The Optimal Design of Insurance Schemes for Preventing Liquidity Runs[1]

by

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(extremely preliminary!)

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[1] To be added

Abstract

We analyze the optimal deposit insurance scheme that can prevent a liquidity run. We obtain a simple characterization of the optimal scheme. Based on this characterization, we examine how the insurance depends on the size of the investment. When comparing agents who are partially insured we show that on a per-dollar basis a larger investor should receive better insurance. However, the relation between size of investment and level of insurance need not be monotone as the largest investor is never fully insured.

I. *Introduction*

The recent financial crisis highlighted the role of a liquidity run. As with most financial crises, a major factor was the lack of coordination that led investors to early demand for fear that others would do the same. However, early demand itself can, and often does, cause banks to become insolvent in a sort of self-fulfilling prophecy. What perhaps distinguishes the recent crisis is the fact that this phenomenon was not restricted to commercial banks and retail investors but occurred in the interdealer market among agents such as Lehman Brothers and J. P. Morgan (see for example Krishnamurthy, Nagel and Orlov 2011, and Gorton Metrick 2011). Hence, while traditionally this is known as a bank run, it might be more appropriate to use a more general term such as liquidity run. [2]

One can think of different ways to avoid such a run. The simplest one is to avoid it ex ante by not using a contract such as term deposits that can be withdrawn on demand. A bank may also temporarily suspend withdrawals; this is often referred to as "suspension of convertibility." The bank can also try to coordinate a joint agreement with all of stakeholders. However, temporary suspension is in many cases infeasible and there might be good reasons why a bank has signed a contract that is equivalent to a short-term loan. After all, one of its roles is to convert illiquid assets into liquid ones. Finally, coordinating among investors is also in many cases infeasible or unlikely.

As a result, a bank may reach a point where there is a significant risk of a run. At this point the only way to avoid a run is to rely on external assets and offer some form of deposit insurance

[2] To make our terminology consistent with previous literature, throughout the rest of the paper we keep the names 'bank' and 'depositors'.

with which an investor is insured that even if all other agents run he will still be entitled to some assets. This can be implemented through collateral using external funds or in other ways that protect an agent in case of a run. Providing full insurance to all agents would clearly eliminate the possibility of a run. However, in reality insurance is costly as the bank owns only a finite amount of resources and almost no bank is able to insure all its creditors. Fortunately, full insurance to all agents may not be necessary. An agent may decide to stay even if he is not fully insured. An agent could stay based on him knowing that other agents are sufficiently insured and hence will not run. In this paper we analyze the size of the required insurance scheme and how to design it in the most efficient way. In particular, how does the optimal allocation depend on investors' size; should a large investor receive more protection on a per-dollar basis? Should the largest investor receive the best coverage or in contrast the optimal coverage is decreasing in size?

Similar to Segal (2003) and Winter (2004), we obtain a simple characterization of the optimal scheme. It can be described through an inductive procedure. At the $k-th$ step, we set the insurance needed for the $k-th$ agent as the level of insurance that will make him stay, assuming that the first $k-1$ agents stay but all other agents run. This implies that optimal scheme discriminates agents even if ex-ante they are all identical.

We focus our attention to the more interesting case where agents are heterogeneous in size. In this case, the solution of the optimal scheme amounts to finding the optimal permutation in the above procedure. We show that on a per-dollar basis coverage is decreasing as in earlier steps we offer better coverage. Hence, the dependence of the optimal permutation on size implies who should get better coverage. We also prove that on a per-dollar basis, the insurance for an agent with $\$x$ is the same as the insurance needed for his marginal dollar assuming that the first $\$(x-1)$ are not demanded. This results in significant cost saving when we insure a larger investor. Intuitively this is because a large investor internalizes a larger fraction of the liquidation costs that follow from mis-coordination among investors.

Based on these two results we describe the relation between the size of the investment and the level of coverage. We find that in general a larger investor should receive better insurance on a per-dollar basis. Nevertheless, this relation can be non-monotone as the largest investor never receives full coverage while smaller investors may. The reason why the larger investor should receive better insurance is based on the observation that on a per-dollar basis it is cheaper to insure the larger investor. The question is, what position do we choose for a large investor so as to best utilize the positive externalities he induces in others? If we place him late in the order, then he will receive low coverage but his positive externality on others will be less effective as it affects only his successors. On the other hand, if we need to fully insure at least one agent, we will not choose the largest investor to be fully insured. We would be better off positioning him at the stage where the insurance starts declining so as to maximize the advantage of his size.

For a concrete example of the second effect, suppose there are two agents who invested a total of $\$3$ at some prior date: A who invested $\$1$, and B who invested $\$2$. The investment will pay a gross return of 1.5 with certainty but each agent can demand his investment immediately. Further assume that liquidity costs are high enough so that if only one agent stays $(A$ or $B)$ then the bank will need to liquidate all its assets.[3] An agent will choose to stay only if he believes that the other agent will. So we will need to secure exactly one of the two agents; the question is which agent should it be? Clearly, the answer is that we should secure the smaller investor, A, as we need to guarantee a future payment of $\$1$.

The rest of the paper is organized as follows. Section II describes the basic model. In Section III we characterize the optimal insurance scheme and present the main results of this paper. Section IV extends the results to the case where agents are hit by random liquidity shocks so that an agent may run even if he is fully insured. Most proofs appear in the Appendix.

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[3] When there are more than two agents we need to assume that if *n-1* agents run that results in complete liquidation, which is a weak condition.

A. *Relation to Prior Literature*

Following Diamond and Dybvig (1983) there is an extensive literature on bank runs. Some have took an approach similar to mechanism design and asked whether one can design contracts so as to avoid a run.[4]

While the focus of most of these papers has been on the ex-ante stage, we perform calculations at the interim stage. Specifically, suppose there are three dates t=0,1,2. At t=0 contracts are signed, at t=2 payoffs of long-term assets are realized, and at t=1 investors have the right to withdraw their deposits. Prior research asked questions from an ex-ante perspective, i.e., t=0. For example, some models solve for the best mechanism or contract, given agents who may have random liquidity needs at t=1. One of the key issues is whether we can avoid a run with the optimal contract.

