CONSISTENCY OF VANISHING SMOOTH FICTITIOUS PLAY

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Abstract. We discuss consistency of *Vanishing Smooth Fictitious Play*, a strategy in the context of game theory, which can be regarded as a *smooth fictitious play procedure*, where the smoothing parameter is time-dependent and asymptotically vanishes. This answers a question initially raised by Drew Fudenberg and Satoru Takahashi.

1. Introduction and background

A recurring question in the theory of repeated games is to define properly a notion of *good strategy* for a player facing an unknown environment. Consequently, in this paper, we are not concerned with the formalisation of strategic interactions between rational players, but rather between a *decision maker* and *nature*. Not much is known about the latter, no assumption is made on its payoff function, its thinking process or its rationality. We take the point of view of the former, whose objective is to maximize his/her average payoff in the long run. A naive approach in this direction is to assume that the game is zero-sum and to look for optimal strategies. However, the fact that his/her opponent is not rational could lead to bad outcomes. A possible definition of *good strategy* for the decision maker has been proposed by Hannan (see [12]). It is closely related to the concept of *regret*. After n stages, the regret of the decision maker is the difference between the payoff that he could have obtained if he knew in advance the empirical moves of nature and the average payoff he actually got. A good strategy for player 1 may then be defined as a strategy which ensures that, regardless of the behaviour of nature, the regret asymptotically goes to zero. Such a strategy is called *consistent*. For instance, *fictitious play* strategies are known to be non-consistent ([11]) while *smooth fictitious play* strategies have been shown to be "almost" consistent by Fudenberg and Levine [10] (see section 1.2 for a rigorous expository). The main objective of this work is to discuss the consistency of *vanishing smooth fictitious play* (VSFP). VSFP is a time-varying smooth fictitious play with a smoothing parameter decreasing to zero. It initially behaves like smooth fictitious play and asymptotically like fictitious play. This answers a question that was raised to us by Drew Fudenberg and Satoru Takahashi.

1.1. **Notation.**

We consider a two-player finite game in normal form. I and L are the (finite) set of moves of respectively player 1 (the decision maker) and player 2 (the nature). The map $\pi : I \times L \to \mathbb{R}$ denotes the payoff function of player 1. The sets of mixed strategies available to players are denoted $X = \Delta(I)$ and $Y = \Delta(L)$, where

$$\Delta(I) := \left\{ x \in \mathbb{R}^I_+ \ \mid \ \sum_{i \in I} x_i = 1 \right\},$$

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and analogously for $\Delta(L)$. As usual π is extended to $X \times Y$ by multilinearity:

$$\forall x \in X,\, y \in Y,\, \pi(x,y) = \sum_{i\in I}\sum_{l\in L} \pi(i,l)x_i y_l.$$

In the following, $(i_1, ..., i_n, ...)$ (respectively $(l_1, ..., l_n, ...)$) will denote the sequence of actions picked by player 1 (resp. player 2 or his/her opponents). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, endowed with a filtration $(\mathcal{F}_n)_n$. Formally, a *strategy* for player 1 is an adapted process $(i_n)_n$ on $(\Omega, (\mathcal{F}_n)_n, \mathbb{P})$. In the whole paper, we assume that there player 1 observes his/her payoff function as well as the actions of his/her opponent. Hence, if we assume that the agents choose their next actions according only to the past actions then a strategy for player 1 can simply be seen as a map from $\cup_n (I \times L)^n$ to $\Delta(I)$, which to a given finite history $h_n = (i_1, l_1, ..., i_n, l_n)$ associates a mixed action $\sigma(h_n)$. Throughout, we assume that the agents play independently: specifically, for $(i,l) \in I \times L$, we have

$$\mathbb{P}\left(i_{n+1} = i,\ l_{n+1} = l \mid \mathcal{F}_n\right) = \mathbb{P}\left(i_{n+1} = i \mid \mathcal{F}_n\right)\mathbb{P}\left(l_{n+1} = l \mid \mathcal{F}_n\right).$$

Finally, we call

$$\overline{x}_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{i_k}$$

the average moves of player 1 at time n, $\overline{y}_n$ the average moves of player 2 and

$$\overline{\pi}_n = \frac{1}{n}\sum_{k=1}^{n} \pi(i_k, l_k)$$

the average payoff to player 1.

1.2. **Consistency, definition and comments.** We now introduce $\Pi : Y \to \mathbb{R}$, defined by

$$\Pi(y) := \max_{x\in X} \pi(x,y).$$

A strategy is *consistent* if, against any strategy of nature, it does at least as much as if their empirical moves was known in advance. More precisely, let us define the *average regret evaluation* along a sequence of moves $h_n = (i_1, l_1, ...i_n, l_n)$:

$$e_n := \max_{x\in X} \pi\left(x, \frac{1}{n}\sum_{m=1}^{n} l_m\right) - \frac{1}{n}\sum_{m=1}^{n} \pi(i_m, l_m) = \Pi(\overline{y}_n) - \overline{\pi}_n.$$

Definition 1.1. *A strategy for player 1 is said to be consistent if, for any strategy of nature,*

$$\limsup_n e_n \leq 0,\ \ \mathbb{P} - \textit{almost surely.}$$

It is η-consistent if

$$\limsup_n e_n \leq \eta,\ \ \mathbb{P} - \textit{almost surely.}$$

For a recent comprehensive overview about consistency in games, see [17] (in french). Given $y \in Y$, we call $br(y)$ the set of best responses of player 1 to y, namely,

$$br(y) = \mathsf{Argmax}_{x\in X}\pi(x,y).$$

The discrete-time *fictitious play* (FP) process has been introduced in [5]. We say that player 1 uses a FP strategy, with prior $\overline{y}_0$ if, for $n \geq 1$,

$$\mathbb{P}(i_{n+1} = \cdot \mid \mathcal{F}_n) \in br(\gamma_n),$$

where $\gamma_n = \frac{1}{n+1}\overline{y}_0 + \frac{n}{n+1}\overline{y}_n$. It is well known that this strategy is not consistent. A simple example is given by the following (see e.g. [11]).

Example 1.2. Assume that the game is matching pennies, i.e. the payoff matrix of player 1 is given by

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

and the prior is $\overline{y}_0 = (1/3, 2/3)$. If player two acts accordingly to the deterministic rule *heads* (H) on odd stages and *tails* (T) on even stages, then player 1 and 2 always play the opposite and the average regret satisfies $\lim_{n\to\infty} e_n = 1/2$.

However, η-consistency can be achieved by small modifications of fictitious play, which are usually called *stochastic fictitious play strategies*. Originally, stochastic fictitious play was introduced by Fudenberg and Kreps in [9] and the concept behind this is that players use fictitious play in a game where payoff functions are perturbed by some random variables in the spirit of Harsanyi [13]. On the subject, see also [10], [11] or [2],. In this paper, we adopt another point of view and assume that player 1 chooses to randomize his/her moves by adding a small perturbation function to his/her initial payoff map π.

