Optimal stopping of a mean reverting diffusion: minimizing the relative distance to the maximum

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Abstract

Considering a diffusion X mean reverting to 0 and starting at $X_0 > 0$, we study the control problem

$$\inf_{\theta} \mathbb{E}\left[f\left(\frac{X_\theta}{\sup\limits_{s\in[0,\tau]} X_s} \right) \right] ,$$

where f is a given function and τ is the next random time where the diffusion X crosses zero. Our motivation is the obtention of optimal selling rules related to the minimization of the relative distance between a stopped mean reverting portfolio and its upcoming maximum. We provide a verification result for this stochastic control problem and derive the solution for different criteria f. For a power utility type criterion $f : y \mapsto -y^\lambda$ with $\lambda > 0$, instantaneous stopping is always optimal. On the contrary, for a relative quadratic error criterion, $f : y \mapsto (1-y)^2$, selling is optimal as soon as the process X crosses a specified function φ of its running maximum X^*. As in [5] and [8], the inverse of φ identifies as the maximal solution of a highly non linear ordinary differential equation. These results reinforce the idea that optimal prediction problems of similar type lead easily to solutions of different nature. Nevertheless, we observe numerically that the continuation region for the relative quadratic error criterion is very small, so that the optimal selling strategy is close to immediate stopping.

Key words: optimal stopping, optimal prediction, running maximum, free boundary PDE, verification, mean reverting diffusion

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1 Introduction

At a first glance, trying to stop a one-dimensional process as close as possible to its ultimate maximum may be viewed as a hopeless ambition. Graversen, Peskir and Shiryaev were the first authors who tackled successfully this challenging problem. Considering a one dimensional Brownian motion B on the time interval $[0,1]$, they solve in [6] the optimal stopping problem $\inf_\theta \mathbb{E}\left[|B_\theta - B_1^*|^2\right]$, where B_1^* denotes the maximum of the process B at time 1 and θ is any stopping time smaller than 1. Stopping is optimal as soon as the drawdown of the Brownian motion, i.e. the gap between its current maximum and its value, goes below the function $t \mapsto c^*\sqrt{1-t}$, for a specified constant c^*. Urusov [10] observes that this strategy leads to a good approximation of the last time τ^* where the Brownian motion reaches its maximum, since it also solves the problem $\inf_\theta \mathbb{E}\left[|\theta - \tau^*|\right]$. For a drifted Brownian motion, this property is no longer satisfied, and Du Toit and Peskir [2, 3] characterize both solutions of these problems. Once again, stopping is optimal as soon as the drawdown of the drifted Brownian motion enters a time-to-horizon dependent region.

Considering instead a geometric Brownian motion $(S_t)_{0\le t\le 1}$, several authors (Shiryaev, Xu and Zhou [9], Du Toit and Peskir [4] or Dai, Jin, Zhong and Zhou [1]) tried to minimize the relative distance between the stopped process S_τ and its ultimate maximum. In particular they solve the problem $\sup_\theta \mathbb{E}\left[S_\theta/S_1^*\right]$. Their purpose is of course the obtention of an optimal selling rule of the stock S as close as possible to its ultimate maximum. As pointed out in [4], the formulation in terms of ratio between the stopped process and its maximum has the effect of stripping away the monetary value of the stock, focusing only on the underlying randomness. Using either probabilistic or deterministic methods, the common interpretation of the solution derived in these papers is that one should "sell bad stocks and keep good ones". Indeed, introducing the "goodness index" α of the stock as the ratio between its excess return rate and its square volatility rate, the optimal strategy appears to be of "bang-bang" type: one should immediately sell the stock if $\alpha \le 1/2$ and keep it until maturity otherwise. Focusing also on the problem $\inf_\theta \mathbb{E}\left[S_1^*/S_\theta\right]$, Du Toit and Peskir [4] observe that one should sell immediately if $\alpha < 0$, keep until the end if $\alpha > 1$ and stop as soon as the ratio S^*/S hits a specified deterministic function of time in the intermediate case. It is worth noticing that these two optimal prediction problems of similar type offer therefore different optimal selling strategies for a large range of parameter set.

Of course, the only consideration of stocks with Black Scholes type dynamics is unrealistic and limitative. A recent paper of Espinosa and Touzi [5] allows for the consideration of more general diffusion dynamics and, as a by product, requires to focus on a stationary version of this problem. Studying an even more realistic infinite time horizon problem and a mean reverting diffusion portfolio X with general dynamics starting at $X_0 > 0$, they treat the problem: $\inf_\theta \mathbb{E}\left[|X_\tau^* - X_\theta|^2\right]$ where τ is the first time where X hits zero and θ is any stopping time smaller than τ. They solve explicitly this problem as a free boundary problem and obtain that one should sell the portfolio whenever the running maximum X^* and the drawdown $X^* - X$ are both large enough.

In a similar framework, the purpose of this paper is to minimize the relative distance between a stopped mean reverting positive scalar diffusion and its upcoming maximum when it reaches zero. The consideration of the ratio of the stopped process with respect to its upcoming maximum allows to capture the scale of the prices themselves and we solve the two following problems

$$V_1 = \sup_\theta \mathbb{E}\left[\left(\frac{X_\theta}{X^*_\tau}\right)^\lambda\right] , \quad \text{for } \lambda > 0 , \qquad \text{and} \qquad V_2 = \inf_\theta \mathbb{E}\left[\left(\frac{X^*_\tau - X_\theta}{X^*_\tau}\right)^2\right] ,$$

where τ is the first time where X hits 0 and θ is any stopping time smaller than τ. For the first problem V_1, we prove that the optimal stopping strategy consists in liquidating the portfolio immediately. This conclusion is in accordance with the results of [1, 4, 9] since keeping the stock until maturity is obviously irrelevant in our framework. Hence, the (bang-)bang type strategy also occurs for general mean reverting diffusion dynamics and for any relative power utility type criterion ($\lambda > 0$). Conversely for V_2, when minimizing the relative quadratic distance between the process and its ultimate maximum, the optimal selling time is the first time where the process X goes below a specified function φ of its running maximum X^*. Similarly to [8] and [5], this function φ (or more precisely its inverse) is characterized as the "biggest" solution of an ordinary differential equation and can be easily approximated numerically.

As already observed by Du Toit and Peskir [4], our results confirm that optimal stopping prediction problems of similar nature can lead to different types of optimal solution. Predicting the maximum of a process is really intricate and the corresponding optimal strategy strongly depends on the criterion choice. However, we shall temper a bit this conclusion since, in our case, numerical experiments provided in the paper show that the function φ is close to the identity function. Hence, even if immediate stopping is not optimal, a portfolio manager will not wait long until the drawdown of his portfolio $X^* - X$ goes below $X^* - \varphi(X^*)$.

The paper is organized as follows. The next section provides the set up of the problem and derives preliminary properties. Section 3 is dedicated to the obtention of a general verification theorem allowing to treat the first and the second optimal stopping problems at once. Sections 4 and 5 tackle successively the power utility type criterion and the quadratic distance one. In both cases, the value function solution is presented and discussed at the beginning of the section, numerical results are presented, and the technical proofs are postponed to the end of it.

2 Problem formulation

2.1 The optimization problem of interest

Let W be a scalar Brownian motion on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ the corresponding augmented canonical filtration. Let X be a diffusion process given by the following dynamics:

$$dX_t \;\; = \;\; -\mu(X_t)dt + \sigma(X_t)dW_t, \qquad t \geq 0 \,, \tag{2.1}$$

together with an initial data $x := X_0 > 0$, where μ and σ are Lipschitz continuous functions. We will assume that the process X mean-reverts towards the origin in the sense that:

$$\mu(x) \geq 0 \,, \quad \text{for } x \geq 0. \tag{2.2}$$

We denote by $\tau := \inf\{t \geq 0, X_t = 0\}$ the first time where the process X hits the origin, $\mathcal{T}$ the set of $\mathbb{F}$-stopping times θ such that $\theta \leq \tau$ a.s and $X^*_. := \sup_{s \leq .} X_s$ the running maximum of X. We consider the following optimization problem:

$$V_0 \;\; := \;\; \inf_{\theta \in \mathcal{T}} \mathbb{E}\, f\left(\frac{X_\theta}{X^*_\tau}\right), \tag{2.3}$$

where f is a continuous function on $[0,1]$, C^1 on $(0,1]$, and such that, there exist two constants $A > 0$ and $\eta > 0$ satisfying

$$|f'(x)| \leq A x^{\eta-2} \,, \qquad 0 < x \leq 1 \,. \tag{2.4}$$

In order to use dynamic programming techniques, we introduce as usual the process Z defined by $Z_t := z \vee X^*_t$ for a given $z > 0$, and define the corresponding value function associated to the optimization problem (2.3) :

$$V(x,z) \;\; := \;\; \inf_{\theta \in \mathcal{T}} \mathbb{E}_{x,z}\, f\left(\frac{X_\theta}{Z_\tau}\right) \,, \qquad (x,z) \in \mathbf{\Delta} \,, \tag{2.5}$$

where $\mathbf{\Delta}$ is defined by

$$\mathbf{\Delta} := \{(x,z),\ 0 \leq x \leq z \text{ and } z > 0\}, \tag{2.6}$$

and corresponds to the domain where (X, Z) lies.

Remark 2.1 Of course, we aim at considering cases where the function f is not increasing, since otherwise a straightforward optimal strategy consists in waiting until time τ.

Remark 2.2 Notice that the definition of $\mathbf{\Delta}$ differs from the one in [5]. Notice also that contrary to [5], the problem is not invariant by translation. More precisely, if one considers the criterion $\inf_\theta f(\frac{b+X_\theta}{b+Z_\tau})$, the problem might not be well-defined for $b < 0$ since we can have $b + Z_\tau = 0$. For $b > 0$ (or $z > b$ if $b < 0$) however, the problem makes sense and could be studied in a similar fashion, but is not a particular case of what we do here.

Defining the reward function g from immediate stopping

$$g(x,z) \quad := \quad \mathbb{E}_{x,z}\, f\left(\frac{x}{Z_\tau}\right)\ , \qquad (x,z)\in\mathbf{\Delta}\ , \tag{2.7}$$

we may re-write this problem in the standard form of an optimal stopping problem

$$V(x,z) \quad = \quad \inf_{\theta\in\mathcal{T}} \mathbb{E}_{x,z}\, g(X_\theta, Z_\theta)\ , \qquad (x,z)\in\mathbf{\Delta}\ . \tag{2.8}$$

2.2 Assumptions and first properties

Let introduce the so-called scale function S defined, for $x \geq 0$, by

$$S(x) \ := \ \int_0^x \exp^{\int_0^u \alpha(r)dr}\, du\ , \qquad \text{with} \qquad \alpha \ := \ \frac{2\mu}{\sigma^2}\ . \tag{2.9}$$

Remark 2.3 Since the process X mean reverts towards 0, the function α is non negative. Therefore, the scale function S is increasing, convex and dominates the Identity function.

By construction, S satisfies $S_{xx} = \alpha S_x$ and is related to the law of Z_τ via the estimate

$$\mathbb{P}_{x,z}\left[Z_\tau \leq u\right] \ = \ \mathbb{P}_x\left[X^*_\tau \leq u\right]\mathbf{1}_{z\leq u} \ = \ \left(1-\frac{S(x)}{S(u)}\right)\mathbf{1}_{z\leq u}\ , \quad (x,z)\in\mathbf{\Delta}\ , \quad u>0\ .$$

Using the scale function S, the reward function g rewrites as

$$g(x,z) = f\left(\frac{x}{z}\right)\left(1-\frac{S(x)}{S(z)}\right) + S(x)\int_z^\infty f\left(\frac{x}{u}\right)\frac{S'(u)}{S(u)^2}du\ , \qquad (x,z)\in\mathbf{\Delta}\ , \tag{2.10}$$

which is well-defined since f is continuous on $[0,1]$ so that we have

$$\int_z^\infty \left|f\left(\frac{x}{u}\right)\right|\frac{S'(u)}{S(u)^2}du \leq \|f\|_\infty \int_z^\infty \frac{S'(u)}{S(u)^2}du = \frac{\|f\|_\infty}{S(z)}\ , \qquad (x,z)\in\mathbf{\Delta}\ .$$

Via an integration by part, we deduce

$$g(x,z) = f\left(\frac{x}{z}\right) - xS(x)\int_z^\infty f'\left(\frac{x}{u}\right)\frac{du}{u^2 S(u)}\ , \qquad (x,z)\in\mathbf{\Delta}\ ,\ \ x>0\ . \tag{2.11}$$

Observe that the previous integral is well defined since, combining Remark 2.3 with estimate (2.4), we compute

$$\left|\frac{f'\left(\frac{x}{u}\right)}{u^2 S(u)}\right| \quad \leq \quad A\,\left(\frac{x}{u}\right)^{\eta-2}\frac{1}{u^3} \quad = \quad A\,\frac{x^{\eta-2}}{u^{1+\eta}}\ , \qquad 0<x\leq u\ .$$

If f is C^1 on $[0,1]$, since $g(0,z)=f(0)$, (2.11) also holds true for $x=0$ and $z>0$.

In this paper, we aim at considering a general framework including the classical types of mean reverting processes. In particular, we intend to treat the following diffusion dynamics:

- Brownian motion with negative drift: α constant, positive and $S(x) = \frac{e^{\alpha x}-1}{\alpha}$;

- Cox Ingersol Ross process: $\alpha(x) = \frac{\alpha x}{x+b}$ and $S'(x) = e^{\alpha x}\left(\frac{x+b}{b}\right)^{\alpha b}$ with $\alpha > 0$ and $b > 0$;
- Ornstein-Uhlenbeck process: $\alpha(x) = \alpha x$ and $S'(x) = e^{\alpha x^2/2}$.

For this purpose, we impose on α similar but less restrictive conditions as in [5] and work under the following standing Assumption:

$\alpha = \frac{2\mu}{\sigma^2} : (0, \infty) \to \mathbb{R}$ is a C^2 positive non-decreasing concave function with $\alpha(0) = 0$.(2.12)

Remark 2.4 As in [5], we observe for later use that the restriction (2.12) implies in particular that the function $2S' - \alpha S - 2$ is a non negative increasing function.

Remark 2.5 One can wonder if the present problem can be solved from [5] using the following change of variable: $Y := \ln(1 + X)$, since Y would also be a process mean-reverting towards 0 if X is. We claim that this is not the case. Indeed, we can observe that $f(\frac{1+X_\theta}{1+Z_\tau}) = f \circ \exp\left(\ln(1+X_\theta) - \ln(1+Z_\tau)\right)$, and define $\ell : x \mapsto f \circ \exp(-x)$. First, as briefly explained in Remark 2.2, the problem considered here where $b = 0$ cannot be deduced from the one with $b = 1$. Moreover, for the functions f that we intend to study, $f : x \mapsto -x^\lambda$ or $f : x \mapsto \frac{1}{2}(1 - x)^2$, the convexity of ℓ required in [5] is not satisfied. Finally, if X is for example an Ornstein-Uhlenbeck process, one can compute that the function α_Y associated to Y is of the form $\alpha_Y : y \mapsto 2\alpha(e^{2y} - e^y) + 1$, which is convex on $\mathbb{R}_+$, and therefore does not satisfy the assumptions of [5].

3 A PDE verification argument

This section is devoted to the obtention of a PDE characterization for the solution of the control problem of interest (2.5). We first derive the corresponding HJB equation and then provide a verification theorem.

3.1 The corresponding dynamic programming equation

The linear second order Dynkin operator associated to the diffusion (2.1) is simply given by

$$\mathcal{L} \, : \, v \;\; \mapsto \;\; v_{xx} - \alpha(x) v_x \, , \quad \text{with} \quad \alpha(x) = \frac{2\mu(x)}{\sigma^2(x)} \, , \qquad \text{for } x \geq 0 \, .$$

By construction, observe that the scale function S satisfies in particular $\mathcal{L}S = 0$. Since the value function of interest V rewrites as the solution of a classical optimal stopping control problem (2.8), we expect V to be solution of the associated dynamic programming equation. Namely, V should be a solution of the Hamilton Jacobi Bellman equation:

$$\min(\mathcal{L}v; g - v) \; = \; 0 \, ; \quad v(0, z) \; = \; f(0) \, , \quad v_z(z, z) = 0 \, , \quad (x, z) \in \mathbf{\Delta} \, . \tag{3.13}$$

The first term indicates that V is dominated by the immediate reward function g and that the dynamics of v in the domain are given by the Dynkin operator of the diffusion X. The second relation manifests that only immediate stopping is possible whenever the diffusion X has reached 0. Finally, the last one is the classical Von Neumann condition encountered whenever the diffusion process hits its maximum.

As in any optimal stopping problem, the domain of definition $\mathbf{\Delta}$ of the value function subdivides into two subsets: the stopping region $\mathcal{S}$ where immediate stopping is optimal and the continuation region where the optimal strategy consists in waiting until the stochastic process enters the stopping region. The optimal stopping time is the first time where the process arrives in the stopping region, and, in order to obtain a stopping time, we expect the region $\mathcal{S}$ to be a closed subset of $\mathbf{\Delta}$. Of course, the stopping region is characterized by the relation $v = g$ since g is the reward function from immediate stopping. Depending on the position of $(x, z) \in \mathbf{\Delta}$ with respect to the region $\mathcal{S}$, we expect the dynamics of (3.13) to rewrite

$$\begin{aligned} &\text{On the stopping region:} && v(x,z) = g(x,z)\,, \qquad \mathcal{L}g(x,z) \geq 0\,;\\ &\text{On the continuation region:} && v(x,z) \geq g(x,z)\,, \qquad \mathcal{L}v(x,z) = 0\,;\\ &\text{Everywhere:} && v_z(x,z)\mathbf{1}_{\{x=z\}} = 0\,. \end{aligned}$$

In the next sections of the paper, we exhibit different shapes of stopping and continuation regions depending on the objective function f. We observe that, although the objective functions may appear rather similar, the optimal strategies can be very different.

