves.) A lump sum dividend is also paid out as the liquidation value upon precautionary exit. Note that if a firm that enters the industry at revenue level x_0 can choose its initial cash holdings then $s_0 = \hat{s}(x_0)$ is the optimal choice.

2.3 Special Case: No Liquidity Premium ($r = \rho$)

Consider now the special case in which there is no liquidity premium: $r = \rho$. Hoarding cash is now costless, so it can never be strictly optimal to pay dividends. The optimal policy is thus defined by dividing the $(x, s)-$ space between the exit region and the continuation region. The qualitative properties of the exit region and the exit threshold $\tilde{s}(x)$ are the same as with $r < \rho$.

Holding cash inside the firm can be strictly optimal only when there is a positive probability of being forced to exit in the future. Of course, no matter how high x_t, falling below x^* remains physically possible. But the firm would become irreversibly unconstrained if it were to accumulate so much cash that it could use the interest

[7] In terms of Dumas (1991), the dividend is "an infinitesimal requlator" (while exit is a discrete regulator) so there must be "super-contact" at $\hat{s}(x)$.

income from its cash holdings to cover what would be the worst-case losses under the optimal unconstrained policy. The worst-case cash flow level under the unconstrained policy is $x^* - c$. This loss can be fully compensated by interst payments once cash holdings reach the *escape level of cash*

$$s^* = \frac{c - x^*}{r}. \tag{20}$$

This means that a fixed boundary condition

$$V(x, s^*) = V^*(x) + s^* \tag{21}$$

now replaces the free boundary $\hat{s}(x)$ seen in the $r < \rho$ case. For $s_t \geq s^*$, the firm is indifferent between paying dividends or not and $V(x, s) = V^*(x) + s$. Above the escape level of cash, the firm can no longer run out of cash before first becoming so unprofitable that it would want to exit even in the absence of a liquidity constraint. As the firm is then in effect unconstrained, its exit policy is the same as for an unconstrained firm: exit if and only if $x \leq x^*$. We summarize these results in the following proposition:

Proposition 2 (Optimal policy when $0 \leq r = \rho$) *If $x_t > x^*$ and $s_t < (c - x^*)/r$, it is strictly optimal to refrain from paying dividends. If $x_t > x^*$ and $s_t > (c - x^*)/r$, the owners are indifferent between paying dividends and continuing without paying dividends. The optimal exit policy is qualitatively the same as when $r < \rho$ (see Proposition 1).*

The special case without a liquidity premium is quite similar to the setup of a financially constrained firm in Mello and Parsons (2000). They study optimal hedging, namely how firms should use futures contracts on an asset that is correlated with their profits to reduce the risk of inefficient exit. They do not take into account that the firm becomes permanently safe from inefficient exit at a finite level of cash holdings, but instead assume that the constrained firm's value reaches that of the unconstrained case only in the limit of infinite cash holdings. The environment faced by the agent in DeMarzo and Sannikov (2008) also features serially correlated cash flow and saving is possible without liquidity costs; there precautionary exit does not arise because expected cash flow is assumed to be always positive (due to a parameter restriction which implies that the exit threshold is always positive).

There is also no liquidity premium in the model of Gryglewicz (2011), so it is never strictly optimal to pay out dividends. To generate predictions about the dividend policy he assumes that the firm pays dividends at the indifference boundary, which is equivalent to our s^* (but is changing over time due to learning). Our numerical results will show that even a tiny liquidity cost can in fact make a large difference to the payout policy (even though not to firm's value).

3 Numerical Analysis

3.1 Solving the Optimal Policy

The PDE defined by (11) and the various free boundary conditions cannot be solved analytically. To solve the firm's problem we turn to a discrete-time approximation of the problem and solve it numerically. In the binomial process approximation of geometric Brownian motion the evolution of x is governed by

$$x(t+\Delta) = \begin{cases} x(t)\, e^{\sigma\sqrt{\Delta}} & \text{with probability } q = \frac{1}{2}\left(1 + \frac{\mu - \frac{\sigma^2}{2}}{\sigma}\sqrt{\Delta}\right) \\ x(t)\, e^{-\sigma\sqrt{\Delta}} & \text{with probability } 1-q \end{cases} \tag{22}$$

where Δ is the length of the time period.[8] The evolution of the cash balance is now

$$s(t+\Delta) = (s(t) - \delta(t))(1 + r\Delta) + (x(t) - c)\,\Delta, \tag{23}$$

where $\delta(t) \in [0, \mathring{s}(t)]$ is the dividend paid at time t. The dividend cannot be so high as to make the cash holdings negative at any point in time, so the maximum feasible dividend is restricted by $\min\{s(t+\Delta), s(t)\} \geq 0$, where $\mathring{s}(t) \equiv s(t) + \min\{0, (x(t) - c)\,\Delta / (1 + r\Delta)\}$.

The value function of the firm, stated in recursive form, is

$$\begin{aligned} & V(x(t), s(t)\,|t) = \\ & \max \left\{ \begin{array}{c} s(t), \\ \max_{\delta\in[0,\mathring{s}]} \left\{\delta + \frac{1}{1+\rho\Delta} E\left[V(x(t+\Delta),\ s(t+\Delta)\,|t+\Delta)\right]\right\}, \end{array} \right\} \end{aligned} \tag{24}$$

where $s(t+\Delta)$ is from (23).

[8] This way of discretizing geometric Brownian motion was inspired by Cox, Ross and Rubinstein (1979).

The recursion in (24) satisfies Blackwell's sufficient conditions so it is a contraction mapping. Thus it can be solved by iterating backwards in time: Starting from an arbitrary $V_T(x, s|T)$ the value function converges to a unique solution that approximates $V(x, s)$.[9]

3.2 Comparative Statics of Optimal Policy

Next we investigate how the firm's optimal policy depends on the parameters r, μ, σ. We do this comparison by varying one parameter at a time from a set of baseline parameters, $r = 0.05$, $\rho = 0.1$, $\mu = 0$, $\sigma = 0.25$, $c = 1$. The results are depicted in Figure 2. The solid lines mark the borders of the continuation region in the liquidity constrained case, and the vertical dashed lines mark the optimal exit threshold in the unconstrained case.[10]

The left hand panel of Figure 2 shows the impact of varying the return on firm's cash holdings, r. As r gets larger it becomes less costly to hold cash so continuation is everywhere more attractive and the continuation region expands. The limiting case $r = \rho = 0.1$ results in the escape level of cash s^*, from (20), that is much higher than the highest cash holdings that the firm would ever keep even at $r = 0.099$. The limiting case is qualitatively different, because the trade-off behind the payout policy (between the liquidity cost of the cash holdings and the expected benefit of preventing exit) is no longer there. While optimal payout policy is very sensitive to r near ρ, the value of the firm is not. When r is very close to ρ the liquidity cost is negligible, and there is a large region in state space where the firm is almost indifferent between retaining and paying out cash. (There the marginal value of cash, V_s, is only very slightly above unity). The high sensitivity of optimal policy to r near ρ means that, even when the liquidity premium is close to zero, the optimal behavior of firms is not well approximated by a model where the liquidity cost is completely assumed away.[11]

The top right panel of Figure 2 shows the relation of the optimal policy and the

[9] A natural starting point for the backward induction is $V(x, s|T) = s$. This means that the problem is turned into a finite-horizon problem with forced exit in the last period. By increasing T the value function at $t = 0$ converges to that of the infinite horizon problem.

[10] The program for solving the optimal policy is available at *http://www.hse-econ.fi/murto*.

[11] Nevertheless, it can be shown that as $r \uparrow \rho$ the optimal policy converges to the limiting case, in the sense that, for every point $(x', s' < s^*)$ in the continuation region of the limiting case $r = \rho$, there exists $r' < \rho$ for which (x', s') is in the continuation region.

volatility of the cash flow process. As is typical, higher volatility makes it optimal to accept bigger losses because it increases the upside potential while the downside is still protected by the exit option. In terms of the optimal policy, the increased option value shows up as an enlarged continuation region. This is already visible in the unconstrained problem, where the exit threshold $x^*$is decreasing in σ. In the constrained problem, the dividend boundary shifts out to the right because, at any given x, higher volatility also increases the risk of facing forced exit within any given period of time.

The bottom right panel shows the effect of varying μ, the percentage drift of the cash flow process. Higher μ increases the option value at any given level of losses, as the firm is more likely to bounce back to positive profits within any given period of time. However, as higher μ also makes the firm safer at any given point—by making it less likely that forced exit would threaten it within any given time—it is not obvious that a higher μ should also shift out the dividend boundary. However, we have found no examples of the opposite.

[Figure 2 here]

Figure 2. Comparative statics of the optimal policy of a liquidity constrained firm. Top Left: $r \in \{0, 0.05, 0.09, 0.099\}$, Bottom Left: same as top left, and $r \in \{0, 0.05, 0.09, 0.099, 0.1\}$, Top Right: $\sigma \in \{0.1, 0.25, 0.4\}$, Bottom Right: $\mu \in \{-0.05, 0, 0.05, 0.09\}$.

4 Debt

We now introduce the assumption that the firm has debt. Endogenizing capital structure would require a significantly different model, but there are a few issues that we can analyze in the present framework by assuming that the debt burden is exogenous. Our main interest lies in explaining the relation of the value of debt with the combinations of profitability and cash holdings—that is, with the firm's position in the state space of our model. The value of debt is inversely related with credit spread and credit ratings. In the end we relate our findings to recent empirical work.

The key simplification is to assume away renegotiation and refinancing. The debt has been incurred in the past, perhaps to cover a part of the (now sunk) entry cost, and the servicing of debt takes the form of a fixed coupon payment to perpetuity. (This way there is no need to introduce another state variable for debt.) In terms of our model, the flow fixed cost c in (9) is now a sum of two components, an operating fixed cost c_f and a fixed cost of servicing the debt c_d. As before, if the firm ever fails to pay the fixed cost c it is forced to go out of business.

We consider three cases, in the order of closest resemblance to the basic model.

