Exponential utility maximization in an incomplete market with defaults

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Abstract

In this paper, we study the exponential utility maximization problem in an incomplete market with a default time inducing a discontinuity in the price of stock. We first consider the case of strategies valued in a *compact* set. Using a verification theorem, we prove that the value function associated with the optimization problem can be characterized as *the solution of a Lipschitz BSDE* (backward stochastic differential equation). Then, we consider the case of non constrained strategies. Using dynamic programming techniques, we prove that the value function is the *maximal solution of a BSDE*. Moreover, the value function is the *limit of a sequence of processes* which are the solutions of Lipschitz BSDEs. These properties can be generalized to the case of several default times or a Poisson process.

Keywords Optimal investment, exponential utility, default time, incomplete market, dynamic programming, backward stochastic differential equation.

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1 Introduction

In this paper, we study the exponential utility maximization problem in an incomplete market with a default time inducing a discontinuity in the price of stock.

Recall that concerning the study of the utility maximization problem from terminal wealth, there exist two possible approaches:

– The first one is the *dual approach* formulated in a *static* way. This dual approach has been largely studied in the literature. Among them, in a Brownian framework, we quote Karatzas *et al.* [17] in a complete market and Karatzas *et al.* [18] in an incomplete market. In the case of general semimartingales, we quote Kramkov and Schachermayer [21], Shachermayer [33] and Delbaen *et al.* [6] for the particular case of the exponential utility function. For the case with a default in a markovian setting we refer to Lukas [24]. Using this approach, these different authors solve the utility maximization problem in the sense of finding the optimal strategy and also give a characterization of the optimal strategy via the solution of the dual problem.

– The second approach is the *direct* study of the primal problem(s) by using stochastic control tools such as *dynamic programming*. Recall that these techniques had been used in finance but only in a markovian setting for a long time. For example the reference paper of Merton [25] uses the well known Hamilton-Jacobi-Bellman verification theorem to solve the utility maximization problem of consumption/wealth in a complete market. The use in finance of stochastic dynamic techniques (presented in El Karoui's course [10] in a general setting) is more recent. One of the first work in finance using these techniques is that of El Karoui and Quenez [11]. Also, recall that the backward stochastic differential equations (BSDEs) have been introduced by Duffie and Epstein [8] in the case of recursive utilities and by Peng [29] for a general Lipschitz coefficient. In the paper of El Karoui *et al.* [12], several applications to finance are provided. One of the achievement of the paper is a *verification* theorem which allows to characterize the dynamic value function of an optimization problem as the solution of a Lipschitz BSDE. This principle has many applications in finance. One of them can be found in Rouge and El Karoui [31] who study the exponential utility maximization problem in the incomplete Brownian case and characterize the dynamic indifference price as the solution of a quadratic BSDE (introduced by Kobylanski [20]). Concerning the exponential utility maximization problem, there is also the work of Hu *et al.* [16] still in the Brownian case. By using a *verification* theorem (different from the previous one), they characterize the logarithm of the dynamic value function as the solution of a quadratic BSDE.

Due to the presence of jumps, the case of a discontinuous framework is much more involved. Concerning the study of the exponential utility maximization problem in this case, we refer to Morlais [26]. In that paper, the price process of stock is driven by an independent Brownian motion and a Poisson point process. The author mainly studies the case of admissible strategies valued in a compact set (not necessarily convex). Using the same approach as in Hu *et al.* [16], she proves that the logarithm of the associated value function is the unique solution of a quadratic BSDE (for which she shows an existence and a uniqueness result). In the non constrained case, she formally obtains that the logarithm of the value function should be a solution of a quadratic BSDE. Concerning this BSDE, she only obtains an existence result but none uniqueness result. Hence, this does not allow to characterize the value function in terms of BSDEs.

In this paper, we first consider the case of strategies valued in a *compact* set. Using a verification theorem, we show quite easily that the value function associated with the exponential utility maximization problem can be characterized as the solution of a *Lipschitz BSDE*. Secondly, we consider the case of *non constrained* strategies. Using dynamic programming techniques, the value function is proved to be the *maximal solution* of a BSDE. Moreover, we provide another characterization of the value function as the *nonincreasing limit of a sequence of processes* which are the solutions of Lipschitz BSDEs. At last, we give some generalizations of the previous results.

The outline of the paper is as follows. In Section 2, we present the market model and the maximization problem in the case of only one risky asset. In Section 3, we study the case of strategies valued in a *compact* set. In Section 4, we consider the *non constrained case*. We first provide a characterization of the value function as the nonincreasing limit of a sequence of processes which are the solutions of Lipschitz BSDEs. Second, the value function is proved to be characterized as the *maximal solution* of a BSDE. In Section 5, we show that some of the previous results still hold in the case of *unbounded coefficients*. Then, we consider the case of coefficients satisfying some *exponential integrability* conditions. In the last section, we generalize the previous results to the case of several assets and several default times and we also extend these results to a Poisson jump model.

2 The market model

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that all processes are defined on a finite time horizon $[0, T]$, with $T < \infty$. Suppose that this space is equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbb{1}_{\tau \leq t}$ for any $t \in [0, T]$, where τ is a random variable which modelizes a default time (see Section 6.1 for several default times). We assume that this default can appear at any time that is $\mathbb{P}(\tau > t) > 0$ for any $t \in [0, T]$. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes. The filtration is supposed to be right-continuous and W is a $\mathbb{G}$-Brownian motion.

We denote by M the compensated martingale of the process N and by Λ its compensator. We assume that this compensator Λ is absolutely continuous with respect to Lebesgue's measure, so that there exists a process λ such that $\Lambda_t = \int_0^t \lambda_s ds$. Hence, the $\mathbb{G}$-martingale M satisfies

$$M_t = N_t - \int_0^t \lambda_s ds \,. \tag{2.1}$$

We introduce the following sets which are used throughout the sequel:

– $\mathcal{S}^{+,\infty}$ is the set of positive $\mathbb{G}$-adapted $\mathbb{P}$-essentially bounded càd-làg processes.

– $\mathcal{S}^2$ is the set of $\mathbb{G}$-adapted càd-làg processes φ such that $\mathbb{E}[\sup_t |\varphi_t|^2] < +\infty$.

– $L^{1,+}$ is the set of positive $\mathbb{G}$-adapted càd-làg processes such that $\mathbb{E}[Y_t] < \infty$ for any $t \in [0,T]$.

– $L^2(W)$ (resp. $L^2_{loc}(W)$) is the set of $\mathbb{G}$-predictable processes with

$$\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty \quad (\text{resp. } \int_0^T |Z_t|^2 dt < \infty \ a.s. \ ).$$

– $L^2(M)$ (resp. $L^2_{loc}(M)$, $L^1_{loc}(M)$) is the set of $\mathbb{G}$-predictable processes such that

$$\mathbb{E}\Big[\int_0^T \lambda_t |U_t|^2 dt\Big] < \infty \quad (\text{resp. } \int_0^T \lambda_t |U_t|^2 dt < \infty\,, \int_0^T \lambda_t |U_t| dt < \infty \ a.s. \ ).$$

We recall the useful martingale representation theorem (see for example Jeanblanc *et al.* [14]) which is paramount in the sequel:

Lemma 2.1. *Any $(\mathbb{P}, \mathbb{G})$-local martingale m has the representation*

$$m_t = m_0 + \int_0^t a_s dW_s + \int_0^t b_s dM_s, \ \forall\, t \in [0,T] \ a.s.\,, \tag{2.2}$$

where $a \in L^2_{loc}(W)$ and $b \in L^1_{loc}(M)$. If m is a square integrable martingale, each term on the right-hand side of the representation (2.2) is square integrable.

We now consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at any date, and one risky asset with price process S which admits a discontinuity at time τ (we give the results for n assets and p default times in Section 6.1). Throughout the sequel, we consider that the price process S evolves according to the equation

$$dS_t = S_{t^-}(\mu_t dt + \sigma_t dW_t + \beta_t dN_t)\,, \tag{2.3}$$

with the classical assumptions:

Assumption 2.1.

(i) λ, μ, σ and β are uniformly bounded $\mathbb{G}$-predictable processes,

(ii) $\sigma_t > 0$ for any $0 \le t \le T$,

(iii) the process β satisfies $\beta_\tau > -1$ a.s.

Note that condition (iii) ensures that the process S is positive. Also, it is equivalent to fact that the process β satisfies $\beta_t > -1$ for any $0 \le t \le T$ a.s. (see Jeanblanc *et al.* [15]).

We also suppose that $\mathbb{E}[\exp(-\int_0^T \alpha_s dW_s - \frac{1}{2}\int_0^T \alpha_t^2 dt)] = 1$ where $\alpha_t = (\mu_t + \lambda_t\beta_t)/\sigma_t$, which ensures the existence of a martingale probability measure and hence the absence of arbitrage.

Throughout the sequel, a process π is called a trading strategy if it is a $\mathbb{G}$-predictable process and if $\int_0^T \frac{\pi_t}{S_{t^-}} dS_t$ is well defined e.g. $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t |\pi_t\beta_t|^2 dt < \infty$ a.s. In this case, π_t describes the amount of money invested in the risky asset at time t. Under the assumption that the trading strategy is self-financing, the wealth process $X^{x,\pi}$ associated with the trading strategy π and an initial capital x satisfies

$$\begin{cases} dX_t^{x,\pi} = \pi_t\big(\mu_t dt + \sigma_t dW_t + \beta_t dN_t\big)\,, \\ X_0^{x,\pi} = x\,. \end{cases} \tag{2.4}$$

For a given initial time t and an initial capital x, the wealth process associated with a trading strategy π is denoted by $X^{t,x,\pi}$.

We assume that the investor in this financial market faces some liability, which is modeled by a random variable ξ (for example, ξ may be a contingent claim written on a default event, which itself affects the price of the underlying asset). We suppose that $\xi \in L^2(\mathcal{G}_T)$ and is non-negative (note that all the results still hold under the assumption that ξ is only bounded from below).

Our aim is to study the classical optimization problem

$$V(x,\xi) = \sup_{\pi\in\mathcal{D}} \mathbb{E}\big[U(X_T^{x,\pi} + \xi)\big]\,, \tag{2.5}$$

where $\mathcal{D}$ is a set of admissible strategies (independent of x) which will be specified throughout the sequel and U is the exponential utility function

$$U(x) = -\exp(-\gamma x),\ \ x \in \mathbb{R}\,,$$

where $\gamma > 0$ is a given constant, which can be seen as a coefficient of absolute risk aversion. The optimization problem (2.5) can be clearly written as

$$V(x,\xi) = e^{-\gamma x} V(0,\xi)\,.$$

Hence, it is sufficient to study the case $x = 0$. To simplify notation, we will denote X^π (resp. $X^{t,\pi}$) instead of $X^{0,\pi}$ (resp. $X^{t,0,\pi}$). Also, note that

$$V(0,\xi) = -\inf_{\pi\in\mathcal{D}} \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi + \xi)\big)\big]\,. \tag{2.6}$$

We stress on that some of the results stated below still hold in the case of unbounded coefficients (see Section 5).

3 Strategies valued in a given compact set

In this section, we study the case where the strategies are constrained to take their values in a given non empty compact set C of $\mathbb{R}$. Thus, the set of admissible strategies denoted by $\mathcal{C}$ is defined as the set of predictable processes π taking their values in C.

This case cannot be solved by using the dual approach because the set of admissible strategies is not necessarily convex. In this context, we address the problem of characterizing dynamically the value function associated with the exponential utility maximization problem. We give a dynamic extension of the initial problem (2.6) (with $\mathcal{D} = \mathcal{C}$). For any initial time $0 \leq t \leq T$, we define the value function $J(t, \xi)$ (also denoted by $J(t)$) by the following random variable

$$J(t,\xi) = \operatorname*{ess\,inf}_{\pi \in \mathcal{C}_t} \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big]\,, \tag{3.1}$$

where $\mathcal{C}_t$ is the set of all restrictions to $[t, T]$ of the strategies of $\mathcal{C}$. Note that $V(0, \xi) = -J(0, \xi)$.