3.2 The verification theorem

As detailed above, we expect the value function V given by (2.5) to be solution of the Hamilton Jacobi Bellman equation (3.13). The solution of this problem is intimately related to the form of the associated stopping region $\mathcal{S}$. Afterwards, we shall not prove that V is indeed a (weak) solution of this PDE but instead try to guess a regular solution to the PDE and verify that it satisfies the assumptions of the following verification theorem.

Theorem 3.1 *Let v be a bounded from below function continuous on $\mathbf{\Delta}$ and piecewise $\mathcal{C}^{2,1}$ on $\mathbf{\Delta} \setminus \{(0,z),\ z>0\}$.*
(i) *If v satisfies $\mathcal{L}v \geq 0$, $v \leq g$ as well as $v_z(z,z) \geq 0$ for $z > 0$, then $v \leq V$.*
(ii) *More precisely, if $v_z(z,z) = 0$ for $z > 0$ and there exists a closed set $\mathcal{S} \subset \mathbf{\Delta}$ containing the axis $\{(0,z)\,, z > 0\}$ such that*

$$v = g \text{ on } \mathcal{S}\,, \qquad \mathcal{L}v \geq 0 \text{ on } \mathcal{S} \setminus \{(0,z),\ z>0\}\,, \quad v \leq g \text{ and } \mathcal{L}v = 0 \text{ on } \mathbf{\Delta} \setminus \mathcal{S}\,, \quad (3.14)$$

then $v = V$ and $\theta^* := \inf\{t \geq 0,\ (X_t, Z_t) \in \mathcal{S}\}$ is an optimal stopping time.
(iii) *If in addition $v < g$ on $\mathbf{\Delta} \setminus \mathcal{S}$, then θ^* is the "smallest" optimal stopping time, in the sense that $\theta^* \leq \nu$ a.s. for any optimal stopping time ν.*

Proof. We prove each assertion separately.

(i) Fix $(X_0, Z_0) := (x, z) \in \mathbf{\Delta}$. Let $\theta \in \mathcal{T}$ and define $\theta_n = n \wedge \theta \wedge \inf\{t \geq 0;\ |Z_t| \geq n \text{ or } |Z_t| \leq \frac{1}{n}\}$ for $n \in \mathbb{N}$. Since (X, Z) takes value in a compact subset of $\mathbf{\Delta}$, a direct application of Itô's formula gives

$$v(x,z) = v(X_{\theta_n}, Z_{\theta_n}) - \int_0^{\theta_n} \mathcal{L}v(X_t, Z_t)\sigma(X_t)^2 dt - \int_0^{\theta_n} v_x(X_t, Z_t)\sigma(X_t) dW_t - \int_0^{\theta_n} v_z(X_t, Z_t) dZ_t \, .$$

Combining estimates $\mathcal{L}v \geq 0$ and $v_z(X_t, Z_t)dZ_t = v_z(Z_t, Z_t)dZ_t \geq 0$ with the fact that (X, Z) lies in a compact subset of $\mathbf{\Delta}$, we deduce

$$v(x,z) \leq \mathbb{E}_{x,z} v(X_{\theta_n}, Z_{\theta_n}) \, . \tag{3.15}$$

Since $v \leq g$, this leads directly to

$$v(x,z) \;\leq\; \mathbb{E}_{x,z} g(X_{\theta_n}, Z_{\theta_n}) \;=\; \mathbb{E}_{x,z}\mathbb{E}_{X_{\theta_n}, Z_{\theta_n}} f\left(\frac{X_{\theta_n}}{Z_\tau}\right) \;=\; \mathbb{E}_{x,z} f\left(\frac{X_{\theta_n}}{Z_\tau}\right).$$

Clearly as $n \to \infty$, $\theta_n \to \theta$ almost surely. Since $0 \leq X_{\theta_n}/Z_\tau \leq 1$ and f is continuous, Lebesgue's dominated convergence theorem gives: $\mathbb{E}_{x,z} f(X_{\theta_n}/Z_\tau) \to_{n\to\infty} \mathbb{E}_{x,z} f(X_\theta/Z_\tau)$, leading to

$$v(x,z) \quad \leq V(x,z) \quad , \quad (x,z) \in \mathbf{\Delta} \, .$$

(ii) Observe that this framework is more restrictive than the previous one, so that $v \leq V$ on $\mathbf{\Delta}$. For $(x,z) \in \mathcal{S}$, we have $v(x,z) = g(x,z) \geq V(x,z)$ by definition of g. We now fix $(x,z) \in \mathbf{\Delta} \setminus \mathcal{S}$ and prove that $v(x,z) \geq V(x,z)$.

Let $\theta^* := \inf\{t \geq 0;\ (X_t, Z_t) \in \mathcal{S}\}$. Observe that $\theta^* \in \mathcal{T}$ since $\mathcal{S}$ contains the axis $\{(0,z),\ z \geq 0\}$. The regularity of v implies $\mathcal{L}v(X_t, Z_t) = 0$ for any $t \in [0, \theta^*)$. As before, we define $\theta_n^* := n \wedge \theta^* \wedge \inf\{t \geq 0;\ |Z_t| \geq n \text{ or } |Z_t| \leq \frac{1}{n}\}$, which is a stopping time since $\mathcal{S}$ is closed. A very similar computation leads directly to

$$v(x,z) \quad = \quad \mathbb{E}_{x,z} v(X_{\theta_n^*}, Z_{\theta_n^*}) \, .$$

Since v is bounded from below and $v \leq g \leq \|f\|_\infty$, v is bounded. Therefore the sequence $\big(v(X_{\theta_n^*}, Z_{\theta_n^*})\big)_n$ is uniformly integrable and we deduce that $v(x,z) = \mathbb{E}_{x,z} v(X_{\theta^*}, Z_{\theta^*})$. Since $(X_{\theta^*}, Z_{\theta^*}) \in \mathcal{S}$ and $v = g$ on $\mathcal{S}$, we get

$$v(x,z) \;=\; \mathbb{E}_{x,z} g(X_{\theta^*}, Z_{\theta^*}) \;=\; \mathbb{E}_{x,z} f\left(\frac{X_{\theta^*}}{Z_\tau}\right) \;\geq\; V(x,z) \, .$$

Thus $v = V$ on $\mathbf{\Delta}$ and θ^* is an optimal stopping time.

(iii) For a given $(x,z) \in \mathbf{\Delta}$, we argue by contradiction and suppose the existence of a stopping time $\nu \in \mathcal{T}$ satisfying $\mathbb{P}(\nu < \theta^*) > 0$ and $V(x,z) = \mathbb{E}_{x,z} f(X_\nu/Z_\tau)$.

By assumption, we have $V(X_\nu, Z_\nu) < g(X_\nu, Z_\nu)$ on $\{\tau < \theta^*\}$, which combined with estimate $V \leq g$ implies

$$V(x,z) = \mathbb{E}_{x,z} f\left(\frac{X_\nu}{Z_\tau}\right) \;=\; \mathbb{E}_{x,z} g(X_\nu, Z_\nu) \;>\; \mathbb{E}_{x,z} V(X_\tau, Z_\tau) \;\geq\; V(x,z)\;,$$

where the last inequality follows from the definition of V. This leads to a contradiction, which guarantees the minimality of θ^*. □

Remark 3.1 From the definition of g, one easily checks that $g_z(z,z) = 0$ for any $z > 0$. Therefore, in the PDE dynamics (3.14), the Neumann boundary condition $v_z(z,z) = 0$ is only necessary for $(z,z) \in \mathbf{\Delta} \setminus \mathcal{S}$, since it is automatically satisfied otherwise.

Remark 3.2 Whenever g is a continuous function on $\mathbf{\Delta}$, $\mathcal{C}^{2,1}$ w.r.t. (x,z) on $\mathbf{\Delta}\setminus\{(0,z),\ z > 0\}$ and $\mathcal{L}g \geq 0$ on $\mathbf{\Delta} \setminus \{(0,z),\ z > 0\}$, then $v = g$ and $\mathcal{S} = \mathbf{\Delta}$ satisfy the assumptions of Theorem 3.1 (ii). In that case, immediate stopping is always optimal. We prove in Proposition 3.1 that the reverse is true. Notice also that expression (2.10) implies that an immediate sufficient condition for g to be in $\mathcal{C}^0(\mathbf{\Delta}) \cap \mathcal{C}^{2,1}(\mathbf{\Delta} \setminus \{(0,z),\ z > 0\})$ is that f is $\mathcal{C}^2$ on $(0,1]$.

Proposition 3.1 *Assume that g is $\mathcal{C}^0$ on $\mathbf{\Delta}$, $\mathcal{C}^{2,1}$ w.r.t. (x,z) on $\mathbf{\Delta} \setminus \{(0,z),\ z > 0\}$, and that there exists $(x_0,z_0) \in \mathbf{\Delta} \setminus \{(0,z),\ z > 0\}$ such that $\mathcal{L}g(x_0,z_0) < 0$. Then, immediate stopping at (x_0,z_0) is not optimal (or equivalently $V(x_0,z_0) < g(x_0,z_0)$).*

Proof. Since $\mathcal{L}g$ is continuous at (x_0,z_0), there exists a neighborhood U_0 of (x_0,z_0) in $\mathbf{\Delta}$ such that $\mathcal{L}g(x,z) < 0$ for any $(x,z) \in U_0$. Without loss of generality, we can assume that U_0 is compact in $\mathbf{\Delta}$. Let $(X_0,Z_0) = (x_0,z_0)$. Since $x_0 > 0$, there exists $\theta_0 \in \mathcal{T}$ such that $\mathbb{E}_{x_0,z_0}\theta_0 > 0$ and let define $\theta_1 := 1 \wedge \theta_0 \wedge \inf\{t \geq 0;\ (X_t,Z_t) \notin U_0\} \in \mathcal{T}$. Since $\{\theta_0 > 0\} = \{\theta_1 > 0\}$, we also have $\mathbb{E}_{x_0,z_0}\theta_1 > 0$. Using Itô's formula, we compute:

$$\begin{aligned} g(x_0,z_0) = g(X_{\theta_1}, Z_{\theta_1}) &- \int_0^{\theta_1} \mathcal{L}g(X_u,Z_u)\sigma(X_u)^2 du \\ &- \int_0^{\theta_1} g_x(X_u,Z_u)\sigma(X_u) dW_u - \int_0^{\theta_1} g_z(X_u,Z_u) dZ_u. \end{aligned}$$

From Remark 3.1, $g_z(z,z) = 0$ for $z > 0$ so that the last term of the previous expression disappears. Since U_0 is compact and $\mathbb{E}_{x_0,z_0}\theta_1 > 0$, taking conditional expectations, we deduce that $g(x_0,z_0) \;>\; \mathbb{E}_{x_0,z_0}\, g(X_{\theta_1}, Z_{\theta_1}) \;\geq\; V(x_0,z_0)$. □

In the next sections, we investigate two particular cases of objective functions, for which we exhibit functions v and stopping regions $\mathcal{S}$, which satisfy the assumptions of Theorem 3.1 and are in general non trivial.

4 The power utility case

Let first examine the case where the function f is given by $f : x \mapsto -\frac{x^\lambda}{\lambda}$, for $\lambda > 0$. In other words, we are computing the following value function

$$V^\lambda(x,z) \quad := \quad -\frac{1}{\lambda} \sup_{\theta \in \mathcal{T}} \mathbb{E}_{x,z} \left(\frac{X_\theta}{Z_\tau} \right)^\lambda , \qquad (x,z) \in \mathbf{\Delta} , \quad \lambda > 0 . \tag{4.16}$$

Consider an investor, whose relative preferences are given by a power utility function and suppose that he detains at time 0 a given portfolio X mean reverting towards 0. The optimal stopping time at which he should liquidate his portfolio is the solution of the previous control problem. With a given finite time horizon T, du Toit and Peskir [4] as well as Shiryaev, Xu and Zhou [9] investigate the case where X is a Geometric Brownian motion. They conclude that the optimal strategy consists in waiting until time T if the portfolio has promising returns (i.e. $1 < 2\mu/\sigma^2 = x\alpha(x)$, $x > 0$, with our notations), and sell immediately otherwise. In our mean reverting framework, waiting until the wealth reaches 0 is obviously a non optimal strategy. For a linear utility function ($\lambda = 1$), we prove in Theorem 4.1 below that immediate stopping is also optimal. Depending on the value of λ, the latter may no longer be the case for the non linear problem (4.16). Nevertheless, we observe that optimal stopping is still optimal for the practical value function of interest $V^\lambda(x,x)$, for $x > 0$.

4.1 The particular case where $\lambda \leq 1$

For $\lambda \leq 1$, we prove hereafter that immediate stopping is always optimal. For $\lambda = 1$, these conclusions are therefore in accordance with those of [4, 9] obtained for the case of an exponential Brownian motion on a fixed time horizon.

A direct application of estimate (2.11) proves that the reward function g^λ associated to problem (4.16) is given by

$$g^\lambda(x,z) \quad = \quad -\frac{x^\lambda}{\lambda z^\lambda} + x^\lambda S(x) \int_z^\infty \frac{du}{S(u) u^{1+\lambda}} , \qquad (x,z) \in \mathbf{\Delta} , \qquad \lambda > 0 .$$

The next theorem indicates that the framework of Remark 3.2 holds for $\lambda \leq 1$, so that g^λ coincides with the value function on $\mathbf{\Delta}$.

Theorem 4.1 *For $\lambda \leq 1$, immediate stopping is optimal for problem* (4.16)*, so that*

$$V^\lambda(x,z) \quad = \quad g^\lambda(x,z) , \qquad (x,z) \in \mathbf{\Delta} , \qquad 0 < \lambda \leq 1 .$$

Proof. For any $\lambda > 0$ and $(x,z) \in \mathbf{\Delta}$ with $x > 0$, we compute

$$g^\lambda_x(x,z) \quad = \quad -\frac{x^{\lambda-1}}{z^\lambda} + \{\lambda x^{\lambda-1} S(x) + x^\lambda S'(x)\} \int_z^\infty \frac{du}{S(u) u^{1+\lambda}} .$$

Differentiating one more time and using the relation $\mathcal{L}S = 0$, we get,

$$g^\lambda_{xx}(x,z) = (1-\lambda)\frac{x^{\lambda-2}}{z^\lambda} + \{\lambda(\lambda-1)x^{\lambda-2}S(x) + (2\lambda x^{\lambda-1} + x^\lambda\alpha(x))S'(x)\}\int_z^\infty \frac{du}{S(u)u^{1+\lambda}} \,,$$

for any $\lambda \neq 1$ and $0 < x \leq z$. Combining the previous estimates, we deduce that

$$\mathcal{L}g^\lambda(x,z) = \; x^{\lambda-2}[x\alpha(x) + 1 - \lambda]\left(\frac{1}{z^\lambda} - \int_z^\infty \frac{S(x)}{S(u)}\frac{\lambda du}{u^{1+\lambda}}\right) + 2x^{\lambda-1}S'(x)\int_z^\infty \frac{\lambda du}{S(u)u^{1+\lambda}}, \tag{4.17}$$

for any $\lambda \neq 1$ and $0 < x \leq z$. In the case, where $\lambda = 1$, we get similarly

$$\mathcal{L}g^1(x,z) \;\; = \;\; \alpha(x)\left(\frac{1}{z} - \int_z^\infty \frac{S(x)du}{u^2S(u)}\right) + 2S'(x)\int_z^\infty \frac{du}{u^2S(u)} \,, \qquad (x,z) \in \Delta \,. \tag{4.18}$$

Furthermore, since S is increasing, we have

$$\int_z^\infty \frac{S(x)}{S(u)}\frac{\lambda du}{u^{1+\lambda}} \;\leq\; \int_z^\infty \frac{S(z)}{S(u)}\frac{\lambda du}{u^{1+\lambda}} \;\leq\; \int_z^\infty \frac{\lambda du}{u^{1+\lambda}} \;=\; \frac{1}{z^\lambda}\,, \qquad (x,z) \in \mathbf{\Delta}\,, \qquad \lambda > 0\,.$$

Plugging this estimate in (4.17) and (4.18), we see that $\mathcal{L}g^\lambda \geq 0$ on $\mathbf{\Delta} \setminus \{(0,z),\ z > 0\}$ for any $\lambda \leq 1$. As detailed in Remark 3.2, since g is C^0 on $\mathbf{\Delta}$ and $C^{2,1}$ on $\mathbf{\Delta} \setminus \{(0,z),\ z > 0\}$, we deduce that $V^\lambda = g^\lambda$ on $\mathbf{\Delta}$ and consequently immediate stopping is optimal for any $\lambda \leq 1$. □

4.2 Construction of the solution when $\lambda > 1$

We now turn to the more interesting and intricate case where $\lambda > 1$. Then, the function $\mathcal{L}g^\lambda$ is still given by expression (4.17) and we observe that:

$$\mathcal{L}g^\lambda(x,z) \;\sim_{x\sim 0}\; (1-\lambda)\frac{x^{\lambda-2}}{z^\lambda} \;<\; 0\,,$$

for any $z > 0$ and $\lambda > 1$. Therefore, $\mathcal{L}g^\lambda$ is not non negative on $\mathbf{\Delta}$ and Proposition 3.1 ensures that the associated continuation region is non empty. Since immediate stopping shall not be optimal close to the axis $\{(0,z);\ z > 0\}$, we expect to have a stopping region of the form $\mathcal{S}^\lambda := \{(x,z) \in \mathbf{\Delta};\ x \geq \varphi^\lambda(z)\}$. Hence, our objective is to find functions v^λ and φ^λ satisfying

$$\mathcal{L}v^\lambda(x,z) \;=\; 0 \qquad \text{for } 0 < x < \varphi^\lambda(z) \quad \text{and} \quad (x,z) \in \mathbf{\Delta}\,, \tag{4.19}$$

$$v^\lambda(x,z) \;=\; g^\lambda(x,z) \;\; \text{and} \;\; \mathcal{L}g^\lambda(x,z) \;\geq\; 0 \qquad \text{for } x \geq \varphi^\lambda(z) \quad \text{and} \quad (x,z) \in \mathbf{\Delta}\,, \tag{4.20}$$

$$v^\lambda(0,z) \;=\; 0 \qquad \text{for } z > 0\,, \tag{4.21}$$

$$v^\lambda_z(z,z) \;=\; 0 \qquad \text{for } z > 0\,. \tag{4.22}$$

Since we look for regular solutions, we complement the above system by the continuity and the smoothfit conditions

$$v^\lambda(\varphi^\lambda(z),z) \;=\; g^\lambda(\varphi^\lambda(z),z) \quad \text{and} \quad v^\lambda_x(\varphi^\lambda(z),z) \;=\; g^\lambda_x(\varphi^\lambda(z),z)\,, \qquad \text{for } z > 0\,. \tag{4.23}$$

The stopping region $\mathcal{S}^\lambda$ will then be defined as

$$\mathcal{S}^\lambda := \{(x,z) \in \mathbf{\Delta};\ x \geq \varphi^\lambda(z)\} \cup \{(0,z);\ z > 0\}. \tag{4.24}$$

Since the optimization problem of practical interest corresponds to the value of V^λ on the diagonal $\{(x,x);\ x > 0\}$, our main concern here is to find out if $\varphi^\lambda(0)$ equals 0 or not, hence indicating if immediate stopping is always optimal on the diagonal. Surprisingly, we verify hereafter that $\varphi^\lambda(0) = 0$ so that immediate stopping is the optimal strategy for the practical problem of interest.