Looting

The first case is the most straightforward: it involves nothing but the reinterpretation of the fixed cost, and leaves the debtors as silent bystanders. The owner does not care how the fixed cost is broken down between the operating cost and the coupon payment. The owner is free to take cash out of the firm, and there is no point in leaving any cash in the firm when it goes out of business. Thus, when the firm exits, no cash is left for the debtors. Now when the owner takes money out of the firm in anticipation of going out business, this is very much against the interests of the debtholders. What we called precautionary exit in the absence of debt is now more aptly called funneling or "looting".[12]

The owner's optimal policy and value function are unchanged from the basic model, but we can now calculate a value function for the debtholders. The value of debt comes from the fact that the firm has to make the coupon payment as long as it stays in business. The value of debt is depicted by the contour lines in the top left panel of Figure 3, with the units measured for a unit coupon. The value of debt is closely related to the expected remaining lifetime of the firm, so it is roughly increasing in the distance from the exit threshold. In the limit case where profits become larger the value of debt approaches that of a risk-free perpetuity, as the firm is expected to stay in business forever.[13]

Performance covenant

In the second case, we add a simple debt covenant: the owner is allowed to take cash out of the firm only if the current cash flow is positive. This means that the owner will not want to exit, no matter how negative the current profits, but will rather run down the funds. There is always some chance of recovering back to profits, but no

[12] We use the term looting broadly in the sense of Akerlof and Romer (1993).

[13] For calculations we assume that debtholders have the same discount rate as the owner; this matters for the value levels but not much for the shapes of the contours.

money to be recovered from precautionary exit.

The performance covenant offers some protection for the debtholders, as the owners cannot loot the remaining cash holdings when the firm is nearing exit. The value of debt comes from continued coupon payments, so delayed exit is good for the debtholders. However, the downside is that owners now have less of an incentive to accumulate cash holdings inside the firm in the first place.

Owner's optimal policy under the performance covenant is depicted in the top right panel of Figure 3, together with the contours for the relative difference between the values of debt with and without the performance covenant. Where payouts are not prevented by the covenant (right of the dashed line) the dividend boundary is now lower than in the looting case. Owners accumulate less cash because they anticipate the possibility of performance covenant becoming binding in the future. Nevertheless, the value of debt is now higher for any given combination of profitability and cash holdings. This is quite obvious where the looting firm would near exit, because with the covenant the debt retains some value due to longer expected lifetime, but even in the region where the covenant is not binding the difference is about 5% at its highest. Here the benefit of preventing looting more than compensates the debtholders for the downside of lower cash holdings, which hurts them by making it harder for the firm to prevent inefficient exit.

Retractable debt with a performance covenant

In the final case we give the debtholders the power to retract the debt (in addition to the performance covenant). This means that the exit policy is now in effect decided by the debtholders. (The owner still has the ability to exit, but, as in the previous case, would never find it optimal.) At redemption the debtors get the remaining cash and the firm is forced to exit. For simplicity we assume that the face value of the debt is sufficiently high so that the firm would never want to accumulate so much cash as to survive a redemption.

Now that debtholders decide the exit policy, firm behavior is determined through strategic interaction between the owner and the debtholders. In equilibrium, the owner maximizes the present value of dividends, taking as given the exit policy used by the debtholders; similarly, debtholders choose the exit policy in order to maximize their present value, taking as given the dividend policy.

A useful simplifying feature of this environment is that the regions where owners and debtholders are active do not overlap. The reason is that the covenant prevents the owners from making payouts when cash flow is negative, and debtholders do not want to force exit when cash flow is positive. Thus there is no true within-period strategic interaction, and it makes no difference whether we assume that decisions within a period are sequential or simultaneous.

We assume that each period begins by everyone observing the new realization of the stochastic revenue x. The owners choose the dividend in order to maximize their value function

$$V_E(x(t), s(t), \tilde{s}_D) = \max_{\delta \in [0, \mathring{s}]} \left\{ \delta + \frac{1}{1+\rho\Delta} E\left[V_E\left(x\left(t+\Delta\right),\ s\left(t+\Delta\right), \tilde{s}_D | t+\Delta\right)\right] \right\} \tag{25}$$

subject to the constraint that the dividend δ does not cause the performance covenant to be violated, $\delta \leq \max\{0, \min\{s, x-c+rs\}\} \equiv \mathring{s}$. Owners anticipate the debtholders' exit policy $\tilde{s}_D$, which is defined analogously to the exit boundary of the basic model in Section 2.

The debtholders can either accept the coupon payment and let the firm continue, or dissolve the firm and take the remaining cash. Their value function is defined recursively as

$$V_D(x(t), s(t), \hat{s}_E) = \max \left\{ s(t), c_d + \frac{1}{1+\rho\Delta} E\left[V_D\left(x\left(t+\Delta\right),\ s\left(t+\Delta\right), \hat{s}_D | t+\Delta\right)\right] \right\} \tag{26}$$

where $\hat{s}_E$ is the next period's dividend policy, anticipated by the debtholders, and defined similarly as the dividend boundary in Section 2. In equilibrium, $\tilde{s}_D$ and $\hat{s}_E$ are mutually consistent. The numerical solution is again obtained by a standard recursive method.

Exit by debtholders and payouts by owners are strategic complements. The point of accumulating cash holdings is to allow the firm to survive a temporary foray into negative profits; if the debtholders force an exit they take the remaining cash and the accumulation was a waste from the owner's point of view. The quicker the debtholders are to pull the plug, the lower the amount of cash that the owners want to accumulate. And vice versa: the lower the dividend boundary, the less valuable is continuation for debtholders because they can expect the firm to arrive at the exit boundary sooner and with less cash.

The value for debtholders consists of the coupon payments during the lifetime of the firm, and of the final cash holdings at exit. Equilibrium policy regions are depicted in the bottom left panel of Figure 3, with the contours for the difference in the value of debt relative to the case with looting superimposed. The value is higher than in the absence of a covenant, and also slightly higher than with a plain performance covenant without retractability.

To solve the equilibrium policies we have to assume some particular fraction for the share of the coupon payment out of total fixed cost, c_d/c. Note that this share was immaterial in previous cases, where it had no impact on firm behavior. The bottom right panel depicts the policy regions under various assumed proportions for the cost of debt service out of total fixed costs; it was set at 25% at the bottom left panel. The continuation region is smaller on both sides when the share of debt is smaller. From the debtholders point of view coupon payments are not a cost but an income. Total fixed cost c is being held constant, so, when the proportion of debt service is smaller, the real operating cost is correspondingly larger, and debtholders have more to lose from continued operation so they choose to exit sooner. The reason for precautionary exit by the debtholders is the same as it was for the owner in the basic setup. Continued operation incurs operating costs and depletes the cash holdings, so, at sufficiently low profitability, the debtholders are better off taking the remaining cash rather than using it to gamble for resurrection.

[Figure 3 here]

Figure 3. Optimal policy in various cases, see Figure 1 for color coding. *Top left*: Looting, with the contours for the value of debt superimposed. *Top right*: Performance covenant, with the difference in the value of debt relative to the looting case. *Bottom*: Performance covenant with retractable debt; on left optimal policy with the difference in the value of debt relative to the looting case superimposed, on right the optimal policy under several assumed values for the share of debt service out of fixed costs.

Discussion

The explicit power of the holders of retractable debt is similar to the power held by the owners of short-term debt, who may choose to stop the rollover before the firm can burn through its remaining cash. However, a model with actual rollover would require us to endogenize both the interest rate on new debt and the capital structure; that is beyond the scope of the current paper. Our point is that when the continued operation of the firm requires the consent of debtholders then they face a similar trade-off as the entrepreneur-owner in our basic model, and the resulting exit behavior is also qualitatively similar. In general, the debtors want the firm to exit while it still has cash but the probability of bouncing back to profits before defaulting is sufficiently low.

The results of our debt-augmented model can be related to recent empirical findings on the relationship between debt, cash holdings, and asset values. Consider the possible life span of a firm in our model. The points along the exit boundary correspond to different combinations of asset value and balance sheet liquidity at the time of exit. From the shape of the exit policy we see that profitability should be negatively correlated with the likelihood of default in the near future. Furthermore, controlling for profitability, the cash holdings should also be negatively correlated with the likelihood of default. This is consistent with Davydenko (2010), who analyzes data on bond and equity values of speculative grade firms (BB+ and below), and finds that the two variables that predict default are the "economic solvency" of the firm, which corresponds to our profitability, and "balance sheet liquidity" that roughly corresponds to our cash holdings. Both our theoretical model and Davydenko's empirical findings suggest that economic value and liquidity are two distinct, yet related, potential triggers of default: some firms exit mainly because of economic insolvency while others exit mainly because of liquidity distress (which may be a result of poor past economic performance).

Our model also has implications for the correlation between cash holdings and bond yields. Consider the regions of state space where a firm can and cannot be located. The shape of the continuation region shows that combinations of low cash holdings and high bond yields are selected out by exit, while combinations of high cash holdings and low bond yields are selected out by payouts. (In the case of performance covenant without early redemption only the latter is true.) Thus a population of firms in the continuation region should show a negative correlation between cash holdings and the value of debt, i.e., a positive correlation between cash holdings and bond yield. This is consistent with Acharya, Davydenko and Strebulaev (2011), who report that there is a positive correlation between cash holdings and credit spread in cross sections of firms, and that this correlation becomes negative if one controls for profitability.

5 New Cash Injections

Next we extend the model by allowing the owners to increase the firm's cash holdings at some transaction cost. Specifically, they can, at any point in time, inject any amount s of cash at cost $\xi + (\gamma + 1)\, s$, where ξ is the fixed and γ the marginal transaction cost. The injection of cash causes the firm to jump directly upwards in the state space (x, s). Paying the transaction cost can only be optimal when the firm would otherwise face immediate forced exit ($s = 0$ and $x < 0$) because otherwise the cost could still be postponed and, with luck, even avoided.[14]

If the firm decides to incur the transaction cost, then its target level of cash is

$$s^{+}(x) = \arg\max_{s} \{V(x, s) - (1 + \gamma)\, s\}\,. \tag{27}$$

The target level s^{+} equalizes the marginal cost of new cash and its marginal value at the firm, $V_s(x, s^{+}(x)) = 1 + \gamma$. Transaction costs are independent of x, so cash is raised on an interval $\{s = 0, x \in [x^{+}_{\min}, 0]\}$, where $x^{+}_{\min} \in (x^{*}, 0)$. The lowest x where the firm replenishes its cash holdings, $x^{+}_{\min}$, is the point where the value of exit (which is zero on the $s = 0$ line) is just equal to the value of continuing from $\{x, s^{+}(x)\}$, net of the transaction cost of moving there:

$$V\left(x^{+}_{\min}, 0\right) = V\left(x^{+}_{\min}, s^{+}\left(x^{+}_{\min}\right)\right) - \xi - (1 + \gamma)\, s^{+} = 0. \tag{28}$$

The left panel of Figure 4 depicts the optimal policy for a firm that faces positive but not prohibitive transaction costs. The qualitative difference to the basic model (recall Figure 1) is the segment of horizontal axis where cash is raised and the associated target curve $s^{+}(x)$ directly above. For sufficiently low cash flow the firm still finds it optimal to exit with positive cash holdings rather than incur the transaction cost.