Let $L : x \in X \mapsto L(x) = -\sum_{i\in I} x_i \log x_i$ be the entropy function. We introduce the perturbed payoff function $\tilde{\pi}$ defined, for $x \in X$, $y \in Y$ and $\beta > 0$ by

$$\tilde{\pi}(x, y, \beta) = \pi(x, y) + \frac{1}{\beta} L(x).$$

Notice that L is a particular case of perturbation function (see [15] for a detailed analysis on the subject). The function $\tilde{\pi}$ enjoys the following properties:

- (i) For all $y \in Y$, $\beta > 0$, $\mathsf{Argmax}_{x\in X} \tilde{\pi}(x, y, \beta)$ reduces to one point and defines a continuous map $\mathbf{br}$ from $Y \times \mathbb{R}_+^*$ to X.
- (ii) $D_1\tilde{\pi}\left(\mathbf{br}(y, \beta), y, \beta\right) \cdot D_1\mathbf{br}(y, \beta) = 0$.

The map $y \in Y \mapsto \mathbf{br}(y, \beta)$ is usually called a *smooth best response* map.

Definition 1.3. *Player* 1 *plays accordingly to a smooth fictitious play strategy, with the parameter* $\beta > 0$ *(SFP(β)) if*

$$\mathbb{P}\left(i_{n+1} = i \mid \mathcal{F}_n\right) = \mathbf{br}(\overline{y}_n, \beta)_i, \;\; \forall n \geq 1.$$

Theorem 1.4 (Fudenberg and Levine, 1995)**.** *For any $\eta > 0$, there exists $\beta_0 > 0$ such that a SFP(β) strategy is η-consistent for any $\beta > \beta_0$.*

Smooth fictitious play is closely related to the so-called *exponential weight algorithm* and also to the *follow the perturbed leader algorithm* (see [6], chapters 4.2 and 4.3), even if the link with the latter is less obvious. In [18], the authors discuss the consistency of continuous-time versions of FP and SFP.

1.3. Vanishing smooth fictitious play.

A related natural strategy is given by the following.

Definition 1.5. *Let $\left(\beta_n\right)_n$ be a sequence going to infinity. A* vanishing smooth fictitious play strategy *induced by (β_n) (VSFP(β_n)) for player* 1 *is given by*

$$\mathbb{P}\left(i_{n+1} = i \mid \mathcal{F}_n\right) = \mathbf{br}\left(\overline{y}_n, \beta_n\right)_i \;\; \forall n \geq 1.$$

Consistency is not verified for any choice of $(\beta_n)_n$. If this sequence increases too fast, then consistency might fail to hold, as shown by the following example.

Example 1.6. Assume that, once again the game is 2-player matching pennies and that nature uses the deterministic strategy described in example 1.2. Then, if player one plays accordingly to a VSFP strategy induced by $\beta_n = n$ and prior $\overline{y}_0 = (1/3, 2/3)$, we have

$$\gamma_{2n} = \left(\frac{1}{2} - \frac{1}{6(2_n+1)}, \frac{1}{2} + \frac{1}{6(2_n+1)}\right) \text{ and } \gamma_{2n+1} = \left(\frac{1}{2} + \frac{1}{6(n+1)}, \frac{1}{2} - \frac{1}{6(n+1)}\right).$$

After a few lines of calculus (left to the reader) one gets:

$$\mathbb{E}\left(\delta_{l_{2n+1}} \mid \mathcal{F}_n\right) = \mathbf{br}(\gamma_{2n}, \beta_{2n}) = \left(\frac{1}{1+\exp\left(\frac{2n}{3(2n+1)}\right)}, \frac{1}{1+\exp\left(-\frac{2n}{3(2n+1)}\right)}\right).$$

Hence $(\pi(i_{2n+1}, l_{2n+1}))_n$ is a sequence of independent random variable taking values in $\{0,1\}$, such that

$$\lim_n \mathbb{P}\left(\pi(i_{2n+1}, l_{2n+1}) = 1\right) = \lim_n \frac{1}{1+e^{2n/3(2n+1)}} = \frac{1}{1+e^{1/3}}.$$

Similarly, $(\pi(i_{2n}, l_{2n}))_n$ is a sequence of independent random variables taking values in $\{0,1\}$ and

$$\lim_n \mathbb{P}\left(\pi(i_{2n}, l_{2n}) = 1\right) = \frac{1}{1+e^{2/3}}.$$

Therefore, consistency is not satisfied for VSFP strategies with $\beta_n = n$. We now can state our main result

Theorem 1.7. *Any VSFP(β_n) strategy, with $\beta_n \leq n^\beta$ for some $\beta < 1$, is consistent.*

In [4], the authors prove the same result as Theorem 1.4 using stochastic approximations methods. Specifically, they consider the state variable $(\overline{x}_n, \overline{y}_n, \overline{\pi}_n)_n$, write it as a stochastic approximation process relative to some differential inclusion, and prove that it almost surely converges to the set

$$\{(x,y,u): \ \Pi(y) - \pi \leq \eta\}.$$

This is the approach taken in this paper. In section 2 we provide some general stability results for non-autonomous differential inclusions, namely we estimate the deviation of so-called *perturbed solutions* from the set of solutions curves. A concept of Lyapunov function for non-autonomous systems is used to derive the main result of this section, Proposition 2.13, which gives a qualitative result on the limit set of *good* perturbed solutions. We then apply these results to stochastic approximation processes relative to a non-autonomous differential inclusion in Section 3. The proof of our main result, Theorem 1.7, is given in Section 4. It consists to show that $(\overline{x}_n, \overline{y}_n, \overline{u}_n)_n$ is almost surely a good perturbed solution and to apply the results of Section 2.

2. Stability of one-sided Lipschitz differential inclusions

Let $M \subset \mathbb{R}^d$. Consider a set-valued map $F : \mathbb{R}_+ \times M \rightrightarrows M$ taking values in the set of non-empty, compact, convex subsets of M. Given $I = [a,b]$, let us consider the *non-autonomous differential inclusion*

$$\dot{\mathbf{x}}(s) \in F(s, \mathbf{x}(s)), \ \ s \in I \tag{1}$$

A map $\mathbf{x} : I \to M$ is a solution of (1) if it is absolutely continuous and, for almost every $s \in [0,T]$, $\dot{\mathbf{x}}(s) \in F(s, \mathbf{x}(s))$. For $A \subset M$ we let $F^{-1}(A) = \{(s,x) \in I \times M : \ F(s,x) \cap A \neq \emptyset\}$. We say that F is measurable if $F^{-1}(A)$ is measurable, for any closed set $A \subset M$. It is *upper semi-continuous* (USC) (resp. *lower semi-continuous* (LSC)) if, for any closed (resp. open) set