Due to the dynamics of (5.41) and since $\mathcal{L}S = 0$, the function v^λ must be of the form

$$v^\lambda(x,z) \quad = \quad A(z) + B(z)S(x)\ , \qquad (x,z) \in \mathbf{\Delta} \setminus \mathcal{S}\,.$$

Combined with the continuity and smooth-fit conditions (4.23), this leads to

$$v(x,z) \quad = \quad g^\lambda(\varphi^\lambda(z),z) + \frac{g_x^\lambda(\varphi^\lambda(z),z)}{S' \circ \varphi^\lambda(z)}[S(x) - S \circ \varphi^\lambda(z)]\ , \qquad (x,z) \in \mathbf{\Delta} \setminus \mathcal{S}.$$

The free boundary φ^λ is then determined by the Dirichlet condition (4.21) and must satisfy:

$$g^\lambda(\varphi^\lambda(z),z)S' \circ \varphi^\lambda(z) \quad = \quad g_x^\lambda(\varphi^\lambda(z),z)S \circ \varphi^\lambda(z)\ , \qquad (x,z) \in \mathbf{\Delta} \setminus \mathcal{S}.$$

The next lemma introduces a free boundary function φ^λ satisfying this required condition. It also provides useful properties of this free boundary function and its technical proof is postponed to Section 4.4.

Lemma 4.1 *For any $\lambda > 1$, the function φ^λ given by*

$$\varphi^\lambda \quad : \quad z \in (0,\infty) \mapsto \arg \min_{x \in [0,z]} \frac{g(x,z)}{S(x)}\ ,$$

is a well defined increasing C^1 function, satisfying:
(i) $0 \leq \varphi^\lambda(z) < z$, *for any $z > 0$;*
(ii) φ^λ *maps* $(0,\infty)$ *onto* $(0,y^\lambda)$, *where y^λ is the unique non null zero of $y \mapsto yS'(y) - \lambda S(y)$.*

Remark 4.1 Observe that, for any fixed $z > 0$ and $\lambda > 1$, $g^\lambda(x,z)/S(x)$ converges to 0 as x goes to 0, since S dominates the Identity function as pointed out in Remark 2.3. Therefore, the function $g^\lambda(.,z)/S(.)$ is well defined on $[0,z]$ for any $z > 0$.

Before providing the value function solution and verifying that it satisfies the requirements of Theorem 3.1, we still need to check that the stopping region $\mathcal{S}^\lambda$ associated to φ^λ is indeed a good candidate, i.e. that the second part of (4.20) holds. This is the purpose of the next lemma, which proof is also postponed to Section 4.4.

Lemma 4.2 *For any $\lambda > 1$, the function $\mathcal{L}g^\lambda$ is non negative on $\{(x,z) \in \mathbf{\Delta},\ x \geq \varphi^\lambda(z)\}$.*

Given the free boundary φ^λ defined above and the corresponding stopping region $\mathcal{S}^\lambda$, we are now in position to provide the optimal strategy and value function solutions of the problem (4.16).

Theorem 4.2 *For any $\lambda > 1$, the value function V^λ solution of problem* (4.16) *is given by*

$$V^\lambda(x,z) \;=\; g^\lambda(\varphi^\lambda(z),z)\frac{S(x)}{S\circ\varphi^\lambda(z)}\,\mathbf{1}_{\{x<\varphi^\lambda(z)\}} + g^\lambda(x,z)\mathbf{1}_{\{x\geq\varphi^\lambda(z)\}}\;,\quad (x,z)\in\mathbf{\Delta}\;. \tag{4.25}$$

The smallest optimal stopping time associated to this stochastic control problem is given by

$$\theta^\lambda \;:=\; \inf\left\{t\geq 0\;,\quad X_t\geq\varphi^\lambda(Z_t)\right\}\;,\qquad \lambda>1\;.$$

Proof. Let denote by v^λ the candidate value function defined by the right-hand side of (4.25). We shall prove that v^λ coincides with the value function (4.16) by checking that it satisfies all the requirements of Theorem 3.1.

It is immediate that v^λ is bounded from below by 0 because $g^\lambda\geq 0$. Since g^λ is C^1 on $\mathbf{\Delta}$ and $C^{2,1}$ w.r.t. (x,z) on $\mathbf{\Delta}\setminus\{(0,z),\ z>0\}$, and φ^λ is C^1 by Lemma 4.1, v^λ is $C^{2,1}$ w.r.t. (x,z) on both $\mathbf{\Delta}\setminus\mathcal{S}$ and $\mathcal{S}\setminus\{(0,z),\ z>0\}$, so that it is piecewise $C^{2,1}$ on $\mathbf{\Delta}\setminus\{(0,z),\ z>0\}$. By construction, v^λ is continuous on $\mathbf{\Delta}$ and we recall from the definition of φ^λ that

$$g^\lambda(\varphi^\lambda(z),z)\frac{S'\circ\varphi^\lambda(z)}{S\circ\varphi^\lambda(z)} \;=\; g^\lambda_x(\varphi^\lambda(z),z)\;,\qquad z>0\;.$$

Therefore, v^λ is C^1 on $\mathbf{\Delta}$.

The closed stopping region associated to the value function v^λ is naturally given by (4.24). By definition, $v^\lambda = g^\lambda$ on $\mathcal{S}^\lambda$ and we deduce from Lemma 4.2 that $\mathcal{L}g^\lambda\geq 0$ on the set $\mathcal{S}^\lambda\setminus\{(0,z),\ z>0\}$. By construction, we have $\mathcal{L}v^\lambda = 0$ on $\mathbf{\Delta}\setminus\mathcal{S}^\lambda$. For any $z>0$, since $g^\lambda(.,z)/S$ achieves its minimum at a unique point $\varphi^\lambda(z)$, we get

$$g^\lambda(\varphi^\lambda(z),z)\frac{S(x)}{S\circ\varphi^\lambda(z)} \;<\; g^\lambda(x,z)\;,\qquad 0\leq x<\varphi^\lambda(z)\;,$$

and we deduce that $v^\lambda < g^\lambda$ on $\mathbf{\Delta}\setminus\mathcal{S}^\lambda$. Finally, since $v^\lambda_z(z,z) = g^\lambda_z(z,z) = 0$ for any $z>0$, all the requirements of (ii)-(iii) in Theorem 3.1 are in force, and the proof is complete. □

4.3 Properties of the solution

We first observe that the two previous cases where λ is above or below 1 seem to be of different natures. However, we prove hereafter that this is not the case and provide via simple arguments the continuity of V^λ with respect to the parameter λ.

Proposition 4.1 *The mapping $\lambda\mapsto V^\lambda$ is continuous on* $(0,\infty)$.

Proof. We fix λ_1 and λ_2 in $(0,\infty)$ such that $\lambda_1 \leq \lambda_2$. First notice that since $X_. \leq Z_\tau$ on $[0,\tau]$, we necessarily have

$$-\lambda_2 V^{\lambda_2}(x,z) \;=\; \sup_{\theta\in\mathcal{T}} \mathbb{E}_{x,z}\left(\frac{X_\theta}{Z_\tau}\right)^{\lambda_2} \;\leq\; \sup_{\theta\in\mathcal{T}} \mathbb{E}_{x,z}\left(\frac{X_\theta}{Z_\tau}\right)^{\lambda_1} \;=\; -\lambda_1 V^{\lambda_1}(x,z)\,, \quad (x,z)\in\mathbf{\Delta}\,. \tag{4.26}$$

Now, using Jensen's inequality, we observe that

$$\left[\mathbb{E}_{x,z}\left(\frac{X_\theta}{Z_\tau}\right)^{\lambda_1}\right]^{\frac{\lambda_2}{\lambda_1}} \;\leq\; \mathbb{E}_{x,z}\left(\frac{X_\theta}{Z_\tau}\right)^{\lambda_2} \;\leq\; -\lambda_2 V^{\lambda_2}(x,z)\,, \qquad \theta\in\mathcal{T}\,,\;\; (x,z)\in\mathbf{\Delta}\,.$$

Bringing this expression to the power λ_1/λ_2 and taking the supremum over θ, we deduce from (4.26) that

$$\big[-\lambda_1 V^{\lambda_1}(x,z)\big]^{\frac{\lambda_2}{\lambda_1}} \;\leq\; -\lambda_2 V^{\lambda_2}(x,z) \;\leq\; -\lambda_1 V^{\lambda_1}(x,z)\,, \qquad (x,z)\in\mathbf{\Delta}\,.$$

Therefore $\lambda_2 V^{\lambda_2} \to \lambda_1 V^{\lambda_1}$ whenever $\lambda_2 \to \lambda_1$ and we deduce the continuity of V^λ with respect to λ. □

For $\lambda > 1$, Theorem 4.2 indicates that the stopping region $\mathcal{S}^\lambda$ associated to problem (4.16) is given by (4.24). Since $\varphi^\lambda(0) = 0$, we see that the stopping region $\mathcal{S}^\lambda$ includes in particular the axis $\{(x,x)\,, x>0\}$. Therefore, if an investor detains a portfolio mean reverting to zero and hopes to get close to its upcoming maximum before it reaches zero according to the criterion (4.16), he should liquidate the portfolio immediately. Theorem 4.1 indicates that this is also the case for $\lambda \in (0,1]$ and these results are in accordance with those of [9] for an exponential Brownian motion on a finite fixed horizon, since waiting until maturity is irrelevant in our framework. Nevertheless, changing the criterion of interest may lead to value functions where immediate stopping is not optimal on the axis $\{(x,x)\,, x>0\}$. This is exactly the purpose of Section 5.

Figure 1 represents the frontier between the stopping and the continuation regions for different values of λ larger than 1 and associated to an Orstein-Uhlenbeck with parameter $\alpha = 1$ and a CIR-Feller process with parameters $\alpha = 1$ and $b = 1$. We first observe that the shape of the free boundary φ^λ is rather similar in both cases, and we observe indeed this feature for a large range of parameter set. Furthermore, the mapping $\lambda \mapsto \varphi^\lambda$ seems to be continuous, property which is easily verified from the definition of φ^λ. Second, we notice that the free boundary φ^λ is decreasing with respect to λ. Indeed, arguing as in Part 2. of the proof of Lemma 4.1, one can easily check that the function $x \in \mathbb{R}^+ \mapsto xS'(x)/S(x)$ is decreasing starting from 1. Hence, by definition of y^λ, the valuation domain $[0, y^\lambda]$ of φ^λ shrinks monotonically to $\{0\}$ as λ decreases to 1, hence leading to the absence of continuation region for the problem V^1.

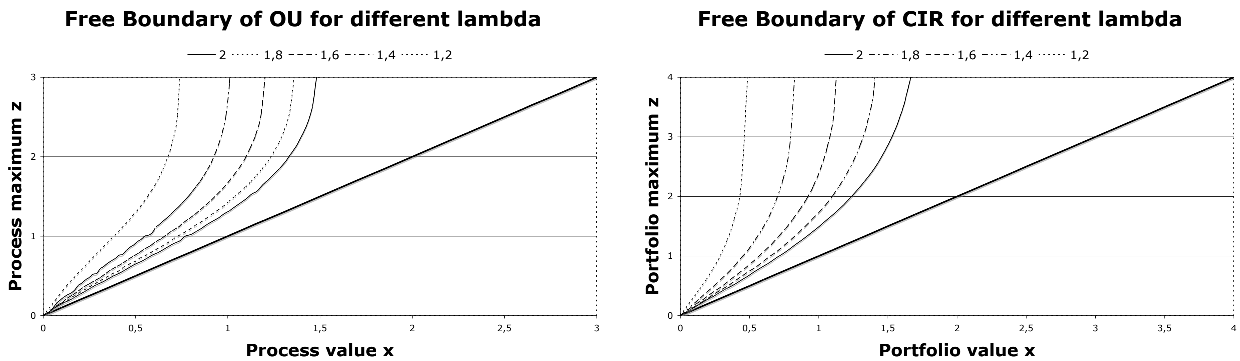

Figure 1: Optimal frontier for an OU ($\alpha = 1$) and a CIR ($\alpha = 1$, $b = 1$) with different parameter λ

Remark 4.2 Considering for example an Ornstein-Uhlenbeck portfolio X, one verifies easily from their definitions that the free boundary φ^λ and the value function v^λ are continuous with respect to the parameter $\alpha \in \mathbb{R}$, characterizing the dynamics of the mean reverting portfolio X. Hence, the continuation and stopping regions are not too sensitive to eventual estimation errors of this parameter of interest.

4.4 Proofs of Lemma 4.1 and Lemma 4.2

This section provides successively the proofs of Lemma 4.1 and Lemma 4.2.

Proof of Lemma 4.1

Fix $\lambda > 1$. Let introduce the functions

$$m \,:\, x \mapsto \frac{xS'(x)}{\lambda S(x)^2} - \frac{1}{S(x)} \qquad \text{and} \qquad \ell \,:\, z \mapsto -\int_z^\infty \frac{\lambda z^\lambda du}{S(u)u^{1+\lambda}} \,,$$

so that the derivative of the function of interest rewrites

$$\frac{\partial}{\partial x}\left[\frac{g(x,z)}{S(x)}\right] \;=\; \frac{x^{\lambda-1}}{z^\lambda}\left\{m(x) - \ell(z)\right\} \,, \qquad (x,z) \in \mathbf{\Delta} \,. \tag{4.27}$$

0. *A useful estimate.*

We will use several times the following expansion as $x \to \infty$:

$$\alpha(x)S(x) \sim_\infty S'(x). \tag{4.28}$$

Indeed, recalling that $\mathcal{L}S = 0$ and integrating by parts, we compute:

$$S(x) = S(1) + \int_1^x \frac{\alpha(u)S'(u)}{\alpha(u)}du = S(1) + \frac{S'(x)}{\alpha(x)} - \frac{S'(1)}{\alpha(1)} - \int_1^x (1/\alpha)'(u)S'(u)du \sim_\infty \frac{S'(x)}{\alpha(x)} \,,$$

since $S(x) \to \infty$ as $x \to \infty$ and (2.12) implies that $(1/\alpha)'(x) \to 0$ as $x \to \infty$.

1. *Definition of* φ^λ.

