# Cournot Competition on a Network of Markets and Firms*

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**Abstract**

Suppose markets and firms are connected in a bi-partite network, where firms can only supply to the markets they are connected to. Firms compete *a la Cournot* and decide how much to supply to each market they have a link with. We assume that markets have linear demand functions and firms have convex quadratic cost functions. We show there exists a unique equilibrium in any given network of firms and markets. We provide a formula which expresses the quantities at an equilibrium as a function of a network centrality measure. We continue to study the effects of a merger between two firms and analyze the behavior of a cartel including all the firms in the network.

**Keywords:** Cournot markets, networks, Nash equilibrium, centrality measures.

**JEL Classification:** C62, C72, D85, L11

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# 1 Introduction

Many of the utilities like water, electricity or natural gas require an infrastructure in the form of a distribution network. This is true both at the wholesale and retail level. Hence the markets for such utilities function differently. The most notable example is that of the market for crude oil and for natural gas on the other hand.

The price of crude oil is determined by many factors from different regions of the world, but as it is relatively easy to transport, we observe a price (e.g. price of Brent or West Texas intermediate) which serves as a reference for all trades of crude oil. Any difference between regional prices would be abated through trade. Market power of an oil exporting country is determined by the capacity and efficiency of its production. The Organization of Petroleum Exporting Countries use their combined market share to influence the price of oil.

The market for natural gas presents a much more complex example. It is carried mainly through pipelines[1]. Other forms of transportation are not economical when compared with pipelines. Which countries can trade natural gas is determined by the structure of the network formed by the natural gas pipelines. This leads to the formation of regional prices. The price for a thousand cubic meters of natural gas ranges almost from zero to 300 (EU Commission Staff Working Document (2006)), depending on the location. An importing country with a single supplier faces a monopoly and pays a higher price while a country which has alternative suppliers will pay a lower price thanks to the competition between. The market power of producers are determined both by their production and their position in the market. The attempts of natural gas exporting countries to mimic OPEC can potentially create a cartel which can decide both the quantity and the destination of supply. Moreover, the transit countries which transport the gas from producers to consumers become strategic actors, independent of whether they produce natural gas or not.

To understand how such markets function we need to go into the details of the network that connects suppliers with consumers. A structural analysis is required to understand the patterns of interaction and to quantify the influence that producers have on each other.

We model a bipartite network, where links connect firms with markets. We look at the Cournot game, where firms decide how much to sell at each market they are connected to.

[1]More than 90 percent of the natural gas imports of the European Union are through pipelines (EU Commission Staff Working Document (2006)). The ratio for global gas imports is around 80 percent (Victor et. al. 2006). The three countries which depend most on maritime transportation of natural gas are Japan, Taiwan and South Korea. It is due to the infeasibility of building long distance pipelines in the ocean.

We assume that firms have convex quadratic costs and markets have linear inverse demand functions. This simplification allows us to focus on the effect of the network structure on market behavior.

We show that there exists a unique the Cournot equilibrium. We write the equilibrium conditions as a linear complementarity problem and provide an interpretation of the equilibrium flows using the Katz-Bonacich centrality (Katz 1953, Bonacich 1987), which reveals the strategic complementarities between links. We then study the effects of a merger between two firms and analyze how a cartel including all firms would segment the markets to maximize their joint profit.

We bridge two branches of the literature. On one side we study Cournot competition. We extend the basic to a network of firms and markets. Given a network, we show how the structure of connections determines firms' supply levels. Bulow *et al.* (1985), which analyze the strategic interactions between the supplies of two firms computing *a la* Cournot in two markets, is the earliest example of a Cournot analysis with multiple markets linked through firms. We extend their model allowing for any number of firms connected through a bipartite network. This generalization in market size and structure requires the use of network centrality measures and graph theoretical techniques to solve for the equilibrium.

The closest model to ours is Nava (2009) which which studies quantity competition in a network of Walrasian agents where agents can simultaneously buy and sell. He provides conditions for the existence of an equilibrium both when sellers make the offers and when buyers make the offers. Nava (2009) holds for very general utility functions, where as in our model the functional restrictions allow us to provide a closed form formula for the equilibrium quantities. Hence we will be able to deepen the market analysis to accommodate for and study mergers and cartel formation.

Our study of mergers is parallel to Farrell and Shapiro (1990). We use similar differential techniques to predict the effect of a merger. We reveal that due to the underlying network the effect of the merger on consumers and rival firms are not uniform. Some consumers and rivals can be hurt by the merger, while others benefit. Next, we assume all the firms in the network form a cartel to maximize their joint profit. We find that their optimal strategy is to segment the markets among themselves and agree to operate only in the markets allocated to them.

Another parallel line of literature is the analysis of behavior on networks. Ballester et al. (2006) analyzes the equilibrium activities at each node of a simple (i.e. not bipartite) non-directed network. Players create externalities on their neighbors. A player has a single level of activity. Her payoff depends on her activity level and of her neighbors'. They show that the equilibrium levels are given by a network centrality index, which is similar to the Katz-Bonacich centrality.

As in Kranton and Minehart (2001) and Corominas-Bosch (2004), we study a bipartite network. Corominas-Bosch (2004) studies the equilibria of a bargaining game in a network of buyers and sellers. In both Kranton and Minehart (2001) and Corominas-Bosch (2004) both buyers and sellers are active agents, where we model only the firms as strategic. Kranton and Minehart (2001) study a similar setup and provides an ascending price mechanism which is strategy-proof and efficient. The graph is decomposed into several submarkets which simultaneously clear and a different price prevails in each of them. Both in Kranton and Minehart (2001) and Corominas-Bosch (2004) buyers and sellers are exchange a single indivisible good. In contrast we assume that the good transferred through the links is perfectly divisible, allowing a firm to supply to many markets.

The basic notation is introduced in Section 2. In Section 3 we define the Cournot game and solve for the equilibrium using in terms of network centrality measures. In Section 4 we analyze the merger of two firms and in Section 5 a cartel formed by all the firms in the network. Section 6 concludes. The proofs are given in the Appendix.

# 2 Notation

There are $m$ markets $m_1, ..., m_n$, and $n$ firms $f_1, ..., f_n$. They are embedded in a network that links markets with firms, and firms can supply to the markets they are connected to. We will represent the network as a graph.

A non-directed *bipartite graph* $g = \langle M \cup F, L \rangle$ consists of a set of *nodes* formed by markets $M = \{m_1, ..., m_m\}$, and firms $F = \{f_1, ..., f_n\}$ and a set of *links* $L$, each link joining a market with a firm. A link from $m_i$ to $f_j$ will be denoted as $(i, j)$. We say that a market $m_i$ is *linked* to a firm $f_j$ if there is a link joining the two. We will use $(i, j) \in g$, meaning that $m_i$ and $f_j$ are connected in $g$. Let $r(g)$ be the number of links in $g$.