Throughout the sequel, we want to characterize this dynamic value function $J(.)$ as the solution of a BSDE.

For that, for any $\pi \in \mathcal{C}$, we introduce the càd-làg process J^π satisfying

$$J_t^\pi = \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big],\ \forall\, t \in [0, T]\,.$$

Since the coefficients are supposed to be bounded and the strategies are constrained to take their values in a compact set, it is possible to solve very simply the problem by using a *verification* principle in terms of Lipschitz BSDEs in the vein of that of El Karoui *et al.* [12].

Note first that for any $\pi \in \mathcal{C}$, by using classical techniques of linear BSDEs (see El Karoui *et al.* [12]), the process J^π can be easily shown to be the solution of a linear Lipschitz BSDE. More precisely, there exist $Z^\pi \in L^2(W)$ and $U^\pi \in L^2(M)$, such that (J^π, Z^π, U^π) is the unique solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the linear BSDE with bounded coefficients

$$-\,dJ_t^\pi = f^\pi(t, J_t^\pi, Z_t^\pi, U_t^\pi)dt - Z_t^\pi dW_t - U_t^\pi dM_t\ ;\ J_T^\pi = \exp(-\gamma\xi)\,, \tag{3.2}$$

where $f^\pi(s, y, z, u) = \frac{\gamma^2}{2}\pi_s^2\sigma_s^2 y - \gamma\pi_s(\mu_s y + \sigma_s z) - \lambda_s(1 - e^{-\gamma\pi_s\beta_s})(y + u)$.

Using the fact that $J(t) = \operatorname{ess\,inf}_{\pi\in\mathcal{C}_t} J_t^\pi$ for any $0 \leq t \leq T$, we state that $J(.)$ corresponds to the solution of a BSDE, whose driver is the essential infimum over π of the drivers of $(J^\pi)_{\pi\in\mathcal{C}}$. More precisely,

Proposition 3.1. *The following properties hold:*

- *Let (Y, Z, U) be the solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the following BSDE*

$$\begin{cases} -\,dY_t = \operatorname*{ess\,inf}_{\pi\in\mathcal{C}} \Big\{\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2 Y_t - \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(Y_t + U_t)\Big\}dt \\ \qquad\qquad -\ Z_t dW_t - U_t dM_t\,, \\ \quad Y_T = \exp(-\gamma\xi)\,. \end{cases} \tag{3.3}$$

  Then, $J(t) = Y_t$ for any $0 \leq t \leq T$ a.s.

- *There exists an optimal strategy for $J(0) = \inf_{\pi\in\mathcal{C}} \mathbb{E}[\exp(-\gamma(X_T^\pi + \xi))]$.*
  Also, a strategy $\hat{\pi} \in \mathcal{C}$ is optimal for J_0 if and only if $\hat{\pi}_t$ attains the essential infimum in (3.3) $dt \otimes d\mathbb{P} - a.e.$

Proof. Let us introduce the driver f which satisfies $ds \otimes d\mathbb{P} - a.e.$

$$f(s,y,z,u) = \operatorname*{ess\,inf}_{\pi\in\mathcal{C}} f^{\pi}(s,y,z,u)\,.$$

Since the driver f is written as an infimum of linear drivers f^π w.r.t (y,z,u) with uniformly bounded coefficients, f is clearly Lipschitz (see Lemma B.1 in Appendix B). Hence, by Tang and Li's results [34], BSDE (3.3) with Lipschitz driver f

$$-\,dY_t = f(t,Y_t,Z_t,U_t)dt - Z_t dW_t - U_t dM_t \ ;\ Y_T = \exp(-\gamma\xi)$$

admits a unique solution $(Y,Z,U) \in \mathcal{S}^2 \times L^2(W) \times L^2(M)$.
Since we have

$$f^{\pi}(t,y,z,u) - f^{\pi}(t,y,z,u') = \lambda_t(u-u')\gamma_t\,, \tag{3.4}$$

with $\gamma_t = e^{-\gamma\pi_t\beta_t} - 1$, and since there exist some constants $-1 < C_1 \le 0$ and $0 \le C_2$ such that $C_1 \le \gamma_t \le C_2$, the comparison theorem in case of jumps (see for example Theorem 2.5 in Royer [32]) can be applied. It implies that $Y_t \le J_t^\pi$, $\forall\, t \in [0,T]$ a.s. As this inequality is satisfied for any $\pi \in \mathcal{C}$, it follows that $Y_t \le \operatorname{ess\,inf}_{\pi\in\mathcal{C}} J_t^\pi$ a.s.
Also, by applying a measurable selection theorem, one can easily show that there exists $\hat\pi \in \mathcal{C}$ such that $dt \otimes d\mathbb{P}$-a.s.

$$\begin{aligned}\operatorname*{ess\,inf}_{\pi\in\mathcal{C}} &\Big\{ \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 Y_t - \gamma\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t(1-e^{-\gamma\pi_t\beta_t})(Y_t+U_t)\Big\}\\ &= \frac{\gamma^2}{2}\hat\pi_t^2\sigma_t^2 Y_t - \gamma\hat\pi_t(\mu_t Y_t + \sigma_t Z_t) - \lambda_t(1-e^{-\gamma\hat\pi_t\beta_t})(Y_t+U_t)\,.\end{aligned}$$

Thus, (Y,Z,U) is a solution of BSDE (3.2) associated with $\hat\pi$. Therefore, by uniqueness of the solution of BSDE (3.2), we have $Y_t = J_t^{\hat\pi}$, $0 \le t \le T$ a.s. Hence, $Y_t = \operatorname{ess\,inf}_{\pi\in\mathcal{C}_t} J_t^\pi = J_t^{\hat\pi}$, $0 \le t \le T$ a.s., and $\hat\pi$ is an optimal strategy.
Note that $Y = J \in \mathcal{S}^{+,\infty}$. □

Remark 3.1. Let us make the following change of variables: $y_t = \frac{1}{\gamma}\log(Y_t)\,,\ z_t = \frac{1}{\gamma}\frac{Z_t}{Y_t}\,,$ $u_t = \frac{1}{\gamma}\log\left(1+\frac{U_t}{Y_{t^-}}\right)$. One can easily verify that the process (y,z,u) is the solution of the following quadratic BSDE

$$-\,dy_t = g(t,z_t,u_t)dt - z_t dW_t - u_t dM_t \ ;\ y_T = -\xi\,, \tag{3.5}$$

where

$$g(s,z,u) = \operatorname*{ess\,inf}_{\pi\in\mathcal{C}}\left(\frac{\gamma}{2}\left|\pi_s\sigma_s - \left(z+\frac{\mu_s+\lambda_s\beta_s}{\gamma}\right)\right|^2 + |u-\pi_s\beta_s|_\gamma\right) - (\mu_s+\lambda_s\beta_s)z - \frac{|\mu_s+\lambda_s\beta_s|^2}{2\gamma}\,,$$

with $|u-\pi\beta_t|_\gamma = \lambda_t\frac{\exp(\gamma(u-\pi\beta_t))-1-\gamma(u-\pi\beta_t)}{\gamma}$. Hence, our result clearly yields the existence and the uniqueness of the quadratic BSDE (3.5) and also gives that the logarithm of the value function is the solution of this BSDE. This corresponds exactly to Morlais's result [26]. Recall that the proof given in [26] consists in showing first an existence and uniqueness result for BSDE (3.5) by using a sophisticated approximation method in the vein of Kobylanski [20] but adapted to the case of jumps. Then, by using a verification theorem quite similar to Hu *et al.*' theorem [16], the logarithm of the value function is proved to be the solution of the quadratic BSDE (3.5).

Note that the short proof given here is based on a simple verification principle (in the vein of El Karoui *et al.* [12]).

4 The non constrained case

We now study the value function in the case where the admissible strategies are no longer required to satisfy any constraints (as in the previous section). Since the utility function is the exponential utility function, the set of admissible strategies is not standard in the literature. The next subsection studies the choice of a suitable set of admissible strategies which will allow to dynamize the problem and to characterize the associated dynamic value function.

4.1 The set of admissible strategies

Recall that in the case of the power or logarithmic utility functions defined (or restricted) on $\mathbb{R}_+$, the admissible strategies are the ones that make the associated wealth positive. Since we consider the exponential utility function which is finitely valued for all $x \in \mathbb{R}$, the wealth process is no longer required to be positive. However, from a financial point of view, it is natural to consider strategies such that the associated wealth process is uniformly bounded by below (see for example Schachermayer [33]) or even such that any increment of the wealth is bounded by below.

More precisely, we introduce the set of admissible trading strategies $\mathcal{A}$ which consists of all $\mathbb{G}$-predictable processes π which satisfy $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t |\pi_t\beta_t|^2 dt < \infty$ a.s. and such that for any fixed π and any $s \in [0,T]$, there exists a real constant $K_{s,\pi}$ such that

$$X_t^\pi - X_s^\pi \geq -K_{s,\pi}, \quad s \leq t \leq T, \quad a.s.$$

Recall that in their paper, Delbaen *et al.* [6] consider the set of strategies Θ_2 defined by

$$\Theta_2 := \Big\{\pi\,,\, \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big] < +\infty \text{ and } X^\pi \text{ is a } \mathbb{Q}-\text{martingale for all } \mathbb{Q}\in\mathbb{P}_f\Big\},$$

where $\mathbb{P}_f$ is the set of absolutely continuous local martingale measures $\mathbb{Q}$ such that its entropy $H(\mathbb{P}|\mathbb{Q})$ is finite.

In general, there is no existence result on the set $\mathcal{A}$ whereas there is one on the set Θ_2. Recall that this existence result has been stated by [6]. More precisely, under the assumption that the price process is locally bounded, using the dual approach, these authors show the existence of an optimal strategy on the set Θ_2.

We stress on the following important point: the value function associated with Θ_2 coincides with that associated with $\mathcal{A}$. More precisely,

Lemma 4.1. *The value function $V(0,\xi)$ associated with $\mathcal{A}$ defined by*

$$V(0,\xi) = -\inf_{\pi\in\mathcal{A}} \mathbb{E}\big[\exp\big(-\gamma(X_T^\pi+\xi)\big)\big]\,, \tag{4.1}$$

is equal to the one associated with Θ_2*.*

Proof. This property follows from the results of Delbaen *et al.* [6]. More precisely, let $V^2(0,\xi)$ be the value function associated with Θ_2. Let us introduce Θ_3 the set of strategies such that the associated wealth process is bounded and let $V^3(0,\xi)$ be the value function associated with Θ_3. By the results of [6], $V^2(0,\xi) = V^3(0,\xi)$. Now, since $\Theta_3 \subset \mathcal{A}$, we have $V(0,\xi) \geq V^3(0,\xi)$. Now, by a classical localization argument (quite similar to the one used in Appendix C), one can easily show that $V(0,\xi) = V^3(0,\xi)$. Hence, $V(0,\xi) = V^2(0,\xi)$. $\square$

In this work, our aim is mainly to characterize and even to approximate the value function $V(0,\xi)$. Our approach consists in giving a dynamic extension of the optimization problem and in using stochastic calculus techniques in order to characterize the dynamic value function. In the compact case (with the set $\mathcal{C}$), the dynamic extension was easy (see Section 3). At any initial time t, the corresponding set $\mathcal{C}_t$ of admissible strategies was simply given by the set of the restrictions to $[t,T]$ of the strategies of $\mathcal{C}$. For the set $\mathcal{A}$, it is also very simple (see below). However, in the case of the set Θ_2, things are not so clear. Actually, this is partly linked to the fact that the set $\mathcal{A}$ is closed by *binding* whereas Θ_2 is not. More precisely, one can easily verify that

Lemma 4.2. *The set* $\mathcal{A}$ *is* closed by *binding that is: if* π^1, π^2 *are two strategies of* $\mathcal{A}$ *and if* $s \in [0,T]$*, then the strategy* π^3 *defined by*

$$\pi^3_t = \begin{cases} \pi^1_t & \textit{if } t \leq s\,, \\ \pi^2_t & \textit{if } t > s\,, \end{cases}$$

belongs to $\mathcal{A}$*.*

On the other hand, the set Θ_2 is clearly *not closed by binding* because of the integrability condition $\mathbb{E}[\exp(-\gamma(X^\pi_T + \xi))] < +\infty$. One could naturally think of considering the space $\Theta_2^{'} := \{\pi\,,\ \ X^\pi \text{ is a } \mathbb{Q} - \text{martingale for all } \mathbb{Q} \in \mathbb{P}_f\}$ (instead of Θ_2) but this set is not really appropriate here: in particular it does not allow to have the dynamic programming principle (in the form of Proposition 4.1 below) because in this case, the Lebesgue theorem cannot be applied (see Remark 4.2).
However, there are some other possible sets which are closed by binding as for example the set $\mathcal{A}^{'}$ defined as the set of $\mathbb{G}$-predictable processes π with $\int_0^T |\pi_t\sigma_t|^2 dt + \int_0^T \lambda_t |\pi_t\beta_t|^2 dt < \infty$ a.s., and such that for any $t \in [0,T]$ and for any $p > 1$, the following integrability condition

$$\mathbb{E}\Big[\sup_{s\in[t,T]} \exp\Big(-\gamma p X^{t,\pi}_s\Big)\Big] < \infty \tag{4.2}$$

holds. Note that $\mathcal{A} \subset \mathcal{A}^{'} \subset \Theta_2$.