$A \subset M$, $F^{-1}(A)$ is closed (resp. open) in $I \times M$. If M is compact, F is upper semi-continuous if and only if its graph

$$Gr(F) := \{(s, x, y) \in I \times M \times M : \; y \in F(s, x)\}$$

is closed. We call d_H the Hausdorff distance, given by

$$d_H(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right\}.$$

Recall that d_H is a pseudo-metric on the set of non-empty subsets of M and a metric if we restrict to the non-empty compact sets of M. We say that F is *Hausdorff continuous* if it is continuous with respect of the Hausdorff metric:

$$\lim_{t'\to t, x'\to x} d_H(F(t, x), F(t', x')) = 0.$$

If F is Hausdorff continuous, we call it L-Lipschitz, for an integrable function $L : I \to \mathbb{R}_+$ if

$$d_H(F(t, x), F(t, x')) \leq L(t)\|x - x'\|, \text{ for a.e. } t \in I, \; \forall\, x, x'$$

We now introduce a weaker regularity condition:

Definition 2.1 (Relaxed One-sided Lipschitz)**.** *we say that the set-valued map F is* Relaxed One-sided Lipschitz *(ROSL) on $I \times M$ if there exists an integrable map $L : I \to M$ such that, for any t, t' in I $x, x' \in M$ and any $y \in F(t, x)$ there exists $y' \in F(t', x')$ with*

$$(x' - x \mid y' - y) \leq L(t)\|x' - x\|^2, \;\; \forall t \in I.$$

Remark 2.2. *If F is $L(\cdot)$-Lipschitz then it is $L(\cdot)$-ROSL.*

The question of existence of solutions to (1) has been studied extensively. One of the first result on the topic was proved by Filippov (see [8]) and says that if $F(\cdot, \cdot)$ is Hausdorff continuous on any closed set of $I \times M$ then, for any $x_0 \in M$, there exists a solution $\mathbf{x}(\cdot)$ of (1), with $\mathbf{x}(a) = x_0$. Under less restrictive assumptions, the same result still holds (see [16])

Theorem 2.3 (Olech, 1975)**.** *Assume that*

(i) $s \mapsto F(s, x)$ is measurable, for each $x \in M$,
(ii) for any $s \in I$, the map $x \mapsto F(s, x)$ has a closed graph,
(iii) The map F is uniformly bounded, i.e., $\sup_{s,x} \sup_{y\in F(s,x)} \|y\| \leq \|F\|_\infty < +\infty$.

Then there exists a solution $\mathbf{x}(\cdot)$ of (1), with $\mathbf{x}(a) = x_0$.

On the topic, see also [14].

In the remaining of this section, we assume that F satisfies the assumptions of the previous theorem. Such a F will be called a *regular* set valued map. The set of solution trajectories on $[a, b]$ (resp. starting in x_0) will be labelled $\mathcal{S}(a, b)$ (resp. $\mathcal{S}(x_0, a, b)$).

Theorem 2.4. *Let $W : I \to M$ be an absolutely continuous function such that there exists a measurable map $\overline{v} : I \to M$ and a bounded measurable map $r : I \to \mathbb{R}_+$ which satisfy, for almost every $s \in I$,*

$$d(W(s), \overline{v}(s)) \leq r(s), \;\; \dot{W}(s) \in F(s, \overline{v}(s)).$$

Then

a) if F is ROSL with respect to the integrable function L, then there exists a solution $\mathbf{x} : I \to M$ of (1) such that $\mathbf{x}(a) = W(a)$ and

$$\sup_{s\in[a,b]} \|\mathbf{x}(s) - W(s)\|^2 \leq \int_a^b \alpha(s) \exp\left(4 \int_s^b L(\tau) d\tau\right),$$

where $\alpha(s) = 4L(s)r^2(s) + 4r(s)\|F\|_\infty$.

b) if we now assume that F is Lipschitz continuous, with respect to L then the conclusions of a) trivially still hold and $\mathbf{x}$ can also be chosen such that

$$\sup_{s\in[a,b]} \|\mathbf{x}(s)-W(s)\| \leq \int_a^b r(s)L(s)\exp\left(\int_s^b L(\tau)d\tau\right).$$

Proof. We prove the first point. Consider the set-valued map $G : I \times M \rightrightarrows M$ given by

$$G(s,x) := \left\{v \in F(s,x) : \ (x-W(s) \mid v-\dot{W}(s)) \leq 2L(s)\|x-W(s)\|^2 + \frac{1}{2}\alpha(s)\right\}.$$

For any (s,x), the set $G(s,x)$ is non-empty. Indeed, by the ROSL condition, since $\dot{W}(s) \in F(s,\overline{v}(s))$, there exists $v \in F(s,x)$ such that

$$(x-\overline{v}(s) \mid v-\dot{W}(s)) \leq L(s)\|x-\overline{v}(s)\|^2.$$

Hence we have

$$\begin{aligned}(x-W(s) \mid v-\dot{W}(s)) &\leq L(s)\|x-\overline{v}(s)\|^2 + \|\overline{v}(s)-W(s)\|(\|v\|+\|\dot{W}(s)\|)\\ &\leq 2L(s)\|W(s)-x\|^2 + 2L(s)r(s)^2 + 2r(s)\|F\|_\infty\\ &= 2L(s)\|W(s)-x\|^2 + \frac{1}{2}\alpha(s).\end{aligned}$$

Now clearly, the set $G(s,x)$ is compact and convex. The map $x \mapsto G(s,x)$ has a closed graph, for any $s \in I$. Finally It is measurable in s since every map involved is measurable. Consequently, there exists a solution to the non-autonomous differential inclusion

$$\dot{\mathbf{x}}(s) \in G(s,\mathbf{x}(s)),$$

with initial condition $\mathbf{x}(a) = W(a)$. In particular, $\mathbf{x}$ is a solution of (1) and we also have, for almost every s

$$(\mathbf{x}(s)-W(s) \mid \dot{\mathbf{x}}(s)-\dot{W}(s)) \leq 2L(s)\|W(s)-\mathbf{x}(s)\|^2 + \frac{1}{2}\alpha(s).$$

Hence , for almost every s, we have

$$\begin{aligned}\frac{d}{ds}\|\mathbf{x}(s)-W(s)\|^2 &= 2(\mathbf{x}(s)-W(s) \mid \dot{\mathbf{x}}(s)-\dot{W}(s))\\ &\leq 4L(s)\|W(s)-\mathbf{x}(s)\|^2 + \alpha(s)\end{aligned}$$

and point a) follows from the differential form of Gronwall's lemma.