A graph $g$ is *connected* if there exists a path linking any two nodes of the graph. Formally, a path linking nodes $m_i$ and $f_j$ will be a collection of $t$ firms and $t$ markets, $t \geq 0$, $m_1, ...m_t, f_1, ..., f_t$ among $M \cup F$ (possibly some of them repeated) such that

$$\{(i, 1), (1, 1), (1, 2), ..., (t, t), (t, j)\} \in g$$

A *subgraph* $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ of $g$ is a graph such that $M_0 \subseteq M, F_0 \subseteq F, L_0 \subseteq L$ and such that each link in $L$ that connects a market in $M_0$ with a firm in $F_0$ is a member of $L_0$. Hence a node of $g_0$ will continue to have the same links it had with the other nodes in $g_0$. We will write $g_0 \subseteq g$ to mean that $g_0$ is a subgraph of $g$. For a subgraph $g_0$ of $g$, we will denote by $g - g_0$, the subgraph of $g$ that results when we remove the set of nodes $M_0 \cup F_0$ from $g$.

Given a subgraph $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ of $g$, let $\overleftrightarrow{g_0}$ be the complete bipartite graph with nodes $M_0 \cup F_0$. We call $\overleftrightarrow{g_0}$ *the completed graph of* $g_0$.

$N_g(m_i)$ will denote the set of firms linked with $m_i$ in $g = \langle M \cup F, L \rangle$, more formally:

$$N_g(m_i) = \{f_j \in F \text{ such that } (i, j) \in g\}$$

and similarly $N_g(f_j)$ stands for the set of markets linked with $f_j$.

For a set $A$, let $|A|$ denote the number of elements in $A$. For $m_i$ in $M$, we denote $|N_g(m_i)|$ by $m_i(g)$. Similarly for $f_j \in F$, let $|N_g(f_j)| = n_j(g)$, be the number of markets connected to $f_j$.

# 3 The Cournot Game

Given a graph $g$, each firm $f_j$ maximizes profit by supplying a non-negative quantities to the markets in $N_g(c_j)$. So, the set of players are the set of firms $F$.

We denote by $q_{ij} \geq 0$ the quantity supplies by firm $f_j$ to the market $m_i$.

Now we define the column vector that shows the quantities flowing at each link. Given a graph $g$, let $Q_g$ be the column vector of quantities supplied and has size $r(g)$.

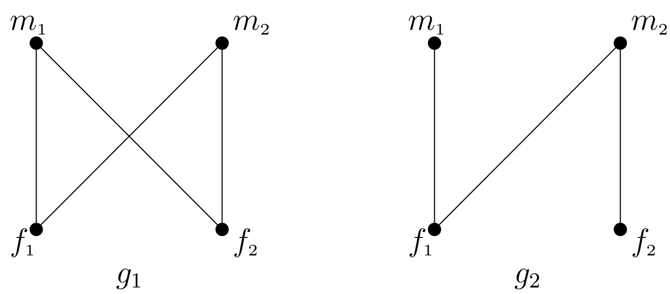



Figure 2

For the two graphs given above

$$Q_{g_1} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{12} \\ q_{22} \end{bmatrix} \qquad\qquad Q_{g_2} = \begin{bmatrix} q_{11} \\ q_{21} \\ q_{22} \end{bmatrix}$$

In the vector $Q_g$, the supply $q_{ij}$ is listed above the supply $q_{kl}$ when $j < l$ or when $j = l$ and $i < k$. We will make use of graphs $g_1$ and $g_2$ in many examples throughout the paper.

Let $\mathbb{Q}^r$ be the set of all non-negative real valued column vectors of size $r$. Given a vector of supplies $Q_g$, for a firm $f_j$, we will denote by $s_j$ the total supply by $f_j$ and for a market $m_i$ we will denote by $c_i$ the total consumption at $m_i$.

The set of strategies of a firm $f_j$ is $\mathbb{Q}_j$. We denote a representative strategy of $f_j$ by $Q_j \in \mathbb{Q}_j$. Given that there are $r(g)$ links in $g$, the strategy space of the game is $\mathbb{Q}_g = \prod_{c_j \in C} \mathbb{Q}_j = \mathbb{Q}^{r(g)}$. We denote a representative strategy profile on a graph $g$ by $Q_g \in \mathbb{Q}_g$.

We assume that markets have linear inverse demand functions. Given a market $m_i$ and a flow vector $Q_g$ the price at $m_i$ is

$$p_i(Q_g) = \alpha_i - \beta_i c_i$$

where $\alpha_i, \beta_i > 0$.

We assume that firms have quadratic costs of production. For firm $f_j$ the total cost of production is

$$T_j(Q_g) = \frac{\gamma_j}{2} s_j^2$$

where $\gamma_j > 0$

Hence, the profit functions of firm $f_j$ is:

$$\pi_j(Q_g) = \sum_{m_i \in N_g(f_j)} \alpha_i q_{ij} - \frac{\gamma_j}{2} s_j^2 - \sum_{m_i \in N_g(f_j)} \beta_i q_{ij} c_i$$

Marginal profit is not separable with respect to each market. The marginal profit from $q_{ij}$ does depend on the supply from $f_j$ to markets other than $m_i$.

The best response $Q'_j$ of firm $f_j$ to $Q_g \in \mathbb{Q}_g$ is such that for all links $(i, j)$

$$q'_{ij} = \begin{cases} \frac{\alpha_i - \gamma_j \sum\limits_{m_l \in N_g(f_j)\setminus\{m_i\}} q_{lj} - \beta_i \sum\limits_{f_k \in N_g(m_i)\setminus\{f_j\}} q_{ik}}{2\beta_i + \gamma_j}, & \text{if } \frac{\partial \pi_j}{\partial q_{ij}}|_{Q_g} \geq 0 \\ 0 & , \text{if } \frac{\partial \pi_j}{\partial q_{ij}}|_{Q_g} < 0 \end{cases}$$

The first order equilibrium conditions of the Cournot game constitutes a linear complementarity problem. Given a matrix $M \in \mathbb{R}^{t \times t}$ and a vector $p \in \mathbb{R}^t$, the linear complementarity problem $LCP(p; M)$ consists of finding a vector $z \in \mathbb{R}^t$ satisfying:

$$z \geq 0, \tag{1}$$
$$p + Mz \geq 0, \tag{2}$$
$$z^T(p + Mz) \geq 0 \tag{3}$$

Samelson *et al*. (1958) shows that a linear complementarity problem $LCP(p; M)$ has a unique solution for all $p \in \mathbb{R}^t$ if and only if all the principal minors of $M$ are positive. We prove this to be true for the linear complementarity problem formed by the first order equilibrium conditions of the Cournot game.

We further check for the second order conditions for each agent, which reveals that the solution of the linear complementarity problem is indeed the equilibrium of the game.