The property of closedness by binding of the set $\mathcal{A}^{'}$ can be easily verified by using the assumption of p-integrability (4.2) and Cauchy-Schwarz inequality (see Appendix D for details). Note that the weaker integrability condition $\mathbb{E}[\exp(-\gamma X^\pi_T)] < +\infty$ is not sufficient to ensure this property.

Remark 4.1. Note first that such a p-exponential integrability condition is not so surprising here. Indeed, it is well-known that the exponential utility maximization problem is related to quadratic BSDEs (see for example Rouge and El Karoui [31]) and that this type of p-exponential integrability condition often appears to solve quadratic BSDEs (see for example Briand and Hu [5]).

Also, note that in the particular case where there is no default, that is in the case of a complete market, the optimal strategy belongs to the set $\mathcal{A}^{'}$ (but of course not to $\mathcal{A}$). Indeed, the optimal terminal wealth is given by $\hat{X}_T = I(\lambda Z_0(T))$, where I is the inverse of $U^{'}$, λ is a fixed parameter, $Z_0(T) := \exp\{-\int_0^T \alpha_t dW_t - \frac{1}{2}\int_0^T \alpha_t^2 dt\}$ and $\alpha_t := \frac{\mu_t + \lambda_t \beta_t}{\sigma_t}$ (supposed to be bounded). However, in the general case , there is no existence result for the set $\mathcal{A}^{'}$.

Let us now give a *dynamic extension* of the initial problem associated with $\mathcal{A}$ given by (4.1). For any initial time $t \in [0,T]$, we define the value function $J(t,\xi)$ by the following random variable

$$J(t,\xi) = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}_t} \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big]\,, \tag{4.3}$$

where the set $\mathcal{A}_t$ is the set of the restrictions to $[t,T]$ of the strategies of $\mathcal{A}$.
Note that $J(0,\xi) = -V(0,\xi)$. Also, for any $t \in [0,T]$, $J(t,\xi)$ is also equal a.s. to the essinf in (4.3) but taken over $\mathcal{A}$ instead of $\mathcal{A}_t$.
For the sake of brevity, we shall denote $J(t)$ instead of $J(t,\xi)$. Note that the random variable $J(t)$ is defined uniquely only up to $\mathbb{P}$-almost sure equivalent. The process $J(.)$ will be called the *dynamic value function*. This process is adapted but not necessarily càd-làg and not even progressive.

Similarly, a dynamic extension of the value function associated with $\mathcal{A}'$ can be easily given. By using a localization argument, one can easily verify (see Appendix C) that

Lemma 4.3. *The dynamic value function $J(.)$ associated with $\mathcal{A}$ coincides a.s. with the one associated with $\mathcal{A}'$.*

Hence, concerning the dynamic study of the value function, it is equivalent to choose $\mathcal{A}$ or $\mathcal{A}'$ as set of admissible strategies. We have chosen the set $\mathcal{A}$ because it appears as a natural set of admissible strategies from a financial point of view. However, all the results in this paper still hold with $\mathcal{A}'$ instead of $\mathcal{A}$.

After this dynamic extension of the value function, the aim is now to characterize the dynamic value function via a BSDE.

4.2 First properties of the dynamic value function

In this section, we will provide a first characterization of the dynamic value function via a BSDE. Note that it is no longer possible to use a verification theorem like the one in Section 3 because the associated BSDE is no longer Lipschitz and there is no existence result for it. One could think to use a verification theorem like that of Hu *et al.* [16]. But because of the presence of jumps, it is no longer possible since again there is no existence and uniqueness results for the associated BSDE as noted by Morlais [26]. In her paper, Morlais proves the existence of a solution of this BSDE by using an approximation method but she does not obtain uniqueness result. Hence, this does not a priori lead to a characterization of the value function in terms of BSDEs.

Therefore, as it seems not possible to derive a *sufficient condition* so that a given process corresponds to the dynamic value function, we will now provide some *necessary* conditions satisfied by $J(.)$ and more precisely a dynamic programming principle. Then, using this property, we will derive a first characterization of the value function via a BSDE.

We first prove the following dynamic programming principle:

Proposition 4.1. *For each $\pi \in \mathcal{A}$, the process $\exp(-\gamma X^{\pi})J(.)$ is a submartingale.*

To prove this proposition, we introduce the random variable J_t^{π} which is defined for any $\pi \in \mathcal{A}_t$ by

$$J_t^{\pi} = \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big]\,.$$

As usual, in order to prove Proposition 4.1, we first prove the following lemma:

Lemma 4.4. *The set $\{J_t^{\pi},\ \pi \in \mathcal{A}_t\}$ is* stable by pairwise minimization *for any $t \in [0,T]$. That is, for every $\pi^1,\ \pi^2 \in \mathcal{A}_t$ there exists $\pi \in \mathcal{A}_t$ such that $J_t^{\pi} = J_t^{\pi^1} \wedge J_t^{\pi^2}$.*
Also, there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{A}_t$ such that

$$J(t) = \lim_{n\to\infty} \downarrow J_t^{\pi^n} \ \textit{a.s.}$$

Proof. Fix $t \in [0,T]$. Let us introduce the set $E = \{J_t^{\pi^1} \leq J_t^{\pi^2}\}$ which belongs to $\mathcal{G}_t$. Let us define π for any $s \in [t,T]$ by $\pi_s = \pi_s^1 \mathbb{1}_E + \pi_s^2 \mathbb{1}_{E^c}$. It is clear that $\pi \in \mathcal{A}_t$ because $X_s^{t,\pi} = X_s^{t,\pi^1}\mathbb{1}_E + X_s^{t,\pi^2}\mathbb{1}_{E^c}$ and the sum of two random variables bounded by below is bounded by below. By construction of π, it is clear that $J_t^{\pi} = J_t^{\pi^1} \wedge J_t^{\pi^2}$ a.s.
The second part of the lemma follows by classical results on the essential infimum (see Appendix A). □

Let us now give the proof of Proposition 4.1.

Proof. Let us show that for $t \geq s$,

$$\mathbb{E}\big[\exp\big(-\gamma(X_t^{\pi} - X_s^{\pi})\big)J(t)\big|\mathcal{G}_s\big] \geq J(s) \ \ \textit{a.s.}$$

Note that $X_t^{\pi} - X_s^{\pi} = X_t^{s,\pi}$. By Lemma 4.4, there exists a sequence $(\pi^n)_{n\in\mathbb{N}} \in \mathcal{A}_t$ such that $J(t) = \lim\limits_{n\to\infty} \downarrow J_t^{\pi^n}$ a.s.
Without loss of generality, we can suppose that $\pi^0 = 0$. For each $n \in \mathbb{N}$, we have $J_t^{\pi^n} \leq J_t^{\pi^0} \leq 1$ a.s. Moreover, the integrability property $\mathbb{E}[\exp(-\gamma X_t^{s,\pi})] < \infty$ holds because $\pi \in \mathcal{A}$. Together with the Lebesgue theorem, it gives

$$\mathbb{E}\big[\lim_{n\to\infty} \exp(-\gamma X_t^{s,\pi})J_t^{\pi^n}\big|\mathcal{G}_s\big] = \lim_{n\to\infty}\mathbb{E}\big[\exp(-\gamma X_t^{s,\pi})J_t^{\pi^n}\big|\mathcal{G}_s\big]. \tag{4.4}$$

Recall that $X_t^{s,\pi} = \int_s^t \frac{\pi_u}{S_{u^-}}dS_u$. Now, we have a.s.

$$\exp\Big(-\gamma\int_s^t \frac{\pi_u}{S_{u^-}}dS_u\Big)J_t^{\pi^n} = \mathbb{E}\Big[\exp\Big(-\gamma\Big(\int_s^T \frac{\tilde{\pi}_u^n}{S_{u^-}}dS_u + \xi\Big)\Big)\Big|\mathcal{G}_t\Big], \tag{4.5}$$

where the strategy $\tilde{\pi}^n$ is defined by

$$\tilde{\pi}^n_u = \begin{cases} \pi_u & \text{if } 0 \le u \le t\,, \\ \pi^n_u & \text{if } t < u \le T\,. \end{cases}$$

Note that by the closedness property by binding (see Lemma 4.2), $\tilde{\pi}^n \in \mathcal{A}$ for each $n \in \mathbb{N}$. By (4.4) and (4.5), we have a.s.

$$\begin{aligned} \mathbb{E}\Big[\exp\Big(-\gamma\int_s^t \frac{\pi_u}{S_{u^-}}dS_u\Big)J(t)\Big|\mathcal{G}_s\Big] &= \lim_{n\to\infty}\mathbb{E}\Big[\exp\Big(-\gamma\Big(\int_s^T \frac{\tilde{\pi}^n_u}{S_{u^-}}dS_u + \xi\Big)\Big)\Big|\mathcal{G}_s\Big] \\ &= \lim_{n\to\infty} J^{\tilde{\pi}^n}_s \quad \ge \quad J(s)\,, \end{aligned}$$

from the definition of $J(s)$. Hence, the process $\exp(-\gamma X^\pi)J(.)$ is a submartingale for any $\pi \in \mathcal{A}$. $\square$

Remark 4.2. The integrability property $\mathbb{E}[\exp(-\gamma X^{s,\pi}_t)] < \infty$ is required in the proof of this property. Indeed, if it is not satisfied, equality (4.4) does not hold since the Lebesgue theorem (and the monotone convergence theorem) cannot be applied. We stress on that the importance of the integrability condition is due to the fact that we study an *essential infimum* of positive random variables. Note that in the case of an *essential supremum* of positive random variables, the dynamic programming principle holds without any integrability condition (see for example the case of the power utility function in Lim and Quenez [23]).

Consequently, the set of $\mathbb{G}$-predictable processes π such that for any $p > 1$, for any $s \in [0,T]$ and for any $t \in [s,T]$, $\mathbb{E}[\exp(-\gamma p X^{s,\pi}_t)] < \infty$, appears as the largest set of strategies which ensures the above dynamic programming principle. Note that the set $\mathcal{A}^{'}$ is nearly the same but with an integrability condition which is uniform with respect to $t \in [s,T]$ (see (4.2)). This uniform integrability in time will be useful to ensure a characterization of the value function via a BSDE (see Remark 4.6).