When the Lipschitz continuity holds, let us consider the set-valued map $H : I \times M \rightrightarrows M$ given by

$$H(s,x) := \left\{v \in F(s,x) : \ \|v-\dot{W}(s)\| \leq L(s)\|x-W(s)\| + L(s)r(s)\right\}.$$

The fact that H has non-empty values follows from Lipschitz continuity: given s and x, since $\dot{W}(s) \in F(s,\overline{v}(s))$, there exists $v \in F(s,x))$ such that

$$\|v-\dot{W}(s)\| \leq L(s)\|x-\overline{v}(s)\| \leq L(s)\left(\|x-W(s)\| + \|W(s)-\overline{v}(s)\|\right).$$

Hence $v \in H(s,x) \neq \emptyset$. Also $H(s,x)$ is convex and compact, the map $x \mapsto H(s,x)$ has a closed graph and $s \mapsto H(s,x)$ is measurable. Thus, there exists a solution $\mathbf{x}$ to the non-autonomous differential inclusion

$$\dot{\mathbf{x}}(s) \in H(s,\mathbf{x}(s)),$$

with initial condition $\mathbf{x}(a) = W(a)$. In particular, $\mathbf{x}$ is a solution of (1) and we also have, for almost every s

$$\|\dot{\mathbf{x}}(s)-\dot{W}(s)\| \leq L(s)\|\mathbf{x}(s)-W(s)\| + L(s)r(s)$$

By Gronwall's corollary 5.1, we then have

$$\sup_{s\in I}\|\mathbf{x}(s)-W(s)\|\le\int_a^b L(s)r(s)\exp\left(\int_s^b L(\tau)d\tau\right)ds$$

and point b) is proved. ∎

Corollary 2.5. *Let $v: I\to M$ be an absolutely continuous map. Assume that there exist measurable maps $\overline{v}: I\to M$, $\delta: I\to\mathbb{R}_+$ bounded and $\overline{U}: I\to M$ integrable such that, for almost every $s\in I$,*

$$\dot{v}(s)-\overline{U}(s)\in F(s,\overline{v}(s)),\ \ \|v(s)-\overline{v}(s)\|\le\delta(s).$$

Then if F is $L(\cdot)$-Lipschitz, there exists a solution $\mathbf{x}$ on I such that $\mathbf{x}(a)=v(a)$ and

$$\sup_{s\in[a,b]}\|v(s)-\mathbf{x}(s)\|\le R(a,b),$$

where

$$R(a,b)=\Delta(a,b)\exp\left(\int_a^b L(\tau)d\tau\right)+\sup_{s\in[a,b]}\delta(s)\left(\exp\left(\int_a^b L(\tau)d\tau\right)-1\right)\tag{2}$$

and $\Delta(a,b):=\sup_{s\in[a,b]}\|\int_a^s\overline{U}(\tau)d\tau\|$.

Proof. Define $W: I\to M$ by

$$W(s):=v(s)-\int_a^s\overline{U}(\tau)d\tau.$$

Clearly, W is absolutely continuous and, for any s for which v is differentiable, we have $\dot{W}(s)=\dot{v}(s)-\overline{U}(s)\in F(s,\overline{v}(s))$. Additionally,

$$\|W(s)-\overline{v}(s)\|=\|v(s)-\overline{v}(s)\|+\|\int_a^s\overline{U}(\tau)d\tau\|\le\delta(s)+\left\|\int_a^s\overline{U}(\tau)d\tau\right\|.$$

By a direct application of Theorem 2.4 with $r(s)=\delta(s)+\left\|\int_a^s\overline{U}(\tau)d\tau\right\|$, we have

$$\sup_{s\in[a,b]}\|v(s)-\mathbf{x}(s)\|\le\Delta(a,b)+\int_a^b L(s)\left(\delta(s)+\|\int_a^s\overline{U}(\tau)d\tau\|\right)\exp\left(\int_s^b L(\tau)d\tau\right)\le R(a,b).$$

The proof is complete. ∎

2.1. **Uniform Lyapunov function and perturbed solutions.**

Definition 2.6. *Let A be a compact set in M and U be an open neighbourhood of A. A smooth map $\Phi:\mathbb{R}_+\times U\to\mathbb{R}_+$ is an* uniform Lyapunov function *for the non-autonomous differential inclusion (1) with respect to A if the following hold:*

a)
$$A=\{x\in U:\ 0\in\mathcal{L}((\Phi(s,x))_s)\},$$
where $\mathcal{L}((\Phi(s,x))_s):=\{u\in M:\ \exists s_n\uparrow+\infty,\ \lim_n\Phi(s_n,x)=u\}$ is the limit set of the map $s\mapsto\Phi(s,x)$.

b) There exists two maps $\lambda:\mathbb{R}_+^*\to]0,1[$ and $\varepsilon:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ with the property that
$$\lim_{T\to+\infty}\lambda(T)=0,\quad\lim_{T\to0,t\to+\infty}\varepsilon(t,T)=0\ \text{ and }\ \lim_{t\to+\infty}\varepsilon(t,T)=0\ \forall T>0;$$
and, for any $t>0,T>0$ and any solution $\mathbf{x}$ on $[t,t+T]$, we have
$$\Phi(t+s,\mathbf{x}(t+s))\le\lambda(s)\Phi(t,\mathbf{x}(t))+\varepsilon(t,T),\ \forall s\in[0,T].$$

If $U=M$ then Φ is called a *global uniform Lyapunov function.*

Definition 2.7. *A compact set A is* asymptotically stable *if it admits an open neighbourhood U such that, for any $\varepsilon > 0$, there exists $\underline{t} > 0$ and $T > 0$ with the property that any solution starting in M at time $t > \underline{t}$ is in $N^\varepsilon(A)$ after time $t + T$.*

Lemma 2.8. *Assume that Φ satisfies the property $b)$ and the following property (stronger than $a)$):*

$a')$ there exists a continuous map $g : U \to \mathbb{R}+$ such that

$$A = \{x \in M : g(x) = 0\}, \ \|g(x) - \Phi(s,x)\| \to_{s\to+\infty} 0,$$

uniformly in $x \in M$.

Then A is asymptotically stable.

Lemma 2.9. *Let $(\Phi_k)_{k\geq k_0}$, $(\lambda_k)_{k\geq k_0}$ and $(\eta_k)_{k\geq k_0}$ be positive sequences of real numbers such that;*

(i) $0 < \lambda_k < 1$ $\forall k \geq k_0$ and

$$\Phi_{k+1} \leq \lambda_k \Phi_k + \eta_{k+1};$$

(ii) Denoting $H_k := \Pi_{i=k_0}^{k-1}\lambda_i$ and $\tilde{H}_k = H_k \sum_{i=0}^{k-1} H_i^{-1}\eta_i$; $\lim_{k\to\infty} H_k = \lim_{k\to\infty} \tilde{H}_k = 0$.

Then $\lim_{k\to\infty} \Phi_k = 0$.

Proof. Without loss of generality, we assume that $k_0 = 0$. A simple recursive argument yields

$$\Phi_k \leq H_k \left(\Phi_0 + \sum_{i=0}^{k-1} H_i^{-1}\eta_i\right)$$

and the proof is complete. ■

Lemma 2.10. *The conditions of previous Lemma are verified in the following cases:*

a) $\lambda_k = \lambda < 1$ and $\lim_{k\to\infty} \eta_k = 0$,

b) $\lim_{k\to\infty} H_k = 0$ and $\sum_i \eta_i < +\infty$.