Instead, we focus on t=1 and start with a situation where there exists an equilibrium in which there is a run on the bank. This is perhaps because an inefficient contract was signed at t=0, which we could not have avoided. Alternatively, it may arise from the fact that agents were optimistic about avoiding the mis-coordination problem when they acted at t=0, but at t=1 they were not so optimistic anymore (perhaps because they witnessed a bank run between the two periods). We ask how a run may still be avoided if we use other assets to insure some investors.

The special feature of the optimal collateral scheme we present here, by which a favorable contract that locks one agent in allows the principal to offer less attractive contracts to others, is related to the "divide and conquer" strategy that has been discussed in the contracting literature. An upstream monopoly can deter the entry of a potential rival by signing exclusionary contracts with downstream buyers. The more buyers are signing the more attractive it is for others to sign as well (see Segal and Whinston (2000)). It is also related to optimal incentives in teams, where a high reward for one agent sufficient enough to induce him to exert effort will increase the

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[4] See for example Jacklin (1987), Postlewaite and Vives (1987), Ennis and Keister (2003), Green and Lin (2003), Peck and Shell (2003), and Ennis, H.M. Keister, (2009)

incentives of other agents to do the same (See Winter (2004)). Finally, it is related to the argument about introductory prices by a monopolist producing goods with positive consumption externalities such as in Bensaid and Lesne (1996) and Cabral, Salant and Woroch (1999) (see also Farrell and Saloner (1985) and Katz and Shapiro (1986)). Berenstein and Winter (2011) discuss a model of contracting with heterogeneous externalities but they require that externalities are bilateral (i.e., for each two players i and j, the externality that i induces on j is independent of the actions taken by other players. Clearly, this assumption does not hold in our framework as the gains that agent i acquires from the fact that agent j is not running strongly depends on who are the other people that run.

Segal (2003) introduced a general model of trade contracts when the principal's trade with one agent generates externalities on other and has shown that with increasing externalities the principal gains by using a divide and conquer strategy, when he cannot coordinate players to play his most preferred equilibrium. Segal's model fits nicely into a variety of IO applications (like takeovers, vertical contracting, exclusive dealing, and network externalities). While Segal defines the divide and conquer strategy in a general contracting setup that allows for asymmetries, he doesn't solve for the optimal mechanism except for special cases such as the symmetric case (although he is able to obtain some comparative static results without deriving the optimal mechanism explicitly).

The contribution of the current paper is driven from the specific coordination problem that arises in liquidity runs. In our framework the asymmetry across players plays a major role and agents' externalities are multilateral. The specific payoff structure in our framework of liquidity runs requires a separate analysis. This analysis enable us characterize the optimal collateral scheme which is our main objective.

II. Setup

A. *Run with No Insurance*

We examine a two-date economy $t = 1, 2$, where agents decide whether to demand their money at $t = 1$ or wait until $t = 2$. There are N such agents and we let □ denote the set of agents. For simplicity, we assume that agents are risk-neutral and do not have a time preference, so that they maximize:

$$u(c_{1,}c_2) = c_1 + E(c_2),$$

where c_1 and c_2 denote the consumption at $t = 1, 2$. In Section IV, we relax this assumption as in Diamond and Dybvig (1983) and allow investors to be subject to random liquidity shocks. An agent who is hit with a liquidity shock may demand his money at $t = 1$ even when his investment is perfectly insured.

There are two assets in the economy: a liquid asset and an illiquid one. The illiquid asset is a long-term asset that yields a gross return of $R > 1$ at $t = 2$. For simplicity, we assume that it is non-random. The illiquidity is represented by liquidation costs of δ, so $\$1$ invested in an illiquid asset yields only $1-\delta$ when liquidated at $t = 1$. Alternatively, if we want to obtain $\$1$ by liquidating risky assets then we need to liquidate $\$\frac{1}{1-\delta}$. The liquid asset yields zero return but has no liquidation cost.

Each agent, say i, has invested x_i with the bank, which in turn invested these funds in the illiquid asset. The amount x_i also equals what an agent has the right to demand. If he demands at $t = 1$ and the bank is solvent then he receives x_i. Following Diamond and Dybvig (1983), we assume that if no agent demands at $t = 1$ then he receives Rx_i at $t = 2$; however, this payoff will be lower if some agents demand early. We let $X \equiv \sum x_i$ and refer to the decision to demand at $t = 1$ as *run*. We denote by BR the subset of agents who run and by NR the set of agents who stay; $BR \cup NR =$ □ . Let $X^{BR} \equiv \sum_{i \in BR} X_i, X^{NR} \equiv \sum_{i \in NR} X_i$. Following Diamond and Dybvig (1983), we assume that withdrawal tenders are served sequentially in random order until the bank runs out of assets.[5] For an agent who does not run, $i \in NR$, we let $h(i, NR, \overrightarrow{X})$ denote his payoff given the decision of others:

$$h(i, NR, \overrightarrow{X}) \equiv \max\{\, 0, R\left(X - \frac{X^{BR}}{1-\delta} \right) \frac{x_i}{X^{NR}} \} = \max\{\, 0, R\left(X - \frac{X - X^{NR}}{1-\delta} \right) \frac{x_i}{X^{NR}} \} \quad (1.1)$$

In the next section when we introduce insurance we will modify this payoff to account for the guarantee an agent may receive. Equilibrium is described by the set of agents who do not run, NR, where

$$i \in NR \Leftrightarrow h(i, NR, \overrightarrow{X}) \geq x_i$$

As a tie-breaking rule we assume that if an agent is indifferent whether to run then he will stay. This simplifies the definition of the optimal insurance scheme.