In order to justify that φ^λ is well defined, we study separately the functions m and ℓ. We observe first that the function ℓ is negative, increasing and, according to (4.28), satisfies

$$\ell(z) \ \sim_\infty \ -\int_z^\infty \frac{\lambda z^\lambda \alpha(u)du}{u^{1+\lambda}S'(u)} \ \sim_\infty \ -\frac{\lambda}{zS'(z)} \ \to_\infty \ 0 \ , \tag{4.29}$$

where the second equivalence comes from the following computation:

$$-\int_z^\infty \frac{\lambda z^\lambda \alpha(u)du}{u^{1+\lambda}S'(u)} = \left[\frac{\lambda z^\lambda}{u^{1+\lambda}S'(u)}\right]_z^\infty + \int_z^\infty \frac{\lambda(1+\lambda)z^\lambda du}{u^{2+\lambda}S'(u)} \ , \qquad z > 0 \ .$$

We now turn to the study of m and compute, for any $x > 0$,

$$m'(x) = \frac{\{\lambda + 1 + x\alpha(x)\}}{\lambda S(x)^3} S'(x)M(x) \ , \qquad \text{with } M : x \ \mapsto \ S(x) - \frac{2x}{\lambda + 1 + x\alpha(x)} S'(x) \ .$$

Differentiating one more time, we obtain

$$M'(x) = \frac{x^2 S'(x)}{(\lambda + 1 + x\alpha(x))^2} \left[\frac{\lambda^2 - 1}{x^2} - \alpha(x)^2 + 2\alpha'(x)\right] \ , \qquad x > 0 \ .$$

Since $\lambda > 1$ while α is non-negative, increasing and concave, the term in between brackets is decreasing. Furthermore $M'(0) = \lambda - 1 > 0$ and, since $x\alpha'(x) \le \alpha(x)$ for $x > 0$, we get

$$M'(x) \le \frac{S'(x)}{(\lambda + 1 + x\alpha(x))^2} \left[\lambda^2 - 1 - x^2\alpha(x)^2 + 2x\alpha(x)\right] \to_{x\to\infty} -\infty \ .$$

Thus M is first increasing and then decreasing. Furthermore, estimate (4.28) implies that

$$M(x) \ \sim_\infty \ \left[1 - \frac{2x\alpha(x)}{\lambda + 1 + x\alpha(x)}\right] S(x) \ = \ \frac{\lambda + 1 - x\alpha(x)}{\lambda + 1 + x\alpha(x)} S(x) \ \sim_\infty \ -S(x) \ .$$

Since $M(0) = 0$, we deduce that m is first increasing and then decreasing. Then we have as $x \to 0$, $m(x) \sim \frac{1-\lambda}{\lambda x} \to -\infty$, and, using (4.28),

$$m(x) \ \sim_{x\to\infty} \ \frac{x\alpha(x) - \lambda}{\lambda S(x)} > 0 \, , \qquad \text{for sufficiently large } x. \tag{4.30}$$

Since the function ℓ is negative, we deduce that, for any $z > 0$, there is a unique point in $(0, \infty)$, denoted $\varphi^\lambda(z)$, such that $m \circ \varphi^\lambda(z) = \ell(z)$, and is the unique minimum of $x \mapsto g(x, z)/S(x)$ on $[0, \infty)$. This point is also the unique solution of

$$g_x(x, z)S(x) - g(x, z)S'(x) = 0. \tag{4.31}$$

for any fixed z. The implicit functions theorem implies that φ^λ is C^1 on $(0, \infty)$. We prove hereafter that $\varphi^\lambda(z) < z$, for any $z > 0$, so that φ^λ corresponds to the definition given in the statement of the lemma.

2. $\varphi^\lambda(z) < z$, *for any* $z > 0$.

For any $z > 0$, since $x \mapsto m(x) - \ell(z)$ is first negative and then positive, on $(0, \infty)$, the property $\varphi^\lambda(z) < z$ will be a direct consequence of the estimate $m(z) - \ell(z) > 0$, that we prove now. First observe that the derivative of $h : z \mapsto [m(z) - \ell(z)]z^{-\lambda}$ is given by

$$h'(z) = \frac{1 + z\alpha(z)}{z^\lambda S(z)^3} S'(z) n(z) \,, \quad z > 0 \,, \qquad \text{with} \quad n : z \mapsto S(z) - \frac{2z}{1 + z\alpha(z)} S'(z).$$

Hence h' has the same sign as n and, differentiating one more time, we compute

$$n'(z) \;=\; S'(z) \left[1 - \frac{2 + 2z\alpha(z)}{1 + z\alpha(z)} + \frac{2z(\alpha(z) + z\alpha'(z))}{|1 + z\alpha(z)|^2} \right] \;=\; -\frac{1 + z^2\alpha(z)^2 - 2z^2\alpha'(z)}{|1 + z\alpha(z)|^2} S'(z) \,,$$

for any $z > 0$. Since α is concave and non-negative, we have $z\alpha'(z) \le \alpha(z)$ for $z > 0$, and, plugging this estimate in the previous expression, we obtain

$$n'(z) \;\le\; -\frac{|1 - z\alpha(z)|^2}{|1 + z\alpha(z)|^2} S'(z) \;\le\; 0 \,, \qquad z > 0 \,.$$

Hence, n is non-increasing starting from $n(0) = 0$, and therefore h is also non-increasing on $(0, \infty)$. Furthermore, we know from (4.29) and (4.30) that $h(z) = [m(z) - \ell(z)]z^{-\lambda} > 0$ for sufficiently large z, so that we have $m(z) - \ell(z) > 0$, for any $z > 0$.

3. φ^λ *is increasing and valued in* $[0, y_\lambda]$.

Recall that $m \circ \varphi^\lambda \;=\; \ell$ and ℓ is increasing and negative. Since m is also increasing when it is negative, we deduce that φ^λ is increasing. Since after crossing zero, the function m remains positive, $\varphi(z)$ must be smaller than the point where m crosses zero, for any $z > 0$. By definition of m, this point y^λ is implicitly defined by $y^\lambda S'(y^\lambda) \;=\; \lambda S(y^\lambda)$. Therefore $\varphi^\lambda(.) \le y^\lambda$ and, since $\ell(z) \to_{z\to\infty} 0$, we even have $\varphi^\lambda(z) \to_{z\to\infty} y^\lambda$. □

Proof of Lemma 4.2

Proof. We fix $\lambda > 1$ and recall from estimate (4.17) in the proof of Theorem 4.1 that $\mathcal{L}g^\lambda$ is given by

$$\mathcal{L}g^\lambda(x, z) = \; x^{\lambda-2}[x\alpha(x) + 1 - \lambda] \left(\frac{1}{z^\lambda} - \int_z^\infty \frac{S(x)}{S(u)} \frac{\lambda du}{u^{1+\lambda}} \right) + 2x^{\lambda-1} S'(x) \int_z^\infty \frac{\lambda du}{S(u) u^{1+\lambda}}, \tag{4.32}$$

for any $0 < x \le z$. Since S is increasing, we first observe that $\mathcal{L}g^\lambda(x, .) \ge 0$ for any $x > 0$ such that $x\alpha(x) + 1 - \lambda \ge 0$. Denoting by x_λ the unique point of $\mathbb{R}^+$ defined implicitly by

$$x_\lambda \alpha(x_\lambda) \;=\; \lambda - 1 \,,$$

we deduce that $\mathcal{L}g^\lambda(x, .) \ge 0$ for any $x \ge x_\lambda$.

It remains to treat the case where $x < x_\lambda$ and we compute

$$\frac{\partial}{\partial z}\mathcal{L}g^\lambda(x,z) = \lambda\frac{x^{\lambda-2}}{z^{1+\lambda}S(z)}\left\{[\lambda-1-x\alpha(x)](S(z)-S(x))-2xS'(x)\right\}, \quad 0<x\le z .$$

For any fixed $x \in (0, x_\lambda)$, the previous expression in between brackets is increasing with respect to z, negative for $z = x$ and positive for z large enough. Hence, for any $x \in (0, x_\lambda)$, $\mathcal{L}g(x, .)$ is first decreasing, then increasing and $\mathcal{L}g(x, z)$ goes to 0 as z goes to infinity. Denoting by γ^λ the inverse of φ^λ, we deduce that

$$\mathcal{L}g(x,z) \ge 0 \text{ , for any } z \le \gamma^\lambda(x) \text{ ,} \qquad \text{if and only if} \qquad \mathcal{L}g(x,\gamma^\lambda(x)) \ge 0 \text{ ,}$$

for any fixed $x \in (0, x^\lambda)$. Since φ^λ and hence γ^λ are increasing, it therefore only remains to verify that $\mathcal{L}g(., \gamma(.)) \ge 0$ on $(0, x_\lambda)$.
We recall from the proof of Lemma 4.1 that γ^λ is defined implicitly by

$$\int_{\gamma^\lambda(x)}^\infty \frac{\lambda[\gamma^\lambda(x)]^\lambda du}{S(u)u^{1+\lambda}} \;=\; \frac{1}{S(x)} - \frac{xS'(x)}{\lambda S(x)^2} , \qquad 0<x<x_\lambda .$$

For a given $x \in (0, x_\lambda)$, plugging this estimate into (4.32), we deduce

$$\mathcal{L}g(x,\gamma^\lambda(x)) \;=\; (x\alpha(x)+1-\lambda)\frac{x^{\lambda-1}S'(x)}{\lambda[\gamma^\lambda(x)]^\lambda S(x)} + \left(\frac{1}{S(x)} - \frac{xS'(x)}{\lambda S^2(x)}\right)\frac{2x^{\lambda-1}S'(x)}{[\gamma^\lambda(x)]^\lambda} ,$$

which after simplifications leads to

$$\mathcal{L}g(x,\gamma^\lambda(x)) \;=\; \frac{(x\alpha(x)+1+\lambda)x^{\lambda-1}S'(x)}{\lambda S(x)^2[\gamma^\lambda(x)]^\lambda}h(x) , \quad \text{with } h: x\mapsto S(x) - \frac{2x}{x\alpha(x)+1+\lambda}S'(x) .$$

In order to get the sign of $\mathcal{L}g(., \gamma^\lambda(.))$, we look for the sign of h and compute

$$\begin{aligned} h'(x) &= S'(x)\left[1 - \frac{2+2x\alpha(x)}{1+\lambda+x\alpha(x)} + \frac{2x(\alpha(x)+x\alpha'(x))}{|1+\lambda+x\alpha(x)|^2}\right] \\ &= \frac{S'(x)}{|1+\lambda+x\alpha(x)|^2}\left[\lambda^2-1-x^2\alpha(x)^2+2x^2\alpha'(x)\right] \\ &\ge \frac{S'(x)}{|1+\lambda+x\alpha(x)|^2}\left[\lambda^2-(1-x\alpha(x))^2\right] , \qquad x<x_\lambda . \end{aligned}$$

since $x\alpha'(x) \le \alpha(x)$ for $x > 0$, due to the concavity of α. By definition of x_λ, we deduce that h is non-decreasing on $(0, x_\lambda)$. But $h(0) = 0$ and therefore $\mathcal{L}g(., \gamma(.)) \ge 0$ on $(0, x_\lambda)$, which concludes the proof. □

5 Minimization of the relative quadratic error

Let us now consider the case where $f : x \mapsto \frac{1}{2}(1-x)^2$. Therefore, we are computing the following value function

$$V(x,z) \;:=\; \frac{1}{2}\inf_{\theta\in\mathcal{T}}\mathbb{E}_{x,z}\left(1-\frac{X_\theta}{Z_\tau}\right)^2 , \qquad (x,z)\in\mathbf{\Delta}. \tag{5.33}$$

With such a criterion, the investor tries to minimize the expected value of the squared relative error between the value of the stopped process and the maximal value of the process up to τ. In other words he wants to minimize the expectation of $[(Z_\tau - X_\theta)/Z_\tau]^2$, whereas in the previous section, $\lambda = 2$ would correspond to the minimization of $1-(X_\theta/Z_\tau)^2$, which is not as natural. In contrast with the previous optimal stopping problem (4.16), we prove that stopping immediately even for $x = z$ is not optimal in general. This suggests that the nature of the stopping region closely depends on the criterion of interest.

5.1 Construction of the solution

From (2.11), we compute the corresponding reward function:

$$g(x,z) \;=\; \frac{1}{2}\left(1-\frac{x}{z}\right)^2 + xS(x)\int_z^\infty \left(1-\frac{x}{u}\right)\frac{du}{u^2S(u)}\,, \qquad (x,z)\in\mathbf{\Delta}\,.$$

In view of Proposition 3.1, we would require $\mathcal{L}g(x,z) \geq 0$ in order for some $(x,z) \in \mathbf{\Delta}$ to be in the stopping region. Let us first compute

$$g_x(x,z) = -\frac{1}{z}\left(1-\frac{x}{z}\right) + [S(x)+xS'(x)]\int_z^\infty \frac{du}{u^2S(u)} - [2xS(x)+x^2S'(x)]\int_z^\infty \frac{du}{u^3S(u)}\,,$$

$$g_{xx}(x,z) = \frac{1}{z^2} + [2+x\alpha(x)]S'(x)\int_z^\infty \frac{du}{u^2S(u)} - [2S(x)+4xS'(x)+x^2\alpha(x)S'(x)]\int_z^\infty \frac{du}{u^3S(u)},$$

for any $(x,z) \in \Delta$. Combining these estimates, we deduce

$$\begin{aligned}\mathcal{L}g(x,z) \;=\;& \frac{1}{z^2}[1+\alpha(x)\,(z-x)] + [2S'(x)-\alpha(x)S(x)]\int_z^\infty \frac{du}{u^2S(u)} \\ &- [S(x)+2xS'(x)-x\alpha(x)S(x)]\int_z^\infty \frac{2du}{u^3S(u)}\,, \qquad (x,z)\in\mathbf{\Delta}\,. \qquad (5.34)\end{aligned}$$

In view of Theorem 3.1 (i), if $\mathcal{L}g \geq 0$ on $\mathbf{\Delta}$, then immediate stopping is optimal, $v = g$ and the problem is trivial. However, the next result gives sufficient conditions such that it is not the case. Consider the following condition:

$$\alpha(0)^2 - 2\alpha'(0) < 0. \qquad (5.35)$$

Remark 5.1 Notice that (5.35) will be satisfied for an Ornstein-Uhlenbeck process as well as a CIR-Feller process with positive "mean", for which we respectively have $\alpha(x) = \alpha x$ and $\alpha(x) = \alpha\frac{x}{x+b}$ respectively, α and b being positive constants. More generally, as soon as $\alpha(0) = 0$, (5.35) is satisfied. However, for a drifted Brownian motion or a degenerated CIR-Feller process with "mean" equal to 0, (5.35) does not hold true.

Proposition 5.1 *Assume that* (5.35) *is satisfied. Then there exists an open subset of* $\mathbf{\Delta}$ *on which* $\mathcal{L}g < 0$.

Proof. Using the asymptotic expansions from Proposition 5.4 in Section 5.4, we compute for z close to 0:

$$\begin{aligned}\mathcal{L}g(z,z) &= [2S'(z)-\alpha(z)S(z)]\int_z^\infty \frac{du}{u^2S(u)} - [S(z)+2zS'(z)-z\alpha(z)S(z)]\int_z^\infty \frac{2du}{u^3S(u)} + \frac{1}{z^2}\\
&= \left(2+\alpha(0)z+O(z^2)\right)\left(\frac{1}{2z^2}-\frac{\alpha(0)}{2z}-\frac{\alpha(0)^2-2\alpha'(0)}{12}\ln z + o(\ln z)\right)\\
&\quad -\left(3z+\frac{3}{2}\alpha(0)z^2+O(z^3)\right)\left(\frac{2}{3z^3}-\frac{\alpha(0)}{2z^2}+O\left(\frac{1}{z}\right)\right)+\frac{1}{z^2}\\
&= -\frac{\alpha(0)^2-2\alpha'(0)}{6}\ln z + o(\ln z)\ .\end{aligned}$$

Since $\ln z \to -\infty$ when $z \to 0$, we see that if (5.35) holds, then $\mathcal{L}g(z,z) < 0$ for z in a neighborhood of 0, so that we have the result by continuity of $\mathcal{L}g$. □

In view of Proposition 3.1, an immediate consequence of Proposition 5.1 is that stopping immediately is not optimal in general, even for initial conditions such that $x = z$, which is the practical case of interest. Hence, the optimal strategy shall be very different from the one in the power utility case. Since we do not have $\mathcal{L}g \geq 0$ on the entire space $\boldsymbol{\Delta}$ but we can exercise only in that region, we first need to study the set

$$\Gamma^+ \;:=\; \{(x,z)\in\boldsymbol{\Delta},\ \mathcal{L}g(x,z)\geq 0\}\ , \tag{5.36}$$

and we define similarly:

$$\Gamma^- \;:=\; \{(x,z)\in\boldsymbol{\Delta},\ \mathcal{L}g(x,z)\leq 0\}\ . \tag{5.37}$$

In fact, observe that (5.34) rewrites as:

$$\begin{aligned}\mathcal{L}g(x,z) \;&=\; \alpha(x)\frac{z-x}{z^2} + [2S'(x)-\alpha(x)S(x)]\int_z^\infty\left(1-\frac{2x}{u}\right)\frac{du}{u^2S(u)}\\
&\quad + \left(\frac{1}{z^2}-S(x)\int_z^\infty \frac{2du}{u^3S(u)}\right)\ , \qquad (x,z)\in\boldsymbol{\Delta}\ .\end{aligned} \tag{5.38}$$

By Remark 2.4, we have $2S'-\alpha S-2\geq 0$ and therefore each of the three terms above are positive if $z\geq 2x$, and so

$$\mathcal{L}g(x,z)>0 \qquad \text{for} \quad z\geq 2x \quad \text{and} \quad (x,z)\in\boldsymbol{\Delta}\ , \tag{5.39}$$

which implies that $\{(x,z)\in\boldsymbol{\Delta},\ z\geq 2x\}\subset \mathrm{Int}(\Gamma^+)$.
Moreover we have the following result, which proof is given in Section 5.4 below.

Lemma 5.1 *For any $x>0$, there exists $\delta_x\in(x,2x)$ such that $\mathcal{L}g(x,.)$ is increasing on $[x,\delta_x)$ and decreasing on $(\delta_x,2x]$.*

In view of (5.39), we can define the following function on $\mathbb{R}_+ \setminus \{0\}$:

$$\Gamma(x) \quad := \quad \inf\{z \geq x,\ \mathcal{L}g(x,z) \geq 0\}. \tag{5.40}$$

Lemma 5.1 and (5.39) imply that, if $z > \Gamma(x)$, then $\mathcal{L}g(x,z) > 0$, while if $z \in (x, \Gamma(x))$, then $\mathcal{L}g(x,z) < 0$. We also deduce that $\Gamma(x) > x$ implies $\mathcal{L}g(x, \Gamma(x)) = 0$. Notice that Γ is continuous, and, from (5.39), we also know that $\Gamma(x) < 2x$.