Proof. For point $a)$, $H_k = \lambda^k$ and we have

$$\begin{aligned}
\tilde{H}_{k+k'} &= \lambda^{k+k'}\left(\sum_{i=0}^{k} H_i^{-1}\eta_i + \sum_{i=k+1}^{k+k'-1} H_i^{-1}\eta_i\right) \\
&\leq \lambda^{k'} \max_{i=0,\dots,k} \eta_i + \eta_{k+1}\sum_{i=0}^{k'-1}\lambda^i \\
&\leq \lambda^{k'} \max_{i=0,\dots,k} \eta_i + \eta_{k+1}\frac{1}{1-\lambda},
\end{aligned}$$

which gives the result.
For the second point, remember that $(H_k)_k$ is a decreasing sequence. Hence

$$\begin{aligned}
\tilde{H}_{k+k'} &= H_{k+k'}\left(\sum_{i=0}^{k} H_i^{-1}\eta_i + H_{k+k'-1}^{-1}\sum_{i=k+1}^{k+k'}\eta_i\right) \\
&\leq H_{k+k'}\left(\sum_{i=0}^{k} H_i^{-1}\eta_i\right) + \sum_{i=k+1}^{+\infty}\eta_i.
\end{aligned}$$

Given $\varepsilon > 0$, by choosing k large enough, the second term is smaller than ε. Then we can pick k' large enough so that the first term is also smaller than ε and the proof is complete. ■

Definition 2.11. *A map $v : \mathbb{R}_+ \to M$ is a* perturbed solution *of the non-autonomous differential inclusion $\dot{\mathbf{x}}(s) \in F(s, \mathbf{x}(s))$ if*

(*i*) *v is absolutely continuous,*

(*ii*) *$s \mapsto \overline{U}(s)$ is a locally integrable function such that*

$$\Delta(t, t+T) := \int_t^{t+T} \overline{U}(s) ds \to_{t \to +\infty} 0,$$

(*iii*) *$\dot{v}(s) - \overline{U}(s) \in F(s, \overline{v}(s))$ for some measurable map $\overline{v} : \mathbb{R}_+ \to M$ such that*

$$\|v(s) - \overline{v}(s)\| \leq \delta(s),$$

with $\delta(s) \downarrow_s 0$.

Remark 2.12. *Notice that, in the autonomous case, this is Definition (II) in [3]*

We say that Φ is uniformly Lipschitz if there exists $L_\Phi > 0$ such that, for any $s \geq 0$ and $v, v' \in M$,

$$\left|\Phi(s, v) - \Phi(s, v')\right| \leq L_\Phi \|v - v'\|.$$

Notice that this condition is verified under the assumptions of Lemma 2.8. We now state the main result of this section

Proposition 2.13. *Assume that v is a perturbed solution relative to a regular Lipschitz map F (with $L : \mathbb{R}_+ \to \mathbb{R}_+$) and that there exists a sequence of positive real numbers $(T_k)_k$ such that*

(*i*) *$S_k := \sum_{i=1}^k T_i \to +\infty$,*

(*ii*) *there exists $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$*

$$R(S_k, S_{k+1}) \leq \gamma_k,$$

with R defined by (2),

(*iii*) *Φ is a global uniform Lyapunov function with respect to a set A such that, denoting $H_k := \Pi_{i=k_0+1}^k \lambda(T_i)$ and $\eta_k := \varepsilon(S_{k-1}, T_k) + \gamma_{k-1}$, we have*

$$\lim_{k \to +\infty} H_k \left(H^{-1} \cdot \eta\right)_k = 0.$$

(*iv*) *the family $\Phi(s, \cdot)$ is uniformly Lipschitz:*

Then the limit set of v is contained in A.

Proof. First recall that, by Corollary 2.5, for any $k \in \mathbb{N}$, there exists a solution $\mathbf{x^k}$ on $[S_k, S_{k+1}]$ such that $\mathbf{x^k}(S_k) = v(S_k)$ and

$$\sup_{s \in [S_k, S_{k+1}]} \|v(s) - \mathbf{x^k}(s)\| \leq R(S_k, S_{k+1}).$$

By (*ii*) the sequence of solutions curves $(\mathbf{x^k})_{k \geq k_0}$ is such that

$$\sup_{s \in [S_k, S_{k+1}]} \|v(s) - \mathbf{x^k}(s)\| \leq \gamma_k.$$

On the other hand, (*iii*) implies that, for any $k \geq k_0$,

$$\Phi(S_{k+1}, \mathbf{x^k}(S_{k+1})) \leq \lambda(T_{k+1}) \Phi(S_k, \mathbf{x^k}(S_k)) + \varepsilon(S_k, T_{k+1}).$$

Hence, by (*iv*),

$$\begin{aligned}
\Phi(S_{k+1}, v(S_{k+1})) &\leq \Phi(S_{k+1}, \mathbf{x^k}(S_{k+1})) + L_\Phi \left\| v(S_{k+1}) - \mathbf{x^k}(S_{k+1}) \right\| \\
&\leq \lambda(T_{k+1}) \Phi(S_k, v(S_k)) + L_\Phi \gamma_k + \varepsilon(S_k, T_{k+1}) \\
&= \lambda(T_{k+1}) \Phi(S_k, v(S_k)) + \eta_{k+1}
\end{aligned}$$

Calling $\Phi_k := \Phi(S_k, v(S_k))$ and $\lambda_k := \lambda(T_{k+1})$ we have $\Phi_k \to 0$ by Lemma 2.9. Now let v_* be a limit point of $v(s)$: $v_* = \lim_n v(s_n)$, for some sequence $s_n \uparrow_n +\infty$. Call $k(n) := \sup\{k \in \mathbb{N} : S_k \leq s_n\}$. For n large enough, $k(n) \geq k_0$ and

$$\Phi(s_n, v(s_n)) \leq \lambda(s_n - S_{k(n)})\Phi(S_{k(n)}, v(S_{k(n)}) + L_\Phi \gamma_{k(n)} + \varepsilon(S_{k(n)}, s_n - S_{k(n)}) \to_{n\to+\infty} 0.$$

We therefore have

$$\Phi(s_n, v_*) \leq \Phi(s_n, v(s_n)) + L_\Phi \|v_* - v_n\| \to_{n\to+\infty} 0.$$

Consequently $0 \in \mathcal{L}(\Phi(s, v_*))$ and the proof is complete. ■

3. Stochastic approximations

Consider a discrete time stochastic process $(v_n)_n$ in M, defined by the recursive formula

$$v_{n+1} - v_n - \gamma_{n+1}U_{n+1} \in \gamma_{n+1}F_n(v_n), \tag{3}$$

where $F_n : M \rightrightarrows M$ is a set-valued map, $(\gamma_n)_n$ is a positive sequence, decreasing to 0 and $(U_n)_n$ a sequence of M-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Set $\tau_n := \sum_{i=1}^n \gamma_i$ and $m(s) := \sup\{j \mid \tau_j \leq s\}$. We make the following additional assumptions:

(i) For all $c > 0$,
$$\sum_n e^{-c/\gamma_n} < \infty,$$

(ii) $(U_n)_n$ is uniformly bounded and
$$\mathbb{E}\left(U_{n+1} \mid \mathcal{F}_n\right) = 0,$$

(iii) The map $F : \mathbb{R}_+ \times M \rightrightarrows M$, given by
$$F(t, v) := F_{m(t)}(v)$$
is regular.