**Theorem 1** *The Cournot game has a unique Nash equilibrium.*

**Example 1** Suppose we have the graph $g_1$. Let $\alpha = \beta = \gamma = 1$. Then the link supplies at equilibrium are $q^*_{11} = q^*_{21} = q^*_{12} = q^*_{22} = 0.2$. The prices and the profits are $p_1 = p_2 = 0.6$ and $\pi_1 = \pi_2 = 0.16$, respectively.

Suppose the graph was $g_2$. Now at equilibrium, $q^*_{11} = 0.2857$, $q^*_{21} = 0.1429$, and $q^*_{22} = 0.2857$. The deletion of the link $(1, 2)$ changes the supply to market $m_2$, and moreover firm $f_1$ supplies less to the market she shares with firm $f_2$. The prices and the profits are $p_1 = 0.7125$, $p_2 = 0.5696$ and $\pi_1 = 0.1936$, $\pi_2 = 0.1224$, respectively.

Let $Q^*_g$ be an equilibrium of the Cournot game. There might be some links in $g$ which carry zero flow at equilibrium $Q^*_g$. Marginal profits of supply via those links need not be zero at $Q_g^*$.

$$\begin{aligned} q_{ij}^* &> 0 \Rightarrow \frac{\partial \pi_j}{\partial q_{ij}} = 0 \\ q_{ij}^* &= 0 \Rightarrow \frac{\partial \pi_j}{\partial q_{ij}} \leq 0 \end{aligned}$$

To calculate the equilibrium quantities, first we need to weed out the links with zero flow. Let $\rho : L \longrightarrow \mathbb{N}_+$ be a lexicographic order on $L$ respecting $\tau$ such that $\rho$ relabels the $(i, j)$ pairs from 1 to $r(g)$ by skipping those links which are not in $g$.[2] Now we delete from $Q_g^*$, the entries that correspond to links with no flow.

Let $Z\left(Q_g^*\right) = \{z \in \mathbb{N}_+ : z = \rho(i, j) \text{ for some } (i, j) \text{ s.t. } q_{ij}^* = 0\}$. Let $\left|Z\left(Q_g^*\right)\right| = t^*$, then $Q^*_{g-Z\left(Q_g^*\right)}$ is a vector of size $r(g) - t^*$ obtained from $Q_g^*$ by deleting the zero entries. It is the vector of equilibrium quantities for links over which there is a strictly positive flow from a firm to a market.

Let $Q_g^*$ be the equilibrium of the Cournot game at network $g$. We denote by $g - Z(Q_g^*)$ the network obtained from $g$ by deleting the links which have zero flow at $Q_g^*$.

**Theorem 2** *Given two networks $g$ and $g'$. Let $Q_g^*$ and $Q_{g'}^*$ be the equilibrium of the Cournot game in $g$ and $g'$, respectively. If $g - Z(Q_g^*) = g' - Z(Q_{g'}^*)$, then $Q^*_{g-Z(Q_g^*)} = Q^*_{g'-Z(Q_{g'}^*)}$.*

At equilibrium there might be links which carry no flows. For the firms of such links, the marginal profits of supplying via them are not positive. They are indifferent between having such a link or not. Theorem 2 tells us such links with zero flow play no role in determining the equilibrium. They are strategically redundant.

[2] Explicitly, $\rho : L \to \mathbb{N}_+$ is such that:

(*i*) $\exists (i, j) \in L$ such that $\rho(i, j) = 1$,

(*ii*) $(i, j) \neq (k, l) \Rightarrow \rho(i, j) \neq \rho(k, l)$,

(*iii*) $j < l \Rightarrow \rho(i, j) < \rho(k, l)$ for all $(i, j), (k, l) \in L$,

(*iv*) $i < k \Rightarrow \rho(i, j) < \rho(k, j)$ for all $(i, j), (k, j) \in L$,

(*v*) if $\exists (i, j)$ s.t. $\rho(i, j) = z > 1$ then $\exists (k, l) \in L$ s.t. $\rho(k, l) = y - 1$.

**Example 2** Take graph $g_3$. Let $\alpha = \beta = \gamma = 1$. Then at equilibrium,

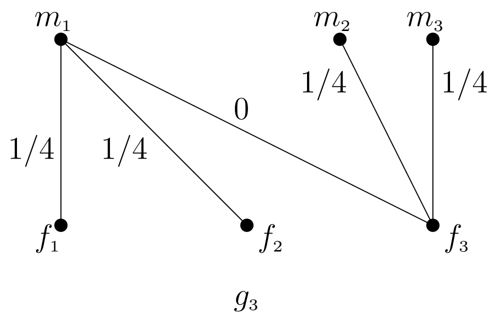



Figure 3

Now we cut the link $(1, 3)$ and denote the new graph by $g_3 - (1, 3)$.

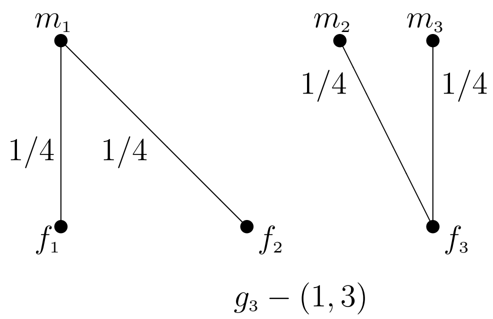



Figure 4

For $\alpha = \beta = \gamma = 1$, according to Theorem 2 the supplies at equilibrium are $q^*_{11} = q^*_{12} = \frac{1}{4}$ and $q^*_{23} = q^*_{33} = \frac{1}{4}$. At the equilibrium in $g_3$, the marginal profit to firm $f_3$ from supplying via $(1, 3)$ was negative. Deleting it does not change the equilibrium quantities on other links, because the marginal profits from them are the same as in graph $g_3$.

We will use the marginal profit argument employed in this example to give a network interpretation for the quantities at equilibrium $Q^*_{g-Z(Q^*_g)}$ on any given graph $g$.