Based on the above functional form, (1.1), we argue that:

LEMMA 1 (i) h satisfies strategic complementarity, i.e.,

$$h(i, NR, \overrightarrow{X}) \leq h(i, NR \cup \{j\}, \overrightarrow{X})$$

(ii) Agents who stay expect the same gross return. For $i, j \in NR$ we have:

$$\frac{h(i, NR, \overrightarrow{X})}{x_i} = \frac{h(j, NR, \overrightarrow{X})}{x_j} \quad (1.2)$$

Based on this we argue that there are only two possible equilibria, one where all agents run and the other where they all stay. Since $R > 1$ the first equilibrium always exists but in many cases both equilibria exist. We also argue that a necessary and sufficient condition for the existence of the equilibrium in which agents run is that the largest investor will run if he believes that all the other agents will not stay.

LEMMA 2 (i) No equilibrium exists where only some of the agents stay, i.e., $NR \neq \square$, and $NR \neq \varnothing$. (ii) A necessary and sufficient condition for the existence of an equilibrium where agents run is:

[5] If the order were deterministic and known to agents, then there would be a unique equilibrium in which agents do not run. This follows from the fact that a sub-game perfect equilibrium is generically unique.

$$h(i_{\max}, \{i_{\max}\}, \overrightarrow{X}) \le x_{\max}, \qquad (1.3)$$

where $x_{\max} = \max\{x_i\}, i_{\max} = \arg\max\{x_i\}$.

Throughout the paper we assume that (1.3) holds. In some cases we will consider a slightly stronger condition, namely,

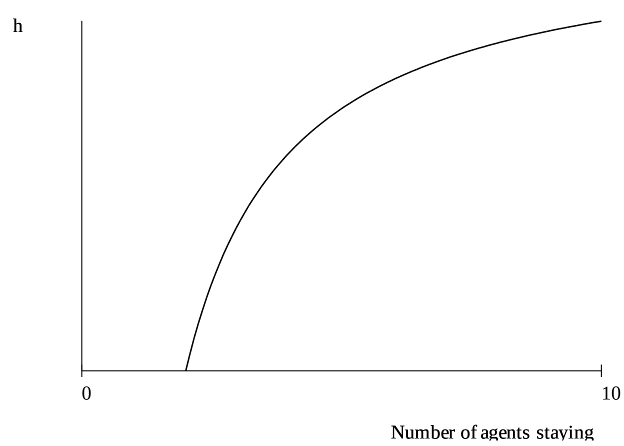
$$h(\{i_{\max}\}, \overrightarrow{X}) = 0$$

This implies that if all but one agent decide to run then all assets will be liquidated, which is equivalent to

$$\forall i : \frac{X - x_i}{1-\delta} > X$$

1. Special Case: \$1 Investors, $h(n, X)$

Of particular importance is the special case where all investors have $x_i = 1$. With some abuse of notation we let $h(n, X)$ stand for $h(i, NR, \overrightarrow{X})$, where exactly $n \le N$ agents stay and $\overrightarrow{X} = \vec{1}, X = \sum x_i$. In the figure below we plot this function for the case with *N*=10 investors, $R = 1.1$, and $\delta = 0.2$.



More generally, a simple calculation reveals that:

LEMMA 3 $h(n, X)$ is non-negative, non-decreasing, and concave in *n* in the interval where it is positive.

B. *An Collateral/Insurance Scheme*

To avoid a run, the bank guarantees future payments to the different investors. A guarantee implies that a certain amount is protected even in the case of a run. We denote by $y_i \geq 0,$ the insurance to agent i. These guarantees can be implemented by posting a collateral with a future value of y_i for investor i. The cost of this insurance scheme is an increasing function of $Y = \sum y_i$, so we would like to examine what is the minimal Y that can insure that there will not be an equilibrium where some agents run. We refer to y_i as the *insurance scheme*. We modify the definition of $h(i, NR, \overrightarrow{X})$ to account for this insurance and we define $i \in NR$ as follows:

$$h^y(i, y_i, \overrightarrow{X}, NR) \equiv \min\{Rx_i, y_i + h(i, NR, \overrightarrow{X})\}$$

The above expression represents a liquidation procedure that may depend on the court's decision. Still, our results are robust as long as this function satisfies the properties we describe in the next section, most notably "strategic complementarity." For example, one can replace the above expression by

$$h^y(i, y_i, \overrightarrow{X}, NR) = \max\{y_i, h(i, NR, \overrightarrow{X})\}$$

In summary, the *run game* is an *N*-person normal form game, where each player has two strategies {stay, run} and the payoff for a player *i* for a given strategy profile is h^y if he stays and x_i if he runs. Similar to the case with no insurance an agent runs when

$$h^y(i, y_i, \overrightarrow{X}, NR) < x_i$$

Definition We say that aggregate collateral Y with allocation $\left\{y_j\right\} : \sum_j y_j = Y$ is sufficient to prevent a run if it is a unique Nash equilibrium for all players to stay.

Given the above, our research question can be formulated as follows:

i) What is the minimal level of *Y* that will prevent a run?

ii) What should be the allocation, $\left\{y_j\right\}$?

C. *Examples*

There are three investors (A, B, and C), each with \$1, and so $X = 3$; we also assume that $R = 1.5$ and $\delta = \frac{1}{3}$. As noted earlier, it is equilibrium for all agents to stay but it is also an equilibrium to run.

If two agents run, then the third one will lose his entire asset. Hence, to eliminate the bank run equilibrium, there should be at least one agent who is fully insured, $y_i = 1$. If contracts have to be symmetric then we need to offer full insurance to all agents, i.e., *Y=3.* Otherwise it is equilibrium for all agents to run.