The liquidity cost of holding cash makes it desirable to limit the cash holdings, so without a fixed transaction cost firms would raise cash only to offset a contemporaneous negative cash flow. The fixed cost makes it optimal to raise new cash in lumps, in order to postpone the prospects of having to incur it again. In the absence of a marginal transaction cost it is optimal to "jump" all the way to the dividend boundary. Any transaction costs reduce the value of continuation and shift the exit boundary to the right.

[14] Hennessy and Whited (2007) estimate that (financial companies excluded) the marginal cost of raising new equity is 0.053 for large companies and 0.12 for small, and that fixed costs are \$38900 and \$95100 respectively.

The value function is now augmented with the additional option of raising more cash. Thus, in solving for the optimal policy, (24) is replaced with

$$V\left(x\left(t\right),s\left(t\right)|t\right)=$$
$$\max\left\{\begin{array}{c} s\left(t\right), \\ \max_{\delta\in[0,\hat{s}]}\left\{\delta+\frac{1}{1+\rho\Delta}\left[EV\left(x\left(t+\Delta\right),\ s\left(t+\Delta\right)|t+\Delta\right)\right]\right\}, \\ \max_{s^{+}\in[s(t),\infty)}\left\{V\left(x\left(t\right),\ s^{+}|t\right)-\xi-\left(1+\gamma\right)\left(s^{+}-s\left(t\right)\right)\right\} \end{array}\right\} \tag{29}$$

where $s\left(t+\Delta\right)$ is from (23). The numerical solution method is otherwise unchanged.

The right panel of Figure 4 shows the optimal policy under different combinations of the transaction cost parameters.[15] In each case the exit boundary is further left than under prohibitive costs, as the threat of forced exit is not as grave with the possibility to raise new capital. The lower the transaction costs, the further the exit boundary shifts towards the unconstrained exit threshold. With low transaction costs it is cheap to add cash whenever necessary, so it is possible to reduce the liquidity cost and never hold very much cash, so the continuation region becomes smaller. In the limiting case the firm holds no cash; it pays out profits as they come in, and raises cash as it makes losses.

The unconstrained case, with the simple exit threshold x^{*} in (3), is the limiting case where both the fixed and the marginal transaction cost are zero. The constrained case, where the firm never raises new cash, is equivalent to assuming that the cost parameters are prohibitively high.[16] Hence this setup encompasses both the constrained and unconstrained cases of the basic model.

One literal interpretation of the model is a risk-neutral owner-entrepreneur who allocates her wealth between two assets; one liquid asset that can be used to pay off possible losses, and another illiquid asset that yields a higher rate of return but can only be turned into liquid form at a transaction cost. The entrepreneur has deep pockets in terms of the illiquid asset, but the transaction cost makes it desirable to hold some liquid assets as well and, in some circumstances, let the firm fold rather than incur another transaction cost.

A related interpretation is a start-up operating with the money of the owner-manager, who can sell the company to new owners at a transaction cost. Now the

[15] The parameter values are $\gamma=\xi=0.005$ in the case of "low" and $\gamma=\xi=0.1$ in the case of "high" costs. For more cases see the working paper version of this paper.

[16] Transaction costs are prohibitively high when $\max_{s}\left\{V\left(0,s\right)-\left(1+\gamma\right)s-\xi\right\}\leq 0$.

owner operates the firm until either hitting the exit threshold $\tilde{s}(x)$ (at which point the firm exits and the owner keeps the remaining cash), or running down the cash reserves (at which point the firm is sold to new owners). In this setup, an alternative formulation would be to assume that the firm is initially owned by a liquidity constrained manager, but the firm can be sold to financially unconstrained owners. This would change the problem slightly: instead of jumping up in state space to point $\{x, s^+(x)\}$, the firm would become permanently unconstrained upon hitting $\{s = 0, x \in [x^+_{\min}, 0]\}$. In that case its continuation value would be given by the unconstrained value function (5), and the point $x^+_{\min}$ would be determined as the unique point along x-axis where this value equals the transaction cost. The qualitative nature of the problem would be otherwise unchanged.

A broad interpretation of the extended model analyzed in this section is a firm that can raise new equity at a transaction cost. This interpretation is similar to Décamps et al (2011) who analalyze the case of non-persistent cash flow risk. Assuming that there is a fixed cost associated with raising equity, the firm delays the recapitalization until it has used up its liquid assets. Then, upon hitting $\{s = 0, x \in [x^+_{\min}, 0]\}$, it will raise new equity in order to increase its cash balance to level $s^+(x)$ that equalizes the marginal value of internal cash with the marginal cost of raising equity. The new owners supply the firm with cash and are compensation with an equally valuable stake in the firm.

[Figure 4 here]

Figure 4. Optimal policy when cash can be raised at a transaction cost.

Left panel: Schematic view. Amount $s^+(x)$ of new cash is raised when $s = 0$ and $x \in [x^+_{\min}, c]$. Right panel: Case with low transaction costs is depicted in red, case with high transaction costs in blue, and case without the possibility to raise new cash (i.e., the basic model) in black.

6 Industry Equilibrium

We saw in Section 2 how a liquidity constraint causes firms to exit at higher levels of current revenue compared to unconstrained firms. It might therefore seem obvious that, at the level of an entire industry, the liquidity constraint would cause there to be fewer but on average more productive firms. However, as we next show, this firm-level reasoning is misleading, because it does not take into account the impact that the liquidity constraint has on output price in competitive equilibrium.

In order to analyze the impact of the liquidity constraint on a competitive industry, we use the definition of industry equilibrium similar to Hopenhayn (1992) and Dixit and Pindyck (1994, Ch 8.4).[17] There is a continuum of firms. We assume that for each firm the revenue x depends on firm-specific output or "productivity" z and an endogenous industry-specific output price p, so that

$$x_t = pz_t. \tag{30}$$

We assume that productivity z follows geometric Brownian motion

$$dz_t = \mu z_t\, dt + \sigma z_t dw_t, \tag{31}$$

with the shocks dw_t independent across firms. New firms of known productivity z_0 can be established by paying an entry cost ϕ. In the constrained case new firms enter with initial cash holdings s_0, which we treat as a parameter of the problem. To guarantee the existence of steady state, we assume an exogenous "death rate" $\lambda > \mu$ at which firms are forced to exit with their cash holdings as the exit value (see the Appendix B for details).[18] In steady state, both the dying and the endogenously exiting firms must be balanced by an equal inflow of new firms of type $\{z_0, s_0\}$.

The industry faces a demand curve $D(p)$ for its output. We assume that the demand curve is everywhere strictly downward sloping. The equilibrating variables are price of output p and mass of firms m. Firms are atomistic, so there is no aggregate uncertainty in steady state. As p is constant, the revenue of individual firms (30) follows the same process (1) that we assumed earlier in Section 2. All firms follow the same optimal policy, which in turn results in a stationary distribution of z. In steady state, m and p must satisfy market clearing

$$D(p) = m\bar{z}(p), \tag{32}$$

[17] Liquidity constraints are introduced to a similar steady-state setting by Gomes (2001) to study the relation of cash flow and investment, and by Cooley and Quadrini (2001) to study the age-conditional relation of growth and firm size. Jones (2003) averages over simulated time series of individual firms to study the impact of liquidity constraints on the propagation of aggregate shocks.

[18] The risk of exogenous exit changes the firm's optimal policy slightly compared to Section 2: the firms discount the future at rate $\lambda + \rho$ instead of ρ and the Bellman equation of the constrained firm includes a term λs on the right hand side of (11).

where $\bar{z}$ denotes the cross-sectional average output of firms in steady state ($\bar{z}$ depends on p because the exit policy in terms of z depends on p). Entry is endogenous, so equilibrium must also satisfy the zero-profit condition for entering firms

$$V(pz_0, s_0) = \phi + s_0. \tag{33}$$

Equilibrium price is fully determined by the entry condition (33): p must adjust to eliminate expected rents to entrants. (If entry were profitable then more firms would enter and m would increase, and if entry resulted in expected loss then no one would enter and m would decrease.) Since the value function V is increasing in revenue, p is uniquely determined by (33); V is obtained numerically as described in the previous section. In the unconstrained case the entry condition (33) is replaced by $V^*(pz_0) = \phi$, where V^* has the closed form seen in (5).

For any p, the mass of firms is determined from (33) as $m = D(p)/\bar{z}(p)$. The role of m is merely to close the model. We are not interested in the number of firms but rather on the cross-sectional distribution of productivity, which is independent of m and of the shape of the demand curve because the model has, at industry-level, constant returns to scale.[19] Thus m and D will not feature in our analysis.[20]

Note that, due to perfect competition, the only component of welfare that can be affected by the liquidity constraint is consumer surplus, which varies in the opposite direction as p. Maximum welfare is, of course, attained in the unconstrained case, so the liquidity constraint can only increase p. In real terms, there are potentially three different components to the distortion: higher aggregate entry cost (due to higher turnover), lower average productivity, and higher liquidity costs. As it turns out, turnover and productivity can move to either direction.