Also, the value function can easily be characterized as follows:

Proposition 4.2. *The process $J(.)$ is the largest $\mathbb{G}$-adapted process such that $e^{-\gamma X^\pi}J(.)$ is a submartingale for any admissible strategy $\pi \in \mathcal{A}$ with $J(T) = \exp(-\gamma\xi)$. More precisely, if $\hat{J}$ is a $\mathbb{G}$-adapted process such that $\exp(-\gamma X^\pi)\hat{J}$ is a submartingale for any $\pi \in \mathcal{A}$ with $\hat{J}_T = \exp(-\gamma\xi)$, then we have $J(t) \ge \hat{J}_t$ a.s., for any $t \in [0,T]$.*

Proof. Fix $t \in [0,T]$. For any $\pi \in \mathcal{A}$, $\mathbb{E}[\exp(-\gamma X^\pi_T)\hat{J}_T|\mathcal{G}_t] \ge \exp(-\gamma X^\pi_t)\hat{J}_t$ a.s. This implies

$$\operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t}\mathbb{E}\big[\exp\big(-\gamma(X^{t,\pi}_T + \xi)\big)\big|\mathcal{G}_t\big] \ge \hat{J}_t \ \ a.s.\,,$$

which gives clearly that $J(t) \ge \hat{J}_t$ a.s. $\square$

With this property, it is possible to show that there exists a càd-làg version of the value function $J(.)$. More precisely, we have:

Proposition 4.3. *There exists a $\mathbb{G}$-adapted càd-làg process J such that for any $t \in [0,T]$,*

$$J_t = J(t) \ a.s.$$

Moreover, the two processes are indistinguishable.

A direct proof is given in Appendix E.

Remark 4.3. Note that Proposition 4.2 can be written under the form: J is the largest $\mathbb{G}$-adapted càd-làg process such that the process $\exp(-\gamma X^\pi)J$ is a submartingale for any $\pi \in \mathcal{A}$ with $J_T = \exp(-\gamma\xi)$.

We now prove that the process J is bounded. More precisely, we have:

Lemma 4.5. *The process J verifies*

$$0 \le J_t \le 1, \ \forall\, t \in [0,T] \ a.s.$$

Proof. Fix $t \in [0,T]$. The first inequality is easy to prove, because it is obvious that

$$0 \le \mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi} + \xi)\big)\big|\mathcal{G}_t\big] \ \ a.s.\,,$$

for any $\pi \in \mathcal{A}_t$, which implies $0 \le J_t$.
The second inequality is due to the fact that the strategy defined by $\pi_s = 0$ for any $s \in [t,T]$ is admissible, which implies $J_t \le \mathbb{E}[\exp(-\gamma\xi)|\mathcal{G}_t]$ a.s. As the contingent claim ξ is supposed to be non negative, we have $J_t \le 1$ a.s. □

Remark 4.4. If ξ is only bounded by below by a real constant $-K$, then J is still upper bounded but by $\exp(\gamma K)$ instead of 1.

Using the previous characterization of the value function (see Proposition 4.2), we now prove that the value function J is characterized by a BSDE. Since we work in terms of *necessary conditions* satisfied by the value function, the study is more technical than in the cases where a verification theorem can be applied.

Since J is a càd-làg submartingale and is bounded (see Lemma 4.5), it admits a unique Doob-Meyer decomposition (see Dellacherie and Meyer [7], Chapter 7)

$$dJ_t = dm_t + dA_t\,,$$

where m is a square integrable martingale and A is an increasing $\mathbb{G}$-predictable process with $A_0 = 0$. From the martingale representation theorem (see Lemma 2.1), the previous Doob-Meyer decomposition can be written under the form

$$dJ_t = Z_t dW_t + U_t dM_t + dA_t\,, \tag{4.6}$$

with $Z \in L^2(W)$ and $U \in L^2(M)$.
Using the dynamic programming principle (Proposition 4.1), we will now precise the process A of (4.6). This will allow to show that the value function J is a subsolution of a BSDE. Let us introduce the set $\mathcal{A}^2$ of the nondecreasing adapted càd-làg processes K with $K_0 = 0$ and $\mathbb{E}|K_T|^2 < \infty$.

Proposition 4.4. – *There exists a process $K \in \mathcal{A}^2$ such that the process $(J, Z, U, K) \in \mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$ is a solution of the following BSDE*

$$\begin{cases} -dJ_t = \operatorname*{ess\,inf}_{\pi \in \mathcal{A}} \Big\{ \dfrac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t - \gamma \pi_t (\mu_t J_t + \sigma_t Z_t) - \lambda_t (1 - e^{-\gamma \pi_t \beta_t})(J_t + U_t) \Big\} dt \\ \qquad\qquad - dK_t - Z_t dW_t - U_t dM_t \,, \\ J_T = \exp(-\gamma \xi) \,. \end{cases} \tag{4.7}$$

– *Furthermore, (J, Z, U, K) is the* maximal solution *in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$ of BSDE (4.7) that is, for any solution $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ of the BSDE in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$, we have $\bar{J}_t \le J_t$, $\forall\, t \in [0, T]$ a.s.*

Remark 4.5. Due to the presence of the nondecreasing process K, the process J is said to be a *subsolution* (and even the maximal one) of the BSDE associated with the terminal condition $\exp(-\gamma\xi)$ and the driver given by the above essinf.

Proof. We prove the first point of this proposition. Applying first Itô's formula to the semimartingale $\exp(-X^\pi)J$, we obtain

$$d(e^{-\gamma X_t^\pi} J_t) = dA_t^\pi + dm_t^\pi \,,$$

with $A_0^\pi = 0$ and

$$\begin{cases} dA_t^\pi = e^{-\gamma X_t^\pi} \Big[dA_t + \Big\{ \dfrac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t - \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(U_t + J_t) - \gamma \pi_t (\sigma_t Z_t + \mu_t J_t) \Big\} dt \Big] \,, \\ dm_t^\pi = e^{-\gamma X_{t-}^\pi} \big[(Z_t - \gamma \pi_t \sigma_t J_t) dW_t + (U_t + (e^{-\gamma \pi_t \beta_t} - 1)(U_t + J_{t-})) dM_t \big] \,. \end{cases}$$

Using then the DP, we argue that $\exp(-X^\pi)J$ is a submartingale for any π which yields

$$dA_t \ge \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(U_t + J_t) + \gamma \pi_t (\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t \Big\} dt \,. \tag{4.8}$$

We then define the process K by $K_0 = 0$ and

$$dK_t = dA_t - \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(U_t + J_t) + \gamma \pi_t (\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t \Big\} dt \,.$$

It is clear that the process K is nondecreasing from (4.8). Since the strategy defined by $\pi_t = 0$ for any $t \in [0, T]$ is admissible, we have

$$\operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(U_t + J_t) + \gamma \pi_t (\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t \Big\} \ge 0 \,.$$

Hence, $0 \le K_t \le A_t$ a.s. As $\mathbb{E}|A_T|^2 < \infty$, we have $K \in \mathcal{A}^2$. Thus, the Doob-Meyer decomposition (4.6) of J can be written as follows

$$\begin{aligned} dJ_t = {} & \operatorname*{ess\,sup}_{\pi \in \mathcal{A}} \Big\{ \lambda_t \big(1 - e^{-\gamma \pi_t \beta_t}\big)(U_t + J_t) + \gamma \pi_t (\sigma_t Z_t + \mu_t J_t) - \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 J_t \Big\} dt \\ & + dK_t + Z_t dW_t + U_t dM_t \,, \end{aligned}$$

with $Z \in L^2(W)$, $U \in L^2(M)$ and $K \in \mathcal{A}^2$. This ends the proof of the first point.

We now prove the second point. Let $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ be a solution of (4.7) in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M) \times \mathcal{A}^2$. Let us prove that the process $\exp(-\gamma X^\pi)\bar{J}$ is a submartingale for any $\pi \in \mathcal{A}$.
From the product rule, we get

$$d\big(e^{-\gamma X_t^\pi}\bar{J}_t\big) = d\bar{M}_t^\pi + d\bar{A}_t^\pi + e^{-\gamma X_t^\pi}d\bar{K}_t\,,$$

with $\bar{A}_0^\pi = 0$ and

$$\begin{cases} d\bar{A}_t = -\operatorname*{ess\,inf}_{\pi\in\mathcal{A}}\Big\{\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big\}dt\,, \\ d\bar{A}_t^\pi = e^{-\gamma X_t^\pi}\Big\{\Big[\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t+\sigma_t\bar{Z}_t) - \lambda_t\big(1-e^{-\gamma\pi_t\beta_t}\big)(\bar{J}_t+\bar{U}_t)\Big]dt + d\bar{A}_t\Big\}\,, \\ d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi}\big[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t)dW_t + \big(\bar{U}_t + (e^{-\gamma\pi_t\beta_t}-1)\big(\bar{U}_t+\bar{J}_{t^-}\big)\big)dM_t\big]\,. \end{cases}$$

Since the strategy π is admissible, there exists a constant C_π such that $\exp(-\gamma X_t^\pi) \le C_\pi$ for any $t \in [0,T]$. With this, one can easily derive that $\mathbb{E}[\sup_{t\in[0,T]}\exp(-\gamma X_t^\pi)\bar{J}_t] < +\infty$ and that $\mathbb{E}[\int_0^T \exp(-\gamma X_t^\pi)d\bar{K}_t] < +\infty$. It follows that the local martingale $\bar{M}^\pi$ is a martingale and that the process $\exp(-\gamma X^\pi)\bar{J}$ is a submartingale.
Now recall that J is the largest process such that $\exp(-\gamma X^\pi)J$ is a submartingale for any $\pi \in \mathcal{A}$ with $J_T = \exp(-\gamma\xi)$ (see Proposition 4.2). Therefore, we get

$$\bar{J}_t \le J_t, \ \forall\, t \in [0,T] \ \ a.s.$$

□

Remark 4.6. Note that the integrability property $\mathbb{E}[\sup_{t\in[0,T]}\exp(-\gamma X_t^\pi)]$ is used in the proof of the second point.

4.3 Approximation of the value function by Lipschitz BSDEs

Throughout the sequel, the value function is characterized as the *limit of a nonincreasing sequence* of processes $(J^k)_{k\in\mathbb{N}}$ as k tends to $+\infty$, where for each $k \in \mathbb{N}$, J^k corresponds to the value function associated with the set of admissible strategies which are bounded by k.

For each $k \in \mathbb{N}$, we denote by $\mathcal{A}_t^k$ the subset of strategies of $\mathcal{A}_t$ uniformly bounded by k, and we consider the associated value function $J^k(.)$ defined for any $t \in [0,T]$ by

$$J^k(t) = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}_t^k}\mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathcal{G}_t\big]\,. \tag{4.9}$$

By similar argument as for J, there exists a càd-làg version of $J^k(.)$ denoted by J^k. As previously, the following dynamic programming principle holds:

Lemma 4.6. *The process* $\exp(-\gamma X^\pi)J^k$ *is a submartingale for any* $\pi \in \mathcal{A}^k$.

We now show that the sequence $(J^k)_{k\in\mathbb{N}}$ converge to J. More precisely, we have:

Theorem 4.1. *(Approximation of the value function)*

- *For any* $t \in [0,T]$*, we have*

$$J_t = \lim_{k\to\infty} \downarrow J^k_t \quad a.s.$$

- *For each* $k \in \mathbb{N}$*, the process* J^k *is the solution of the* Lipschitz *BSDE (3.3) with* $\mathcal{C}$ *replaced by* $\mathcal{B}^k$*, where* $\mathcal{B}^k$ *is the set of all strategies (not necessarily in* $\mathcal{A}$*) taking their values in* $[-k,k]$*.*

Proof. Let us prove the first point of the theorem.
Fix $t \in [0,T]$. It is obvious with the definitions of sets $\mathcal{A}_t$ and $\mathcal{A}^k_t$ that $\mathcal{A}^k_t \subset \mathcal{A}_t$ for each $k \in \mathbb{N}$ and hence

$$J_t \leq J^k_t \quad a.s.$$

Moreover, since $\mathcal{A}^k_t \subset \mathcal{A}^{k+1}_t$ for each $k \in \mathbb{N}$, the sequence of positive random variables $(J^k_t)_{k\in\mathbb{N}}$ is nonincreasing. Let us define the random variable

$$\bar{J}(t) = \lim_{k\to\infty} \downarrow J^k_t \quad a.s.$$

From the previous inequality we get that $J_t \leq \bar{J}(t)$ a.s., and this holds for any $t \in [0,T]$.
It remains to prove that $J_t \geq \bar{J}(t)$ a.s. for any $t \in [0,T]$.
Step 1: Let us now prove that the process $\bar{J}(.)$ is a submartingale. Fix $0 \leq s < t \leq T$. From Lemma 4.6, J^k is a submartingale, which gives for each $k \in \mathbb{N}$

$$\mathbb{E}\big[J^k_t\big|\mathcal{G}_s\big] \geq J^k_s \geq \bar{J}(s) \quad a.s.$$

The dominated convergence theorem (which can be applied since $0 \leq J^k_t \leq 1$ for each $k \in \mathbb{N}$) gives

$$\mathbb{E}\big[\bar{J}(t)\big|\mathcal{G}_s\big] = \lim_{k\to\infty} \mathbb{E}\big[J^k_t\big|\mathcal{G}_s\big] \geq \bar{J}(s) \quad a.s.\,,$$

which gives step 1.