Based on the above we can see that a symmetric scheme is not optimal even when agents are identical. If two agents are offered full insurance then the third investor will stay even if he is offered no insurance. In this case he is certain that the other two are staying, so he prefers to stay.

In fact the optimal scheme is simple to compute in this case. To eliminate the bank run equilibrium one agent needs to be offered full insurance, $y_i = 1$. Otherwise, it is equilibrium strategy for all agents to run. Suppose now that one of the two other remaining agents runs. In this case we need to liquidate 3/2. This will leave 1.5 invested in the illiquid asset, which yields a payoff of 2.25 at t=2. Hence, even if we offer no insurance to the other two agents.

III. Analysis

A. *Required Insurance*

Consider that $i \notin NR$ and $h^y(i, 0, NR, \overrightarrow{X}) < x_i$, which implies that this agent will run if he does not have any coverage. Let $y_i^*(\overrightarrow{X}, NR)$ denote the unique solution to $h^y(i, y_i, NR, \overrightarrow{X}) = x_i$; $y_i^*(\overrightarrow{X}, NR)$ is well defined as $h^y(i, y_i, NR, \overrightarrow{X})$ is increasing in y_i, and corresponds to the minimal coverage that will prevent i from running; we refer to this as the required insurance. If $h^y(i, 0, NR, \overrightarrow{X}) \geq x_i$ then we define $y_i^*(\overrightarrow{X}, NR)$ to be zero.

Similar to the case of no insurance, II.A, also $h^y(i, y_i, \overrightarrow{X}, NR)$ satisfies strategic complementarity. If an agent *i* stays when *j* runs then he will prefer to do so also if agent *j* stays:

$$h^y(i, y_i, NR, \overrightarrow{X}) \leq h^y(i, y_i, NR \cup \{j\}, \overrightarrow{X}) \qquad (1.4)$$

It follows that one needs less insurance if we increase the set of agents who stay:

$$y_i^*(\overrightarrow{X}, NR \cup \{j\}) \leq y_i^*(\overrightarrow{X}, NR) \qquad (1.5)$$

A more interesting property is that on a per-dollar basis, the collateral needed for each agent equals the collateral needed for his marginal dollar assuming that he decided to keep his first $x_i - 1$ invested. Formally, suppose that we split agent i into two agents i' and i'' while keeping everything else the same. We divide their holdings to $x_{i'} = x_i - 1, x_{i''} = 1$ so that

$$\overrightarrow{X} = (x_1, .., x_{i-1}, x_{i'} = x_i - 1, x_{i''} = 1, .., x_n)$$

Therefore

LEMMA 4 $\frac{1}{x_i} y_i^*(\overrightarrow{X}, NR) = y_{i''}^*(\overrightarrow{X}, NR \cup \{i'\})$,

This property is key to our analysis. It implies that a large investor internalizes a larger part of the liquidation cost. On a per-dollar basis the needed collateral is lower by the fact that when deciding to keep his last dollar he takes into account the fact that he also keeps the first $x_i - 1$ dollars.

Based on this and (1.5) we argue that if we convince an agent to stay then on a per-dollar basis the collateral that is needed to convince the next agent decreases.

LEMMA 5 $\frac{1}{x_i} y_i^*(\overrightarrow{X}, NR) \geq \frac{1}{x_j} y_j^*(\overrightarrow{X}, NR \cup \{i\})$

1. Example

Suppose there are four agents, $\{a, b, c, d\}$ where $x_a = 2, x_b = x_c = x_d = 1$ and $R = 1.5, \delta = 0.5$ and assume that $NR = \{b\}$ so we assume that only agent b decides to stay. We are interested in the collateral that would keep agent a from running. Note that in this case if a also stays we have $X-\frac{X^{BR}}{1-\delta}=1$. So he will receive $R\ \ X-\frac{X^{BR}}{1-\delta}\ \ \frac{2}{3}=1$. Hence, we need a coverage of $\$1$ or $\$0.5$ on a per-dollar basis.

Now consider the case where we split agent a into two agents a' and a'' where $x_{a'}=x_{a''}=1$. We compute the coverage needed for a' and a'' assuming we treat them separately. We begin with agent a' and compute the needed coverage assuming that only agent b stays. Since $X-\frac{X^{BR}}{1-\delta}<0$, it follows that to keep him we would need $\$1$. Assuming now that both a' and b stay, we would only need a coverage of $\$0.5$ to convince a'' to stay. So in the original setup on a per-dollar basis the needed collateral for agent a is the collateral needed for his marginal dollar, a''.

2. *Required Insurance for $1 investors*

Consider again the special case where all *n* investors have $x_i=1$. We let $y^{*,1}(n,X)$ denote the required optimal collateral scheme that will keep all investors in the bank in any Nash equilibrium.

In the figure below we plot this function for the case when *n=10* investors, $R=1.1$, and $\delta=0.2$.

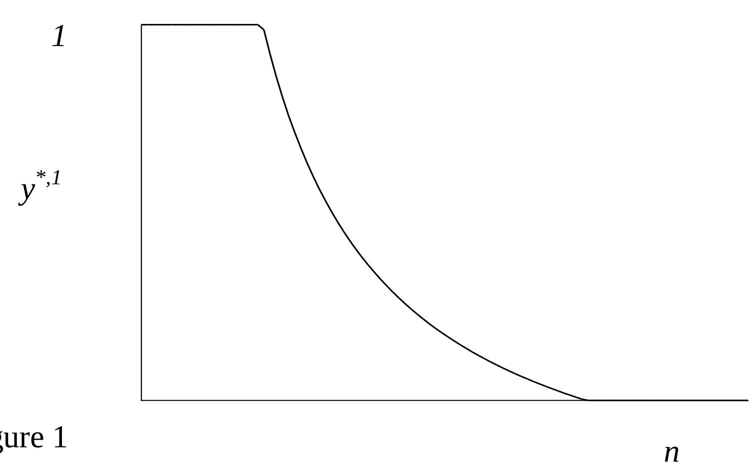


figure 1

B. *The Optimal Scheme in the General Case*

We first argue that the optimal collateral scheme takes a very specific form. One can describe it as a sequential algorithm, but it is important to note that the game itself does not follow this order. The procedure depends on a permutation π which we take as given. We let $\vec{Y}(\pi)$ be defined as follows:

- $\vec{Y}_{\pi(1)} = y^*(\vec{X},\emptyset)$ is the collateral of the first agent that will keep him staying even if all agents run.