To understand why the impact of the liquidity constraint on mean productivity is ambiguous, consider, for simplicity, a world where entering firms have no cash holdings ($s_0 = 0$). The position of firms in (z, s)-space is illustrated in Figure 5. Entry level z_0 is at the point to the right of the zero-profit level ($z = c/p$) where the continuation value matches the entry cost. As price is distorted upwards, the lowest type to ever continue ($z_{\min}$) is below the unconstrained exit threshold (z^*), even though the associated revenue level is higher (Recall $x_{\min} > x^*$ in Figure 1). The price distortion

[19] Doubling of entry flow will double the steady state industry output.

[20] For a more detailed exposition of this industry equilibrium concept, see Miao (2005), who studies capital structure (in the absence of liquidity constraints).

makes it optimal for firms with sufficient cash reserves to continue at productivity levels that would trigger exit in the unconstrained world. The light shaded region (*inefficient survival*) covers firms that would exit in the unconstrained solution but stay in under the liquidity constraint. The dark region (*inefficient exit*) covers firms that are more productive than the unconstrained exit threshold z^* but exit due to the liquidity constraint. Whether mean productivity is increased or decreased by a liquidity constraint depends on which of these two effects dominates.[21]

[Figure 5 here]

Figure 5. Liquidity constraint and average productivity in industry equilibrium.

Numerical Results To analyze the effect of the liquidity constraint on market equilibrium, we calculated the steady state firm distributions for a wide range of combinations of entry cost ϕ and starting cash s_0. For the unconstrained case those distributions can be calculated analytically, but for the constrained case we first have to solve numerically the optimal firm policy (as explained in Section 3.1). The steady state distribution is then obtained by iterating the firm distribution according to this policy until the distribution converges (see the Appendix B for more details). Once the firm distributions are calculated, various statistics are readily computed.

Selected steady state outcomes are reported in Figure 6. Each outcome is reported for those combinations of entry cost ϕ and starting cash s_0 that result in firms entering inside the continuation region. Other parameters are held at the baseline levels used in Section 3.1.[22] The assumption that transaction costs are prohibitively high is made in order to obtain a clear contrast between the constrained and unconstrained cases: Varying γ and ξ between zero and prohibitive levels covers the entire ground between the two cases in a continuous manner, as seen in Section 5. Blank regions correspond to s_0 so high that entering firms would be in the dividend region; the outcomes for points in the blank region are thus exactly the same as in the highest colored point

[21]If s_0 is sufficiently high and ϕ not too high then $z_0 \in (z^*, c/p)$ and the picture is more complicated, as some of inefficiently exiting firms are replaced by less productive firms.

[22]Baseline parameters are $\mu = 0$, $\sigma = 0.25$, $\rho = 0.1$, $r = 0.05$, $\lambda = 0.1$, $c = 1$, $z_0 = 1$. Note that $z_0 = 1$ merely normalizes the units of output. The combinations $\{\gamma, \xi\}$ that result in prohibitive costs can be obtained by solving $\xi(\gamma)$ implicitly the equality in footnote 16. For example, $\gamma = 0.15$, $\xi = 0.25$ results (just barely) in prohibitive costs.

directly below. Values of ϕ that are outside the figures result in such high p that, in terms of Figure 1, the position of entrants is to the right of $x_{\max}$.

The top panels of Figure 6 show the output price and mean productivity of firms; the middle panels show the same values relative to the unconstrained benchmark. The liquidity constraint is harsher when s_0 is small, so the relative distortion is always decreasing in s_0 as the constraint becomes milder. However, there is a subtle interaction with the entry cost ϕ. If ϕ is small then p is low and the profit level of entering firms is low or even negative, so newborn firms enter near the exit boundary and immediately face an acute threat of exit. By contrast, when ϕ is high then entrants must have a large safety margin in terms of initial revenue making any liquidity constraint less important. The relative impact of the constraint is highest when both s_0 and ϕ are low: the constraint is harsh and the safety margin low. At high values of ϕ the level contours are almost vertical, reflecting the safety margin effect that reduces the impact of the liquidity constraint.

[Figure 6 here]

Figure 6. Impact of liquidity constraint on industry equilibrium.

Mean productivity is shown in the top right panels of Figure 6. The liquidity constraint has a negative impact on mean productivity at low levels of ϕ. Thus we find a case of "survival of the fattest" when the entry cost is sufficiently low, with a magnitude of up to a 15% decrease in mean productivity. At higher levels of ϕ the impact is positive but eventually the impact of the constraint is attenuated as the safety margin effect becomes overwhelming. Output is increasing in s_0 at low levels of ϕ and decreasing at high levels of ϕ. This means that, regardless of its sign, the magnitude of the output distortion generally gets smaller as the liquidity constraint gets milder.

Average cash holdings are depicted in the bottom-left panel of Figure 6. An increase in initial cash holdings naturally tends to increase the mean cash holdings of all firms in steady state, but, surprisingly, not always. When both ϕ and s_0 are low then an increase in s_0 decreases average cash holdings. This is possible because entering firms have a narrow safety margin. When entrants' profit level is negative then young firms tend to have cash holdings further below s_0. The decrease in p caused by higher s_0 further reduces the cash holdings of young firms, which have a high steady state population share precisely because many firms exit soon after entry.

For simplicity, we have treated initial cash holdings s_0 as a parameter, but our setup allows it to be endogenized as the entering firms' optimal response to the transaction cost parameters. The lower-right panel of Figure 6 maps the implicit marginal transaction cost γ_0 that would result in the given s_0 being the optimal choice of the entering firms, assuming that entering firms can choose any $s_0 \geq 0$ at a cost $(1+\gamma_0)s_0$, while the cost of raising more cash post-entry is still prohibitive. The dark shaded region covers the points that do not arise endogenously under any $\{\gamma_0, \phi\}$.

The cross section of firms in our setup bears a resemblance to that in Gomes (2001), who analyzes industry equilibrium with a model where firms face a mean reverting productivity process and a cost of raising external funds. In his model firms are not able to hold cash, but use an excessive stock of physical capital in effect as a form of precautionary savings, in order to reduce the need for external finance in the future. Gomes shows that the nonlinearity of the optimal investment rule generates a spurious correlation between investment and cash flow, irrespective of whether there are liquidity constraints. In our model cash holdings have a purely precautionary motive while physical capital is fixed (and sunk). Now suppose that the observed value of capital includes assets that are held for precautionary reasons. It is clear from our results that the contribution of the precautionary motive to the relation of cash flow and accumulation of capital is then necessarily non-monotone. To see this, recall Figure 1. Firms with lowest x are spending their reserves on covering losses (and thus have $E[dS|x] < 0$), firms with intermediate x are on average accumulating cash ($E[dS|x] > 0$), while at $x > x_{\max}$ no cash is held and $dS = 0$.[23] Gomes' point is that the power of a cash flow variable in classic investment regressions arises spuriously when the data is generated in a structural model. Our model implies that, if the capital stock includes assets held for precautionary reasons, then the relation between "investment" and cash flow is nonlinear (indeed non-monotone) even if the relation of physical investment and cash flow were linear (as it is in our model).

[23] The same non-monotonicity applies to $E[dS|V]$ because the contour lines of V are downward-sloping in (x, s)-space.

7 Conclusion

We have analyzed the problem of a liquidity constrained firm that faces persistent cash flow uncertainty. The firm may be forced to exit due to inability to absorb a negative cash flow, even when the possibility to rebound into profits conveys option value that would make continuation (socially) optimal. To prevent such inefficient exit, the firm engages in precautionary saving out of retained earnings, and to preempt it the firm will exit before actually running out of cash. Our main contribution is to show how profitability and liquidity jointly influence the firm's exit and payout policies. We have also analyzed extensions to the model and showed that our findings are not an artifact of ignoring debt or equity financing.

The obvious selection effect of pre-entry liquidity constraints is to increase the average productivity of firms in market equilibrium, because the standard for profitable entry is set too high. Similarly, for a fixed output price, the post-entry liquidity constraint would seem to distort the average productivity upwards, by weeding out firms with upside potential that are currently unproductive. We showed that, taking into account endogenous entry and exit, post-entry liquidity constraints lead also to an opposite phenomenon where unproductive firms that have a lot of cash (from earlier success) do not exit soon enough and end up reducing the average productivity below the efficient benchmark level. Our steady state calculations showed that the negative effect dominates when entry costs are sufficiently low.

Appendix A: Proof of Proposition 1

Preliminaries. We begin by three lemmas that collect together the key properties of $V(x,s)$ utilized in the proof. The first one merely records properties of $V(x,s)$ that are discussed in more detail in Section 2.2 of the main text:

Lemma 1 *$V(x,s)$ is continuous and increasing in both arguments, and $V(x,s) \geq s$ for all (x,s). Depending on the optimal policy at (x,s):*

- *If it is optimal to exit, then $V(x,s) = s$.*
- *If it is optimal to continue without paying dividends, then $V(x,s) > s$, $V_s(x,s) > 1$, and the following partial differential equation holds locally at (x,s):*

$$\rho V(x,s) = (x - c + rs)\, V_s(x,s) + \mu x V_x(x,s) + \frac{\sigma^2}{2} x^2 V_{xx}(x,s)\,. \tag{34}$$

- *If it is optimal to pay dividends and continue thereafter, then $V(x,s) > s$ and $V_s(x,s) = 1$.*

Proof. Choosing dividend $dD_t = s_t$ and exiting immediately thereafter is a feasible policy at every point in state space and gives value s_t. It follows immediately that $V(x,s) \geq s$ for all (x,s). In particular $V(x,s) = s$ whenever it is optimal to exit and $V(x,s) > s$ whenever it is strictly optimal to continue. The application of Bellman's principle and Ito's lemma imply that if it is optimal to continue without paying dividends, then the value function must satisfy the Hamilton-Jacobi-Bellman equation (34) locally at (x,s), and $V_s(x,s) > 1$ (see Section 2.2 in the main text). Finally, if it is optimal to pay a positive dividend $dD > 0$ and continue thereafter, the principle of dynamic programming gives $V(x,s) = dD + V(x, s - dD)$, which implies that $V_s(x,s') = 1$ for all $s' \in [s - dD, s]$. Continuity and monotonicity of $V(x,s)$ follow from the properties of state transition dynamics and monotonicity of cash flow with respect to x. ■

Lemma 2 establishes lower and upper bounds for $V(x,s)$:

Lemma 2 *For all (x,s), we have*

$$\underline{V}(x) + s \leq V(x,s) \leq V^*(x) + s, \tag{35}$$

where

$$\underline{V}(x) = \begin{cases} \left(\frac{c}{\rho} - \frac{c}{\rho-\mu}\right)\left(\frac{x}{c}\right)^{\beta} + \frac{x}{\rho-\mu} - \frac{c}{\rho} \text{ for } x > c \\ 0 \text{ for } x \leq c \end{cases}, \tag{36}$$

and where $V^*(x)$ *is given by (5) and* β *is given by (4) in the main text.*

Proof. Consider the following policy: pay out immediately any positive cash reserves, and thereafter keep cash holdings at $s_t = 0$ by immediately paying out any incoming cash. This leads to forced exit as soon as $x_t \leq c$. The unique value function that satisfies the appropriate differential equation (equation (2) in the main text) together with the boundary condition $\underline{V}(c) = 0$ is given by (36). Since this policy is feasible, it gives a lower bound for the value of the optimally managed firm. On the other hand, the net value of a firm that faces no liquidity constraint is $V^*(x)$, and this must be an upper bound for the liquidity constrained firm. ■

Finally, Lemma 3 states that a firm that is at the edge of being profitable ($x_t = c$) is more valuable to its owners than its cash holdings. This lemma guarantees that positive cash holdings are optimal at least under some conditions:

Lemma 3 $V(c, s) > s$ *for all* $s > 0$.