Step 2: Let us show that the process $\exp(-\gamma X^\pi)\bar{J}(.)$ is a submartingale for any bounded strategy $\pi \in \mathcal{A}$.
Let π be a bounded admissible strategy. Then, there exists $n \in \mathbb{N}$ such that π is uniformly bounded by n. For each $k \geq n$, since $\pi \in \mathcal{A}^k$, $\exp(-\gamma X^\pi)J^k$ is a submartingale from Lemma 4.6. Then, by the dominated convergence theorem, the process $\exp(-\gamma X^\pi)\bar{J}(.)$ can be easily proved to be a submartingale.

Step 3: Note now that the process $\bar{J}(.)$ is a submartingale not necessarily càd-làg. However, by a theorem of Dellacherie-Meyer [7] (see VI.18), we know that the nonincreasing limit of a sequence of càd-làg submartingales is indistinguishable from a càd-làg adapted process. Hence, there exists a càd-làg version of $\bar{J}(.)$ which will be denoted by $\bar{J}$. Note that $\bar{J}$ is still a submartingale.

Step 4: Let us show that $\bar{J}_t \le J_t,\ \forall\, t \in [0,T]$ a.s. Since by steps 1 and 3, $\bar{J}$ is a càd-làg submartingale of class D, it admits the following Doob-Meyer decomposition

$$d\bar{J}_t = \bar{Z}_t dW_t + \bar{U}_t dM_t + d\bar{A}_t\,,$$

where $\bar{Z} \in L^2(W)$, $\bar{U} \in L^2(M)$ and $\bar{A}$ is a nondecreasing $\mathbb{G}$-predictable process with $\bar{A}_0 = 0$. As before, we use the fact that the process $\exp(-\gamma X^\pi)\bar{J}$ is a submartingale for any bounded strategy $\pi \in \mathcal{A}$ to give some necessary conditions satisfied by the process $\bar{A}$.
Let $\pi \in \mathcal{A}$ be a uniformly bounded strategy. The product rule gives

$$d(e^{-\gamma X_t^\pi}\bar{J}_t) = d\bar{M}_t^\pi + d\bar{A}_t^\pi\,,$$

with $\bar{A}_0^\pi = 0$ and

$$\begin{cases} d\bar{A}_t^\pi = e^{-\gamma X_t^\pi}\Big\{d\bar{A}_t + \Big[\dfrac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t + \lambda_t(e^{-\gamma\pi_t\beta_t}-1)(\bar{U}_t+\bar{J}_t) - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t)\Big]dt\Big\}\,, \\ d\bar{M}_t^\pi = e^{-\gamma X_{t^-}^\pi}\big[(\bar{Z}_t - \gamma\pi_t\sigma_t\bar{J}_t)dW_t + (\bar{U}_t + (e^{-\gamma\pi_t\beta_t}-1)(\bar{U}_t + \bar{J}_{t^-}))dM_t\big]\,. \end{cases}$$

Let $\bar{\mathcal{A}}$ be the set of uniformly bounded admissible strategies. Since the process $e^{-\gamma X^\pi}\bar{J}$ is a submartingale for any $\pi \in \bar{\mathcal{A}}$, we have $d\bar{A}_t^\pi \ge 0$ a.s. for any $\pi \in \bar{\mathcal{A}}$. Hence, there exists a process $\bar{K} \in \mathcal{A}^2$ such that

$$d\bar{A}_t = -\operatorname*{ess\,inf}_{\pi\in\bar{\mathcal{A}}}\Big\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t(1-e^{-\gamma\pi_t\beta_t})(\bar{J}_t + \bar{U}_t)\Big\}dt + d\bar{K}_t\,.$$

Now, the following equality holds $dt \otimes d\mathbb{P} - a.e.$ (see Appendix F for details)

$$\begin{aligned} &\operatorname*{ess\,inf}_{\pi\in\bar{\mathcal{A}}}\Big\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t(1-e^{-\gamma\pi_t\beta_t})(\bar{J}_t + \bar{U}_t)\Big\} \\ &\qquad = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}}\Big\{\frac{\gamma^2}{2}\pi_t^2\sigma_t^2\bar{J}_t - \gamma\pi_t(\mu_t\bar{J}_t + \sigma_t\bar{Z}_t) - \lambda_t(1-e^{-\gamma\pi_t\beta_t})(\bar{J}_t + \bar{U}_t)\Big\}\,. \end{aligned} \tag{4.10}$$

Hence, $(\bar{J}, \bar{Z}, \bar{U}, \bar{K})$ is a solution of BSDE (4.7), and Theorem 4.4 implies that

$$\bar{J}_t \le J_t,\ \forall\, t \in [0,T] \quad a.s.\,,$$

which ends the proof of the first point.

Let us prove the second point that is, for each $k \in \mathbb{N}$, J^k is characterized as the solution of a *Lipschitz BSDE*.
For each $k \in \mathbb{N}$ and for any $t \in [0,T]$, let $\mathcal{B}_t^k$ be the set of the restrictions to $[t,T]$ of the strategies of $\mathcal{B}^k$. By a localization argument (see Appendix G for details), one can show that the value function associated with $\mathcal{A}^k$ coincides with that associated with $\mathcal{B}^k$, that is

$$J_t^k = \operatorname*{ess\,inf}_{\pi\in\mathcal{B}_t^k}\mathbb{E}\big[\exp(-\gamma(X_T^{t,\pi} + \xi))\big|\mathcal{G}_t\big] \quad a.s.\,, \tag{4.11}$$

with J^k defined by (4.9).

It follows that by Proposition 3.1, for each $k \in \mathbb{N}$, the process J^k is the solution of the *Lipschitz* BSDE (3.3) with $\mathcal{C}$ replaced by $\mathcal{B}^k$. This ends the proof of the theorem. □

4.4 Characterization of the dynamic value function as the maximal solution of a BSDE

In this section, we add the following assumption:

Assumption 4.1. *ξ is bounded*

In this case, it is possible to prove that the dynamic value function J is the maximal solution of a BSDE (and not only the maximal subsolution). More precisely,

Theorem 4.2. *(Characterization of the value function)*
The value function (J, Z, U) is the maximal solution *in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the following BSDE:*

$$\begin{cases} -dJ_t = \operatorname*{ess\,inf}_{\pi\in\mathcal{A}} \Big\{ \frac{\gamma^2}{2}\pi_t^2\sigma_t^2 J_t - \gamma\pi_t(\mu_t J_t + \sigma_t Z_t) - \lambda_t(1 - e^{-\gamma\pi_t\beta_t})(J_t + U_t) \Big\} dt \\ \qquad\qquad - Z_t dW_t - U_t dM_t\,, \\ J_T = \exp(-\gamma\xi)\,. \end{cases} \tag{4.12}$$

Remark 4.7. (i) Recall that $\mathcal{A}$ can be replaced by $\mathcal{A}'$ in the above essential infimum.

(ii) Let $\hat{\pi} \in \Theta_2$ be the optimal strategy for J_0 (which exists by Delbaen *et al.* [6]). If $\hat{\pi} \in \mathcal{A}'$, then $\hat{\pi}_t$ attains the above essential infimum.

(iii) Recall that if there is no default, the optimal strategy $\hat{\pi}$ for J_0 belongs to $\mathcal{A}'$ and that our result corresponds to that of Hu *et al.* [16] in the complete case (by making the exponential change of variable $y_t = \frac{1}{\gamma}\log(J_t)$).

Proof. : For each $k \in \mathbb{N}$, let us denote by (J^k, Z^k, U^k) the solution of the associated *Lipschitz* BSDE (3.3) with $\mathcal{C}$ replaced by $\mathcal{B}^k$. We make the following change of variables

$$\begin{cases} y_t^k = \frac{1}{\gamma}\log(J_t^k)\,, \\ z_t^k = \frac{1}{\gamma}\frac{Z_t^k}{J_t^k}\,, \\ u_t^k = \frac{1}{\gamma}\log\Big(1 + \frac{U_t^k}{J_{t^-}^k}\Big)\,. \end{cases}$$

It is clear that the process (y^k, z^k, u^k) is a solution of the following quadratic BSDE

$$-dy_t^k = g^k(t, z_t^k, u_t^k)dt - z_t^k dW_t - u_t^k dM_t\ ;\ y_T^k = -\xi\ ,$$

where

$$g^k(s,z,u) = \operatorname*{ess\,inf}_{\pi\in\mathcal{B}^k}\Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s + \lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u - \pi_s\beta_s|_\gamma\Big) - (\mu_s + \lambda_s\beta_s)z - \frac{|\mu_s + \lambda_s\beta_s|^2}{2\gamma}$$

with $|u - \pi\beta_t|_\gamma = \lambda_t\frac{\exp(\gamma(u-\pi\beta_t)) - 1 - \gamma(u-\pi\beta_t)}{\gamma}$.

Recall now that by a result of Morlais [26], the sequence $(y^k, z^k, u^k)_{k\in\mathbb{N}}$ converges to (y, z, u) in the following sense

$$\mathbb{E}\Big(\sup_{t\in[0,T]} |y_t^k - y_t|\Big) + |z^k - z|_{L^2(W)} + |u^k - u|_{L^2(M)} \to 0\,,$$

where (y, z, u) is solution of

$$-\,dy_t = g(t, y_t, z_t, u_t)dt - z_t dW_t - u_t dM_t \ ;\ y_T = -\xi\,,$$

with

$$g(s, z, u) = \operatorname*{ess\,inf}_{\pi\in\bar{\mathcal{B}}} \Big(\frac{\gamma}{2}\Big|\pi_s\sigma_s - \Big(z + \frac{\mu_s + \lambda_s\beta_s}{\gamma}\Big)\Big|^2 + |u - \pi_s\beta_s|_\gamma\Big) - (\mu_s + \lambda_s\beta_s)z - \frac{|\mu_s + \lambda_s\beta_s|^2}{2\gamma}\,,$$

where $\bar{\mathcal{B}} = \cup_k \mathcal{B}^k$. Note that the proof of this result (see [26]) is based on similar arguments as those used in the proof of the monotone stability convergence theorem for quadratic BSDEs of Kobylanski [20].
Note now that by localization arguments (as in Appendix F or G), one can easily show that in the above essinf, the set $\bar{\mathcal{B}}$ can be replaced by $\mathcal{A}$ or even by $\mathcal{A}'$.