**Definition 1** *Given a graph $g$, a line graph $I(g)$ of $g$ is a graph obtained by denoting each link in $g$ with a node in $I(g)$ and connecting two nodes in $I(g)$ if and only if the corresponding links in $g$ meet at one endpoint.*

Given a network $g$, let $r^*(g) = r(g) - t^*$. Let $G^* = [g_{ij}]_{r^*(g) \times r^*(g)}$ be the weighted adjacency matrix of the line graph of $g - Z(Q^*_g)$ such that

$$g_{ij} = \begin{cases} \gamma_l, \text{ if } \rho^{-1}(i) \text{ and } \rho^{-1}(j) \text{ share firm } f_l \\ \beta_l, \text{ if } \rho^{-1}(i) \text{ and } \rho^{-1}(j) \text{ share market } m_l \\ 0, \text{ otherwise} \end{cases}$$

For example for graph $g_2$ all links have positive flows at equilibrium. Then,

$$G^*_{g2} = \begin{bmatrix} 0 & \gamma_1 & 0 \\ \gamma_1 & 0 & \beta_2 \\ 0 & \beta_2 & 0 \end{bmatrix}$$

For any graph $g$, $G^*$ has diagonal entries as 0 and non-diagonal entries are either $0, \gamma$ or $\beta$. We will use $G^*$ to denote both the line graph of $g - Z(Q^*_g)$ and the weighted adjacency matrix of this graph. Similarly, we define $A$, a diagonal matrix with the same size as $G^*$ such that

$$a_{kl} = \begin{cases} \frac{1}{2\beta_i+\gamma_j}, & \text{if } k = l \text{ and } \rho^{-1}(k) = (i,j) \\ 0, & \text{otherwise} \end{cases}$$

For $a \geq 0$, and a network adjacency matrix $G^*$, let

$$M(G^*, a) = [I - aG^*]^{-1} = \sum_{k=0}^{\infty} (aG^*)^k$$

If $M(a, G^*)$ is non-negative, its entries $m_{ij}(G^*, a)$ counts the number of paths in the network, starting at node $i$ and ending at node $j$, where paths of length $k$ are weighted by $a^k$.

**Definition 2** *For a network adjacency matrix $G$, and for scalar $a > 0$ such that $M(G, a) = [I - aG]^{-1}$ is well-defined and non-negative, the vector Katz-Bonacich centralities of parameter $a$ in $G$ is:*

$$\boldsymbol{b}(G, a) = [I - aG]^{-1} .\mathbf{1}$$

In a graph with $z$ nodes, the Katz-Bonacich centrality of node $i$,

$$b_i(G, a) = \sum_{j=1}^{z} m_{ij}(G, a)$$

counts the total number of paths in $G$ starting from $i$.

**Theorem 3** *Given a network of Cournot markets and firms $g$, the Nash equilibrium flow vector is*

$$Q^*_{g-Z(Q^*_g)} = \left[\sum_{k=0}^{\infty}(AG^*)^{2k} - \sum_{k=0}^{\infty}(AG^*)^{2k+1}\right] A\boldsymbol{\alpha}$$

*where $\boldsymbol{\alpha}$ is a column vector such that for $t = \rho(i,j)$, $\alpha_t = \alpha_i$.*

The first summation counts the total number of even paths that start from the corresponding node in $G^*$, and the second summation counts the total number of odd paths that start from it.

The first sum tells that the equilibrium flows from a link is positively related with the number of even length paths that start from it. The links which have an even distance between them are complements. In contrast, the negative sign on the second summation means the equilibrium supply from a link is negatively related with the number of odd length paths that start from it. The links which have an odd distance between them are substitutes.

For example, in graph $g_1$,

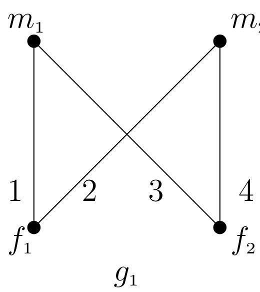



Figure 5

links $(1,1)$ and $(2,2)$ are complements. The supply to market $m_2$ by firm $f_2$ increases incentives for firm $f_1$ to supply more to market $m_1$, because the former decreases the marginal revenue on $m_2$. This makes $m_1$ a better option. Links $(1,1)$ and $(2,1)$ are substitutes, because supply through one decreases the marginal revenue to firm $f_1$. This decreases firm's incentives to supply more.

In general, the links of a firm are substitutes for each other (e.g. $(1,1)$ and $(2,2)$ at graph $g_1$). Similarly, the links of a market are substitutes for each other, too (e.g. $(1,1)$ and $(1,2)$ at graph $g_1$). If two firms are sharing a market, then their links to markets they don't share are complements (e.g. $(1,1)$ and $(2,2)$ at graph $g_1$). Moreover, if a link $(i_1, j_1)$ is a substitute of a link $(i_2, j_2)$ and $(i_2, j_2)$ is a substitute of $(i_3, j_3)$, then $(i_1, j_1)$ and $(i_3, j_3)$ are complements. Therefore, the effect depends on the parity of the distance between two links.

In the Cournot game the adjacency matrix $G^*$ does not necessarily have binary entries, neither its non-zero entries are all equal. Each link in $G^*$ has a weight. While counting the number of paths, these weights are taken into account as well. The total supply a firm $f_j$ is calculated by summing up the link centralities of the elements in $N_g(f_j)$.

# 4 Merger

Given a network $g$, let $Q_g$ be the Cournot equilibrium. Suppose two firms $f_j$ and $f_k$ merge to maximize their joint profit. Let $\tilde{Q}_g$ be the new Cournot equilibrium after the merger. The joint profit of firms $f_j$ and $f_k$, $\pi_j(\tilde{Q}_g) + \pi_k(\tilde{Q}_g)$, is

$$\Pi_{jk} = \sum_{m_i \in N_g(f_j)} \alpha_i \tilde{q}_{ij} + \sum_{m_i \in N_g(f_k)} \alpha_i \tilde{q}_{ik} - \sum_{m_i \in N_g(f_j)} \beta_i \tilde{q}_{ij} \tilde{c}_i - \sum_{m_i \in N_g(f_k)} \beta_i \tilde{q}_{ik} \tilde{c}_i - \frac{\gamma_j}{2} \tilde{s}_j^2 - \frac{\gamma_k}{2} \tilde{s}_k^2$$

**Proposition 4** *i) If two firms do not share a market, then the Cournot equilibrium after the merger is equivalent to the no-merger situation.*
*ii) If the firms share markets, then they decrease their supply to some of the markets they share and they increase their supply to any markets which are not shared. Their total supply decreases.*

**Example 3** Let there be 2 markets and 5 firms connected as in the graph $g_4$ below.

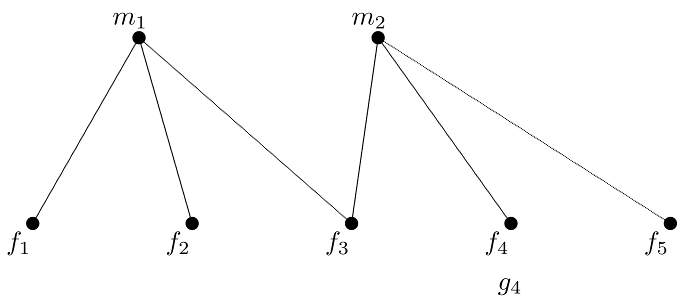


Figure 6

Let $\alpha_i = \beta_i = \gamma_j = 1$ for all markets $m_i$ and firms $f_j$. Then the equilibrium quantities, prices and profits are