- $\vec{Y}_{\pi(2)} = y^*(\vec{X},\{\pi(1)\})$ is the collateral of the second agent that would keep him staying assuming that the first agent stays and everybody else runs.

- $\vec{Y}_{\pi(3)} = y^*(\vec{X},\{\pi(1),\pi(2)\})$ is the collateral of the third agent that would keep him staying assuming that the first two agents stay and everybody else runs.

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Let Π be the set of n! permutations and denote it by:

$$\pi^* = \text{argmin}_{\pi\in\Pi} \sum_j \vec{Y}_{\pi(j)}$$

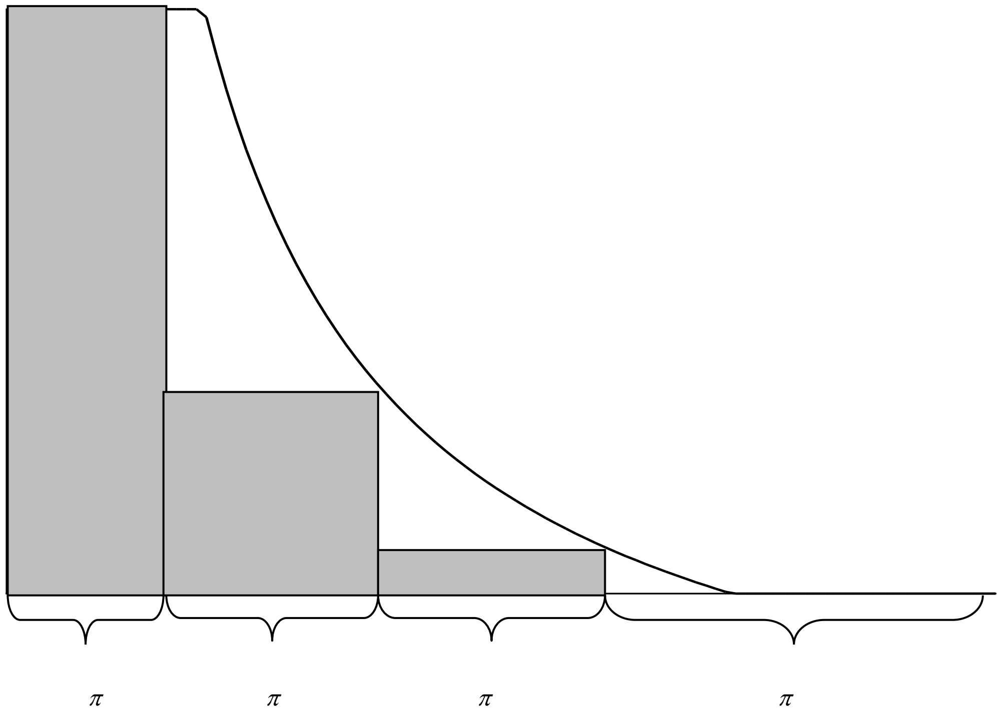
We argue that:

THEOREM 1: $\vec{Y}_{\pi^*}$ is the optimal insurance scheme.

Based on Theorem 1 it follows that in optimizing we should optimize over all possible permutations. Recall Lemma 3 that implies that the cost of insuring an agent matches the cost of insuring the marginal dollar. Based on this we can compare the cost of insuring $\mathrm{X} = \sum x_i$ agents each with \$1. For example, instead of insuring 5 agents with a total of \$100 we look at the case of 100 agents each with \$1. Lemma 3 implies that the cost of our scheme is given by:

$$\sum_i y^{*,1}(\sum_{j\leq i} x_{\pi(j)}, X)x_i \qquad (1.6)$$

This can be expressed as the sum of the area in the rectangle in the figure below. In this graph we proceed as in the previous example where $R=1.1$ and $\delta=0.2$. There are 4 investors and a total of *X=10* and so the cost of a given permutation can be computed based on the function $y^{*,1}$ that we plotted in figure1.



So finding the right permutation can be described as:

$$\min_{\pi\in\Pi}\sum_i y^{*,1}(\sum_{j\leq i} x_{\pi(j)},X)x_i \qquad (1.7)$$

Intuitively, we would like to reap the potential cost saving from insuring large investors. As we shall discuss in the Appendix, the combinatorial problem in (1.7) is NP-compete and thus there is no simple solution. Nevertheless, we can characterize the optimal solution.

THEOREM 2: Let $i_{\max}$ be the agent with the largest investment and assume that he is unique so that $\forall i : x_{i_{\max}} > x_i$. Then in the optimal scheme, $\vec{Y}_{\pi^*}$, this agent is not fully insured, that is, $x_{i_{\max}} > y_{\pi^*,i_{\max}}$.

We next show that among almost all agents who are partially insured agents with higher investment are treated better not only in terms of their total collateral but also in terms of their collateral pre-dollar of investment.

THEOREM 3: Consider the order π according to which an optimal collateral scheme is derived (in the inductive algorithm presented in the first part of this section). Consider all agents who are not fully insured except the first among them. Then an agent with a higher investment receives higher per-dollar collateral.