Proof. The key to this result is the kink in the value function $\underline{V}(x)$ at $x = c$. Take an arbitrary $s > 0$, and let $x_t = c$, $s_t = s$. Take a sequence $\{\Delta_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \Delta_n = 0$ and $\Delta_n > 0$ for each n. Denote by V_n the expected payoff of a feasible (but suboptimal) policy, according to which the firm continues without paying dividends for a period of length Δ_n, and thereafter pays out all incoming cash:[24]

$$V_n = e^{-\rho\Delta_n}\mathbb{E}\left(\underline{V}(x_{t+\Delta_n}) + s_{t+\Delta_n}\right).$$

Since x_t is a geometric Brownian motion, we have:

$$\frac{x_{t+\Delta_n} - x_t}{x_t} \sim \mathcal{N}\left(\mu x_t \Delta_n, \sigma^2 \Delta_n\right).$$

Standard properties of Normal distribution imply:

$$\mathbb{E}\left|\frac{x_{t+\Delta_n} - x_t}{x_t} - \mu x_t \Delta_n\right| = \sqrt{\frac{2}{\pi}}\sigma\sqrt{\Delta_n}.$$

[24]Note that $s_t > 0$ and $x_t - c = 0$, so that the firm is not under threat of immediate forced exit. Therefore, as we consider short intervals Δ_n, we can safely ignore the possibility that $s_{t'} = 0$ for some $t' \in [0, \Delta_n]$.

Since Normal distribution is symmetric around its mean, we have

$$\mathbb{E}\left[\max\left(0;\frac{x_{t+\Delta_n}-x_t}{x_t}-\mu x_t\Delta_n\right)\right]=\frac{1}{2}\mathbb{E}\left|\frac{x_{t+\Delta_n}-x_t}{x_t}-\mu x_t\Delta_n\right|=\sigma\sqrt{\frac{\Delta_n}{2\pi}},$$

so that

$$\mathbb{E}\left[\max\left(0;x_{t+\Delta_n}-x_t\right)\right]=\frac{\sigma x_t}{\sqrt{2\pi}}\sqrt{\Delta_n}+o\left(\Delta_n\right),$$

where $o\left(\Delta_n\right)$ denotes terms that go to zero at least linearly in Δ_n. Denoting by ζ the derivative from right of $\underline{V}\left(x\right)$ at the kink:

$$\zeta:=\lim_{x\downarrow c}\underline{V}\left(x\right)>0,$$

and noting that

$$\mathbb{E}s_{t+\Delta_n}=s_t+\mathbb{E}\int_{t'=t}^{t+\Delta_n}\left(x_{t'}-c+rs_{t'}\right)dt'=s_t+o\left(\Delta_n\right),$$

we have

$$\begin{aligned}V_n &= e^{-\rho\Delta_n}\mathbb{E}\left(\underline{V}\left(x_{t+\Delta_n}\right)+s_{t+\Delta_n}\right)\\ &= e^{-\rho\Delta_n}\left(\max\left(0;\zeta\frac{\sigma x_t}{\sqrt{2\pi}}\sqrt{\Delta_n}\right)+s_t+o\left(\Delta_n\right)\right)\\ &= \zeta\frac{\sigma x_t}{\sqrt{2\pi}}\sqrt{\Delta_n}+s_t+o\left(\Delta_n\right).\end{aligned}$$

Therefore, for n large enough, $V_n>s_t$. But since the optimal policy is at least weakly better than this strategy, we have $V\left(c,s_t\right)\geq V_n$ for any n, and it follows that

$$V\left(c,s_t\right)>s_t.$$

■

Proof of Proposition 1

Part 1: We want to show that there is some $x'>x^*$ such that stopping is optimal for all $x\leq x'$, $s\geq 0$. Suppose the contrary. Then we can find a sequence $\{x_n,s_n\}_{n=1}^{\infty}$ with $x_n>x^*$ for all n, $\lim_{n\to\infty}s_n=\underline{s}>0$ and $\lim_{n\to\infty}x_n=x^*$, such that all points (x_n,s_n) are within the continuation region so that (34) holds by Lemma 1.[25] Since $V^*\left(x^*\right)=0$, it follows from Lemma 2 that $V\left(x^*,s\right)=s$ for all s. Therefore

$$\begin{aligned}V\left(x_n,s_n\right) &\to s_n \text{ and}\\ V_s\left(x_n,s_n\right) &\to 1\end{aligned}$$

[25] Part 2 of the Proposition, which we will prove shortly, states that it is optimal to exit whenever s is small enough for all $x<c$, and therefore we can assume a limit point $\underline{s}>0$ for s_n.

as $n \to \infty$. By the smooth-pasting condition of the unconstrained firm, we have $V_x^* (x^*) = 0$, and therefore we must have

$$V_x (x_n, s_n) \to 0.$$

(Otherwise we would have either $V (x_n, s_n) < s_n$ or $V (x_n, s_n) > V^* (x_n) + s_n$ for n large enough, hence violating Lemma 2.)

Since (34) must hold at all points in the sequence $\{x_n, s_n\}_{n=1}^{\infty}$, we have:

$$\frac{\sigma^2}{2} x_n^2 V_{xx} (x_n, s_n) \to (\rho - r) \underline{s} + c - x^*.$$

On the other hand, from the corresponding Hamilton-Jacobi-Bellman equation of the unconstrained firm (equation (2) in the main paper) we have

$$\frac{\sigma^2}{2} x^2 V_{xx}^* (x^*) = c - x^* < (\rho - r) \underline{s} + c - x^*,$$

and therefore

$$\lim_{n \to \infty} V_{xx} (x_n, s_n) > V_{xx}^* (x^*) .$$

But since $V (x_n, s_n) \to V^* (x^*) + s_n$ and $V_x (x_n, s_n) \to V_x^* (x^*)$, this implies that $V (x_n, s_n) > V^* (x_n) + s_n$ for n large enough. This is a contradiction with Lemma 2. We can conclude that $V (x, s) = s$ for all s for some $x > x^*$. We let

$$x_{\min} := \sup \{x \,|V (x, s) = s \text{ for all } s \geq 0\} . \tag{37}$$

Part 2: By Lemma 3, we have $V (c, s) > s$ for all $s > 0$. It follows from continuity of the value function that $V (c - \varepsilon, s_t) > s_t$ for some $\varepsilon > 0$, so that $x_{\min}$ defined in (37) satisfies $x_{\min} < c$.

Next, we show that for all $x \in (x_{\min}, c)$, there is some $s' > 0$ such that $V (x, s) = s$ for all $s \leq s'$. Suppose, by contrast, that there is some $x' \in (x_{\min}, c)$ such that $V (x', s) > s$ for all $s > 0$. Since $V (x, s)$ is increasing in x, this implies that $V (x, s) > s$ for all $x \in (x', c)$, $s > 0$. Therefore, there is a continuation region that reaches all the way down to $s = 0$ for the interval (x', c), and by Lemma 1, (34) must hold for all s sufficiently small. However, since a cashless firm is forced to exit at $s = 0$ for $x < c$, the boundary condition $V (x, 0) = 0$ must hold for the whole interval, and therefore also $V_x (x, 0) = V_{xx} (x, 0) = 0$ for all $x \in (x', c)$. Substituting these into (34) yields $V_s (x, 0) = 0$ for $x \in (x', c)$. But since $V (x, s) \geq s$ for all (x, s) by Lemma 1, this is a contradiction. It follows that $V(x,s) = s$ for all $x \in (x_{\min}, c)$ and for all $s \leq s'$ for some $s' > 0$. Define for all $x \in (x_{\min}, c)$:

$$\widetilde{s}(x) := \max\{s\,|V(x,s) = s\}.$$

It remains to show that $\widetilde{s}(x_t)$ is decreasing in x_t and $\lim_{x_t \to c} \widetilde{s}(x_t) = 0$. The former property follows from the monotonicity of $V(x,s)$ in x: suppose on the contrary that $\widetilde{s}(x'') > \widetilde{s}(x')$ for some $x'' > x'$. But then, $V(x', \widetilde{s}(x'')) > s = V(x'', \widetilde{s}(x''))$ which violates the property that $V(x,s)$ is increasing in x. The latter property follows from the continuity of $V(x,s)$: suppose that there is some $s' > 0$ such that $\widetilde{s}(x) > s'$ for all x in some open neighbourhood of c. But this means that $V(x,s) = s$ for all $0 < s < s'$ when x is arbitrarily close to c, and this is in contradiction with continuity of $V(x,s)$ and our previous finding that $V(c,s) > s$ for all $s > 0$.

Part 3: One available (non-optimal) policy is to pay-out all incoming cash and keep cash balance at $s_t = 0$. When $x > c$, this policy gives value $\underline{V}(x) + s > s$, so it cannot be optimal to exit.