Let us now define the following processes

$$\begin{cases} J_t^* = e^{\gamma y_t}\,, \\ Z_t^* = \gamma J_t^* z_t\,, \\ U_t^* = (e^{\gamma u_t} - 1)J_{t^-}^*\,. \end{cases}$$

Note that (J^*, Z^*, U^*) is clearly a solution of BSDE (4.12).
Also, using the above convergence property and our characterization of J as the *nonincreasing limit* of $(J^k)_{k\in\mathbb{N}}$ (see Theorem 4.1), we have

$$J_t = \lim_{k\to\infty} J_t^k = \lim_{k\to\infty} e^{\gamma y_t^k} = e^{\gamma y_t} = J_t^* \ \text{ a.s.}$$

Moreover, the uniqueness of the Doob-Meyer decomposition (4.6) of J implies that $Z_t^* = Z_t$ and $U_t^* = U_t\ dt \otimes d\mathbb{P} - a.s.$ Hence, (J, Z, U) is a solution of BSDE (4.12). In other words, J is not only a subsolution but a solution of this BSDE. Since by Theorem 4.4, J is the maximal subsolution of BSDE (4.12), it follows that J is the maximal solution of BSDE (4.12). This makes the proof ended. □

5 Case of unbounded coefficients

In this section, we consider the case of unbounded coefficients and condition (i) of Assumption 2.1 is replaced by

$$\int_0^T \Big(|\mu_t| + |\sigma_t|^2 + \lambda_t|\beta_t|^2\Big)dt < \infty \ \ a.s.$$

5.1 Case of unbounded coefficients

In this part, we consider the general case of unbounded coefficients. We have:

Proposition 5.1. *The two following properties hold:*

- *The dynamic value function J is the maximal subsolution of BSDE (4.7).*
- *For any $t \in [0,T]$, we have*
$$J_t = \lim_{k\to\infty} \downarrow J_t^k \quad a.s.$$

For the proof, it is sufficient to note that the proofs of Proposition 4.4 and of the first point of Theorem 4.1 still hold in the case of unbounded coefficients because the arguments used do not require any assumption of boundedness on the coefficients.

Remark 5.1. If β is bounded, then the price process is locally bounded and hence, by Delbaen *et al.*'s result [6] the value function J_0 is equal to the value function associated with the set Θ_2. Also, under this assumption, the dynamic value functions associated respectively with $\mathcal{A}$ and $\mathcal{A}'$ coincide. Note that the proofs of these two assertions are based on the same arguments as those used in the proofs of Lemmas 4.1 and 4.3.

5.2 Case of coefficients which satisfy some exponential integrability conditions

In this section, we consider the case of coefficients not necessarily bounded but satisfying some integrability conditions. We first study the particular case of strategies valued in a convex-compact set and second the non constrained case.

5.2.1 Case of strategies valued in a convex-compact set

Suppose that the set of admissible strategies is given by $\mathcal{C}$ (see Section 3) where C is a convex-compact set with $0 \in C$. Here, it simply corresponds to a closed interval of $\mathbb{R}$ since we are in the one dimensional case. However, the following results clearly still hold in the multidimensional case (see Section 6.1). Let $J(.)$ be the associated dynamic value function to $\mathcal{C}_t$ defined as in Section 3 (see (3.1)). Using some classical results of convex analysis (see for example Ekeland and Temam [9]), we easily derive the following existence property:

Proposition 5.2. *There exists a unique optimal strategy $\hat{\pi} \in \mathcal{C}$ for the optimization problem (2.5), that is*
$$J(0) = \inf_{\pi\in\mathcal{C}} \mathbb{E}\big[\exp\big(-\gamma(X_T^{\pi}+\xi)\big)\big] = \mathbb{E}\big[\exp\big(-\gamma(X_T^{\hat{\pi}}+\xi)\big)\big]\,.$$

Proof. Note that $\mathcal{C}$ is strongly closed and convex in $L^2([0,T]\times\Omega)$. Hence, $\mathcal{C}$ is closed for the weak topology. Moreover, since $\mathcal{C}$ is bounded, $\mathcal{C}$ is compact for the weak topology.
We define the function $\phi(\pi) = \mathbb{E}[\exp(-\gamma(X_T^{\pi}+\xi))]$ on $L^2([0,T]\times\Omega)$. This function is clearly convex and continuous for the strong topology in $L^2([0,T]\times\Omega)$. By classical results of convex analysis, it is s.c.i for the weak topology. Now, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{C}$ such that $\phi(\pi^n) \to \min_{\pi\in\mathcal{C}}\phi(\pi)$ as $n\to\infty$.

Since $\mathcal{C}$ is weakly compact, there exists an extracted sequence still denoted by (π^n) which converges for the weak topology to $\hat{\pi}$ for some $\hat{\pi} \in \mathcal{C}$. Now, since ϕ is s.c.i for the weak topology, it implies that

$$\phi(\hat{\pi}) \leq \liminf \phi(\pi^n) = \min_{\pi \in \mathcal{C}} \phi(\pi).$$

Therefore, $\phi(\hat{\pi}) = \inf_{\pi \in \mathcal{C}} \phi(\pi)$. The uniqueness of the optimal strategy derives from the convexity property of the set C and the strict convexity property of the function $x \mapsto \exp(-\gamma x)$. □

We now want to characterize the value function $J(.)$ as the unique solution of a BSDE. For that, we cannot apply the same techniques as in the case of bounded coefficients. Indeed, since the coefficients are not necessarily bounded, the drivers of the associated BSDEs are no longer Lipschitz. Hence, the existence and uniqueness properties do not a priori hold. Therefore, in order to show the desired characterization of $J(.)$, we will use the dynamic programming principle and also the existence of an optimal strategy.
In order to have a dynamic programming principle similar to Proposition 4.2, we suppose that the coefficients satisfy the following integrability condition:

Assumption 5.1. *β is uniformly bounded and*

$$\mathbb{E}\Big[\exp\Big(a\int_0^T |\mu_t|dt\Big)\Big] + \mathbb{E}\Big[\exp\Big(b\int_0^T |\sigma_t|^2 dt\Big)\Big] < \infty\,,$$

with $a = 2\gamma||\mathcal{C}||_\infty$ and $b = 8\gamma^2||\mathcal{C}||_\infty^2$.

By classical computations, one can easily derive that for any $t \in [0,T]$ and any $\pi \in \mathcal{C}_t$, the following integrability property holds

$$\mathbb{E}\big[\sup_{s\in[t,T]} \exp\big(-\gamma X_s^{t,\pi}\big)\big] < \infty\,. \tag{5.1}$$

Using this integrability property, the process $J(.)$ can be proved to satisfy the following dynamic programming principle: $J(.)$ is the largest $\mathbb{G}$-adapted process such that $\exp(-\gamma X^\pi)J(.)$ is a submartingale for any $\pi \in \mathcal{C}$ with $J(T) = \exp(-\gamma\xi)$.

We now show the following characterization of the dynamic value function:

Theorem 5.1. *(Characterization of the value function)*
There exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is the unique solution *in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (3.3).*
Also, the optimal strategy $\hat{\pi} \in \mathcal{C}$ for J_0 is characterized by the fact that $\hat{\pi}_t$ attains the essential infimum in (3.3), $dt \otimes d\mathbb{P} - a.e.$

Note first that the two following lemmas hold:

Lemma 5.1. *(Optimality criterion)*
Fix $\hat{\pi} \in \mathcal{C}$. The strategy $\hat{\pi} \in \mathcal{C}$ is optimal for $J(0)$ if and only if the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ *is a martingale.*

Lemma 5.2. *There exists a càd-làg version of $J(.)$ which will be denoted by J.*

Proof of Theorem 5.1

Step 1: Let us prove that there exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is a solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (3.3).

Note first that since $0 \in \mathcal{C}$, the process J satisfies $0 \le J_t \le 1$, $\forall\, t \in [0,T]$ a.s. From the Doob-Meyer decomposition, since the process J is a bounded càd-làg submartingale, there exist $Z \in L^2(W)$, $U \in L^2(M)$ and A a nondecreasing process with $A_0 = 0$ such that

$$dJ_t = Z_t dW_t + U_t dM_t + dA_t\,.$$

Since for any $\pi \in \mathcal{C}$ the process $\exp(-\gamma X^\pi)J(.)$ is a submartingale, one can easily derive that

$$dA_t \ge \operatorname*{ess\,sup}_{\pi \in \mathcal{C}} \Big\{ \gamma \pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t(1 - e^{-\gamma \pi_t \beta_t})(J_t + U_t) - \frac{\gamma^2}{2}\pi_t^2 \sigma_t^2 J_t \Big\} dt\,.$$

Now, by Proposition 5.2, there exists an optimal strategy $\hat{\pi} \in \mathcal{C}$. The optimality criterion (Lemma 5.1) gives

$$dA_t = \Big\{ \gamma \hat{\pi}_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t(1 - e^{-\gamma \hat{\pi}_t \beta_t})(J_t + U_t) - \frac{\gamma^2}{2}\hat{\pi}_t^2 \sigma_t^2 J_t \Big\} dt\,,$$

which implies

$$dA_t = \operatorname*{ess\,sup}_{\pi \in \mathcal{C}} \Big\{ \gamma \pi_t(\mu_t J_t + \sigma_t Z_t) + \lambda_t(1 - e^{-\gamma \pi_t \beta_t})(J_t + U_t) - \frac{\gamma^2}{2}\pi_t^2 \sigma_t^2 J_t \Big\} dt\,.$$

Hence, (J, Z, U) is solution of BSDE (3.3).

Using similar arguments as in the proof of Theorem 4.4, one can derive that (J, Z, U) is the *maximal solution* in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of BSDE (3.3).

Step 2: Let us show that (J, Z, U) is the unique solution of BSDE (3.3). Let $(\bar{J}, \bar{Z}, \bar{U})$ be a solution of BSDE (3.3). By a measurable selection theorem, we know that there exists at least a strategy $\bar{\pi} \in \mathcal{C}$ such that $dt \otimes d\mathbb{P} - a.e.$

$$\begin{aligned}
&\operatorname*{ess\,inf}_{\pi \in \mathcal{C}} \Big\{ \frac{\gamma^2}{2}\pi_t^2 \sigma_t^2 \bar{J}_t - \gamma \pi_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t(1 - e^{-\gamma \pi_t \beta_t})(\bar{J}_t + \bar{U}_t) \Big\} \\
&\qquad\qquad = \frac{\gamma^2}{2}\bar{\pi}^2 \sigma_t^2 \bar{J}_t - \gamma \bar{\pi}_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t(1 - e^{-\gamma \bar{\pi}_t \beta_t})(\bar{J}_t + \bar{U}_t)\,.
\end{aligned}$$

Hence, BSDE (3.3) can be written under the form

$$d\bar{J}_t = \Big\{ \gamma \bar{\pi}_t(\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) + \lambda_t(1 - e^{-\gamma \bar{\pi}_t \beta_t})(\bar{J}_t + \bar{U}_t) - \frac{\gamma^2}{2}\bar{\pi}^2 \sigma_t^2 \bar{J}_t \Big\} dt + \bar{Z}_t dW_t + \bar{U}_t dM_t\,.$$

Let us introduce by $B_t = \exp(-\gamma X_t^{\bar{\pi}})$. Itô's formula and rule product give

$$d(B_t \bar{J}_t) = \big(B_t \bar{Z}_t - \gamma \sigma_t \bar{\pi}_t B_t \bar{J}_t\big) dW_t + \big[(e^{-\gamma \beta_t \bar{\pi}_t} - 1)B_{t^-}\bar{J}_t + e^{-\gamma \beta_t \bar{\pi}_t} B_{t^-}\bar{U}_t\big] dM_t\,.$$

By Assumption 5.1 and since $\bar{J}$ is bounded, one can derive that the local martingale $B\bar{J}$ satisfies $\mathbb{E}[\sup_{0 \le t \le T}|B_t \bar{J}_t|] < \infty$. Hence, $B\bar{J}$ is a martingale. Thus,

$$\bar{J}_t = \mathbb{E}\Big[\frac{B_T}{B_t} e^{-\gamma \xi} \Big| \mathcal{G}_t\Big] = \mathbb{E}\big[\exp(-\gamma(X_T^{t,\bar{\pi}} + \xi)) \big| \mathcal{G}_t\big]\,.$$

Hence,

$$\bar{J}_t \geq \operatorname*{ess\,inf}_{\pi\in\mathcal{C}} \mathbb{E}\big[\exp(-\gamma(X_T^{t,\pi}+\xi))\big|\mathcal{G}_t\big] = J_t\,.$$

Now, by step 1, J is the maximal solution of BSDE (3.3). This yields that for any $t\in[0,T]$, $J_t \leq \bar{J}_t$, a.s. Hence, $J_t=\bar{J}_t$, $\forall\, t\in[0,T]$ a.s. and $\bar{\pi}$ is optimal and the proof is ended. □

For completeness, the proofs of the two above lemmas are given.