The preferential treatment of agents with a large investment in the per-dollar collateral they receive reflects the fact that these agents are critical in preventing a run that would turn the bank insolvent. This also means that investors could be made better off if they pool their investments and present themselves as a single investor vis-a-vis the bank. This makes sense as by pooling the investments they internalize some of the negative externalities associated with mis-coordination and eventually an inefficient bank run. However this internalization of the externalities plays a role only after the bank managed to "lock in" a sufficiently large group of investors so that if all, but one agent outside this group, are running it would still pay off for this last player to stay. At an early stage of the algorithm when this condition fails to hold it doesn't matter how large my investment is, I will be treated (on a per-dollar basis) exactly the same as someone who is investing one dollar only.

IV. Random Liquidity Shocks

An important extension is the case where investors are hit by random liquidity shocks (Diamond and Dybvig 1983). This provides a natural motivation for giving agents demand deposits despite the fact that the bank invests in illiquid assets where early liquidation is inefficient.

Such shocks pose a challenge to our scheme. An investor who is given high collateral as an incentive to stay and thereby convince others to follow suit may nevertheless decide to leave. Still, we argue that our scheme can be modified to account for this uncertainty. We refer to an agent who decides to demand his money at $t=1$ but is not hit by a liquidity shock as an agent who runs. An agent who demands early only because he is hit with a liquidity shock is considered as an agent who does not run. We examine the insurance that is needed to ensure that all agents will not run. Our scheme will guarantee that those who are not hit with liquidity shocks will stay.

We let ζ_j denote a binary random variable where $\zeta_j = 1$ in the event that agent j is hit with a liquidity shock and it equals zero otherwise. We assume liquidity shocks are independent and that $\Pr(\zeta_j = 1) = \rho_i$. The set of agents who actually stay is a random set and can be defined as a function of those who do not run and the realized liquidity shocks.

$$ST(NR, \vec{\xi}) \equiv NR \setminus \{i : \xi_i = 1\}$$

Let Z^{NR} denote the amount of funds that are not demanded; this is a random variable as it depends on the realization of the liquidity shocks, $Z^{NR} \equiv \sum_{j \in NR} (1 - \zeta_j) x_j$. With some abuse of notation we let $h^{\bar{\rho},y}(\bar{\rho}, i, y_i, \overrightarrow{X}, NR)$ denote the expected payoff for an agent who stays given the decisions of others:

$$h^{\bar{\rho},y}(\bar{\rho},, i, y_i, \overrightarrow{X}, NR) \equiv E_{ST} h^y(i, y_i, \overrightarrow{X}, ST(NR, \vec{\xi})),$$

Note that we assume that agents know the probability that other agents are hit with a liquidity shock but are uncertain regarding the actual realization. If instead we had assumed that the realizations of liquidity shocks are common knowledge then we would be back at a similar case to what we have examined before. The main difference would be that we will start the process with lower future payoffs as a result of the realized liquidity shocks.

A key property that we have used in the case of no random liquidity shocks was the fact that the there is a strategic complementarity in the decision to stay. If one agent decides to stay then it is easier for other agents to follow. A similar property holds when we introduce random liquidity shocks:

Lemma 6 $h^{\bar{\rho},y}(\bar{\rho},, i, y_i, \overrightarrow{X}, NR)$ satisfies strategic complementarity:

$$h^{\bar{\rho},y}(\bar{\rho},, i, y_i, \overrightarrow{X}, NR \cup \{j\}) \geq h^{\bar{\rho},y}(\bar{\rho},, i, y_i, \overrightarrow{X}, NR)$$

Proof of Lemma 1 Based on our analysis in earlier sections we know that for any realization of shocks which cause some agents to leave with their investments the payoff function defined on the remaining players satisfies strategic complementarity. By Lemma 6 this property applies also on the the expected payoff function with respect to the distribution of shocks. An implication of the above claim is that the procedure that we described before is still valid.

Based on this we conclude:

THEOREM 4: (i) The inductive procedure yields the optimal scheme also for the case with ransom liquidity shocks, (ii) in the optimal scheme the largest investor is not be fully insured

Proof: Both results rely only on the shape of the functions $g(NR)$ and $h(NR)$ as specified in the proof of these two results. By Lemma 6 above the properties of concavity and convexity of these functions are preserved under the random liquidity shocks. Hence, these results still hold true. The set of agents who demand at $t=1$ is the union of agents who run and agents who do not run but are hit with liquidity shocks.
where the expectation is taken with respect to the distribution of liquidity shocks.

A more interesting question concerns how the properties of the optimal scheme change as a result of this noise. We argue that randomness favors small investors since securing small investors becomes even more attractive. Suppose that $\rho = 0.5,$ and consider the difference between securing an investor who has $\$10$ and 10 investors who each have $\$1$. If it costs us the same then the difference results from the reaction of other investors. In both cases we get the same expected value of their investment which will not be demanded early, $\$5.$ But in the case of a single large investor we have a more variable outcome as it equals $\$10$ or zero with equal probability. How does this affect other agents? For this we need to consider $h(\rho, i, y_i, \overrightarrow{X}, NR)$ and examine the effect of a more variable outcome. To see this, consider the expression

$$R\left(X - \frac{X - \tilde{Z}^{NR}}{1-\delta}\right)\frac{x_i}{\tilde{Z}^{NR}}\} = a + \frac{b}{\tilde{Z}^{NR}},$$

where $a = \frac{Rx_i}{1-\delta} > 0, b = Rx_i X(1 - \frac{1}{1-\delta}) < 0$ which is concave in $\tilde{Z}^{NR}$

V. Conclusions

VI. References

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VII. Appendix

PROOF OF LEMMA 1

We argue that

$$\left(X - \frac{X - X^{NR}}{1-\delta} \right) \frac{x_i}{X^{NR}} < \left(X - \frac{X - (X^{NR} + x_j)}{1-\delta} \right) \frac{x_i}{X^{NR} + x_j}$$