The next result provides the main properties of Γ: it is increasing and equal to the Identity function for sufficiently large x. Again the proof is postponed to Section 5.4.

Proposition 5.2 *We have the two following properties:*
(i) Γ *is increasing on* $(0, +\infty)$*;*
(ii) *Denoting* $\Gamma^\infty := \sup\{x \geq 0;\ \Gamma(x) > x\}$*, we get* $\Gamma^\infty < \infty$.

Notice that $\Gamma^+ \neq \mathbf{\Delta}$ implies directly $\Gamma^\infty > 0$.

Now that we have a better understanding of the set Γ^+, we expect to have a stopping region of the form $\{(x,z) \in \Delta\,;\ z \geq \gamma(x)\}$ and, our objective is then to find functions v and γ, satisfying the following free-boundary problem:

$$\mathcal{L}v(x,z) \;=\; 0 \qquad \text{for } 0 < z < \gamma(x) \quad \text{and} \quad (x,z) \in \mathbf{\Delta}\ , \tag{5.41}$$

$$v(x,z) \;=\; g(x,z) \ \text{ and } \ \mathcal{L}g(x,z) \;\geq\; 0 \qquad \text{for } z \geq \gamma(x) \quad \text{and} \quad (x,z) \in \mathbf{\Delta}\ , \tag{5.42}$$

$$v(0,z) \;=\; \frac{1}{2} \qquad \text{for } z > 0\ , \tag{5.43}$$

$$v_z(z,z) \;=\; 0 \qquad \text{for } z > 0\ . \tag{5.44}$$

In order to allow for the application of Itô's formula, the verification step requires a value function which is $C^{1,0}$ and piecewise $C^{2,1}$ with respect to (x,z). Therefore, as in the previous section, we complement the above system by the continuity and the smooth-fit conditions

$$v(x,\gamma(x)) \;=\; g(x,\gamma(x)) \quad \text{and} \quad v_x(x,\gamma(x)) \;=\; g_x(x,\gamma(x))\,, \qquad \text{for } x > 0\ . \tag{5.45}$$

The stopping region $\mathcal{S}$ will then be defined as:

$$\mathcal{S} \quad := \quad \{(x,z) \in \mathbf{\Delta};\ z \geq \gamma(x)\} \cup \{(0,z);\ z > 0\}. \tag{5.46}$$

First by (5.41), on the continuation region, v is of the form:

$$v(x,z) \;=\; A(z) + B(z)S(x)\ , \qquad (x,z) \in \mathbf{\Delta} \setminus \mathcal{S}\,.$$

Then, on the interval where γ is one-to-one, the continuity and smoothfit conditions (5.45) imply that

$$v(x,z) \;=\; g(\gamma^{-1}(z), z) + \frac{g_x(\gamma^{-1}(z), z)}{S' \circ \gamma^{-1}(z)}[S(x) - S \circ \gamma^{-1}(z)]\ , \qquad (x,z) \in \mathbf{\Delta} \setminus \mathcal{S}.$$

Finally, the Neumann condition (5.44), implies that we expect the boundary γ to satisfy on its domain of definition the following ODE:

$$\gamma'(x) \quad = \quad \frac{\gamma(x)^2 \mathcal{L}g(x,\gamma(x))}{\left(\frac{2x}{\gamma(x)} - 1\right)\left(1 - \frac{S(x)}{S\circ\gamma(x)}\right)}. \tag{5.47}$$

As in [5], there is no a priori initial condition for this ODE. In the sequel, we take this ODE (with no initial condition) as a starting point to construct the boundary γ. Notice that this ODE has infinitely many solutions, as the Cauchy-Lipschitz condition is locally satisfied whenever (5.47) is complemented with the condition $\gamma(x_0) = z_0$ for any $0 < x_0 < z_0$ and $z_0 \neq 2x_0$. We will follow the ideas of [5], however in our case, (5.47) is not well-defined for $\gamma(x) = 2x$, so that our framework requires to be more cautious. Notice also that we encounter here a similar feature as in Peskir [8]. The following result selects an appropriate solution of (5.47), and its proof is given in Section 5.5.

Proposition 5.3 *Let* $Int(\Gamma^-)$ *be non empty. Then, there exists an increasing continuous function* γ *defined on* $\mathbb{R}_+$ *with graph* $\{(x,\gamma(x)) : x > 0\} \subset \mathbf{\Delta}$*, such that:*
(i) *On the set* $\{x > 0 : \ \gamma(x) > x\}$*,* γ *is a* C^1 *solution of the ODE* (5.47)*,*
(ii) $\{(x,\gamma(x)) : \ x > 0\} \subset \Gamma^+$*, and* $\{(x,\gamma(x)) : \ x > 0$ *and* $\gamma(x) > x\} \subset Int(\Gamma^+)$*,*
(iii) $\gamma(x) = x$ *for all* $x \geq \Gamma^\infty$*.*

Since γ is increasing, we can define:

$$\varphi := \gamma^{-1}. \tag{5.48}$$

Now that we have constructed the free-boundary φ, we are able to state the following result.

Theorem 5.1 *Let* $Int(\Gamma^-)$ *be non empty,* γ *be given by Proposition 5.3 and* φ *be defined by* (5.48)*. Then the value function* V *solution of problem* (5.33) *is given, for* $(x,z) \in \mathbf{\Delta}$*, by:*

$$V(x,z) := \begin{cases} g(x,z)\ , & \text{if } x \leq \varphi(z) \\ g(\varphi(z),z) + g_x(\varphi(z),z)\frac{S(x) - S\circ\varphi(z)}{S'\circ\varphi(z)}\ , & \text{if } x > \varphi(z) \end{cases} . \tag{5.49}$$

Moreover, the smallest optimal stopping time associated to this stochastic control problem is given by $\theta^* := \inf\{t \geq 0\ , \quad X_t \leq \varphi(Z_t)\}$.

Proof. Let v be defined by (5.49) and recall that $\mathcal{S}$ is defined by (5.46). The result follows from verifying that all the assumptions of Theorem 3.1 (ii) and (iii) are satisfied.

1. *Regularity of* v.
We know from Proposition 5.3 that γ and therefore φ are continuous and hence v is continuous on $\mathbf{\Delta}$ by construction. Furthermore, by Proposition 5.3 (i) and (ii) together with the dynamics of the ODE (5.47), γ is a C^1 function with positive derivative on the set

$\{x > 0;\ \gamma(x) > x\}$. Therefore φ is C^1 as well on $\{z > 0;\ \varphi(z) < z\}$ so that it is immediate that v is C^0 and piecewise $C^{2,1}$ w.r.t. (x, z). Furthermore, since $\Gamma^\infty < \infty$ by Proposition 5.2, $\mathbf{\Delta} \setminus \mathcal{S}$ is bounded. Since v is continuous and $g \geq 0$, v is bounded from below.

2. *Dynamics of v.*
By definition, we have $\mathcal{L}v = 0$ on $\mathbf{\Delta} \setminus \mathcal{S}$. By Proposition 5.3 (ii), $\mathcal{L}g(x, \gamma(x)) \geq 0$ for $x > 0$, and we deduce from Lemma 5.1 and (5.39) that $\mathcal{L}g(x, z) \geq 0$ for any $(x, z) \in \mathbf{\Delta}$ such that $z \geq \gamma(x)$. Hence, (5.39) ensures that $\mathcal{L}g \geq 0$ on $\mathcal{S}$.
It remains to prove that $v_z(z, z) = 0$ for $z > 0$. We fix $z > 0$. If $\varphi(z) \geq z$, since $g_z(z, z) = 0$, we have $v_z(z, z) = 0$ as well. Suppose now that $\varphi(z) < z$. Then, by Proposition 5.3 (i), γ satisfies (5.47) in a neighborhood of $\varphi(z)$, and by Proposition 5.3 (ii), $\mathcal{L}g\big(\varphi(z), z\big) > 0$, which implies $\gamma' \circ \varphi(z) > 0$, so that:

$$\varphi'(z)\mathcal{L}g(\varphi(z), z) \quad = \quad \frac{1}{z^2}\left(\frac{2\varphi(z)}{z} - 1\right)\left(1 - \frac{S \circ \varphi(z)}{S(z)}\right) .$$

We then compute from the definitions of v and g that

$$\begin{aligned} v_z(z, z) &= g_z(\varphi(z), z) + g_{xz}\frac{S(z) - S \circ \varphi(z)}{S' \circ \varphi(z)} + \varphi'(z)\mathcal{L}g(\varphi(z), z)\frac{S(z) - S \circ \varphi(z)}{S' \circ \varphi(z)} \\ &= \left[\frac{1}{z^2}\left(1 - \frac{2\varphi(z)}{z}\right)\left(1 - \frac{S \circ \varphi(z)}{S(z)}\right) + \varphi'(z)\mathcal{L}g(\varphi(z), z)\right]\frac{S(z) - S \circ \varphi(z)}{S' \circ \varphi(z)} = 0 . \end{aligned}$$

3. *Comparing v and g.*
Finally, the fact that $v \leq g$ on $\mathbf{\Delta}$ and $v < g$ on $\mathbf{\Delta} \setminus \mathcal{S}$ follows from similar arguments as in the proof of Proposition 6.2 in [5] but the demonstration is simpler in our context since $\Gamma^\infty < \infty$. For the sake of completeness, we detail this proof. For $(x, z) \in \mathbf{\Delta}$ such that $x > \varphi(z)$, we compute

$$v(x, z) - g(x, z) \quad = \quad g(\varphi(z), z) + g_x(\varphi(z), z)\frac{S(x) - S \circ \varphi(z)}{S' \circ \varphi(z)} - g(x, z) ,$$

and, differentiating twice w.r.t. x and using (5.45), we verify that

$$v_x(x, z) - g_x(x, z) \quad = \quad -S'(x)\int_{\varphi(z)}^{x} \frac{\mathcal{L}g(u, z)}{S'(u)}du . \tag{5.50}$$

Therefore, from Lemma 5.1 and Proposition 5.2 (i), for any fixed z, the function $x \mapsto (v - g)(x, z)$ is either decreasing on $[\varphi(z), z]$, or decreasing on $[\varphi(z), \delta)$ and then increasing on $(\delta, z]$ for a given $\delta \in (\varphi(z), z)$. For any $z > 0$, since $v(\varphi(z), z) = g(\varphi(z), z)$, we only need to prove that $n(z) := v(z, z) - g(z, z) < 0$ if $\varphi(z) < z$.

Since $v_z(z, z) = g_z(z, z) = 0$ for $z > 0$, we compute:

$$n'(z) \quad = \quad v_x(z, z) - g_x(z, z) \quad = \quad -S'(z)\int_{\varphi(z)}^{z} \frac{\mathcal{L}g(u, z)}{S'(u)}du , \quad z > 0 .$$

We assume the existence of a fixed $z < \Gamma^\infty$ such that $n(z) \geq 0$ and $\varphi(z) < z$ and work towards a contradiction. We first observe that necessarily $n'(z) > 0$. If not, $\int_{\varphi(z)}^{z} \frac{\mathcal{L}g(u,z)}{S'(u)} du \geq 0$ implies that $\int_{\varphi(z)}^{x} \frac{\mathcal{L}g(u,z)}{S'(u)} du > 0$ for any $x \in (\varphi(z), z)$, and (5.50) combined with $v(\varphi(z), z) = g(\varphi(z), z)$ leads to $n(z) < 0$ which is impossible. Since n is continuous, this implies that n is increasing on any connected subset of $\{z' \geq z,\ \varphi(z') < z'\}$. Defining $a := \inf\{z' > z;\ \varphi(z') = z'\} \leq \Gamma^\infty < \infty$, we get $n(a) = v(a,a) - g(a,a) > 0$, which contradicts the definition of v.

Therefore, $n(z) < 0$ for any $z > 0$ such that $\varphi(z) < z$ and we deduce that $v \leq g$ on $\mathbf{\Delta}$ and $v < g$ on $\mathbf{\Delta} \setminus \mathcal{S}$. □

5.2 Properties of the value function

Theorem 5.1 and Proposition 5.1 indicate that, at least for processes satisfying (5.35), such as the Ornstein-Uhlenbeck process or the CIR-Feller process, the diagonal $\{(x,x);\ x > 0\}$ is not included in the stopping region $\mathcal{S}$. In other words, it is not always optimal to stop immediately, even when starting from points such that $x = z$. Therefore, the form of the solution and the nature of the optimal strategy to apply in order to be as close as possible to the maximum using this criterion is very different from the ones obtained in Section 4 or in [9].

The Ornstein-Uhlenbeck process as well as the CIR-Feller process are two examples for which the coefficient α satisfies Conditions (2.12) and $\mathrm{Int}(\Gamma^-) \neq \emptyset$. Indeed we have $\alpha(x) = \alpha x$ and $\alpha(x) = \alpha \frac{x}{x+b}$ respectively, where α and b are two positive constants. Therefore, Condition (5.35) is satisfied, ensuring that $\mathrm{Int}(\Gamma^-) \neq \emptyset$ by Proposition 5.1. Hence, Theorem 5.1 can be applied. Figure 2 represents the boundary φ for those two processes, with $\alpha = 1$ for the OU process and $(\alpha, b) = (0.1, 0.1)$ for the CIR process. We observe that the continuation region is in fact pretty small since the free boundary is very close to the diagonal axis. Therefore, even if immediate stopping is not optimal, an investor should not wait long until the process (X, X^*) enters the stopping region.

Remark 5.2 Similarly to Proposition 7.3 of [5], an homogeneity result can be derived for the OU process, so that the free boundary for any $\alpha > 0$ can be deduced by a change of scale from the one for $\alpha = 1$.

The Brownian motion with negative drift is another example for which α satisfies Condition (2.12). However, since $\alpha(x) = \alpha > 0$ is constant, Condition (5.35) does not hold. Although we did not verify it, numerical computations suggest that $\mathcal{L}g \geq 0$ on $\mathbf{\Delta}$.

Finally, we can also consider the case of a Brownian motion. In this case, $\alpha(x) = 0$, so that α does not satisfy Condition (2.12). However, for any $(x,z) \in \mathbf{\Delta}$, we can compute from

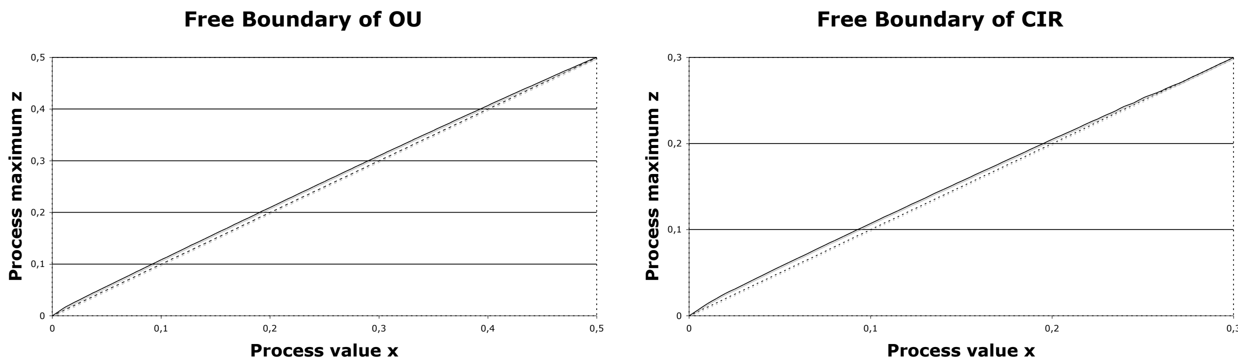

Figure 2: Optimal frontier for an OU ($\alpha = 1$) and a CIR ($\alpha = 0.1$, $b = 0.1$)

(5.34) that $\mathcal{L}g(x, z) = 2\frac{z-x}{z^3} \geq 0$ on $\mathbf{\Delta}$. Since the proofs of Theorem 3.1 and Remark 3.2 do not require Condition (2.12), we deduce that immediate stopping is always optimal.

Remark 5.3 Let α be associated to an Ornstein-Uhlenbeck or a CIR process and hence be parametrized by a possibly bi-dimensional parameter set a. Since the parameter set a may be badly estimated, let consider a sequence of parameter set (a_n) converging to a and denote by (α_n) the corresponding sequence of functions. Then, S_n, g_n and all their derivatives converge respectively to S, g and their derivatives in the sense of the uniform norm on the compact sets. Moreover, Γ_n converges to Γ in the same sense so that for n sufficiently large, $\text{Int}(\Gamma_n^-) \neq \emptyset$. ODE (5.47) also depends continuously on a_n, so that $z_n^*(x_0)$ defined by (5.63) converges to $z^*(x_0)$, and γ_n given by Proposition 5.3 converges pointwise to γ. Since γ_n is increasing for any n and γ is continuous, Dini's theorem implies that the convergence is uniform on any compact set of $\mathbb{R}_+ \setminus \{0\}$. Let us prove that φ_n converges to φ in the same sense. Let $y > 0$ be fixed, we define $x_n := \varphi_n(y)$ and $x := \varphi(y)$. We shall prove that $x_n \to x$. Indeed, since $\varphi_n(y) \in [\frac{y}{2}, y]$ for any n, $\{x_n;\ n \in \mathbb{N}\}$ is relatively compact in $\mathbb{R}_+ \setminus \{0\}$. Now let x' be the limit of a subsequence of (x_n). For notational reasons, let us write $x_n \to x'$, forgetting that it is a subsequence. Since (γ_n) converges uniformly on compact sets of $\mathbb{R}_+ \setminus \{0\}$, $\gamma_n(x_n) \to \gamma(x')$. Recalling that $\gamma_n(x_n) = y$ for any n, we get $\gamma(x') = y$ and therefore $x' = x$. In consequence, $x_n \to x$, or in other words, (φ_n) converges pointwise to φ on $\mathbb{R}_+ \setminus \{0\}$. Noticing that $\varphi_n(0) = 0$ for any n and $\varphi(0) = 0$ and using again Dini's theorem, we see that (φ_n) converges to φ uniformly on the compact sets of $\mathbb{R}_+$. This finally implies that (V_n) converges pointwise to V. As a consequence, if one makes a small mistake estimating the parameters of the model, the induced mistake on the free boundary as well as the mistake on the value function will be small as well.