We call $v(\cdot)$ the continuous time affine interpolated process induced by $(v_n)_n$ and $\overline{\gamma}(\cdot)$ (resp. $\overline{U}(\cdot)$) the piecewise constant deterministic processes induced by $(\gamma_n)_n$ (resp. $(U_n)_n$):

$$v(\tau_i + s) = x_i + s\frac{v_{i+1} - v_i}{\gamma_{i+1}} \text{ for } \; s \in [0, \gamma_{i+1}], \;\; \overline{\gamma}(\tau_i + s) := \gamma_{i+1} \text{ for } \; s \in [0, \gamma_{i+1}[,$$

and analogously for $\overline{U}$.

Lemma 3.1. *For almost every $s \in \mathbb{R}_+$, $v(\cdot)$ is differentiable and we have*

$$\dot{v}(s) - \overline{U}(s) \in F(s, v_{m(s)}).$$

Proof. We have

$$v(s) = v_{m(s)} + \frac{v_{m(s)+1} - v_{m(s)}}{\gamma_{m(s)+1}}(t - \tau_{m(s)})$$

Hence, if $s \notin \{\tau_n,\, n \in \mathbb{N}^*\}$, $v(\cdot)$ is differentiable and

$$\dot{v}(s) = \frac{v_{m(s)+1} - v_{m(s)}}{\gamma_{m(s)+1}}.$$

Consequently

$$\dot{v}(s) - \overline{U}(s) \in F_{m(s)}(v_{m(s)}) = F(s, v_{m(s)}).$$

The proof is complete ■

In the sequel, we use the notation $\overline{v}(s) := v_{m(s)}$. Notice that $\overline{v}$ is a piecewise constant map on $\mathbb{R}_+$.

3.1. **Particular case $\gamma_n = 1/n$.** We focus here on the classical case where the step size is $\frac{1}{n}$. We then have $\tau_n \sim \log n$ and $m(s) = \mathcal{O}(e^s)$[1]. Given positive real numbers t and T, we call $\Delta(t, t+T)$ the random variable

$$\int_t^{t+T} \overline{U}(s)ds.$$

The following lemma is classical (it is proved in [7] or [1] for instance)

Lemma 3.2. *There exists positive constants C and C' (depending on $\|U\|_\infty$) such that, for any $\alpha > 0$,*

$$\mathbb{P}\left(\Delta(t, t+T) \geq \alpha\right) \leq C \exp\left(\frac{-\alpha^2 e^t}{C'T}\right).$$

Notice that, by Lemmas 3.1 and 3.2 and a Borel-Cantelli argument, v is almost surely a perturbed solution, with $\delta(s) = A\overline{\gamma}(s) \leq 2Ae^{-s}$, where $A = \|F\|_\infty + \|U\|_\infty$.

Proposition 3.3. *Assume that F is Lipschitz, with Lipschitz function L such that $L(s) \leq Ls$. Then there exist $T > 0$, and $\gamma > 0$ such that, with probability one, there exists $k_0 \in \mathbb{N}$ with the property that points (i) and (ii) of Proposition 2.13 are verified for v, with $T_k = T$ and $\gamma_k = e^{-\gamma k}$.*

Proof. Pick $T < 1/2L$. Point (i) is trivially satisfied, as $S_k = kT$. We have $\int_{kT}^{(k+1)T} L\tau d\tau = kLT^2 + LT^2/2$. Hence

$$A\overline{\gamma}(S_k)\exp\left(\int_{kT}^{(k+1)T} L\tau d\tau\right) \leq 2A\exp\left(kT(LT-1) + LT^2/2\right).$$

On the other hand, by previous lemma,

$$\begin{aligned}
\mathbb{P}\left(\Delta(kT,(k+1)T)\exp\left(\int_{kT}^{(k+1)T} L\tau d\tau\right) \geq \frac{1}{2}e^{-\gamma k}\right) &\leq C\exp\left(\frac{-e^{-2\gamma k}e^{kT}}{4C'\exp\left(2kLT^2 + LT^2\right)T}\right) \\
&\leq C\exp\left(\frac{-\exp\left(k(T - 2LT^2 - 2\gamma)\right)}{4C'T\exp(LT^2)}\right).
\end{aligned}$$

Choose γ in $]0, T(1-2LT)/2[$. Then, for k large enough

$$A\overline{\gamma}(S_k)\exp\left(\int_{kT}^{(k+1)T} L\tau d\tau\right) \leq \frac{1}{2}e^{-\gamma k}.$$

Consequently, if we call A_k the event

$$\left\{(\Delta(S_k, S_{k+1}) + A\overline{\gamma}(S_k))\exp\left(\int_{S_k}^{S_{k+1}} L(\tau)d\tau\right) \geq e^{-\gamma k}\right\},$$

then

$$\mathbb{P}\left(A_k\right) \leq C\exp\left(\frac{-\exp\left(k(T - 2LT^2 - 2\gamma)\right)}{4C'T\exp(LT^2)}\right).$$

By an application of Borel-Cantelli lemma, with probability one, there exists $k_0 \in \mathbb{N}$ such that, for any $k \geq k_0$,

$$(\Delta(S_k, S_{k+1}) + A\overline{\gamma}(S_k))\exp\left(\int_{S_k}^{S_{k+1}} L(\tau)d\tau\right) \leq e^{-\gamma k},$$

and the proof is complete. ∎

[1]more precisely, $\frac{e-1}{e}e^s \leq m(s) \leq e^s - 1$

Proposition 3.4. *Assume that F is Lipschitz, with Lipschitz function L such that $L(s) \leq e^{\beta s}$, for some $\beta \in (0,1)$. Then if we call $T_k := (\beta k)^{-1}$ there exist some constant $\gamma > 1$ such that, with probability one, there exists $k_0 \in \mathbb{N}$ such that points (i) and (ii) of Proposition 2.13 are verified for v, with $\gamma_k = \gamma^{-k}$*

Proof. By our choice of the sequence T_k, $\exp(\beta S_k) \leq \exp(1 + \log k) \leq 3k$. Hence

$$\exp\left(\int_{S_k}^{S_{k+1}} L(\tau) d\tau\right) \leq \exp(T_{k+1} e^{\beta S_{k+1}}) \leq C_0,$$

for some constant C_0 which depends on β. Additionally, $\overline{\gamma}(S_k) \leq 2e^{-S_k} \leq 2k^{-1/\beta}$. Hence

$$A\overline{\gamma}(S_k) \exp\left(T_{k+1} e^{\beta S_{k+1}}\right) \leq \frac{3A}{k^{1/\beta}}.$$