If we let $a = X - \frac{X - X^{NR}}{1-\delta}$ this can be expressed as:

$$\Leftrightarrow a[X^{NR} + x_j] < \left(a + \frac{x_j}{1-\delta} \right) (X^{NR})$$

$$\Leftrightarrow a x_j < X^{NR} \frac{x_j}{1-\delta}$$

This holds as $a = X - \frac{X - X^{NR}}{1-\delta} < X - (X - X^{NR}) = X^{NR}$ and $\frac{1}{1-\delta} > 1$ **QED**

PROOF OF LEMMA 2

(i) Suppose by contradiction that there exists an equilibrium where agent *i* stays but *j* runs. So we have $\frac{h(i, NR, \vec{X})}{x_i} \geq 1, \frac{h(j, NR \cup \{j\}, \vec{X})}{x_j} < 1$. Lemma 1 implies that $\frac{h(i, NR \cup \{j\}, \vec{X})}{x_i} \geq 1$ but then implies that $\frac{h(j, NR \cup \{j\}, \vec{X})}{x_j} \geq 1$, which is a contradiction.

(ii) We first note that

$$\max\{0, R\left(X - \frac{X - x_i}{1-\delta} \right)\} - x_i$$

is increasing in x_i so that if $h(i_{\max}, \{i_{\max}\}, \vec{X}) < x_{\max}$ then for all agents $h(i, \{i\}, \vec{X}) < x_i$. In this case each agent would prefer to run if he believed that the others would do likewise, and hence it is equilibrium for all agents to run. Now suppose that $h(i_{\max}, \{i_{\max}\}, \vec{X}) \leq x_{\max}$, which implies that it is a dominant strategy for the largest investor to stay. Hence, based on (i) the only equilibrium is for all agents to stay. **QED**

PROOF OF LEMMA 3

The claim follows from the fact that the payoff for agent *i*” with collateral $y_{i''} = \frac{y_i}{x_i}$ is exactly $1/x_i$ the payoff for agent i who has collateral y_i

$$\max\{0, E[\min\{D, \frac{y_i}{x_i} + \tilde{R}\left(X - \frac{X - X^{NR}}{1-\delta}\right)\frac{1}{X^{NR}}\}]\}$$
$$= \frac{1}{x_i}\max\{0, E[\min\{x_i D, y_i + \tilde{R}\left(X - \frac{X - X^{NR}}{1-\delta}\right)\frac{x_i}{X^{NR}}\}]\}$$

QED

PROOF OF LEMMA 4

We replace agent i and j each with two agents. Agent i is replaced with i', i'' and agent j with agents j', j'' so that $x_{i''} = y_{i''} = 1$ and $x_{i'} = x_i - 1,\ x_{j'} = x_j - 1$

$$\overrightarrow{X} = (x_1, .., x_{i-1}, x_{i'} = x_i - 1, x_{i''} = 1, .., x_{j'} = x_j - 1, x_{j''} = 1, .., x_n)$$

As we have shown in Lemma 2 the per-dollar collateral for agents i and j is equal to that of i'' and j''. That is,

$$\frac{1}{x_i} y_i^*(\overrightarrow{X}, A) = y_{i''}^*(\overrightarrow{X}, A \cup \{i'\})$$
$$\frac{1}{x_j} y_j^*(\overrightarrow{X}, A \cup \{i\}) = y_{j''}^*(\overrightarrow{X}, A \cup \{i', i'', j'\})$$

So we need to argue that $y_{i''}^*(\overrightarrow{X}, A \cup \{i'\}) \geq y_{j''}^*(\overrightarrow{X}, A \cup \{i', i'', j'\})$. Based on strategic complementarity we conclude that the collateral for agent j'' is not higher than that of i''. **QED**

PROOF OF THEOREM 1

We consider some collateral scheme Y^* that prevents a run. We first note that there must be an agent (call him $\pi(1)$) who is offered a collateral sufficiently high as to induce him to stay even if he believes that all other agents are running. If in contrast all agents receive coverage lower than this threshold then there exists a Nash equilibrium in which all agents run. If $\pi(1)$ is offered this level of collateral, then because of strategic complementarity $\pi(1)$ has a dominant strategy to stay. Now there must be another agent, $\pi(2)$, who is offered a collateral high enough to make him stay even if he believes that all agents but $\pi(1)$ will run. If no such agent exists then there is an equilibrium in which all agents except player $\pi(1)$ run. Because of strategic complementarity, if $\pi(1)$ stays, then the dominant strategy of $\pi(2)$ is to stay as well. More generally, assume by induction that we designated $\pi(1), \pi(2), ..., \pi(j-1)$. There must be an agent $\pi(j)$ who is offered a collateral high enough so that he prefers to stay assuming that $\pi(1), \pi(2), ..., \pi(j-1)$ are staying and no matter what the rest of the agents are doing. If such an agent does not exist, then there is a Nash equilibrium in which $\pi(1), \pi(2), ..., \pi(j-1)$ stay and all the rest run. Designating $\pi(1), \pi(2), ..., \pi(n)$ with the appropriate collateral yields a scheme Y_π that admits a unique equilibrium in which all agents choose to stay and more over $\sum Y_{\pi(i)} \leq \sum Y^*(i)$. The fact that $\vec{Y}(\pi^*)$ is the most efficient scheme now follows immediately from the definition of π^* as the permutation that minimizes the total collateral. **QED**