Part 4: Fix $s > 0$, and suppose that it is not optimal to pay dividends even at high values of x so that (34) holds for all x. Let $x \to \infty$. From equation (5) in the main text and (36), $V^*(x) - \underline{V}(x) \to 0$, and therefore it follows from Lemma 2 that $V(x,s) \to V^*(x) + s$. This means that $V_s(x,s) \to 1$, $V_x(x,s) \to V_x^*(x)$, and $V_{xx}(x,s) \to V_{xx}^*(x)$, so that

$$\rho V(x,s) - \mu x V_x(x,s) - \frac{\sigma^2}{2}x^2 V_{xx}(x,s) \to \rho(V^*(x) + s) - \mu x V_x^*(x) - \frac{\sigma^2}{2}x^2 V_{xx}^*(x).$$

But then, combining (34) and equation (2) in the main text,

$$rs \to \rho s,$$

which is a contradiction because we have $\rho > r$ and $s > 0$. It follows that the continuation region must be bounded from the right: it is optimal to pay dividends for high enough x. We let

$$x_{\max} := \inf\{x > c\,|V(x,s) = V(x,0) + s \text{ for all } s \geq 0\}.$$

Finally, fix $x > x_{\min}$ and suppose that it is not optimal to pay dividends even at high values of s. But then, as $s \to \infty$, it follows from (34) that

$$\mu x V_x(x,s) + \frac{\sigma^2}{2}x^2 V_{xx}(x,s) \to \infty.$$

But this is in contradiction with (35) holding for all x and s, and the fact that $V_x^*(x)$, $V_{xx}^*(x)$, $\underline{V}_x(x)$, and $\underline{V}_{xx}(x)$ are all bounded and independent of s. We can therefore conclude that the continuation region must be bounded from above: it is optimal to pay dividends for high enough s. We let

$$s_{\max} := \inf\{s > 0 \,|V(x, s') = V(x, s) + s' - s \text{ for all } x \text{ and for all } s' \geq s\}.$$

Appendix B: Stationary distributions

Unconstrained Case

In the unconstrained case, the steady-state firm distribution and its properties reported in Section 6 can be derived analytically as follows. Denote $y \equiv \log z$. The exit threshold is $y^* = \log z^*$ and new firms are born at $y_0 > y^*$. Taking a discrete time approximation, y follows the binomial process:

$$y(t + \Delta) = \begin{cases} y(t) + \Delta y & \text{with probability } q \\ y(t) - \Delta y & \text{with probability } 1 - q \end{cases}$$

where Δ is the length of a period, $q = \frac{1}{2}\left(1 + \frac{\mu - \sigma^2/2}{\sigma}\sqrt{\Delta}\right)$, and $\Delta y = \sigma\sqrt{\Delta}$. The steady state condition gives a difference equation for the mass of firms located at an arbitrary state point y,

$$(1 - \lambda\Delta)\left[qf(y - \Delta y) + (1 - q) f(y + \Delta y)\right]\Delta y + g(y)\Delta y = f(y)\Delta y,$$

where $f(y)\Delta y$ is the mass of all firms and $g(y)\Delta y$ is the mass of newborn firms at state point y. Taking the limit $\Delta \to 0$ leads to a differential equation for the stationary firm density:[26]

$$\frac{1}{2}\sigma^2 f''(y) - \left(\mu - (1/2)\sigma^2\right) f'(y) - \lambda f(y) + g(y) = 0, \tag{38}$$

with $f(y^*) = 0$ and $\lim_{y\to\infty} f(y) = 0$ as boundary conditions. In our setup $g(y)$ is positive at y_0 and zero elsewhere. The point y_0 splices the differential equation into two regions, with the $f(y_0) = f_0$ as a boundary condition in the middle. (f is finite but not differentiable at y_0). The value of f_0 can be solved from the condition that

[26] See Dixit and Pindyck (1993), chapter 8, section 4.c for more details.

total probability density integrates to one. Combining the boundary conditions with (38) yields the closed-form solution:

$$f(y) = \begin{cases} 0 & y \le y^* \\ f_0 e^{-\frac{(\xi+\eta)(y-y_0)}{2\sigma^2}} \frac{\left(e^{\frac{\eta y}{\sigma^2}} - e^{\frac{\eta y^*}{\sigma^2}}\right)}{\left(e^{\frac{\eta y_0}{\sigma^2}} - e^{\frac{\eta y^*}{\sigma^2}}\right)} & y^* < y \le y^0 \\ f_0 e^{-\frac{(\xi+\eta)(y-y_0)}{2\sigma^2}} & y^0 < y \end{cases} \tag{39}$$

where $\xi \equiv \sigma^2 - 2\mu$, $\eta = \sqrt{8\lambda\sigma^2 + \xi^2}$, and

$$f_0 = \frac{2\lambda}{\eta} \frac{\left(e^{\frac{\eta y}{\sigma^2}} - e^{\frac{\eta y^*}{\sigma^2}}\right)}{\left(e^{\frac{\eta y_0}{\sigma^2}} - e^{-\frac{(\eta-\xi)y^* + (\eta+\xi)y_0}{2\sigma^2}}\right)}. \tag{40}$$

There is no economically sensible steady state unless $z = e^y$ has a finite mean. Here $\int_{y_0}^{\infty} e^y f(y)\, dy < \infty$ is a necessary and a sufficient condition for the finite mean. Taking out the terms that are independent of y in (39), the finite mean requirement becomes

$$\int_{y_0}^{\infty} e^{y - \frac{(\xi+\eta)y}{2\sigma^2}} dy < \infty. \tag{41}$$

This holds if $2\sigma^2 - \xi - \eta < 0$, which simplifies to $\lambda > \mu$.

Constrained Case

The stationarity proof in the unconstrained case is sufficient for the stationarity of the distribution of z in the constrained process. As s is endogenously bounded by the optimal dividend policy and, firm by firm, depends deterministically on the history of z, the fact that z has a stationary distribution suffices for the stationarity of the joint distribution (z, s). However, now the optimal policy has no closed-form solution so the steady state distribution must be computed numerically. In the discrete time approximation the life span of each individual firm is a Markov chain in the discretized state space. Therefore, the steady state distribution is obtained by first computing the optimal policy of an individual firm, and then, starting from some initial firm distribution, iterating the firm distribution according to the state transition equations associated with the policy (where a constant mass of new firms are established at the birth point within each iteration) until the firm distribution converges to the steady state.

References

Acharya, Viral; Sergei Davydenko and Ilya Strebulaev (2011): "Cash Holdings and Credit Risk." *NBER Discussion Paper 16955.*

Akerlof, George A and Paul M Romer (1993). Looting: The Economic Underworld of Bankruptcy for Profit. *Brookings Papers on Economic Activity*, Vol 2 , pp. 1-73.

Albuquerque, Rui and Hugo Hopenhayn (2004): "Optimal Lending Contracts and Firm Dynamics." *Review of Economic Studies*, 71, pp. 285–315.

Boyle, Glenn W and Graeme A Guthrie (2003): "Investment, Uncertainty, and Liquidity." *Journal of Finance*, 58, pp. 2143–2166.

Caballero, Ricardo and Robert Pindyck (1996): "Uncertainty, Investment, and Industry Evolution." *International Economic Review*, 37, pp. 641-662.

Cooley, Thomas F and Vincenzo Quadrini (2001): "Financial Markets and Firm Dynamics." *American Economic Review*, 91(5), pp. 1286-1310.

Cox, John D; Stephen A Ross and Mark Rubinstein (1979): "Option Pricing: A Simplified Approach." *Journal of Financial Economics*, 97, pp. 229–263.

Davydenko, Sergei (2010): "What Triggers Default? A Study of the Default Boundary." *Working paper.*

Deaton, Angus (1991): "Saving and Liquidity Constraints." *Econometrica*, 59, pp. 1221–1248.

Décamps, Jean-Paul and Stéphane Villeneuve (2007), "Optimal Dividend Policy and Growth Option." *Finance and Stochastics,* 11, pp. 3-27.

Décamps, Jean-Paul; Thomas Mariotti; Jean-Charles Rochet and Stéphane Villeneuve (2011), "Free Cash-Flow, Issuance Costs and Stock Price Volatility." *Journal of Finance*, 66, pp. 1501-1544.

DeMarzo, Peter M and Yuliy Sannikov (2006): " Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model." *Journal of Finance, 61, pp.* 2681-2724.

DeMarzo, Peter M and Yuliy Sannikov (2008): "Learning in Dynamic Incentive Contracts." *Unpublished paper.*

Dixit, Avinash (1989): "Entry and Exit Decisions under Uncertainty." *Journal of Political Economy*, 97, pp. 620–638.

Dixit, Avinash and Robert Pindyck (1994): "Investment Under Uncertainty."

Princeton University Press, Princeton NJ.

Dumas, Bernard (1991): "Super Contact and Related Optimality Conditions." *Journal of Economic Dynamics and Control*, 15, pp. 675–685.

Dutta, Prajit K and Roy Radner (1999): "Profit Maximization and the Market Selection Hypothesis." *Review of Economic Studies*, 66, pp. 769–798.

Evans, David S and Boyan Jovanovic (1989): "An Estimated Model of Entrepreneurial Choice under Liquidity Constraints." *Journal of Political Economy,* 97(4), pp. 808–27.

Gomes, Joao F (2001): "Financing Investment." *American Economic Review,* 91(5), pp. 1263–1285.

Gryglewicz, Sebastian (2011): "A Theory of Financial Decisions with Liquidity and Solvency Concerns." *Journal of Financial Economics,* 99, pp. 365–384.

Hennessy, Christopher A and Toni A Whited (2007): "How Costly is External Financing? Evidence from a Structural Estimation." *Journal of Finance*, 62, pp. 1705–1745.

Holmström, Bengt and Jean Tirole (2011): Inside and Outside Liquidity." *MIT Press.*

Holtz-Eakin, Douglas; David Joulfaian and Harvey S Rosen (1994): "Sticking It Out: Entrepreneurial Survival and Liquidity Constraints." *Journal of Political Economy*, 102(1): pp. 53–75.

Hopenhayn, Hugo (1992): "Entry, Exit, and Firm Dynamics in Long Run Equilibrium." *Econometrica*, 60, pp. 1127–1150.