$(q_{11}, q_{21}, q_{31}, q_{32}, q_{42}, q_{52}) = (\frac{3}{14}, \frac{3}{14}, \frac{1}{7}, \frac{1}{7}, \frac{3}{14}, \frac{3}{14})$
$(p_1, p_2) = (\frac{3}{7}, \frac{3}{7})$
$(\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (0.069, 0.069, 0.082, 0.069, 0.069)$

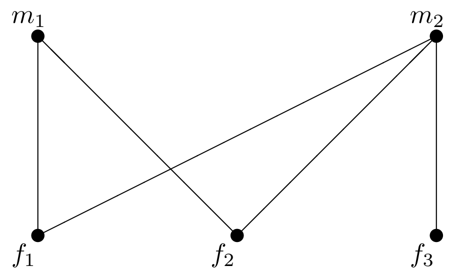
Suppose firms 2 and 3 form a cartel. Now, the equilibrium quantities, prices and profits are

$(q'_{11}, q'_{21}, q'_{31}, q'_{32}, q'_{42}, q'_{52}) = (\frac{12}{49}, \frac{11}{49}, \frac{2}{49}, \frac{9}{49}, \frac{10}{49}, \frac{10}{49})$
$(p'_1, p'_2) = (\frac{24}{49}, \frac{20}{49})$
$(\pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5) = (0.090, 0.085, 0.070, 0.062, 0.062)$

The collusion benefits firms 1, while it hurts firms 4 and 5. Consumers in market 1 are worse off, while consumers in market 2 benefit.

In Example 3 the merged firms decrease their supply to the market they share. Though this is not a general feature. If there are several markets shared by the merger, they might decrease their supply to some while increasing to others.

**Example 4** Let there be 2 markets and 3 firms connected as in the graph $g_5$ below.



$g_5$

Figure 7

Let $\alpha_i = \beta_i = \gamma_j = 1$ for all markets $m_i$ and firms $f_j$. The Cournot equilibrium supplies are

$$(q_{11}, q_{21}, q_{12}, q_{22}, q_{23}) = (\frac{8}{37}, \frac{8}{37}, \frac{5}{37}, \frac{5}{37}, \frac{9}{37})$$

Suppose firms $f_1$ and $f_2$ merge. The supplies after the merger are

$$(q'_{11}, q'_{21}, q'_{12}, q'_{22}, q'_{23}) = (\frac{5}{32}, \frac{5}{32}, \frac{7}{32}, \frac{7}{32}, \frac{6}{32})$$

Hence, the merger decreases its supply to market $m_1$ which is captive, but increase its supply market $m_2$ where it is competing with firm $f_3$. The merger decreases its supply in a less competitive market and increases it in a more competitive one.

A merger in a simple Cournot market would have benefited all outsider producers and hurt all consumers. In a networked market the effect is not symmetric, and its sign is determined by the network.

# 5 The Perfect Cartel

We will study the case where only one cartel including all the firms is formed. To focus on the effect of the network structure we will simplify our model by assuming that all the markets and all the firms are homogenous among themselves. Hence, for the rest of the paper, given a market $m_i$ and a flow vector $Q_g$ the price at $m_i$ is, for $\alpha, \beta > 0$,

$$p_i(Q_g) = \alpha - \beta c_i$$

and for a firm $f_j$ the total cost of production, for $\gamma > 0$, is

$$T_j(Q_g) = \frac{\gamma}{2} s_j^2$$

Hence, the profit function of a firm $f_j$ is:

$$\pi_j(Q_g) = \sum_{m_i \in N_g(f_j)} \alpha q_{ij} - \frac{\gamma}{2} s_j^2 - \sum_{m_i \in N_g(f_j)} \beta q_{ij} c_i$$

Suppose all the firms in the network form a cartel which maximize the total profit of the firms. Given a supply vector $Q_g$, the profit of the cartel is

$$\Pi(Q_g) = \sum_{f_j \in F} \pi_j(Q_g) = \alpha \sum_{(i,j) \in g} q_{ij} - \frac{\gamma}{2} \sum_{f_j \in C} (s_j)^2 - \beta \sum_{m_i \in S} (d_i)^2$$

First, we will characterize the optimal cartel supply in Proposition 5. In a complete bipartite network, due to its symmetry, it is easy to calculate the cartel supply. We next establish that for a class of networks, the cartel supply is equal to those in their completed bi-partite graphs (Propositions 6 & 7). In Proposition 8, we provide a network decomposition to calculate the cartel supply. Proposition 9 reveals the cartel supply is less than the Cournot equilibrium supply.

**Proposition 5** *Given a graph $g$, the supply vector $Q_g$ maximizes the cartel's profit if and only if*

$$\text{for all } (i,j) \in g \qquad \begin{cases} \text{if } q_{ij} \neq 0, \text{ then } \alpha = \gamma f_j + 2\beta m_i \\ \text{if } q_{ij} = 0, \text{ then } \alpha < \gamma f_j + 2\beta m_i \end{cases}$$

The conditions in Proposition 5 are the first order conditions to maximize $\Pi(Q_g)$. Since the profit functions of firms are strictly concave in their supply, the cartel maximizes its profit by distributing the markets among its members as equally as possible within the graph $g$. This means smoothing out both the supplies by firms, and consumptions in markets. If $\tilde{Q}_g$ is a vector of supplies which maximizes the cartel's profit, then for a firm $f_j$ and any two different markets $m_i, m_k \in N_g(f_j)$

$$\begin{aligned} \tilde{q}_{ij}, \tilde{q}_{kj} &\neq 0 \Rightarrow \tilde{q}_i = \tilde{q}_k \\ \tilde{q}_{ij} &= 0 \text{ and } \tilde{q}_{kj} \neq 0 \Rightarrow \tilde{q}_i > \tilde{q}_k \end{aligned}$$

Similarly, for a market $m_i$ and any two different firms $f_j, f_l \in N_g(m_i)$

$$\begin{aligned} \tilde{q}_{ij}, \tilde{q}_{il} &\neq 0 \Rightarrow \tilde{q}_j = \tilde{q}_l \\ \tilde{q}_{ij} &= 0 \text{ and } \tilde{q}_{il} \neq 0 \Rightarrow \tilde{q}_j > \tilde{q}_l \end{aligned}$$

We are not guaranteed a unique solution. Indeed, we will see that, in general, there exists a continuum of solutions to the problem of maximizing the cartel's profit. But all such supply vectors will lead to the same supply by all firms and the same consumption at each market.