5.3 Generalization

As in the previous section, we may also consider, for any $\lambda > 0$, the following extension of the previous problem:

$$V_\lambda(x,z) \quad := \quad \frac{1}{\lambda} \inf_{\theta \in \mathcal{T}} \mathbb{E}_{x,z} \left(1 - \frac{X_\theta}{Z_\tau}\right)^\lambda , \qquad (x,z) \in \mathbf{\Delta} . \tag{5.51}$$

In that case, (2.10) rewrites

$$g_\lambda(x,z) \quad = \tfrac{1}{\lambda} \left(1 - \tfrac{x}{z}\right)^\lambda + xS(x) \int_z^\infty \left(1 - \tfrac{x}{u}\right)^{\lambda - 1} \tfrac{du}{u^2 S(u)} du , \quad (x,z) \in \mathbf{\Delta} , \quad \lambda > 0 . \tag{5.52}$$

If $\lambda = 2$, $\mathcal{L}g_2$ is given by (5.34). If $\lambda = 1$, the control problem has already been solved in Section 4 and $\mathcal{L}g_1$ is given by (4.18). For any $\lambda > 0$ such that $\lambda \notin \{1,2\}$, we compute :

$$\begin{aligned}\mathcal{L}g_\lambda(x,z) = (\lambda-1)\left\{\frac{1}{z^2}\left(1-\frac{x}{z}\right)^{\lambda-2} - \int_z^\infty \frac{S(x)}{S(u)}\left[\frac{2}{u^3}\left(1-\frac{x}{u}\right)^{\lambda-2} - \frac{(\lambda-2)x}{u^4}\left(1-\frac{x}{u}\right)^{\lambda-3}\right]du\right\} \\ + \left[2S'(x) - \alpha(x)S(x)\right]\int_z^\infty \left(1-\frac{x}{u}\right)^{\lambda-2}(u-\lambda x)\frac{du}{u^3 S(u)} + \alpha(x)\frac{(z-x)^{\lambda-1}}{z^\lambda}\end{aligned} , \tag{5.53}$$

for $0 \le x < z$. In this case, the sign of $\mathcal{L}g_\lambda$ is hardly identifiable analytically, and we shall restrict our analysis to simple remarks and guesses on the solution of the problem (5.51).

Noticing that $\int_z^\infty \left(1 - \frac{x}{u}\right)^{\lambda-2} \frac{2}{u^3} - (\lambda - 2)\left(1 - \frac{x}{u}\right)^{\lambda - 3} \frac{x}{u^4} du = \frac{1}{z^2}\left(1 - \frac{x}{z}\right)^{\lambda-2}$ for $0 < x < z$, we deduce from (4.18), (5.39) and (5.53) that

$$\mathcal{L}g^\lambda(x,z) \geq 0 , \qquad \text{for } z \geq \lambda x > 0 \quad \text{and } 1 \leq \lambda \leq 2 . \tag{5.54}$$

Therefore, for $1 \le \lambda \le 2$, we expect to obtain as for $\lambda = 2$ a free boundary γ_λ in between the axis $\{(x,x)\,;x>0\}$ and $\{(x,\lambda x)\,;x>0\}$. We verify easily as in Proposition (4.1) that $\lambda \mapsto V_\lambda$ is continuous and, as expected, we observe a disappearance of the free boundary γ_λ for $\lambda = 1$.

On the other hand, for $\lambda < 1$, we observe that $\mathcal{L}g(x,z) < 0$ for x small enough and z large enough. Indeed, recalling from (4.28) that $S'(z) \sim \alpha(z)S(z)$ when $z \to \infty$, an integration by parts leads to $\int_z^\infty \frac{du}{u^2 S(u)} \sim_{z\to\infty} \frac{1}{z^2 S'(z)}$. Assuming moreover that $\alpha(0) = 0$ and plugging this estimate in (5.53), we get

$$\mathcal{L}g(0,z) \quad \sim_{z\to\infty} \quad \frac{\lambda - 1}{z^2} \;<\; 0 , \qquad \text{for any } \lambda < 1 .$$

In view of Proposition 3.1, this implies that the stopping region cannot have the same form as the one in the quadratic case $\lambda = 2$. It even suggests that the nature of the stopping region could be similar to the one of Section 4.2.

5.4 Proofs of Lemma 5.1 and Proposition 5.2

This section is dedicated to the proofs of Lemma 5.1 and Proposition 5.2, but we first state the asymptotic expansions used in Proposition 5.1.

Proposition 5.4 *As $z \to 0$, we have the following expansions:*

$$\begin{aligned}
S'(z) &= 1 + \alpha(0)z + (\alpha'(0) + \alpha(0)^2)\frac{z^2}{2} + o(z^2) \; ; \\
S(z) &= z + \alpha(0)\frac{z^2}{2} + (\alpha'(0) + \alpha(0)^2)\frac{z^3}{6} + o(z^3) \; ; \\
\alpha(z)S(z) &= z\alpha(0) + \frac{z^2}{2}\big(\alpha(0)^2 + 2\alpha'(0)\big) + \frac{z^3}{6}\big(\alpha(0)^3 + 4\alpha(0)\alpha'(0) + 3\alpha''(0)\big) + o(z^3) \; ; \\
\int_z^\infty \frac{du}{u^2 S(u)} &= \frac{1}{2z^2} - \frac{\alpha(0)}{2z} - \frac{\alpha(0)^2 - 2\alpha'(0)}{12}\ln(z) + o\big(\ln(z)\big) \; ; \\
\int_z^\infty \frac{2du}{u^3 S(u)} &= \frac{2}{3z^3} - \frac{\alpha(0)}{2z^2} + \frac{\alpha(0)^2 - 2\alpha'(0)}{6z} + o\left(\frac{1}{z}\right) \; .
\end{aligned}$$

Proof. As $z \to 0$, we directly compute the expansion:

$$S'(z) = S'(0) + zS''(0) + \frac{z^2}{2}S^{(3)}(0) + o(z^2) = 1 + \alpha(0)z + (\alpha'(0) + \alpha(0)^2)\frac{z^2}{2} + o(z^2) \; .$$

The exact same reasoning also leads to

$$\begin{aligned}
S(z) &= z + \frac{\alpha(0)}{2}z^2 + \frac{\alpha'(0) + \alpha(0)^2}{6}z^3 + o(z^3) \; ; \\
\alpha(z)S(z) &= z\alpha(0) + \frac{z^2}{2}\big(\alpha(0)^2 + 2\alpha'(0)\big) + \frac{z^3}{6}\big(\alpha(0)^3 + 4\alpha(0)\alpha'(0) + 3\alpha''(0)\big) + o(z^3) \; .
\end{aligned}$$

Using one of the previous estimates, we get

$$\begin{aligned}
\int_z^\infty \frac{du}{u^2 S(u)} &= \int_z^\infty \frac{du}{u^3\left(1 + \frac{\alpha(0)}{2}u + \frac{\alpha'(0)+\alpha(0)^2}{6}u^2 + o(u^2)\right)} \\
&= \int_z^\infty \left(1 - \frac{\alpha(0)}{2}u - \frac{\alpha'(0) + \alpha(0)^2}{6}u^2 + \left(\frac{\alpha(0)u}{2}\right)^2 + o(u^2)\right)\frac{du}{u^3} \\
&= \int_z^\infty \left(\frac{1}{u^3} - \frac{\alpha(0)}{2u^2} + \frac{\alpha(0)^2 - 2\alpha'(0)}{12u} + o\left(\frac{1}{u}\right)\right)du \\
&= \frac{1}{2z^2} - \frac{\alpha(0)}{2z} - \frac{\alpha(0)^2 - 2\alpha'(0)}{12}\ln(z) + o\big(\ln(z)\big) \; ,
\end{aligned}$$

which is justified since all the non-zero terms go to infinity when $z \to 0$. Similarly, we compute

$$\begin{aligned}
\int_z^\infty \frac{2du}{u^3 S(u)} &= \int_z^\infty \left(\frac{2}{u^4} - \frac{\alpha(0)}{u^3} + \frac{\alpha(0)^2 - 2\alpha'(0)}{6u^2} + o\left(\frac{1}{u^2}\right)\right)du \\
&= \frac{2}{3z^3} - \frac{\alpha(0)}{2z^2} + \frac{\alpha(0)^2 - 2\alpha'(0)}{6z} + o\left(\frac{1}{z}\right) \; .
\end{aligned}$$

$\square$

Proof of Lemma 5.1 Differentiating (5.34) w.r.t. z, we compute

$$\begin{aligned}\frac{\partial}{\partial z}\mathcal{L}g(x,z) &= -\frac{2S'(x)-\alpha(x)S(x)}{z^2S(z)} + \frac{[2-2x\alpha(x)]S(x)+4xS'(x)}{z^3S(z)} - \frac{2-2x\alpha(x)}{z^3} - \frac{\alpha(x)}{z^2}\\ &= [(2x-z)\alpha(x)-2]\frac{S(z)-S(x)}{z^3S(z)} + (2x-z)\frac{2S'(x)}{z^3S(z)}\,, \qquad (x,z)\in\mathbf{\Delta}\,.\end{aligned}$$

Let us introduce x_α as the unique solution of:

$$x_\alpha\alpha(x_\alpha) = 2\,. \tag{5.55}$$

If $x \le x_\alpha$, then $z \mapsto (2x-z)\alpha(x)-2$ is negative on $[x,2x)$, whereas if $x > x_\alpha$, then there exists $z_x \in (x,2x)$ such that $z \mapsto (2x-z)\alpha(x)-2$ will be positive on (x,z_x), zero at z_x and negative on $(z_x,2x)$.

Let x be fixed and let us introduce

$$F \quad : \quad z \mapsto S(z)-S(x) + \frac{2S'(x)(2x-z)}{(2x-z)\alpha(x)-2}\,,$$

which is well defined and continuous on $[x,2x]$ if $x < x_\alpha$, on $[x,2x)$ if $x = x_\alpha$ and on $[x,2x]\setminus\{z_x\}$ if $x > x_\alpha$. Furthermore F is increasing, since we compute on the domain of definition of F:

$$F'(z) = S'(z) + \frac{4S'(x)}{\big((2x-z)\alpha(x)-2\big)^2} \ > \ 0\,.$$

We consider first the case where $x \le x_\alpha$. Then F and $\frac{\partial}{\partial z}\mathcal{L}g(x,.)$ have opposite signs on $[x,2x)$. Since F is increasing, $F(x) < 0$ while $F(2x) = S(2x)-S(x) > 0$, $\mathcal{L}g(x,.)$ is increasing on $[x,\delta_x)$ and decreasing on $(\delta_x,2x]$, for a certain $\delta_x \in (x,2x)$.

We now turn to the case where $x > x_\alpha$. Then F and $\frac{\partial}{\partial z}\mathcal{L}g(x,.)$ have the same sign on $[x,z_x)$ and opposite signs on $(z_x,2x]$. Since F is increasing, $F(x) > 0$, $F(z_x^+) = -\infty$ and $F(2x) > 0$, we see that again $\mathcal{L}g(x,.)$ is increasing on $[x,\delta_x)$ and decreasing on $(\delta_x,2x]$, for a certain $\delta_x \in (z_x,2x) \subset (x,2x)$. $\square$

Proof of Proposition 5.2 We prove the two assertions separately.

(i) Γ is increasing on $(0,+\infty)$

We fix $x > 0$ such that $\Gamma(x) > x$. Then $\mathcal{L}g(.,\Gamma(.)) = 0$ in a neighborhood of x, and using the implicit functions theorem, Γ is C^1 in a neighborhood of x and we have:

$$\Gamma'(x)\frac{\partial}{\partial z}\mathcal{L}g(x,\Gamma(x)) + \frac{\partial}{\partial x}\mathcal{L}g(x,\Gamma(x)) = 0\,. \tag{5.56}$$

We will prove that $\Gamma'(x) > 0$. Denoting $m := 2S' - \alpha S$ which is increasing and positive, we get combining $\mathcal{L}g(x,\Gamma(x)) = 0$ and (5.38):

$$m(x)\int_{\Gamma(x)}^{\infty}\frac{u-2x}{u^3S(u)}du \;=\; S(x)\int_{\Gamma(x)}^{\infty}\frac{2du}{u^3S(u)} - \frac{1+\alpha(x)(\Gamma(x)-x)}{\Gamma(x)^2} \;\le\; -\frac{\alpha(x)(\Gamma(x)-x)}{\Gamma(x)^2}\,,$$

since S is increasing. Differentiating (5.38) with respect to x, we also compute

$$\frac{\partial}{\partial x}\mathcal{L}g(x,z) = \frac{\alpha'(x)(z-x)-\alpha(x)}{z^2} + m'(x)\int_z^\infty \frac{u-2x}{u^3S(u)}du - \Big[m(x)+S'(x)\Big]\int_z^\infty \frac{2du}{u^3S(u)}\,,$$

for $z \geq x$. Denoting $A := (\alpha m' - \alpha' m)(\Gamma - Id) + \alpha m$, the two previous estimates lead to

$$\frac{\partial}{\partial x}\mathcal{L}g(x,\Gamma(x)) \leq -\frac{A(x)}{m(x)\Gamma(x)^2} - \Big[m(x)+S'(x)\Big]\int_{\Gamma(x)}^\infty \frac{2du}{u^3S(u)}\,. \tag{5.57}$$

Introducing $B := \alpha m' - \alpha' m$ and observing that $x \leq \Gamma(x) \leq 2x$, we obtain

$$A(x) \;\geq\; \alpha(x)m(x)\mathbf{1}_{\{B(x)\geq 0\}} + \big(xB(x)+\alpha(x)m(x)\big)\mathbf{1}_{\{B(x)<0\}}\,. \tag{5.58}$$

Introducing finally $C : x \mapsto xB(x) + \alpha(x)m(x)$, we compute $C(0) = 2\alpha(0) \geq 0$ and

$$C'(x) \;=\; 2\alpha(x)m'(x) + x(\alpha(x)m''(x) - \alpha''(x)m(x)) \geq 0\,,$$

because $m'' \geq 0$ and $\alpha'' \leq 0$. Therefore C is non-negative, and, according to (5.58), A is also non-negative. As a consequence, combining $m > 0$ and (5.57), we deduce that $\frac{\partial}{\partial x}\mathcal{L}g(x,\Gamma(x)) < 0$. Using Lemma 5.1, we have $\frac{\partial}{\partial z}\mathcal{L}g(x,\Gamma(x)) > 0$, and (5.56) implies that $\Gamma'(x) > 0$.

Therefore Γ is increasing on the set $\{x > 0,\ \Gamma(x) > x\}$. But it is also increasing on the interior of the set $\{x > 0,\ \Gamma(x) = x\}$. Since Γ is continuous, it is increasing on $(0, +\infty)$.

(ii) We have $\Gamma^\infty := \sup\{x \geq 0;\ \Gamma(x) > x\} < \infty$.
The arguments used here are very close to the ones in the proof of Proposition 4.3 in [5]. However, our conclusions cannot be deduced form theirs since the involved computations are different and we need to detail this proof.

From the definition of the scale function (2.9), we compute:

$$S(x) = S(1) + \frac{S'(x)}{\alpha(x)} - \frac{S'(1)}{\alpha(1)} - \int_1^x \left(\frac{1}{\alpha}\right)'(u)S'(u)du\,, \qquad x > 0\,.$$

We then distinguish two cases depending on the explosion of the last term in the previous expression.