Proof of Lemma 5.1. Suppose that $\hat{\pi}$ is optimal for $J(0)$. Hence,

$$J(0)=\inf_{\pi\in\mathcal{A}}\mathbb{E}\big[\exp\big(-\gamma(X_T^{\pi}+\xi)\big)\big]=\mathbb{E}\big[\exp\big(-\gamma(X_T^{\hat{\pi}}+\xi)\big)\big].$$

Since the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a submartingale and since $J(0)=\mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}}+\xi))]$, the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a martingale.

Suppose now that the process $\exp(-\gamma X^{\hat{\pi}})J(.)$ is a martingale. Then, $\mathbb{E}[\exp(-\gamma X_T^{\hat{\pi}})J(T)]=J(0)$. Also, since for any $\pi\in\mathcal{A}$, the process $\exp(-\gamma X^{\pi})J(.)$ is a submartingale and $J(T)=\exp(-\gamma\xi)$, it is clear that $J(0)\leq \inf_{\pi\in\mathcal{A}}\mathbb{E}[\exp(-\gamma(X_T^{\pi}+\xi))]$. Consequently,

$$J(0)=\inf_{\pi\in\mathcal{A}}\mathbb{E}\big[\exp\big(-\gamma(X_T^{\pi}+\xi)\big)\big]=\mathbb{E}\big[\exp\big(-\gamma(X_T^{\hat{\pi}}+\xi)\big)\big].$$

In other words, $\hat{\pi}$ is an optimal strategy. □

Proof of Lemma 5.2. The proof is simple here because we have an existence result. More precisely, by Proposition 5.2, there exists $\hat{\pi}\in\mathcal{C}$ which is optimal for J_0. Hence, by the optimality criterium (Proposition 5.1), we have $J(t)=\exp(-\gamma X_t^{\hat{\pi}})\mathbb{E}[\exp(-\gamma(X_T^{\hat{\pi}}+\xi))|\mathcal{G}_t]$ for any $t\in[0,T]$ (in other words, $\hat{\pi}$ is also optimal for $J(t)$). By classical results on the conditional expectation, there exists a càd-làg version denoted by J. *Box*

5.2.2 The non constrained case

In this part, the set of admissible strategies is given by $\mathcal{A}$. Under some exponential integrability conditions on the coefficients, we can also precise the characterization of the value function J as the limit of $(J^k)_{k\in\mathbb{N}}$ as k tends to $+\infty$.

Assumption 5.2. β is uniformly bounded, $\mathbb{E}[\int_0^T\lambda_t dt]<\infty$ and for any $p>0$ we have

$$\mathbb{E}\Big[\exp\Big(p\int_0^T|\mu_t|dt\Big)\Big]+\mathbb{E}\Big[\exp\Big(p\int_0^T|\sigma_t|^2dt\Big)\Big]<\infty\,.$$

Again, for each $k\in\mathbb{N}$, we consider the set $\mathcal{B}_t^k$. Since Assumption 5.2 is satisfied, the integrability condition (G.1) holds and hence, for each $k\in\mathbb{N}$,

$$J_t^k=\operatorname*{ess\,inf}_{\pi\in\mathcal{B}_t^k}\mathbb{E}\big[\exp\big(-\gamma(X_T^{t,\pi}+\xi)\big)\big|\mathcal{G}_t\big]\quad a.s.$$

In this case, for each $k\in\mathbb{N}$, the process J^k is characterized as the unique solution of BSDE (3.3) with $\mathcal{C}=\mathcal{B}^k$. Therefore, we have:

Theorem 5.2. *(Characterization of the value function)*
The value function J is characterized as the nonincreasing limit *of the sequence $(J^k)_{k\in\mathbb{N}}$ as k tends to $+\infty$, which are* the unique solutions *of BSDEs (3.3) with $\mathcal{C}=\mathcal{B}^k$ for each $k\in\mathbb{N}$.*

6 Generalizations

In this section, we give some generalizations of the previous results. The proofs are not given, but they are identical to the proofs of the case with a default time and a stock. In all this section, elements of $\mathbb{R}^n$, $n \geq 1$, are identified to column vectors, the superscript $'$ stands for the transposition, $||.||$ the square norm, $\mathbb{1}$ the vector of $\mathbb{R}^n$ such that each component of this vector is equal to 1. Let U and V two vectors of $\mathbb{R}^n$, $U * V$ denotes the vector such that $(U * V)_i = U_i V_i$ for each $i \in \{1, \ldots, n\}$. Let $X \in \mathbb{R}^n$, $diag(X)$ is the matrix such that $diag(X)_{ij} = X_i$ if $i = j$ else $diag(X)_{ij} = 0$.

6.1 Several default times and several stocks

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two stochastic processes: an n-dimensional Brownian motion W and a p-dimensional jump process $N = (N^i, 1 \leq i \leq p)$ with $N^i_t = \mathbb{1}_{\tau^i \leq t}$, where $(\tau^i)_{1 \leq i \leq p}$ are p default times. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by these processes. This filtration is supposed to be right-continuous and W is a $\mathbb{G}$-Brownian motion. We make the following assumptions on the default times:

Assumption 6.1. (i) The defaults do not appear simultaneously: $\mathbb{P}(\tau^i = \tau^j) = 0$ for $i \neq j$.

(ii) Each default can appear at any time: $\mathbb{P}(\tau^i > t) > 0$.

We denote for each $j \in \{1, \ldots, p\}$ by M^j the compensated martingale of N^j and Λ^j its compensator. We assume that Λ^j is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a process λ^j such that $\Lambda^j_t = \int_0^t \lambda^j_s ds$.

We consider a financial market which consists of one risk-free asset, whose price process is assumed for simplicity to be equal to 1 at any time, and n risky assets, whose price processes $(S^i)_{1 \leq i \leq n}$ admit p discontinuities at times $(\tau^j)_{1 \leq j \leq p}$. Throughout the sequel, we consider that the price process $S := (S^i)_{1 \leq i \leq n}$ evolves according to the equation

$$dS_t = diag(S_{t^-})(\mu_t dt + \sigma_t dW_t + \beta_t dN_t)\,, \tag{6.1}$$

with the classical assumptions:

Assumption 6.2.

(i) μ, σ, β and λ are uniformly bounded $\mathbb{G}$-predictable processes such that σ is nonsingular for any $t \in [0, T]$,

(ii) there exist d coefficients $\theta^1, \ldots, \theta^d$ that are $\mathbb{G}$-predictable processes such that

$$\mu^i_t + \sum_{j=1}^{p} \lambda^j_t \beta^{i,j}_t = \sum_{j=1}^{d} \sigma^{i,j}_t \theta^j_t,\ \forall\, t \in [0, T]\ a.s.,\ 1 \leq i \leq n\,,$$

we suppose that θ^j is bounded,

(iii) the process β satisfies $\beta^{i,j}_{\tau_j} > -1$ a.s. for each $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$.

Using the same techniques as in the previous sections, all the results stated in the previous sections can be generalized to this framework. In particular, we have:

Theorem 6.1. *There exist $Z \in L^2(W)$ and $U \in L^2(M)$ such that (J, Z, U) is the maximal solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(M)$ of the BSDE*

$$\begin{cases} -\,dJ_t = \ \underset{\pi \in \mathcal{A}}{\operatorname{ess\,inf}} \Big\{ \dfrac{\gamma^2}{2} ||\pi'_t \sigma_t||^2 J_t - \gamma \pi'_t(\mu_t J_t + \sigma_t Z_t) - (\mathbb{1} - e^{-\gamma \pi'_t \beta_t})(\lambda_t J_t + \lambda_t * U_t) \Big\} dt \\ \qquad\qquad - \ Z_t dW_t - U_t dM_t \,, \\ \ J_T = \ \exp(-\gamma \xi) \,. \end{cases}$$

6.2 Poisson jumps

We consider a market defined on the complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$ equipped with two independent processes: a unidimensional Brownian motion W and a real-valued Poisson point process p defined on $[0, T] \times \mathbb{R}\backslash\{0\}$, we denote by $N_p(ds, dx)$ the associated counting measure, such that its compensator is $\hat{N}_p(ds, dx) = n(dx)ds$ and the Levy measure $n(dx)$ is positive and satisfies $n(\{0\}) = 0$ and $\int_{\mathbb{R}\backslash\{0\}} (1 \wedge |x|)^2 n(dx) < \infty$. We denote by $\mathbb{G} = \{\mathcal{G}_t, 0 \leq t \leq T\}$ the completed filtration generated by the two processes W and N_p. We denote by $\tilde{N}_p(ds, dx)$ ($\tilde{N}_p(ds, dx) = N_p(ds, dx) - \hat{N}_p(ds, dx)$) the compensated measure, which is a martingale random measure.

The financial market consists of one risk-free asset, whose price process is assumed to be equal to 1, and one single risky asset, whose price process is denoted by S. In particular, the stock price process satisfies

$$dS_t = S_{t^-} \Big(\mu_t dt + \sigma_t dW_t + \int_{\mathbb{R}\backslash\{0\}} \beta_t(x) N_p(dt, dx) \Big) \,.$$

μ, σ and β are assumed to be uniformly bounded $\mathbb{G}$-predictable processes. Moreover, the process σ (resp. β) satisfies $\sigma_t > 0$ (resp. $\beta_t(x) > -1$ a.s.). Note that this case corresponds to that studied in Morlais [26].

Using the same techniques as in the previous sections, all the results stated in the previous sections can be generalized to this framework. In particular, we have:

Theorem 6.2. *There exist $Z \in L^2(W)$ and $U \in L^2(\tilde{N}_p)$ such that (J, Z, U) is the maximal solution in $\mathcal{S}^{+,\infty} \times L^2(W) \times L^2(\tilde{N}_p)$ of the BSDE*

$$\begin{cases} -\,dJ_t = \ \underset{\pi \in \mathcal{A}}{\operatorname{ess\,inf}} \Big\{ \dfrac{\gamma^2}{2} |\pi_t \sigma_t|^2 J_t - \gamma \pi_t(\mu_t J_t + \sigma_t Z_t) - \displaystyle\int_{\mathbb{R}\backslash\{0\}} (1 - e^{-\gamma \pi_t x})(J_t + U_t(x)) n(dx) \Big\} dt \\ \qquad\qquad - \ Z_t dW_t - \displaystyle\int_{\mathbb{R}\backslash\{0\}} U_t(x) \tilde{N}_p(dt, dx) \,, \\ \ J_T = \ \exp(-\gamma \xi) \,. \end{cases}$$

Appendix

A Essential supremum

Recall the following classical result (see Neveu [27]):

Theorem A.1. *Let F be a non empty family of measurable real valued functions $f : \Omega \to \bar{\mathbb{R}}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, there exists a measurable function $g : \Omega \to \bar{\mathbb{R}}$ such that*

(i) for all $f \in F$, $f \leq g$ a.s.,

(ii) if h is a measurable function satisfying $f \leq h$ a.s., for all $f \in F$, then $g \leq h$ a.s.

This function g, which is unique a.s., is called the essential supremum of F and is denoted $\operatorname{ess\,sup}_{f \in F} f$.
Moreover, there exists at least one sequence $(f_n)_{n\in\mathbb{N}}$ in F such that $\operatorname{ess\,sup}_{f\in F} f = \lim_{n\to\infty} f_n$ a.s. Furthermore, if F is filtrante croissante *(i.e. f, $g \in F$ then there exists $h \in F$ such that both $f \leq h$ a.s., and $g \leq h$ a.s.), then the sequence $(f_n)_{n\in\mathbb{N}}$ may be taken nondecreasing and $\operatorname{ess\,sup}_{f\in F} f = \lim_{n\to\infty} \uparrow f_n$ a.s.*

B A classical lemma of analysis

Lemma B.1. *The supremum of affine functions, whose coefficients are bounded by a constant $c > 0$, is Lipschitz and the Lipschitz constant is equal to c.*
More precisely, let $\mathcal{A}$ be the set of $[-c, c]^n \times [-k, k]$. Then, the function f defined for any $y \in \mathbb{R}^n$ by

$$f(y) = \sup_{(a,b)\in\mathcal{A}} \{a.y + b\}$$

is Lipschitz with Lipschitz constant c.