Choose $\gamma \in (1, \frac{\beta+1}{2\beta})$. By Lemma 3.2,

$$\begin{aligned}
\mathbb{P}\left(\Delta(S_k, S_{k+1}) \exp\left(\int_{S_k}^{S_{k+1}} L(\tau d\tau\right) \geq \frac{1}{2k^\gamma}\right) &\leq C \exp\left(\frac{-k^{-2\gamma} e^{S_k}}{4C' C_0 T_{k+1}}\right) \\
&\leq C \exp\left(\frac{-k^{-2\gamma+1/\beta}}{C' C_0 \beta^{-1} (k+1)^{-1}}\right) \\
&\leq C \exp\left(\frac{-k^{-2\gamma+1+1/\beta}}{C_1'}\right)
\end{aligned}$$

for some positive constant C_1'. Now, since $\gamma < 1/\beta$, we have for k large enough

$$A\overline{\gamma}(S_k) \exp\left(T_{k+1} e^{\beta S_{k+1}}\right) \leq \frac{1}{2k^\gamma}.$$

Consequently, if we call A_k the event

$$\left\{(\Delta(S_k, S_{k+1}) + A\overline{\gamma}(S_k)) \exp\left(\int_{S_k}^{S_{k+1}} L(\tau) d\tau\right) \geq \frac{1}{k^\gamma}\right\},$$

then

$$\mathbb{P}(A_k) \leq C \exp\left(\frac{-k^{-2\gamma+1+1/\beta}}{C_1'}\right).$$

By an application of Borel-Cantelli lemma, with probability one, there exists $k_0 \in \mathbb{N}$ such that, for any $k \geq \mathbb{N}$,

$$(\Delta(S_k, S_{k+1}) + A\overline{\gamma}(S_k)) \exp\left(\int_{S_k}^{S_{k+1}} L(\tau) d\tau\right) \leq \frac{1}{k^\gamma},$$

and the proof is complete. ■

4. Proof of Theorem 1.7

4.1. **Vanishing perturbed best response dynamics.** Consider the map

$$\tilde{\Pi} : Y \times \mathbb{R}_+^* \to \mathbb{R},\ y \mapsto \max_{x \in X} \pi(x, y, \beta) = \pi\left(\mathbf{br}(y, \beta), y, \beta\right).$$

Our state variable is $v_n := (\overline{x}_n, \overline{y_n}, \overline{\pi}_n) \in M := X \times Y \times [-\|\pi\|_\infty, \|\pi\|_\infty]$. We have

$$\overline{x}_{n+1} - \overline{x}_n - \frac{1}{n+1}\left(\delta_{i_{n+1}} - \mathbb{E}_\sigma(\delta_{i_{n+1}} \mid \mathcal{F}_n)\right) = \frac{1}{n+1}\left(-\overline{x}_n + \mathbf{br}(\overline{y}_n, \varepsilon_n)\right).$$

Notice that

$$v_{n+1} - v_n - \frac{1}{n+1} U_{n+1} \in \frac{1}{n+1} F_n(v_n),$$

where

- the noise sequence

$$U_{n+1} = (v_{n+1} - v_n) - \mathbb{E}(v_{n+1} - v_n \mid \mathcal{F}_n)$$

is a bounded martingale difference,
- the set valued map F_n is given by

$$F_n(x, y, \pi) := \{(\mathbf{br}(y, \beta_n) - x, \tau - y, \pi(\mathbf{br}(y, \beta_n), \tau) - \pi, \ \ \tau \in Y\}$$

Let $F : \mathbb{R}_+ \times M \rightrightarrows M$ be the map given by $F(s, v) := F_{m(s)}(v)$.

Lemma 4.1. *F is a regular set-valued map.*

Proof. The fact that F has non-empty compact convex values is straightforward, as well as measurability. Also, the map F takes values in M, which is compact. Thus F is uniformly bounded. Given $s \in \mathbb{R}_+$, we now need to check upper semi-continuity of $v \mapsto F(s, v)$, which is equivalent to $\{(v, w), \ w \in F(s, v)\}$ being closed. Let (x_n, y_n, π_n) converge to (x, y, π). We then have $\mathbf{br}(y_n, \beta_{m(s)}) \to \mathbf{br}(y, \beta_{m(s)})$. Hence,

$$\left(\mathbf{br}(y_n, \beta_{m(s)}), \tau_n, \pi\left(\mathbf{br}(y_n, \beta_{m(s)}), \tau_n\right)\right) \to \left(\mathbf{br}(y, \beta_{m(s)}), \tau, \pi\left(\mathbf{br}(y, \beta_{m(s)}), \tau\right)\right) \in F(s, x, y, \pi).$$

The proof is complete. ■

Theorem 4.2. *Let $A = \{(x, y, \pi) \in M \mid \Pi(y) - \pi \leq 0\}$. There exist a global uniform Lyapunov function Φ relative to the compact set A and the non-autonomous differential inclusion*

$$\dot{\mathbf{v}}(s) \in F(s, \mathbf{v}(s)).$$

Proof: We prove that properties $a')$ and $b)$ hold. Let $\Phi : \mathbb{R}_+ \times M \to \mathbb{R}_+$, defined by

$$\Phi(s, x, y, \pi) = \begin{cases} \tilde{\Pi}(y, \beta_{m(s)}) - \pi & \text{if } \tilde{\Pi}(y, \beta_{m(s)}) \geq \pi \\ 0 & \text{if } \tilde{\Pi}(y, \beta_{m(s)}) < \pi. \end{cases}$$

Notice that

$$A = \{(x, y, \pi) : g(x, y, \pi) = 0\} \quad \text{and} \quad \|g(v) - \Phi(s, v)\| \to_{s \to +\infty} 0$$

uniformly, where $g(x, y, \pi) := \max\{0, \Pi(y) - \pi\}$. Let t and T be positive real numbers and $(x(s), y(s), \pi(s))$ be a solution of the non-autonomous differential inclusion (1) on $[t, t+T]$, such that and $\pi(s) \leq \tilde{\pi}(y(s), \beta_{m(s)})$. Let

$$\Psi(s) := \Phi(s, x(s), y(s), \pi(s)) = \tilde{\pi}\left(\mathbf{br}(y(s), \beta_{m(s)}), y(s), \beta_{m(s)}\right) - \pi(s).$$

Recall that $\beta_{m(s)}$ is piecewise constant on $[t, t+T]$. For almost every s, by definition of $\mathbf{br}(y(s), \beta_{m(s)})$, we have $D_1\tilde{\pi}\left(\mathbf{br}(y(s), \beta_{m(s)}), y(s), \beta_{m(s)}\right) D_1\mathbf{br}(y(s), \beta_{m(s)}) = 0$. Hence, for almost every $s \in [t, t+T]$,

$$\begin{aligned} \dot{\Psi}(s) &= \pi\left(\mathbf{br}(y(s), \beta_{m(s)}), \dot{y}(s)\right) - \dot{\pi}(s) \\ &= \pi\left(\mathbf{br}(y(s), \beta_{m(s)}), \tau(s), \beta_{m(s)}\right) - \pi\left(\mathbf{br}(y(s), \beta_{m(s)}), y(s), \beta_{m(s)}\right) \\ &\quad -\pi(\mathbf{br}(y(s), \beta_{m(s)}), \tau(s), \beta_{m(s)}) + \pi(s) \\ &\leq -\Psi(s) + \frac{1}{\beta_{m(s)}}. \end{aligned}$$