Case 1: $\int_1^\infty (1/\alpha)'(u)S'(u)du > -\infty$.
Then $S(x) = \frac{S'(x)}{\alpha(x)} + O(1)$ for x large enough. Recalling that $\mathcal{L}S = 0$, we compute

$$\int_x^\infty \frac{du}{u^2S(u)} = \int_x^\infty \frac{du}{u^2\left(\frac{S'(u)}{\alpha(u)} + O(1)\right)} = \int_x^\infty \frac{\alpha(u)}{u^2S'(u)}\frac{du}{1+O\left(\frac{\alpha(u)}{S'(u)}\right)}$$
$$= \int_x^\infty \frac{\alpha(u)du}{u^2S'(u)} + O\left(\int_x^\infty \frac{\alpha^2(u)}{u^2[S'(u)]^2}\right)\,,$$

for x large enough. Integrating by parts, we observe that

$$\int_x^\infty \frac{\alpha(u)du}{u^2 S'(u)} = \frac{1}{x^2 S'(x)} - 2\int_x^\infty \frac{du}{u^3 S'(u)} \,, \qquad x > 1 \,.$$

We now prove that $\frac{x\alpha^2(x)}{S'(x)} \to 0$ when $x \to \infty$.
Indeed, since $\alpha(1) > 0$ by (2.12), and since α is non-decreasing, we get $S'(x) \geq e^{(x-1)\alpha(1)}$, for any $x \geq 1$. On the other hand, since α is concave, we also have $0 \leq \alpha(x) \leq x\alpha'(0)$, so that:

$$0 \;\leq\; \frac{x\alpha^2(x)}{S'(x)} \;\leq\; \frac{x^3\big[\alpha'(0)\big]^2}{e^{(x-1)\alpha(1)}} \;\to\; 0 \qquad \text{when} \quad x \to \infty\,.$$

As a consequence, we get

$$\int_x^\infty \frac{du}{u^2 S(u)} = \frac{1}{x^2 S'(x)} - 2\int_x^\infty \frac{du}{u^3 S'(u)} + \circ\left(\int_x^\infty \frac{du}{u^3 S'(u)}\right) \,.$$

Integrating by parts again, we finally compute

$$\int_x^\infty \frac{du}{u^2 S(u)} = \frac{1}{x^2 S'(x)} - \frac{2}{\alpha(x) x^3 S'(x)} + \circ\left(\frac{1}{\alpha(x) x^3 S'(x)}\right) \,,$$

and similarly we get

$$\int_x^\infty \frac{du}{u^3 S(u)} = \frac{1}{x^3 S'(x)} - \frac{3}{\alpha(x) x^4 S'(x)} + \circ\left(\frac{1}{\alpha(x) x^4 S'(x)}\right) \,,$$

Plugging these estimates in the expression of $\mathcal{L}g$ given by (5.34) leads to:

$$\begin{aligned}
\mathcal{L}g(x,x) &= \frac{1}{x^2} + [2S'(x) - \alpha(x)S(x)]\int_x^\infty \frac{du}{u^2 S(u)} - [S(x) + 2xS'(x) - x\alpha(x)S(x)]\int_x^\infty \frac{2du}{u^3 S(u)} \\
&= \frac{1}{x^2} + \big(S'(x) + O(\alpha(x))\big)\left(\frac{1}{x^2 S'(x)} - \frac{2}{\alpha(x)x^3 S'(x)} + \circ\left(\frac{1}{\alpha(x)x^3 S'(x)}\right)\right) \\
&\quad - 2\left(xS'(x) + \frac{S'(x)}{\alpha(x)} + O(x\alpha(x))\right)\left(\frac{1}{x^3 S'(x)} - \frac{3}{\alpha(x)x^4 S'(x)} + \circ\left(\frac{1}{\alpha(x)x^4 S'(x)}\right)\right).
\end{aligned}$$

where the third term in the previous expansion might be negligible or not (depending on α). Similarly, we compute:
Using the fact that $\frac{x\alpha^2(x)}{S'(x)} \to 0$ as $x \to \infty$, we get:

$$\mathcal{L}g(x,x) = \frac{2}{\alpha(x)x^3} + \circ\left(\frac{1}{\alpha(x)x^3}\right) \,.$$

Hence $\mathcal{L}g(x,x) > 0$ and therefore $\Gamma(x) = x$ for x large enough, so that $\Gamma_\infty < \infty$.

Case 2: $\int_1^\infty (1/\alpha)'(u)S'(u)du = -\infty$.
For x large enough, we have

$$S(x) \;=\; \frac{S'(x)}{\alpha(x)}\left[1 - \left(\frac{1}{\alpha}\right)'(x) + \circ\left(\left(\frac{1}{\alpha}\right)'(x)\right)\right] \,,$$

so that

$$\int_x^\infty \frac{du}{u^2S(u)} = \int_x^\infty \frac{\alpha(u)}{u^2S'(u)}\left[1+\left(\frac{1}{\alpha}\right)'(u)+\circ\left(\left(\frac{1}{\alpha}\right)'(u)\right)\right]du$$
$$= \frac{1}{x^2S'(x)} - \int_x^\infty \frac{2du}{u^3S'(u)} - \int_x^\infty \frac{\alpha'(u)du}{u^2\alpha(u)S'(u)} + \circ\left(\int_x^\infty \frac{\alpha(u)+u\alpha'(u)}{u^3\alpha(u)S'(u)}\,du\right).$$

Noticing that $0 \le x\alpha'(x) \le \alpha(x)$ for $x > 0$, since α is concave, we have:

$$\circ\left(\int_x^\infty \frac{\alpha(u)+u\alpha'(u)}{u^3\alpha(u)S'(u)}\,du\right) \quad = \quad \circ\left(\int_x^\infty \frac{du}{u^3S'(u)}\right).$$

Integrating by parts, we finally get

$$\int_x^\infty \frac{du}{u^2S(u)} \quad = \quad \frac{1}{x^2S'(x)} - \frac{2}{x^3\alpha(x)S'(x)} - \frac{\alpha'(x)}{x^2\alpha^2(x)S'(x)} + \circ\left(\frac{1}{x^3\alpha(x)S'(x)}\right),$$

where the third term in the previous expansion might be negligible or not (depending on α). Similarly, we compute:

$$\int_x^\infty \frac{du}{u^3S(u)} = \frac{1}{x^3S'(x)} - \frac{3}{x^4\alpha(x)S'(x)} - \frac{\alpha'(x)}{x^3\alpha^2(x)S'(x)} + \circ\left(\frac{1}{x^4\alpha(x)S'(x)}\right),$$

so that:

$$\begin{aligned}\mathcal{L}g(x,x) =& \frac{1}{x^2} + [2S'(x)-\alpha(x)S(x)]\int_x^\infty \frac{du}{u^2S(u)} - [S(x)+2xS'(x)-x\alpha(x)S(x)]\int_x^\infty \frac{2du}{u^3S(u)}\\
=& \frac{1}{x^2} - 2\left(xS'(x) - x\frac{\alpha'(x)}{\alpha^2(x)}S'(x) + \frac{S'(x)}{\alpha(x)}\right)\left(\frac{1}{x^3S'(x)} - \frac{3}{x^4\alpha(x)S'(x)} - \frac{\alpha'(x)}{x^3\alpha^2(x)S'(x)}\right)\\
&+ \left(S'(x) - \frac{\alpha'(x)}{\alpha^2(x)}S'(x)\right)\left(\frac{1}{x^2S'(x)} - \frac{2}{x^3\alpha(x)S'(x)} - \frac{\alpha'(x)}{x^2\alpha^2(x)S'(x)}\right) + \circ\left(\frac{1}{x^3\alpha(x)}\right)\\
=& \frac{2}{x^3\alpha(x)} + \frac{2\alpha'(x)}{x^2\alpha^2(x)} + \circ\left(\frac{1}{x^3\alpha(x)}\right),\end{aligned}$$

where the second term might be or not negligible. In any case, we see that for sufficiently large x, $\mathcal{L}g(x,x) > 0$, so that $\Gamma(x) = x$. Therefore, $\Gamma_\infty < \infty$ also holds in this case. □

5.5 Proof of Proposition 5.3

This section is dedicated to the proof of Proposition 5.3. As already explained, this proof uses the same ideas as the one developed in [5]. However, because of the specificity of (5.47), the properties of the flow are different and the analysis needs to be adapted to our framework. We will try to follow their notations and point out in the proofs the parts that are identical to their paper, but we choose to rewrite them for the sake of completeness.

First, for the convenience of the reader, we recall ODE (5.47) that γ needs to satisfy:

$$\gamma'(x) = \frac{\gamma(x)^2\mathcal{L}g(x,\gamma(x))}{\left(\frac{2x}{\gamma(x)}-1\right)\left(1-\frac{S(x)}{S\circ\gamma(x)}\right)}, \qquad x > 0. \tag{5.59}$$

Let us first define

$$\mathbf{D}^- \ := \ \{x > 0 : \ \mathcal{L}g(x,x) < 0\}, \tag{5.60}$$

and, for all $x_0 \in \mathbf{D}^-$, we introduce

$$d(x_0) := \sup\{x \le x_0 : \ \mathcal{L}g(x,x) \ge 0\} \quad \text{and} \quad u(x_0) := \inf\{x \ge x_0; \ \mathcal{L}g(x,x) \ge 0\}, \tag{5.61}$$

with the convention that $d(x_0) = 0$ if $\{x \le x_0 : \ \mathcal{L}g(x,x) \ge 0\} = \emptyset$. Observe that Proposition 5.2 ensures that $u(x_0) \le \Gamma^\infty < \infty$. Since $\mathcal{L}g$ is continuous and $x_0 \in \mathbf{D}^-$ we must have $d(x_0) < x_0 < u(x_0) < \infty$.

For any $x_0 \in \mathbf{D}^-$ and $z_0 > x_0$, we denote by $\gamma_{x_0}^{z_0}$ the maximal solution of the Cauchy problem (5.59) complemented by the additional condition $\gamma(x_0) = z_0$, and we denote by $I_{x_0}^{z_0} := \left(\ell_{x_0}^{z_0}, r_{x_0}^{z_0}\right)$ the corresponding (open) interval of definition of $\gamma_{x_0}^{z_0}$. Since the right-hand side of ODE (5.59) is locally Lipschitz on either one of the sets $\{(x,\gamma), \ 0 < 2x < \gamma\}$ or $\{(x,\gamma), \ x < \gamma < 2x\}$ but is not defined on the set $\{(x,\gamma), \ 2x = \gamma\}$, the maximal solution will be defined as long as $(x,\gamma(x))$ remains in one of those two sets. Since $\Gamma(x_0) < 2x_0$, we restrict our attention to conditions $\gamma(x_0) = z_0$ satisfying $x_0 < z_0 < 2x_0$.

The next lemma provides useful additional properties of the maximal solutions described above and their respective domains of definitions.

Lemma 5.2 *Assume that α satisfies Conditions* (2.12) *and let $x_0 \in \mathbf{D}^-$ be fixed.*
(i) *For all $z_0 \in (x_0, 2x_0)$, $\ell_{x_0}^{z_0} \le d(x_0)$, we have $\lim_{x \to \ell_{x_0}^{z_0}} \gamma_{x_0}^{z_0}(x) = \ell_{x_0}^{z_0}$ and, if $\ell_{x_0}^{z_0} > 0$, we get $\mathcal{L}g(\ell_{x_0}^{z_0}, \ell_{x_0}^{z_0}) \ge 0$;*
(ii) *for all $z_0 \in (x_0, \Gamma(x_0)]$, $\mathcal{L}g(x, \gamma_{x_0}^{z_0}(x)) < 0$ for any $x \in (x_0, r_{x_0}^{z})$;*
(iii) *there exists $a_0 \in (x_0, 2x_0)$ such that for any $z_0 \in [a_0, 2x_0)$, $\mathcal{L}g(x, \gamma_{x_0}^{z_0}(x)) > 0$ for any $x \in (x_0, r_{x_0}^{z_0})$.*

Proof. We fix $x_0 \in \mathbf{D}^-$ and prove each property separately.
(i) Let us fix $z_0 \in (x_0, 2x_0)$. The right-hand side of (5.59) is locally Lipschitz as long as $0 < x < \gamma_{x_0}^{z_0}(x) < 2x$, so that this last estimate holds for any $x \in I_{x_0}^{z_0}$. We intend to prove that $\gamma_{x_0}^{z}$ hits the diagonal $\{(x,z); \ x = z\}$ at the left hand side $\ell_{x_0}^{z_0}$ of $I_{x_0}^{z_0}$.

For this purpose, let us first prove that, for any $\zeta \in (0, x_0)$, the graph of $\gamma_{x_0}^{z_0}$ restricted to $[\zeta, x_0]$ cannot come too close to $\{(x,z); \ 2x = z\}$. Since $\Gamma(x) < 2x$ for $x > 0$ and Γ and $\mathcal{L}g$ are continuous, there exist $\varepsilon > 0$ and $\delta > 0 \in (0,\zeta)$ such that $\mathcal{L}g \ge \varepsilon$ on the compact set $\{(x,z); \ x \in [\zeta, x_0] \text{ and } z \in [2x-\delta, 2x]\}$. Observe that, for $x \in [\zeta, x_0]$ such that $\gamma_{x_0}^{z_0}(x) \in \left[\max\left(2x-\delta, \frac{4x}{2+\zeta^2\varepsilon}\right), 2x\right)$, we get from (5.59) that

$$(\gamma_{x_0}^{z_0})'(x) \ \ge \ \frac{\gamma_{x_0}^{z_0}(x)^2 \mathcal{L}g(x, \gamma_{x_0}^{z_0}(x))}{\frac{2x}{\gamma_{x_0}^{z_0}(x)} - 1} \ \ge \ \frac{2\gamma_{x_0}^{z_0}(x)^2 \varepsilon}{\zeta^2 \varepsilon} \ \ge \ 2 \, ,$$

where, for the last inequality, we used $\gamma(x) \ge x \ge \zeta$. Hence, the function $x \mapsto 2x - \gamma_{x_0}^{z_0}(x)$ is non-increasing on the set $\left\{x \in [\zeta, x_0] \cap I_{x_0}^{z_0}; \ \gamma_{x_0}^{z_0}(x) \in \left[\max\left(2x-\delta, \frac{4x}{2+\zeta^2\varepsilon}\right), 2x\right)\right\}$. Therefore, by arbitrariness of $\zeta > 0$, the graph of $\gamma_{x_0}^{z_0}$ restricted to $(\ell_{x_0}^{z_0}, x_0]$ stays away from

$\{(x,z);\ 2x=z\}$ and $\gamma_{x_0}^{z_0}$ necessarily hits the diagonal $\{(x,z);\ x=z\}$ at the left hand side $\ell_{x_0}^{z_0}$ of the maximal interval $I_{x_0}^{z_0}$.
On the other hand, we observe from (5.59) that $\gamma_{x_0}^{z_0}$ is non-increasing at the points x satisfying $(x,\gamma_{x_0}^{z_0}(x))\in\Gamma^-$ and therefore $\ell_{x_0}^{z_0}\notin\mathbf{D}^-$ by minimality of $I_{x_0}^{z_0}$. Since $\big(d(x_0),u(x_0)\big)\subset\mathbf{D}^-$, we get $\ell_{x_0}^{z_0}\le d(x_0)$ and $\mathcal{L}g(\ell_{x_0}^{z_0},\ell_{x_0}^{z_0})\ge 0$, or equivalently $\Gamma(\ell_{x_0}^{z_0})=\ell_{x_0}^{z_0}$.
It still remains to prove properly that $\lim_{x\to\ell_{x_0}^{z_0}}\gamma_{x_0}^{z_0}(x)=\ell_{x_0}^{z_0}$. Assume first that $\ell_{x_0}^{z_0}>0$. Notice from (5.59) that $\gamma_{x_0}^{z_0}$ is non-decreasing if $(x,\gamma_{x_0}^{z_0}(x))\in\Gamma^+$, and $x\le\gamma_{x_0}^{z_0}(x)\le\Gamma(x)$ otherwise. Since Γ is also non-decreasing, $\tilde\gamma_{x_0}^{z_0}:=\max(\gamma_{x_0}^{z_0},\Gamma)$ is a non-decreasing function defined on $(\ell_{x_0}^{z_0},x_0]$, which therefore admits a limit at $\ell_{x_0}^{z_0}$. Since $\ell_{x_0}^{z_0}>0$, we have $\lim_{x\to\ell_{x_0}^{z_0}}\tilde\gamma_{x_0}^{z_0}(x)<2\ell_{x_0}^{z_0}$ as observed above, so that, combining the maximality of $I_{x_0}^{z_0}$ with $\Gamma(\ell_{x_0}^{z_0})=\ell_{x_0}^{z_0}$, we obtain $\lim_{x\to\ell_{x_0}^{z_0}}\tilde\gamma_{x_0}^{z_0}(x)=\ell_{x_0}^{z_0}$. Since $x\le\gamma_{x_0}^{z_0}(x)\le\tilde\gamma_{x_0}^{z_0}(x)$ for $x\in(\ell_{x_0}^{z_0},x_0]$, we have $\lim_{x\to\ell_{x_0}^{z_0}}\gamma_{x_0}^{z_0}(x)=\ell_{x_0}^{z_0}$. Finally, if $\ell_{x_0}^{z_0}=0$, since $x<\gamma_{x_0}^{z_0}(x)<2x$, we also have the result.

(ii) Let us fix $z_0\in\big(x_0,\Gamma(x_0)\big)$. As already observed, the dynamics of (5.59) imply that $\gamma_{x_0}^{z_0}$ is non-increasing in the neighborhood of any point x such that $(x,\gamma_{x_0}^{z_0}(x))\in\mathrm{Int}(\Gamma^-)$. On the other hand, Proposition 5.2 tells us that the function Γ is increasing on $[x_0,+\infty)$. Hence $x\mapsto(x,\gamma_{x_0}^{z_0}(x))$ remains in $\mathrm{Int}(\Gamma^-)$ on $\big[x_0,r_{x_0}^{z_0}\big)$.
We consider now the case where $z_0=\Gamma(x_0)$. Since $\Gamma(x_0)>x_0$, the proof of Proposition 5.2 (i) tells us that Γ' is positive on a neighborhood of x_0. Since $z_0=\Gamma(x_0)$, we deduce from (5.59) that $(\gamma_{x_0}^{z_0})'(x_0)=0$, and the exact same reasoning as above implies that $x\mapsto\big(x,\gamma_{x_0}^{z_0}(x)\big)\in\mathrm{Int}(\Gamma^-)$ on $\big(x_0,r_{x_0}^{z_0}\big)$.