Jones, John B (2003): "The Dynamic Effects of Firm-Level Borrowing Constraints." *Journal of Money, Credit, and Banking,* 35(5), pp. 743–762.

McDonald, Robert and Daniel Siegel (1986): "The Value of Waiting to Invest." *Quarterly Journal of Economics*, 101(4), pp. 707–728.

Mello, Antonio S and John E Parsons (2000): "Hedging and Liquidity." *Review of Financial Studies*, 13, pp. 127–153.

Miao, Jianjun (2005): "Optimal Capital Structure and Industry Dynamics." *Journal of Finance*, 60, pp. 2621–2659.

Milne, Alistair and Donald Robertson (1996): "Firm Bahviour under the Threat of Liquidation." *Journal of Economic Dynamics and Control*, 20, pp. 1427–1449.

Murto, Pauli and Marko Terviö (2010): "Exit Options and Dividend Policy under Liquidity Constraints." *HECER Discussion Paper 254*.

PEURA, SAMU AND JUSSI KEPPO (2006): "Optimal Bank Capital with Costly Recapitalization." *Journal of Business*, 79, pp. 2163–2201.

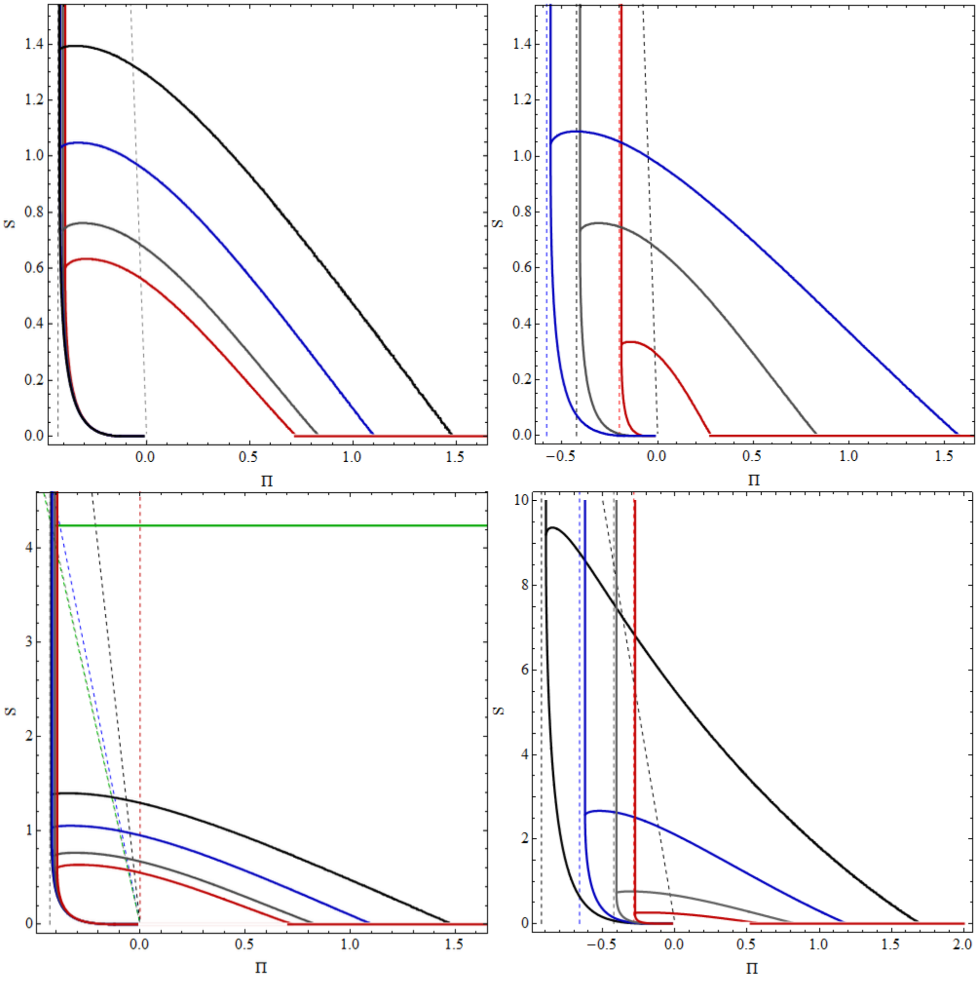
RADNER, ROY AND LARRY SHEPP (1996): "Risk vs Profit Potential: A Model For Corporate Strategy." *Journal of Economic Dynamics and Control*, 20, pp. 1373–1393.

ROCHET, JEAN-CHARLES AND STÉPHANE VILLENEUVE (2005): "Corporate Portfolio Management." *Annals of Finance*, 1, pp. 225–243.

ZELDES, STEPHEN P (1989): "Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence." *Quarterly Journal of Economics*, 104(2), pp. 275–298.

ZINGALES, LUIGI (1998): "Survival of the Fittest or the Fattest? Exit and Financing in the Trucking Industry." *Journal of Finance*, 53(3), 905–38.

For Figure 1, see page 8.



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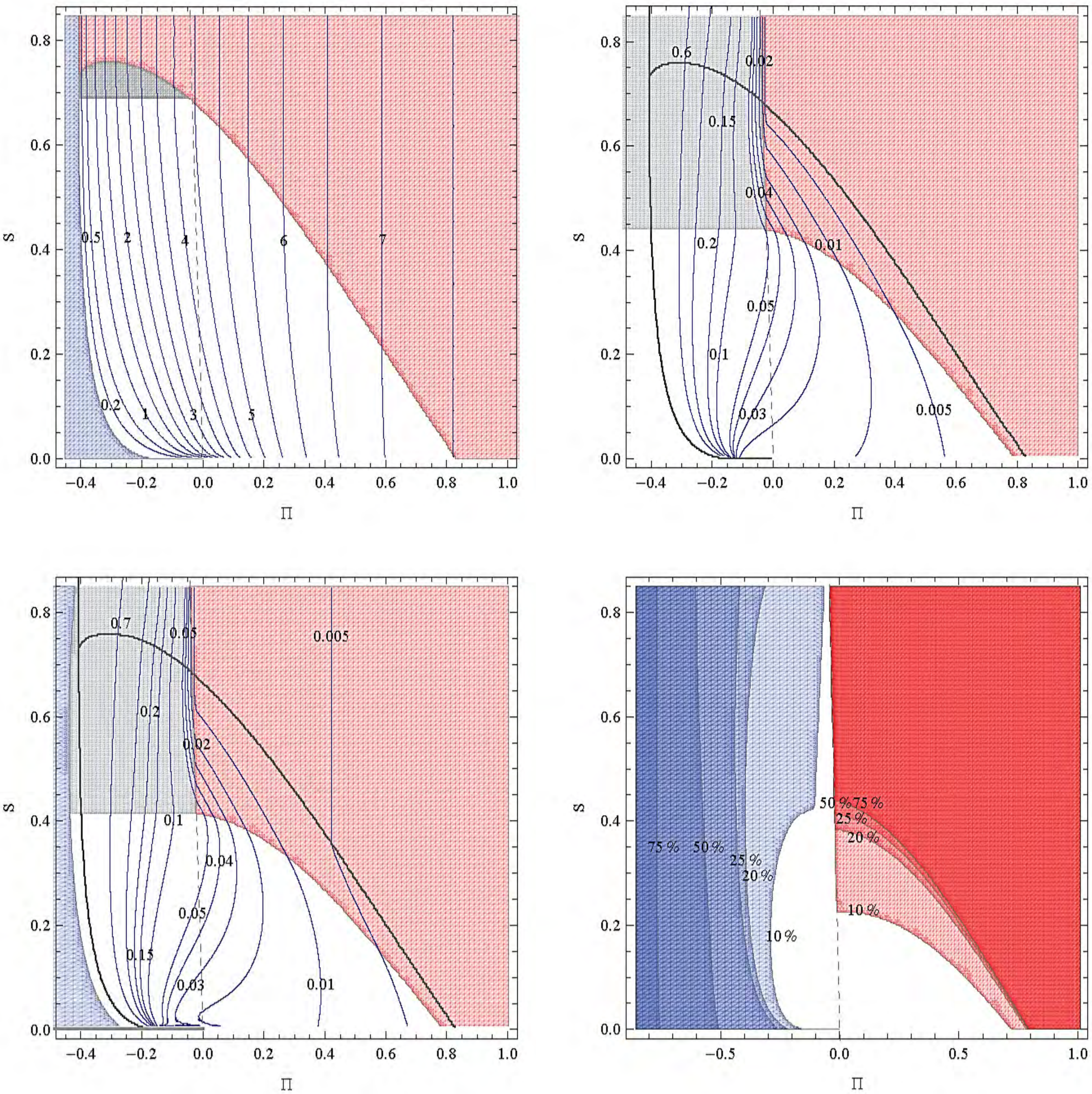