By Gronwall's Lemma we therefore have

$$\Psi(t+T) \leq e^{-T}\Psi(t) + \frac{T}{\beta_{m(t)}}$$

Consequently, Φ is a global uniform Lyapunov function with respect to A, which proves the result. ■

4.2. **VSFP and external consistency, proof of Theorem 1.7.** The set-valued map F is regular and $L(\cdot)$-Lipschitz, with $L(s) = L\beta_{m(s)}$, for some constant L depending on the payoff function π (see Lemma 5.2). Hence we can assume, without loss of generality that $L(s) \leq e^{\beta s}$. We call v the piecewise linear interpolated process relative to $(v_n)_n$. By Proposition 3.4, almost surely, points (i) and (ii) of Proposition 2.13 are satisfied for $k \geq k_0$, with $T_k = (\beta k)^{-1}$ and $\gamma_k = k^{-\gamma}$, $\gamma > 0$. We now need to check points (iii) and (iv).

By Theorem 4.2, Φ is a global uniform function Lyapunov relative to

$$A = \{(x, y, \pi) \in M \mid \Pi(y) - \pi \leq 0\},$$

with $\lambda(T) = e^{-T}$ and $\varepsilon(t, T) = \frac{T}{\beta_{m(t)}}$, for some positive constant c. Hence $\eta_k = k^{-\gamma} + c\frac{T_{k+1}}{k}$. Clearly, by point $b)$ of Remark 2.10, point (iii) is checked because $\sum_i \eta_i < \infty$ and $H_k = e^{-\sum_{i=k_0}^{k} T_i} = \mathcal{O}(k^{-1/\beta})$

Now let b be a positive constant and consider the map $\phi : Y \times [-\|\pi\|_\infty, \|\pi\|_\infty] \to \mathbb{R}_+$, given by

$$\phi(y, \pi) = \begin{cases} \tilde{\Pi}(y, b) - \pi & \text{if } \tilde{\Pi}(y, b) \geq \pi \\ 0 & \text{if } \tilde{\Pi}(y, b) < \pi. \end{cases}$$

Let (y, π) be such that $\tilde{\Pi}(y, b) > \pi$. Then clearly

$$\begin{aligned} \frac{\partial}{\partial y}\phi(y, \pi) &= \frac{\partial}{\partial y}\tilde{\pi}(\mathbf{br}(y, b), y, b) \\ &= D_1\tilde{\pi}(\mathbf{br}(y, b), y, b) \cdot \frac{\partial}{\partial y}\mathbf{br}(y, b) + D_2\tilde{\pi}(\mathbf{br}(y, b), y, b) \cdot I_d \\ &= \pi(\mathbf{br}(y, b), \cdot) \end{aligned}$$

and

$$\frac{\partial}{\partial \pi}\phi(y, \pi) = Id.$$

Thus ϕ is Lipschitz with some constant that does not depend on b. As a consequence, point (iv) is verified for Φ and we have

$$\mathcal{L}((v_n)_n) \subset A.$$

■

5. Appendix

5.1. A Gronwall's Lemma.

Lemma 5.1. *Let y be a continuously differentiable function on $I = [a, b]$ and α, β be non-negative, continuous maps. If, for every $s \in I$, $\|\dot{y}(s)\| \leq \alpha(s)\|y(s)\| + \beta(s)$ then*

$$\|y(s)\| \leq \|y(a)\| \exp\left(\int_a^s \alpha(\tau)d\tau\right) + \int_a^s \beta(u) \exp\left(\int_u^s \alpha(\tau)d\tau\right) ds$$

Proof. Notice that

$$\|y(s)\| \leq \|y(a)\| + \int_a^s \|\dot{y}(u)\|du \leq \|y(a)\| + \int_a^s \beta(u)du + \int_a^s \alpha(u)\|y(u)\|du$$

and apply Gronwall's integral form. ■

5.2. **Some remarks on the Logit function.**

Lemma 5.2. *The logit map with parameter $\beta > 0$:*

$$\sigma : \Delta^N \to \Delta^N, \ \ x = (x_1, ..., x_N) \mapsto (\sigma^1, ..., \sigma^N),$$

where

$$\sigma^i(x) := \frac{\exp(\beta x_i)}{\sum_{j=1}^N \exp(\beta x_j)}$$

is Lipschitz continuous for the infinte norm, with Lipschitz constant 2β.

Proof. We have

$$\|\sigma(x) - \sigma(x')\|_1 = \left(u \mid \sigma(x) - \sigma(x')\right),$$

where $u = (sg(\sigma^1(x) - \sigma^1(x')), .., sg(\sigma^N(x) - \sigma^N(x')))$. Let $\Phi : [x, x'] \to \Delta^N : \ \ \Phi(y) = (u \mid \sigma(y))$. By the mean value Theorem, there exists $y \in [x, x']$ such that

$$\|\sigma(x) - \sigma(x')\|_1 = \Phi(x) - \Phi(x') = (\nabla \Phi(y) \mid x - x') = \sum_{i=1}^N u_i (\nabla \sigma^i(y) \mid x - x').$$

Therefore we have

$$\|\sigma(x) - \sigma(x')\|_1 \leq \|x - x'\|_\infty \sum_{i=1}^N u_i \|\nabla \sigma^i(y)\|_1.$$

Now,

$$\begin{aligned} \frac{\partial \sigma^i(y)}{\partial y_j} &= \frac{\beta \exp(\beta y_i) \mathbb{I}_{i=j} \sum_k \exp(\beta y_k) - \beta \exp(\beta y_j) \exp(\beta y_i)}{(\sum_k \exp(\beta y_k))^2} \\ &= \beta \left(\sigma^i(y) \mathbb{I}_{i=j} - \sigma^i(y) \sigma^j(y)\right). \end{aligned}$$

Hence

$$\begin{aligned} \|\nabla \sigma^i(y)\|_1 &= \beta \left(\sigma^i(y) - (\sigma^i(y))^2 + \sum_{j \neq i} \sigma^i(y) \sigma^j(y) \right) \\ &= 2\beta \sigma^i(y)(1 - \sigma^i(y)). \end{aligned}$$

Finally

$$\|\sigma(x) - \sigma(x')\|_1 \leq 2\beta \|x - x'\|_\infty \sum_{i=1}^N u_i \sigma^i(y)(1 - \sigma^i(y)) \leq 2\beta \|x - x'\|_\infty$$

and the proof is complete since $\|\sigma(x) - \sigma(x')\|_\infty \leq \|\sigma(x) - \sigma(x')\|_1$. ■

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