(iii) Recall that $\Gamma^\infty<\infty$. Therefore, as in (i), there exist $\varepsilon>0$ and $\delta\in(0,1)$ such that $\mathcal{L}g\ge\varepsilon$ on $\{(x,z);\ x\in[x_0,\Gamma^\infty]$ and $z\in[(2-\delta)x,2x]\}$. Let $b:=\min(x_0^2\varepsilon,\delta)$. From (5.59), we see that if $x\in[x_0,\Gamma^\infty]$ and $\gamma_{x_0}^{z}(x)\in[(2-b)x,2x)$, then $(\gamma_{x_0}^{z})'(x)\ge\frac{2-b}{b}x_0^2\varepsilon\ge 2-b$. We denote $a_0:=(2-b)x_0$ and fix $z\in[a_0,2x_0)$. We deduce from the previous reasoning that we must have

$$\gamma_{x_0}^{z}(x)\ge z+\int_{x_0}^{x}(\gamma_{x_0}^{z})'(u)du\ \ge\ z+(2-b)(x-x_0)\ >\ (2-b)x\ >\ \Gamma(x)\ ,$$

for $x\in\big[x_0,\min(r_{x_0}^{z},\Gamma^\infty)\big)$. If ever $r_{x_0}^{z}\le\Gamma^\infty$, we just obtained the announced result and, if ever $r_{x_0}^{z}>\Gamma^\infty$, we complete the proof noticing that the maximality of $I_{x_0}^{z}$ implies that $\gamma_{x_0}^{z}(x)>x=\Gamma(x)$ for $x\ge\Gamma^\infty$. □

We now construct the stopping boundary γ by selecting one of the previous maximal solutions. For a given $x_0\in\mathbf{D}^-$, let

$$\mathbf{Z}(x_0):=\Big\{z\in(x_0,2x_0);\ \mathcal{L}g\big(x,\gamma_{x_0}^{z}(x)\big)<0 \text{ for some } x\in\big[x_0,r_{x_0}^{z}\big)\Big\}\,, \tag{5.62}$$

$$z^*(x_0):=\sup\mathbf{Z}(x_0). \tag{5.63}$$

Moreover, whenever $z^*(x_0) < 2x_0$, we denote

$$\gamma^*_{x_0} := \gamma^{z^*(x_0)}_{x_0},\ \ell^*_{x_0} := \ell^{z^*(x_0)}_{x_0},\ r^*_{x_0} := r^{z^*(x_0)}_{x_0}, \quad \text{and} \quad I^*_{x_0} := \big(\ell^*_{x_0}, r^*_{x_0}\big). \tag{5.64}$$

The next lemma provides useful properties on the function γ^* and its domain of definition. In particular, it discusses its dependance with respect to the starting point x_0.

Lemma 5.3 *Assume that* α *satisfies Conditions* (2.12) *and let* x_0 *be arbitrary in* $\mathbf{D}^-$. *Then, the following holds.*
(i) $z^*(x_0) \in (\Gamma(x_0), 2x_0)$ *and* $\gamma^*_{x_0}$ *has a positive derivative on the interval* $I^*_{x_0}$.
(ii) $\big(d(x_0), u(x_0)\big) \subset I^*_{x_0}$ *and* $\lim_{x\to r^*_{x_0}} \gamma^*_{x_0}(x) = r^*_{x_0} \le \Gamma^\infty$ *with equality if* $u(x_0) = \Gamma^\infty$.
(iii) *For* $x_0, x_1 \in \mathbf{D}^-$, *we have either* $I^*_{x_0} \cap I^*_{x_1} = \emptyset$, *or* $I^*_{x_0} = I^*_{x_1}$ *and* $\gamma^*_{x_0} = \gamma^*_{x_1}$.

Proof. We fix $x_0 \in \mathbf{D}^-$ and prove each assertion separately. The proofs of points (i) and (iii) are very close to the proof of Lemma 5.2 in [5], but we rewrite and adapt them here.

(i) Lemma 5.2 (iii) ensures the existence of $a_0 < 2x_0$ such that $\mathcal{L}g\big(x, \gamma^z_{x_0}(x)\big) > 0$ for any $x \ge x_0$ and $z \ge a_0$. By definition of $z^*(x_0)$, we deduce that $z^*(x_0) \le a_0 < 2x_0$. Since $x_0 \in \mathbf{D}^-$, we obtain from Lemma 5.2 (ii) that $\Gamma(x_0) \in \mathbf{Z}(x_0)$ and deduce that $\Gamma(x_0) \le z^*(x_0)$.
In order to prove that $z^*(x_0) \in (\Gamma(x_0), 2x_0)$, we now assume that $z^*(x_0) = \Gamma(x_0)$ and work towards a contradiction. Since $\mathcal{L}g$ is continuous, $\mathbf{D}^-$ is an open set and there exists $\varepsilon > 0$ such that $(x_0, x_0 + 2\varepsilon) \subset \mathbf{D}^- \cap (x_0, r^*_{x_0})$ and $d(x) = d(x_0)$ for any $x \in (x_0, x_0 + \varepsilon)$. Let us denote $x_\varepsilon := x_0 + \varepsilon \in \mathbf{D}^-$ and $z_\varepsilon := \Gamma(x_\varepsilon) > \Gamma(x_0)$. By Lemma 5.2 (i), we have $\ell^{z^\varepsilon}_{x_\varepsilon} \le d(x_0) < x_0$, and it follows from Lemma 5.2 (ii) and the dynamics of (5.59) that $\gamma^{z_\varepsilon}_{x_\varepsilon}$ is decreasing on $\big(x_0, r^{z^\varepsilon}_{x_\varepsilon}\big)$. Therefore, we compute

$$\gamma^{z_\varepsilon}_{x_\varepsilon}(x_0) \ > \ \gamma^{z_\varepsilon}_{x_\varepsilon}(x_\varepsilon) \ = \ \Gamma(x_\varepsilon) \ > \ \Gamma(x_0) \ = \ z^*. \tag{5.65}$$

On the other hand, since $\gamma^{\gamma^{z_\varepsilon}_{x_\varepsilon}(x_0)}_{x_0}(x_\varepsilon) = z^\varepsilon = \Gamma(x_\varepsilon)$, Lemma 5.2 (ii) ensures that $\gamma^{z^\varepsilon}_{x_\varepsilon}(x_0) \in \mathbf{Z}(x_0)$, leading to $z^* \ge \gamma^{z^\varepsilon}_{x_\varepsilon}(x_0) \in \mathbf{Z}(x_0)$, which contradicts (5.65).

The same line of argument implies also that $(x, \gamma^*(x)) \in \mathrm{Int}(\Gamma^+)$ for any $x \in I^*_{x_0}$. We deduce from the dynamics of (5.59) that γ^* has a positive derivative on $I^*_{x_0}$, and in particular $\lim_{x\to r^*_{x_0}} \gamma^*(x)$ exists.

(ii) For any $z \in \mathbf{Z}(x_0)$, since $\gamma^z_{x_0}$ is non-increasing in Γ^-, we deduce that $\lim_{x\to r^z_{x_0}} \gamma^z_{x_0}(x) = r^z_{x_0} \le \Gamma^\infty$. Let us write $r_0 := \sup\{r^z_{x_0};\ z \in \mathbf{Z}(x_0)\} \le \Gamma^\infty$. Let us first prove that $r_0 \ge u(x_0)$. Assume on the contrary that $r_0 < u(x_0)$, so that $\Gamma(r_0) > r_0$. Let fix $z \in (r_0, \Gamma(r_0))$. By Lemma 5.2 (i), $\ell^z_{r_0} \le d(x_0)$, so that $x_0 \in I^z_{r_0}$ and, since Lemma 5.2 (ii) implies that $\mathcal{L}g(x, \gamma^z_{r_0}(x)) < 0$ for $x > r_0$, we deduce that $\gamma^z_{r_0}(x_0) \in \mathbf{Z}(x_0)$. This contradicts the definition of r_0 since $z \in (r_0, \Gamma(r_0))$ implies that $r^{\gamma^z_{r_0}(x_0)}_{x_0} = r^z_{r_0} > r_0$. In conclusion, $r_0 \ge u(x_0)$.

Besides, Lemma 5.2 (i) implies that $\ell^*_{x_0} \le d(x_0)$, and we intend to prove that $r_0 = r^*_{x_0}$ in order to derive $(d(x_0), u(x_0)) \subset I^*_{x_0}$.

First, we derive the existence of a sequence $(z_n) \in \mathbf{Z}(x_0)$ such that $z_n \to z^*(x_0)$ and $r^{z_n}_{x_0} \to r_0$. Combining the one-to-one property of the flow with the property that $\lim_{x\to r^z_{x_0}} \gamma^z_{x_0}(x) = r^z_{x_0}$ for $z \in \mathbf{Z}(x_0)$, we deduce that $z \mapsto r^z_{x_0}$ is non-decreasing on $\mathbf{Z}(x_0)$. Hence, if $z^*(x_0) \notin \mathbf{Z}(x_0)$, any sequence (z_n) valued in $\mathbf{Z}(x_0)$ such that $z_n \to z^*(x_0)$ satisfies also $\sup\{r^{z_n}_{x_0};\ n \in \mathbb{N}\} = \sup\{r^z_{x_0};\ z \in \mathbf{Z}(x_0)\} = r_0$ and thus the required property. If ever $z^*(x_0) \in \mathbf{Z}(x_0)$, we simply pick the sequence $z_n := z^*(x_0)$, for any $n \in \mathbb{N}$.

We now prove that $r_0 = r^*_{x_0}$. Let $z \in (r_0, 2r_0)$ be arbitrary. Up to a subsequence, we have by construction $I^{z_n}_{x_0} \cap I^z_{r_0} \neq \emptyset$ for any $n \in \mathbb{N}$. We know that $\lim_{x\to r^{z_n}_{x_0}} \gamma^{z_n}_{x_0}(x) = r^{z_n}_{x_0}$ and $\gamma^z_{r_0}(r^{z_n}_{x_0}) > r^{z_n}_{x_0}$ since $r^z_{r_0} > r_0 \ge r^{z_n}_{x_0}$, for any $n \in \mathbb{N}$. Hence, the one-to-one property of the flow ensures that $\gamma^z_{r_0}(x) > \gamma^{z_n}_{x_0}(x)$ for any $x \in I^{z_n}_{x_0} \cap I^z_{r_0}$ and $n \in \mathbb{N}$. By Lemma 5.2 (i), $\lim_{x\to \ell^z_{r_0}} \gamma^z_{r_0}(x) = \ell^z_{r_0}$, so that $x_0 \in I^{z_n}_{x_0} \cap I^z_{r_0}$. Since $(z_n) \in \mathbf{Z}(x_0)$ converges to $z^*(x_0) = \sup \mathbf{Z}(x_0)$, we deduce that $\gamma^z_{r_0}(x_0) \ge \gamma^*_{x_0}(x_0) = z^*(x_0) \ge z^n = \gamma^{z_n}_{x_0}(x_0)$, for any $n \in \mathbb{N}$. Hence, the one-to-one property of the flow implies that

$$2x > \gamma^z_{r_0}(x) \ge \gamma^*_{x_0}(x) \ge \gamma^{z_n}_{x_0}(x)\ , \qquad x \in [x_0, r^{z_n}_{x_0} \wedge r^*_{x_0})\ , \quad n \in \mathbb{N}\ . \tag{5.66}$$

Therefore $r^*_{x_0} \ge r^{z_n}_{x_0}$ for $n \in \mathbb{N}$, and, passing to the limit, we get $r^*_{x_0} \ge r_0$. Besides, (5.66) implies that $\limsup_{x\to r_0} \gamma^*_{x_0}(x) \le \gamma^z_{r_0}(r_0) = z$, and the arbitrariness of $z \in (r_0, 2r_0)$ leads to $\limsup_{x\to r_0} \gamma^*_{x_0}(x) \le r_0$. Since $\gamma^*_{x_0}(x) \ge x$ for $x \in I^*_{x_0}$, we get $\lim_{x\to r_0} \gamma^*_{x_0}(x) = r_0$ and $r^*_{x_0} \le r_0$. Hence, $r^*_{x_0} = r_0 \le \Gamma^\infty$, and, if $u(x_0) = \Gamma^\infty$, $r_0 \ge u(x_0)$ implies that $r^*_{x_0} = \Gamma^\infty$.

(iii) Let x_1 in $\mathbf{D}^-$. Suppose that $x_0 < x_1$ and that there exists $x_2 \in I^*_{x_0} \cap I^*_{x_1}$. If ever $\gamma^*_{x_0}(x_2) = \gamma^*_{x_1}(x_2)$, the one-to-one property of the flow combined with the maximality of I^* imply that $I^*_{x_0} = I^*_{x_1}$ and $\gamma^*_{x_0} = \gamma^*_{x_1}$ and conclude the proof. It therefore only remains to prove that $\gamma^*_{x_0}(x_2) = \gamma^*_{x_1}(x_2)$.
We assume on the contrary that $\gamma^*_{x_0}(x_2) < \gamma^*_{x_1}(x_2)$, the case where $\gamma^*_{x_0}(x_2) > \gamma^*_{x_1}(x_2)$ being treated similarly. The one-to-one property of the flow implies that $\gamma^*_{x_0} < \gamma^*_{x_1}$ on all the interval $I^*_{x_0} \cap I^*_{x_1}$. Furthermore, Lemma 5.3 (i) and Lemma 5.2 (i) ensure that $\lim_{r^*_{x_1}} \gamma^*_{x_1} = r^*_{x_1}$ and $\lim_{\ell^*_{x_1}} \gamma^*_{x_1} = \ell^*_{x_1}$. Hence, we deduce from the maximality of $I^*_{x_1}$ that $I^*_{x_0} \subset I^*_{x_1}$. Combining the definition of $z^*(x_1)$ with the continuity of the flow with respect to initial data, we obtain the existence of $z \in \mathbf{Z}(x_1)$ such that $z < z^*(x_1)$ and $\gamma^*_{x_0}(x_2) < \gamma^z_{x_1}(x_2) < \gamma^*_{x_1}(x_2)$. Once again, the one-to-one property of the flow implies that $I^*_{x_0} \subset I^z_{x_1}$ and $\gamma^*_{x_0} < \gamma^z_{x_1} < \gamma^*_{x_1}$ on $I^*_{x_0}$. Since $z \in \mathbf{Z}(x_1)$, we deduce that $\gamma^z_{x_1}(x_0) \in \mathbf{Z}(x_0)$ while $\gamma^z_{x_1}(x_0) > z^*(x_0) = \gamma^*_{x_0}(x_0)$, which contradicts the definition of $z^*(x_0)$. □

Finally, we are in position to provide the proof of Proposition 5.3:

Proof of Proposition 5.3 This construction follows similar ideas as in the proof of Proposition 5.1 in [5], but turns out to be simpler since $\Gamma^\infty < \infty$.
Let

$$\mathcal{D} \ := \bigcup_{x_0 \in \mathbf{D}^-} I^*(x_0) \quad \supset \quad \mathbf{D}^-. \tag{5.67}$$

Lemma 5.3 (iii) ensures that , for any x_0 and x_1 in $\mathbf{D}^-$, we either have $I^*_{x_0} = I^*_{x_1}$ or $I^*_{x_0} \cap I^*_{x_1} = \emptyset$. Hence, there exists a subset $\mathbf{D}_0^-$ of $\mathbf{D}^-$ such that $\mathcal{D} = \bigcup_{x_0 \in \mathbf{D}_0^-} I^*(x_0)$ and, for any $x_0, x_1 \in \mathbf{D}_0^-$, $x_0 \neq x_1$ implies that $I^*_{x_0} \cap I^*_{x_1} = \emptyset$.
We now define the function γ on $\mathbb{R}_+ \setminus \{0\}$ by:

$$\gamma(x) := \begin{cases} \gamma^*_{x_0}(x) \text{ if } x \in I^*_{x_0}, \text{ for } \ x_0 \in \mathbf{D}_0^- \\ x \text{ otherwise.} \end{cases} \tag{5.68}$$

According to Lemma 5.3, this definition does not depend on the choice of $\mathbf{D}_0^-$.

Lemmata 5.2 and 5.3 imply that γ is continuous at the endpoints $\ell^*_{x_0}$ and $r^*_{x_0}$, for any $x_0 \in \mathbf{D}_0^-$. Hence, setting $\gamma(0) := 0$, we obtain a continuous function γ on $\mathbb{R}_+$. For any $x_0 \in \mathbf{D}_0^-$, $\gamma^*_{x_0}$ is increasing on $I^*_{x_0}$ and the identity function is increasing as well, so that γ is increasing on $\mathbb{R}_+$. We now justify each assertion of the proposition separately. (i) is immediate from the definition of γ.
To prove (ii), we first notice that $\{x \geq 0 : \ \gamma(x) = x\} = \mathbb{R}_+ \setminus \mathcal{D} \subset \mathbb{R}_+ \setminus \mathbf{D}^-$, so that $\mathcal{L}g(x,x) \geq 0$ on the set $\{x > 0 : \ \gamma(x) = x\}$. On the set $\{x > 0 : \ \gamma(x) > x\}$, since γ has a positive derivative by Lemma 5.3 (ii) and satisfies (5.59), we have $\mathcal{L}g(x, \gamma(x)) > 0$. Finally, (iii) can be deduced from Lemma 5.3 (ii), since $r^*_{x_0} \leq \Gamma^\infty$ for any $x_0 \in \mathbf{D}_0^-$. □

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