Proof.

$$\sup_{(a,b)\in\mathcal{A}} \{a.y + b\} \leq \sup_{(a,b)\in\mathcal{A}} \{a.(y - y')\} + \sup_{(a,b)\in\mathcal{A}} \{a.y' + b\}.$$

Which implies

$$f(y) - f(y') \leq c||y - y'||.$$

By symmetry, we have also

$$f(y') - f(y) \leq c||y - y'||,$$

which gives the desired result. □

C Proof of Lemma 4.3

We have to prove that the dynamic value function $J(.)$ associated with $\mathcal{A}$ coincides a.s. with the one associated with $\mathcal{A}'$.
Fix $t \in [0, T]$. Put $J^{'}(t) := \operatorname{ess\,inf}_{\pi\in\mathcal{A}^{'}_t} \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathcal{G}_t]$, where $\mathcal{A}^{'}_t$ is the set of the restrictions to $[t, T]$ of the strategies of $\mathcal{A}'$. Since $\mathcal{A}_t \subset \mathcal{A}^{'}_t$, it follows that $J^{'}(t) \leq J(t)$ a.s.

To prove the other inequality, it is sufficient to show that for any $\pi \in \mathcal{A}_t^{'}$, there exists a sequence $(\pi^n)_{n\in\mathbb{N}}$ of $\mathcal{A}_t$ such that $\pi^n \to \pi,\ dt\otimes d\mathbb{P}$ a.s. Let us define π^n by

$$\pi_s^n = \pi_s \mathbb{1}_{s\leq\tau_n},\ \forall\ s\in[t,T]\,,$$

where τ_n is the stopping time defined by $\tau_n = \inf\{s\geq t, |X_s^{t,\pi}|\geq n\}$.
It is clear that for each $n\in\mathbb{N}$, $\pi^n\in\mathcal{A}_t$. Thus, $\exp(-\gamma X_T^{t,\pi^n}) = \exp(-\gamma X_{T\wedge\tau_n}^{t,\pi}) \xrightarrow{a.s.} \exp(-\gamma X_T^{t,\pi})$ as $n\to+\infty$. By definition of $\mathcal{A}_t^{'}$, $\mathbb{E}[\sup_{s\in[t,T]}\exp(-\gamma X_s^{t,\pi})] < \infty$. Hence, by the Lebesgue Theorem, $\mathbb{E}[\exp(-\gamma(X_T^{t,\pi^n}+\xi))|\mathcal{G}_t] \to \mathbb{E}[\exp(-\gamma(X_T^{t,\pi}+\xi))|\mathcal{G}_t]$ a.s. as $n\to+\infty$. Therefore, we have $J(t)\leq J^{'}(t)$ a.s. which ends the proof.

D Proof of the closedness by binding of $\mathcal{A}'$

Lemma D.1. *Let π^1,π^2 be two admissible strategies of $\mathcal{A}^{'}$ and $s\in[0,T]$. The strategy π^3 defined by*

$$\pi_t^3 = \begin{cases}\pi_t^1 & \text{if } t\leq s,\\ \pi_t^2 & \text{if } t>s,\end{cases}$$

belongs to $\mathcal{A}^{'}$.

Proof. For any $u\in[0,T]$, we have for any $p>1$

(i) if $u>s$, then

$$\mathbb{E}[\sup_{r\in[u,T]}\exp(-\gamma p X_r^{u,\pi^3})] = \mathbb{E}[\sup_{r\in[u,T]}\exp(-\gamma p X_r^{u,\pi^2})] < \infty,$$

(ii) if $u\leq s$, then

$$\mathbb{E}[\sup_{r\in[u,T]}\exp(-\gamma p X_r^{u,\pi^3})] \leq \mathbb{E}[\sup_{r\in[u,T]}\exp(-\gamma p X_r^{u,\pi^1})] + \mathbb{E}[\sup_{r\in[s,T]}\exp(-\gamma p(X_s^{u,\pi^1}+X_r^{s,\pi^2}))].$$

By Cauchy-Schwarz inequality,

$$\mathbb{E}[\sup_{r\in[s,T]}\exp(-\gamma p(X_s^{u,\pi^1}+X_r^{s,\pi^2}))] \leq \mathbb{E}[\sup_{r\in[u,T]}\exp(-2\gamma p X_r^{u,\pi^1})]^{1/2} \times \mathbb{E}[\sup_{r\in[s,T]}\exp(-2\gamma p X_r^{s,\pi^2})]^{1/2}.$$

Hence, $\mathbb{E}[\sup_{r\in[u,T]}\exp(-\gamma p X_r^{u,\pi^3})] < \infty$.

□

E Proof of the existence of a càd-làg modification of J

The proof is not so simple since we do not know if there exists an optimal strategy in $\mathcal{A}$. Let $\mathbb{D} = [0,T] \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers. Since $J(.)$ is a submartingale, the mapping $t \to J(t,\omega)$ defined on $\mathbb{D}$ has for almost every $\omega \in \Omega$ and for any t of $[0,T[$ a finite right limit

$$J(t^+,\omega) = \lim_{s\in\mathbb{D}, s\downarrow t} J(s,\omega),$$

(see Karatzas and Shreve [19], Proposition 1.3.14 or Dellacherie and Meyer [7], Chapter 6). Note that it is possible to define $J(t^+,\omega)$ for any $(t,\omega) \in [0,T]\times\Omega$ by $J(T^+,\omega) := J(T,\omega)$ and

$$J(t^+,\omega) := \limsup_{s\in\mathbb{D}, s\downarrow t} J(s,\omega), \ \ t \in [0,T[.$$

From the right-continuity of the filtration $\mathbb{G}$, the process $J(.^+)$ is $\mathbb{G}$-adapted. It is possible to show that $J(.^+)$ is a $\mathbb{G}$-submartingale and even that the process $\exp(-\gamma X^\pi)J(.^+)$ is a $\mathbb{G}$-submartingale for any $\pi \in \mathcal{A}$. Indeed, from Proposition 4.2, for any $s \le t$ and for each sequence of rational numbers $(t_n)_{n\in\mathbb{N}}$ converging down to t, we have

$$\mathbb{E}\big[\exp(-\gamma X^\pi_{t_n}) J(t_n)\big|\mathcal{G}_s\big] \ge \exp(-\gamma X^\pi_s) J(s) \ a.s.$$

Let n tend to $+\infty$. By the Lebesgue theorem, we have that for any $s \le t$,

$$\mathbb{E}\big[\exp(-\gamma X^\pi_{t}) J(t^+)\big|\mathcal{G}_s\big] \ge \exp(-\gamma X^\pi_s) J(s) \ a.s. \tag{E.1}$$

This clearly implies that for any $s \le t$, $\mathbb{E}[\exp(-\gamma X^\pi_t)J(t^+)|\mathcal{G}_s] \ge \exp(-\gamma X^\pi_s)J(s^+)$ a.s., which gives the submartingale property of the process $\exp(-\gamma X^\pi)J(.^+)$. Using the right-continuity of the filtration $\mathbb{G}$ and inequality (E.1) applied to $\pi = 0$ and $s = t$, we get

$$J(t^+) = \mathbb{E}\big[J(t^+)\big|\mathcal{G}_t\big] \ge J(t) \ \ a.s.$$

On the other hand, by the characterization of $J(.)$ (see Proposition 4.2), and since the process $\exp(-\gamma X^\pi)J(.^+)$ is a $\mathbb{G}$-submartingale for any $\pi \in \mathcal{A}$, we have that for any $t \in [0,T]$,

$$J(t^+) \le J(t) \ \ a.s.$$

Thus, for any $t \in [0,T]$,

$$J(t^+) = J(t) \ \ a.s.$$

Furthermore, the process $J(.^+)$ is càd-làg. The result follows by taking $J_t = J(t^+)$.

F Proof of equality (4.10)

For any $\pi \in \mathcal{A}$, we define the strategy $\pi^k_t = \pi_t \mathbb{1}_{|\pi_t|\le k}$ for each $k \in \mathbb{N}$. The strategy π^k is uniformly bounded but not necessarily admissible. For that we define for each $(k,n) \in \mathbb{N}\times\mathbb{N}$ the stopping time

$$\tau_{k,n} := \inf\{t, |X^{\pi^k}_t| \ge n\}\,,$$

and the strategy $\pi_t^{k,n} := \pi_t^k \mathbb{1}_{t \leq \tau_{k,n}}$. By construction, it is clear that the strategy $\pi^{k,n} \in \mathcal{A}^k$ for each (k,n). Since $\pi_t = \lim_k \lim_n \pi_t^{k,n}$ $dt \otimes d\mathbb{P}$ a.s., the following equality

$$\operatorname*{ess\,inf}_{\pi \in \bar{\mathcal{A}}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t - \gamma \pi_t (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t (1 - e^{-\gamma \pi_t \beta_t})(\bar{J}_t + \bar{U}_t) \Big\} =$$
$$\operatorname*{ess\,inf}_{\pi \in \mathcal{A}} \Big\{ \frac{\gamma^2}{2} \pi_t^2 \sigma_t^2 \bar{J}_t \gamma \pi_t - (\mu_t \bar{J}_t + \sigma_t \bar{Z}_t) - \lambda_t (1 - e^{-\gamma \pi_t \beta_t})(\bar{J}_t + \bar{U}_t) \Big\}$$

holds $dt \otimes d\mathbb{P}$ a.s.

G Proof of equality (4.11)

Fix $k \in \mathbb{N}$ and $t \in [0,T]$. Note first that for each $k \in \mathbb{N}$, $\forall\, p > 1$ and $\forall\, t \in [0,T]$, the following integrability property is satisfied

$$\sup_{\pi \in \mathcal{B}^k} \mathbb{E}\big[\exp(-\gamma p X_t^{\pi})\big] < \infty\,. \tag{G.1}$$

Put $\bar{J}_t^k := \operatorname{ess\,inf}_{\pi \in \mathcal{B}_t^k} \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathcal{G}_t]$. Since $\mathcal{A}_t^k \subset \mathcal{B}_t^k$, we get $\bar{J}_t^k \leq J_t^k$. To prove the other inequality, we state that there exists a sequence $(\pi^n)_{n \in \mathbb{N}}$ of $\mathcal{A}_t^k$ such that $\pi^n \to \pi,\ dt \otimes d\mathbb{P}$ a.s., for any $\pi \in \mathcal{B}_t^k$. Let us define π^n by

$$\pi_s^n = \pi_s \mathbb{1}_{s \leq \tau_n},\ \forall\ s \in [t,T]\,,$$

where τ_n is the stopping time defined by $\tau_n = \inf\{s \geq t, |X_s^{t,\pi}| \geq n\}$.
It is clear that for each $n \in \mathbb{N}$, $\pi^n \in \mathcal{A}_t^k$. Thus, $\exp(-\gamma X_T^{t,\pi^n}) = \exp(-\gamma X_{T \wedge \tau_n}^{t,\pi}) \xrightarrow{a.s.} \exp(-\gamma X_T^{t,\pi})$ as $n \to +\infty$. By (G.1), the set of random variables $\{\exp(-\gamma X_T^{t,\pi}),\ \pi \in \mathcal{B}_t^k\}$ is uniformly integrable. Hence, $\mathbb{E}[\exp(-\gamma(X_T^{t,\pi^n} + \xi))|\mathcal{G}_t] \to \mathbb{E}[\exp(-\gamma(X_T^{t,\pi} + \xi))|\mathcal{G}_t]$ a.s. as $n \to +\infty$. Therefore, we have $J_t^k \leq \bar{J}_t^k$ a.s. which ends the proof.

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