**Example 2** Suppose we have graph $g_1$. Let $\alpha = \beta = \gamma = 1$. The supplies which maximize the profit of the cartel are such that

$$\{\tilde{q}_{11}, \tilde{q}_{21}, \tilde{q}_{12}, \tilde{q}_{21} \geq 0 : \tilde{q}_{11} + \tilde{q}_{12} = \frac{1}{3}, \tilde{q}_{21} + \tilde{q}_{22} = \frac{1}{3}, \tilde{q}_{11} + \tilde{q}_{21} = \frac{1}{3} \text{ and } \tilde{q}_{12} + \tilde{q}_{22} = \frac{1}{3}\}$$

There exists a continuum of supplies which maximize the cartel's profit. The total supply by each firm and the total consumption at each market are the same for all those supplies.

Now we will find a vector of supplies that satisfies the first order conditions. Given a subgraph $g_0 = \langle S_0 \cup C_0, L_0 \rangle$ of $g$, consider the cartel's profit maximizing supplies and market consumptions in its completed graph $\overleftrightarrow{g_0}$. Clearly the levels are identical across firms and across markets. Let $\tilde{s_0}$ be the supply by a firm in $\overleftrightarrow{g_0}$ and $\tilde{c_0}$ the consumption at a market in $\overleftrightarrow{g_0}$. If $|M_0| = m_0$ and $|F_0| = n_0$, then direct calculation shows that

$$\tilde{s_0} = \frac{\alpha m_0}{\gamma m_0 + 2\beta n_0} \text{ and } \tilde{c_0} = \frac{\alpha n_0}{\gamma m_0 + 2\beta n_0}.$$

These values depend only on the market/firm ratio. For two graphs $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ and $g_1 = \langle M_1 \cup F_1, L_1 \rangle$,

$$\frac{|M_0|}{|F_0|} = \frac{|M_1|}{|F_1|} \Rightarrow \tilde{s_0} = \tilde{s_1} \text{ and } \tilde{c_0} = \tilde{c_1}.$$

We will use the quantities at the complete graph as benchmarks while calculating the amounts at incomplete bipartite graphs.

Given $g$, we say that a supply vector $Q_g$ is *feasible* if all supplies in $Q_g$ are non-negative. The set of feasible flow vectors in $g_0$ is a subset of the set of feasible flow vectors in its completed graph $\overleftrightarrow{g_0}$. Then given efficient levels of supply $\tilde{s_0}$ and consumption $\tilde{c_0}$ at $\overleftrightarrow{g_0}$, if these amounts are possible at $g_0$, then they must be maximize the cartel's profit at $g_0$ also.

**Proposition 6** *Let $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ be a subgraph of $g$. If the supply of $\tilde{s_0}$ by each firm in $F_0$ is possible without exceeding the consumption $\tilde{c_0}$ in any market in $M_0$, then these levels maximize the cartel's profit in $g_0$.*

To calculate the cartel supply we introduce two graphical definitions.

An *inclusive subgraph* $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ *of* $g$ is such that $g_0$ is connected and

$$M_0 = \bigcup_{f_j \in F_0} N_g(f_j).$$

An inclusive subgraph[3] includes all the markets to which its firms were connected in graph $g$. Let $W(g) = \{g_0 \subseteq g : g_0 \text{ is inclusive}\}$ be the set of inclusive subgraphs in $g$. Since $g$ is an inclusive subgraph of itself $W(g) \neq \emptyset$. In graph $g_3$ in Figure 5, the subgraph $g_3^0$ that we encircle is inclusive. It includes $f_1$ and all the markets that $f_1$ is connected to.

[3]See Bochet *et al.* (2010) for the relationship between inclusive subgraphs and the Gallai-Edmonds decomposition (Ore 1962) of a bipartite graph.

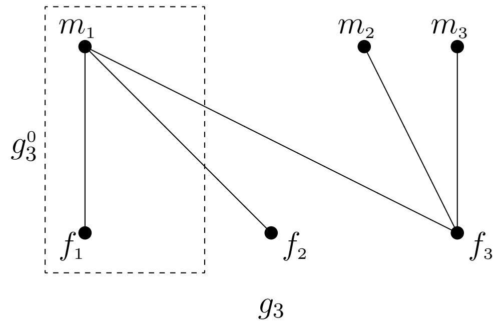



Figure 5

Given a subset of markets $M_0 \subseteq M$ and a subset of firms $F_0 \subseteq F$, $\frac{|M_0|}{|F_0|}$ is the average number of markets per firm. A *least inclusive subgraph* $\widehat{g} = \left\langle \widehat{M} \cup \widehat{F}, \widehat{L} \right\rangle$ *of* $g$ is such that

$$\frac{\left|\widehat{M}\right|}{\left|\widehat{F}\right|} < \frac{|M|}{|F|} \text{ and } \left\langle \widehat{M} \cup \widehat{F}, \widehat{L} \right\rangle \in \underset{\langle M_0 \cup F_0, L_0 \rangle \in W(g)}{\operatorname{argmin}} \frac{|M_0|}{|F_0|}$$

The first requirement for $\widehat{g}$ to be a least inclusive subgraph of $g$ is for it to have a strictly smaller market/firm ratio than $g$. This means that a graph does not necessarily have a least inclusive subgraph. For example a complete bipartite graph has no least inclusive subgraphs. The second requirement is for $\widehat{g}$ to have the smallest market/firm ratio among the inclusive subgraphs of $g$. A least inclusive subgraph is inclusive and formed by a set of the least connected firms. There should be no firms in $g$ which are strictly worse than them with respect to connectedness.

In Figure 5, the subgraph $g_3^0$ is not least inclusive, because the ratio of markets to firms in it is 1. This ratio for graph $g_3$ is also 1. The subgraph $g_3^1$ of $g_3$, as encircled Figure 6 below, is a least inclusive subgraph. Its market/firm ratio is lower than that of $g_3$, and there is no other inclusive subgraph of $g_3$ with a lower ratio.

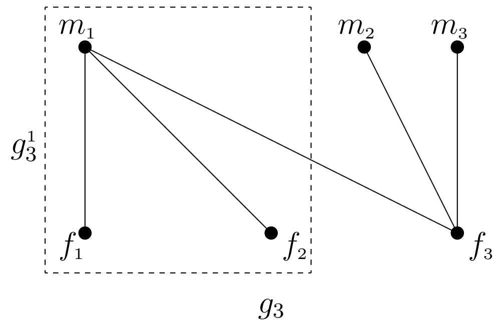



Figure 6

If $\widehat{g}$ is a least inclusive subgraph of $g$, then $\widehat{g}$ cannot have a least inclusive subgraph of its own. Any inclusive subgraph of $\widehat{g}$ is also inclusive in $g$. If $\widehat{g}$ had a least inclusive subgraph with a smaller market/firm ratio than $\widehat{g}$, this would have contradicted $\widehat{g}$ having the smallest market/firm ratio in $g$.