PROOF OF THEOREM 2

Let $g(NR)$ be the function that specifies the level of collateral that solves $h^y(i, y_i, \vec{X}, NR) = 1$, as a function of the number of players that stay NR, where h^y stands for the symmetric problems with n investors with one dollar each and $n = \sum_i x_i$. Since h is concave g is a convex function (see figure XX). Consider the inductive algorithim described in the first part of this section, and let π be the optimal order according to which the scheme is derived. Let k denote the horizontal section of g, i.e., the number of agents in the symmetric problem that need to be fully insured. If $x_1 > k$, agent 1 has a sufficiently large investment so that he need not be fully insured and therefore all subsequent players are not fully insured either including the agent with the maximal investment. Suppose now that $x_1 < k$; then some agents may be fully insured. Let $i_{\max}$ denote the agent with the largest investment, and assume by way of contradiction that $i_{\max}$ is one of the agents who are fully insured. Let j be the first player in the order π who is not fully insured. Such a player always exists since the last agent in the algorithm is never fully insured as he receives zero collateral. Clearly j appears after $i_{\max}$ in the order π because all agents who are fully insured appear before all the agents who are partially insured. Consider now an alternative order denoted π' in which the positions of j and $i_{\max}$ are swapped so that now j appears before $i_{\max}$. We first note that the agents who precede $i_{\max}$ in π are receiving the same full collateral in both π and π'. Likewise, the agents who appear after j in π receive the same (partial) collateral in π and π'. So it is enough to compare the collateral of those appearing between $i_{\max}$ and j under the order π (including $i_{\max}$ and j) when moving from π to π'. Consider the collateral offered to players between $i_{\max}$ and j in the order π. By the definition of j these players are all fully insured under the scheme that corresponds to the order π, and their collateral under π' is at most 1. On the other hand the total collateral offered to $i_{\max}$ and j under π is strictly greater than their total collateral under π'. This is because the per-dollar collateral to an agent equals the marginal collateral of his last dollar of investment. Hence if y is the per-dollar collateral of j in π and z is the per-dollar collateral of $i_{\max}$ in π' we must have $z < y$. Now the total collateral paid to these two players under π is given by $yx_j + x_{i_{\max}}$ and under π' it is $zx_{i_{\max}} + x_j$, which is smaller since $x_{i_{\max}} > x_j$. This point is a contradiction to the assumption that π is the optimal order and completes the proof. **QED**

PROOF OF THEOREM 2

Consider a symmetric problem with $n = X$ agents and with one dollar investment for each agent. Let $h(NR) = \tilde{R} \;\; X - \frac{X-NR}{1-\delta} \;\; \frac{1}{NR} = \frac{1}{1-\delta}(1-\delta\frac{X}{NR})$ be the residual payment for an agent in this problem, where NR is the set of agents who stay. Since δ and X and $\tilde{R}$ are constants h is an increasing and concave function in NR. Consider again the inductive procedure for deriving the optimal collateral scheme as described earlier in this section and let π be the corresponding optimal order. The per-dollarcollateral $\frac{y_i}{x_i}$ offered to an agent in the optimal scheme must satisfy $\frac{y_i}{x_i} + h(X^{NR}) = D$. Hence, as a function of X^{NR} the per-dollar collateral is a declining and convex function. We shall denote this function by g (see figure XX).) Suppose by way of contradiction that the statement of the result is false. Then there must exist three agents i, j, and k such that

k is the first agent to receive partial collateral, $\pi(k) < \pi(i) = \pi(j) - 1$ (i.e., i and j adjacent and appear after k in the optimal order of the inductive scheme) and furthermore i is investing x, j is investing y, and $y > x$. Let x^* be the total investment over all agents preceding agent i in the optimal order. We denote this set of agents Q. We know that by iterative elimination of dominated strategies all the agents in Q stay even when $N \setminus Q$ run. We now argue that by reversing the order of i and j the bank can save on the collateral while maintaining a unique equilibrium in which all players stay. We first recall from the proof of Proposition 1 that whenever an agent i is placed in the order after a set of agents Q when the total investment for the agents in Q is k, and the investment of i is $x_{i,}$ the optimal per-dollarcollateral for agent i is $g(x^* + x_i)$, which equals the collateral we would have offered to the last among $x^* + x_i$ agents each with an investment of \$1 only. Hence, in the given order of i and j the total collateral for i and j is given by $xg(x^* + x) + yg(x^* + x + y)$. If we reverse the order of the two agents the total collateral will be $yg(x^* + y) + xg(x^* + y + x)$. Since $\pi(k) < \pi(i) < \pi(j)$ both points $x^* + x$ and $x^* + x + y$ are in a region in which g is declining and convex. We note that by swapping the positions of i and j and keeping the rest of the order intact we will not affect the collaterals of the rest of the players. Hence we have to show that $yg(x^* + y) + xg(x^* + y + x) < xg(x^* + x) + yg(x^* + x + y)$. which is equivalent to $\frac{g(x^*+x)-g(x^*+x+y)}{y} > \frac{g(x^*+y)-g(x^*+x+y)}{x}$. The nominator of the LHS equals $(1)\ \sum_{i=x^*+x+1}^{x^*+x+y}(g(i+1) - g(i))$, where the LHS is $(2)_{i=x^*+y+1}^{x^*+x+y}(g(i+1) - g(i))$ Since $y > x$, the (1) contains more terms than (2). By the convexity of g every term in the (1) which is not in (2) is smaller than every term in (2). Hence the average of the terms in (1) is greater than that of (2), which implies the desired inequality.

Suppose now that the bank remains solvent even when all agents but the one with the smallest investment runs. Then regardless of the identity of the agent who appears first in the optimal order of the collateral scheme this agent is not fully insured. Furthermore, by the arguments provided in the first part of the proof all the remaining players receive a per-dollar collateral that is less than that of the first agent. Hence all these agents are also not fully insured, which establishes the result. **QED**

PROOF OF ERREUR ! SOURCE DU RENVOI INTROUVABLE.

Since both functions are continuously differentiable the integral is identical to the integral of the derivative of the integrand. Hence the sign of the second derivative of $y_\theta(x)$ is positive (negative) for every value of θ, so this must also be the case for $E_\theta(y_\theta(x))$. If the random variable is discrete the result follows directly from the fact that the sum of two convex (concave) functions is concave (convex) and from the fact that multiplication by a positive number preserves convexity(concavity).