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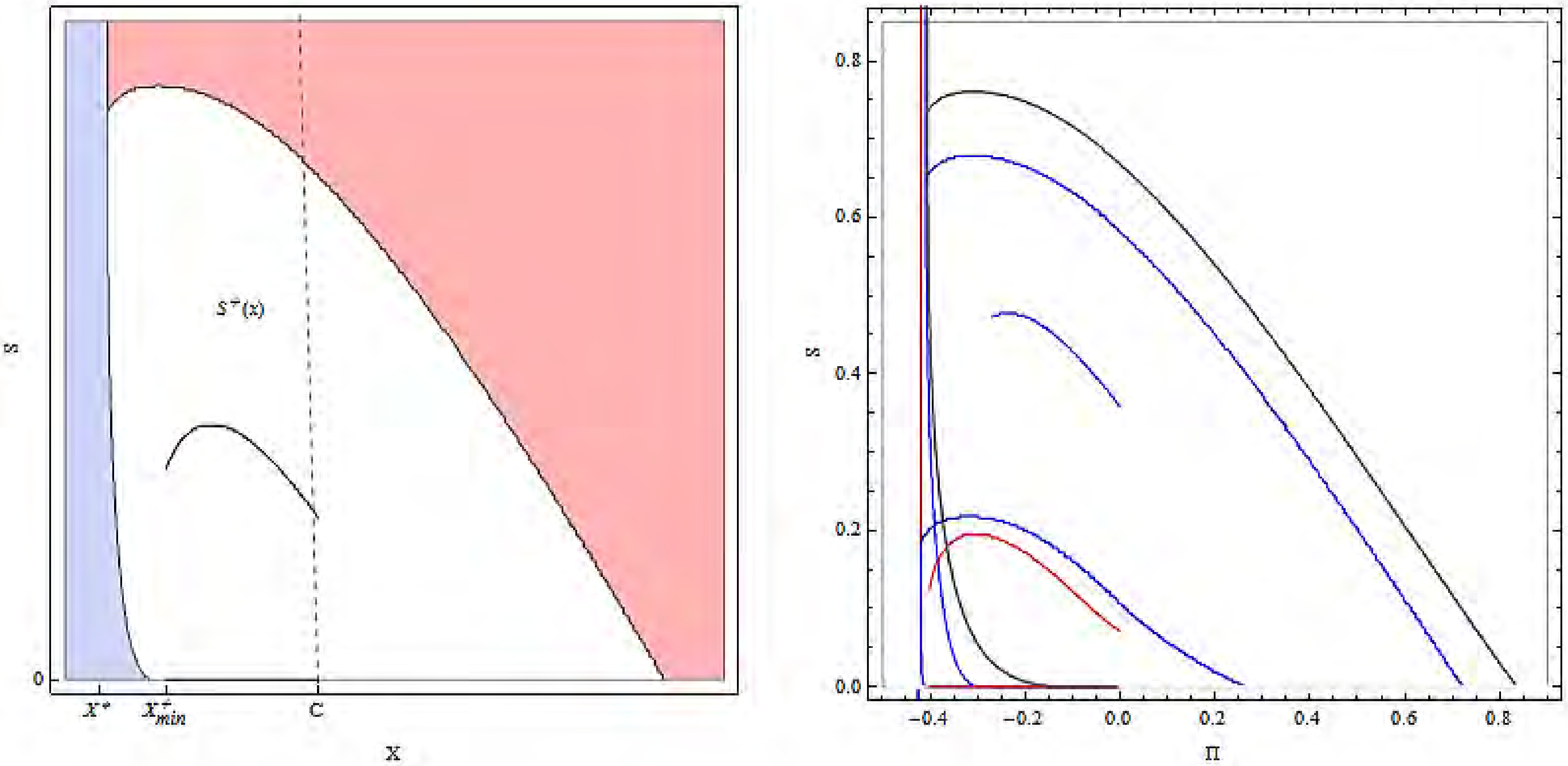

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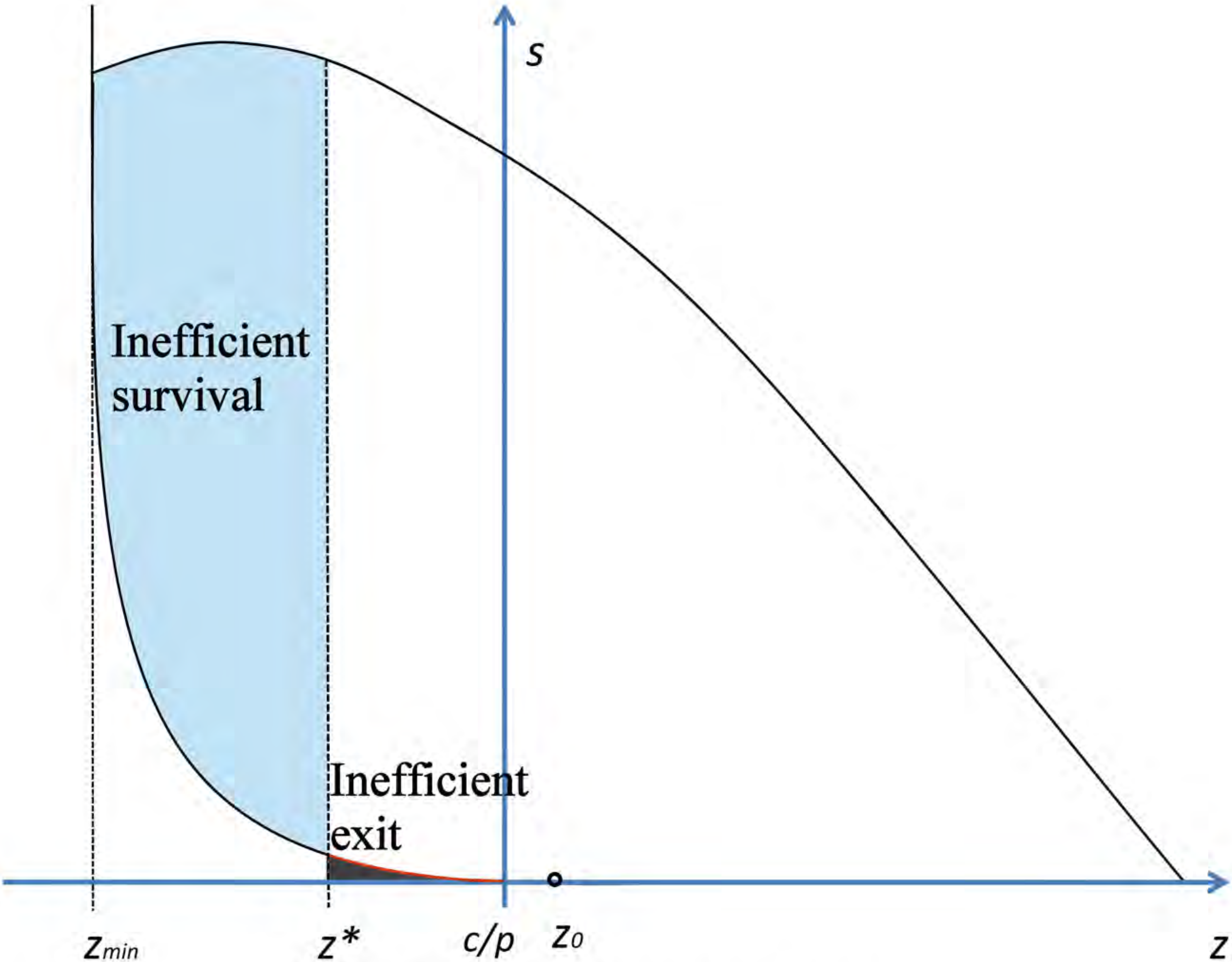

Figure 5. Liquidity Constraint and Average Productivity

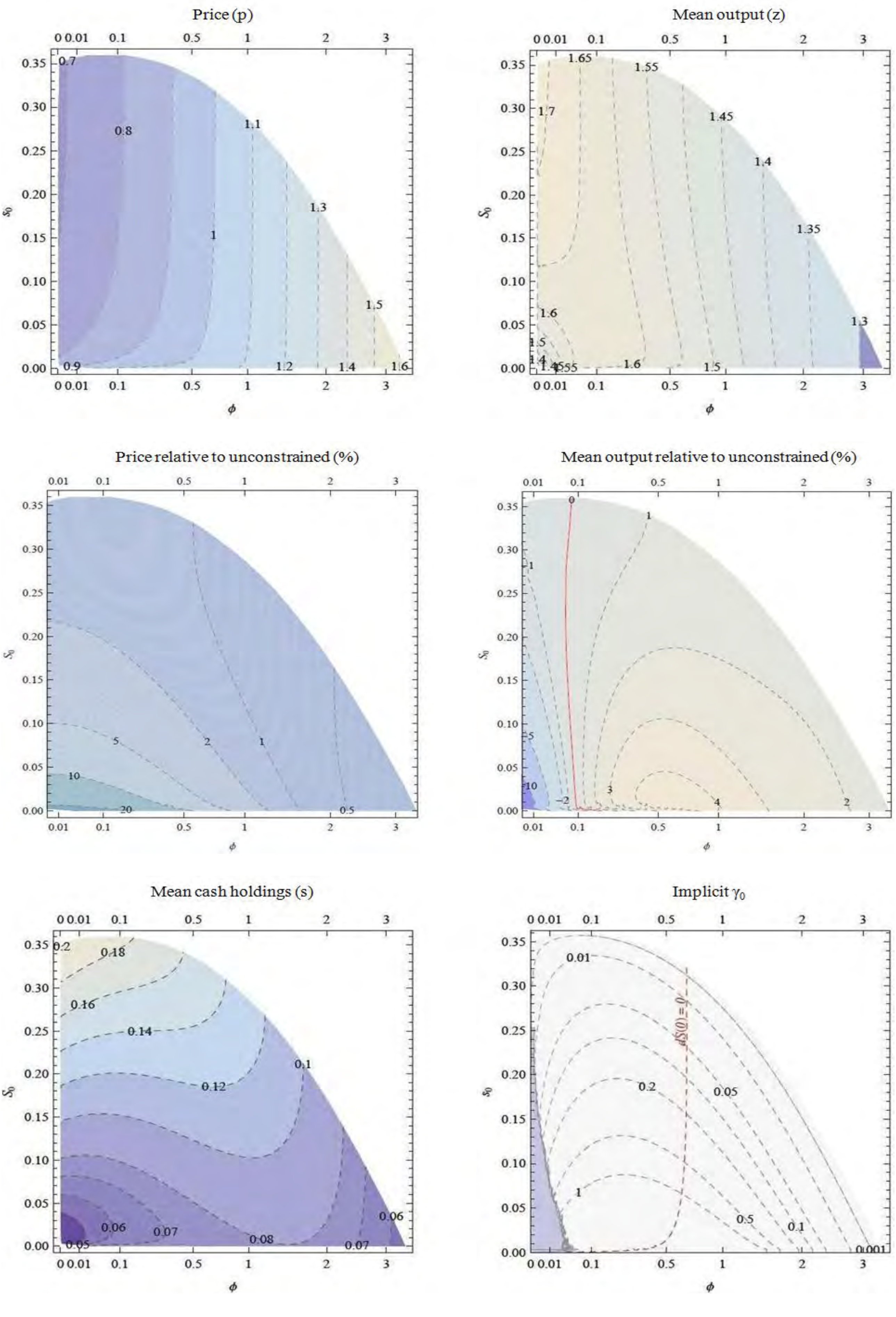

Figure 6. Impact of liquidity constraint on industry equilibrium.