Now we show that if a subgraph $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ of $g$ has no least inclusive subgraph, then the supply of $\tilde{s_0}$ by each firm in $F_0$ is possible without exceeding the consumption $\tilde{c_0}$ in any market in $M_0$

**Proposition 7** *Let $g_0 = \langle M_0 \cup F_0, L_0 \rangle$ of $g$ be an inclusive subgraph. If $g_0$ has no least inclusive subgraph, then the supply of $\tilde{s_0}$ by each firm in $F_0$ is possible without exceeding the consumption $\tilde{c_0}$ in any market in $M_0$.*

The result means that if a network has no least inclusive subgraph, it can be treated as a complete network. All the firms are symmetric under efficiency. Hence there is no difference between this problem and the simple Cournot with a single market.

To prove Proposition 7 we start with a firm $f_j$ of a graph $g_0$ with no inclusive subgraphs. This firm must be able to supply $\tilde{s_0}$, without exceeding the consumption $\tilde{c_0}$ in any of its markets. If not, that firm with its markets would have formed a least inclusive subgraph in $g_0$. Next, we add a new firm to this subgraph and iteratively show that such supply levels must be possible for all inclusive subgraphs of $g_0$ that contain $f_j$. As $g_0$ is an inclusive subgraph of itself, this proves that such supply levels are possible in $g_0$.

**Decomposing the network** Now we will break down the network $g$, so that the cartel's optimization problem in each subnetwork is independent from the other ones. We will sequentially cut out least inclusive subgraphs. Hence, they will not have any least inclusive subgraphs of their own. We will continue until we reach a subgraph which has no least inclusive subgraphs. Then in each subgraph, the cartel optimal supplies at each firm and consumptions at each market will be equal to the amounts in their completed graphs. The next result follows from Propositions 6 and 7.

**Proposition 8** *Given a network of commons $g$, the following algorithm calculates the optimal cartel supply by each firm and consumption from each market.*

*Step 1: Take $g$. Suppose $g = \langle M \cup F, L\rangle$ has no least inclusive subgraph. Then the supply by a firm $f_j$ and consumption at a market $m_i$ are equal to the levels in a complete bipartite graph with nodes $M \cup F$, and we are done.*

*Suppose $g = \langle M \cup F, L\rangle$ has a least inclusive subgraph. Let $g_0 = \langle M_0 \cup F_0, L_0\rangle$ be the largest least inclusive subgraph[4] in $g$. Then, the supply by a firm $f_j \in F_0$ is $\tilde{s_0}$, and the consumption at a market $m_i \in M_0$ is $\tilde{c_0}$.*

*Step 2: Now, for the rest of the firms and markets apply Step 1 to $g - g_0$.*

*In this way we obtain a series of regions out of $g$, with a strictly increasing market per firm ratio. In each of them, the supplies would equal to the levels in their respective completed graphs.*

So, given a subgraph $g_0 = \langle M_0 \cup F_0, L_0\rangle$ obtained from the above decomposition, supply by a firm in $g_0$ is

$$\tilde{s_0} = \frac{\alpha m_0}{\gamma m_0 + 2\beta n_0}$$

and the efficient outflow from each market in $g_0$ is

$$\tilde{c_0} = \frac{\alpha n_0}{\gamma m_0 + 2\beta n_0}$$

These levels satisfy the first order conditions within each region. Moreover, less connected firms have lower supplies and less connected markets have lower consumptions. Since there are no flows between different regions the first order conditions hold for graph $g$ as well.

The link redundancies reappear with the cartel. Take two graphs $g$ and $g'$ such that their decomposition yields the same regions. The optimal amounts of supplies at each firm and consumptions at each market are the same for both $g$ and $g'$.

[4] The ratio $\frac{|N_g(F_0)|}{|F_0|}$ is a submodular function of $F_0$, where $N_g(F_0)$ is the set of markets connected to $F_0$. Then at any graph $g$, there exists a unique largest least inclusive subgraph.

**Example 3** Suppose we have graph $g_3$. Let $\alpha = \beta = \gamma = 1$. The decomposition would give us two regions, $g_3^1$ and $g_3 - g_3^1$. Then the cartel supplies are

$$\{\tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{13}, \tilde{q}_{23}, \tilde{q}_{33} \geq 0 : \tilde{q}_{11} = \frac{1}{5}, \tilde{q}_{12} = \frac{1}{5}, \tilde{q}_{13} = 0, \tilde{q}_{23} = \frac{1}{4} \text{ and } \tilde{q}_{33} = \frac{1}{4}\}$$

Suppose the graph was $g_3 - (1,3)$. The decomposition leads to the same regions. The supplies are

$$\{\tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{23}, \tilde{q}_{33} \geq 0 : \tilde{q}_{11} = \frac{1}{5}, \tilde{q}_{12} = \frac{1}{5}, \tilde{q}_{23} = \frac{1}{4} \text{ and } \tilde{q}_{33} = \frac{1}{4}\}$$

The link $(1,3)$ is redundant for the cartel, just as it was at equilibrium. The supply levels are below the equilibrium for $m_1$, which is shared by $f_1$ and $f_2$ and equal to the equilibrium for $m_2$ and $m_3$, which are used only by $f_3$.

# 6 Conclusion

We have analyzed a situation where firms embedded in a network with markets compete *a la* Cournot. We have shown that the equilibrium flows will depend on the whole structure. The quantity supplied by a firm to a market depends on the centrality of the links it has. The centrality index which determines the quantities is calculated using the line graph of the positive flow network. The quantity flowing through a link is positively proportional with the number of even paths and negatively proportional with the number of odd paths starting from it.

We further study the effects of a merger between two firms on the network. Different from the simple Cournot model the effect of the merger on consumers and outside firms are not symmetric. Some consumers and firms suffer, while others benefit from the merger.

We also study how a cartel formed by all the firms in the network would maximize its joint profit by segmenting the markets. The firms in the cartel would operate only within their assigned markets and refuse to supply to others.

Although the network in our model is fixed, the analysis paves way for further research on strategic network formation in competitive markets. The results we provide can be used to calculate the benefit of each potential link to a firm. Once players know the payoff they would obtain in each network, they could manipulate their connections to maximize their profits.

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## Appendix

We first need to introduce additional notation for the proofs.

**Labeling of pairs (i,j)** We will order all possible links such that the links of a firm $f_j$ are assigned a lower number than any firm $f_i$ for $i > j$, and the links of a firm are ordered according to the indices of the markets they are connected. The label of a possible link $(i, j)$ will be denoted by $\tau(i, j)$. For example for 2 firms and 2 markets, we will order the links starting from the first firm and the first market, $\tau(1, 1) = 1$. The second link is between the first firm and the second market, $\tau(2, 1) = 2$. Now, as all links of firm $f_1$ are ranked, $\tau$ will next rank the link between $f_2$ and $m_1$, $\tau(1, 2) = 3$. Then comes the link between firm $f_2$ and market $m_2$, $\tau(2, 2) = 4$.

For a network $g$, let $Y(g) = \{1 \leq y \leq (m \times n) : y = \tau(i, j) \text{ for some } (i, j) \notin g\}$ be the set of indices that $\tau$ assigns to links which are not in $g$. For 2 firms and 2 markets, for a graph $g$, if the only missing link is $(1, 2)$, then $Y(g) = \{3\}$ and $r(g) = 3$.

$\tau$ orders all possible links, independent of $g$, where as $Y(g)$ does depend on $g$. We can see how this works on an example. Suppose that 2 firms and 2 markets form a completely connected bipartite graph $g_1$. For graph $g_1$, $Y(g_1) = \emptyset$.

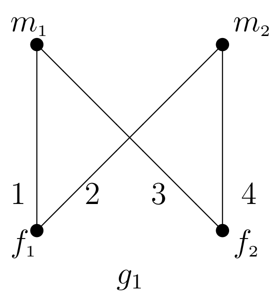


Figure 7

Now we cut the link between $f_2$ and $m_1$, to obtain $g_2$.

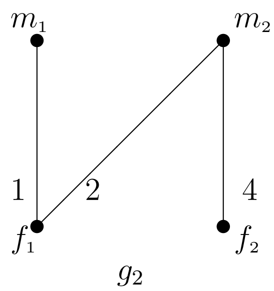


Figure 8

Although link $(1, 2)$ does not exist in $g_2$ it is still labeled equally by $\tau$. $\tau(1, 2) = 3$, meaning that $Y(g_2) = \{3\}$.

Let $\mathbb{N}_+$ be the set of positive integers. Let $\rho : L \to \mathbb{N}_+$ be a lexicographic order on $L$ respecting $\tau$ such that $\rho$ relabels the $(i, j)$ pairs from 1 to $r(g)$ by skipping those links which are not in $g$.

Explicitly, $\rho : L \to \mathbb{N}_+$ is such that:

$(i)$ $\exists(i, j) \in g$ such that $\rho(i, j) = 1$,

$(ii)$ $(i, j) \neq (k, l) \Rightarrow \rho(i, j) \neq \rho(k, l)$,

$(iii)$ $j < l \Rightarrow \rho(i, j) < \rho(k, l)$ for all $(i, j), (k, l) \in g$,

$(iv)$ $i < k \Rightarrow \rho(i, j) < \rho(k, j)$ for all $(i, j), (k, j) \in g$,

$(v)$ if $\exists(i, j)$ s.t. $\rho(i, j) = z > 1$ then $\exists(k, l) \in g$ s.t. $\rho(k, l) = y - 1$.

Let $Z(Q_g) = \{1 \leq z \leq r(g) : z = \rho(i, j)$ for some $(i, j)$ s.t. $q_{ij} = 0\}$. Let $|Z(Q_g)| = t$, then $Q_{g-Z(Q_g)}$ is a vector of size $r(g) - t$ obtained from $Q_g$ by deleting the zero entries. It is the vector of quantities for links over which there is a strictly positive flow.

Let $Q_g^*$ be the equilibrium of the Cournot game at network $g$. We denote by $g - Z(Q_g^*)$ the network obtained from $g$ by deleting the links which have zero flow at $Q_g^*$.

Given a network $g$, let $r^*(g) = r(g) - t^*$. Let $G^* = [g_{ij}]_{r^*(g) \times r^*(g)}$ be the weighted adjacency matrix of the line graph of $g - Z(Q_g^*)$ such that

$$g_{ij} = \begin{cases} \gamma_l, & \text{if } \rho^{-1}(i) \text{ and } \rho^{-1}(j) \text{ share firm } f_l \\ \beta_l, & \text{if } \rho^{-1}(i) \text{ and } \rho^{-1}(j) \text{ share market } m_l \\ 0, & \text{otherwise} \end{cases}$$

**Proof of Theorem 1** Given a graph $g$, at any the equilibrium of the Cournot game the flows cannot be negative

$$Q_g^* \geq 0 \tag{4}$$

For each link $(i,j) \in g$, at equilibrium $\frac{\partial \pi_j}{\partial q_{ij}}|_{q_{ij}^*} \leq 0$. More explicitly

$$\frac{\partial \pi_j}{\partial q_{ij}}|_{q_{ij}^*} = \alpha_i - \beta_i q_{ij}^* - \gamma_j \sum_{m_k \in N_g(f_j)} q_{kj}^* - \beta_i \sum_{f_k \in N_g(m_i)} q_{ik}^* \leq 0$$

These set of equations can be written in matrix form

$$-\boldsymbol{\alpha} + D_g Q_g^* \geq 0 \tag{5}$$

where $\boldsymbol{\alpha} = [\alpha_t]_r$ such that for $t = \tau(i,j)$, $\alpha_t = \alpha_i$ and $D_g = [d_{tz}]_{r \times r}$ such that

$$d_{tz} = \begin{cases} 2\beta_i + \gamma_j, & \text{if } t = z = \tau(i,j) \text{ for some } m_i \in M, f_j \in F \\ \gamma_j & , \text{ if } t \neq z, t = \tau(i,j), z = \tau(k,j) \text{ for some } m_i, m_k \in M, f_j \in F \\ \beta_i & , \text{ if } t \neq z, t = \tau(i,j), z = \tau(i,k) \text{ for some } m_i \in M, f_j, f_k \in F \\ 0 & , \text{ otherwise} \end{cases}$$

Lastly, for each link $(i,j) \in g$, at equilibrium $\frac{\partial \pi_j}{\partial q_{ij}}|_{q_{ij}^*} q_{ij}^* < 0$. In matrix form

$$(Q_g^*)^T(-\boldsymbol{\alpha} + D_g Q_g^*) \geq 0 \tag{6}$$

The first order equilibrium conditions $(4), (5), (6)$ of the Cournot game constitute a $LCP(-\boldsymbol{\alpha}; D_g)$.

Samelson *et al.* (1958) shows that a linear complementarity problem $LCP(p; M)$ has a unique solution for all $p \in \mathbb{R}^t$ if and only if all the principal minors of $M$ are positive. Positive definite matrices satisfy this condition and we will now that $D_g$[5] is positive definite for any graph $g$.

[5] The interpretation, when we use it to find the equilibrium quantities flowing from markets to firms, is that the column $z$ and the row $z$ in $D_g$ corresponds to the link $(i,j)$ in $g$ such that $\tau(i,j) = z$. Hence, column 1 and row 1 corresponds to the link $(1,1)$, column 2 and row 2 corresponds to the link $(2,1)$, column 3 and row 3 corresponds to the link $(1,2)$, and column 4 and row 4 corresponds to the link $(2,2)$.

We show that for any matrix $D_g$ we can find a matrix $R$ with independent columns such that $D_g = R^T R$. [6]

For example for graph $g_1$,

$$D_{g_1} = \begin{pmatrix} 2\beta_1+\gamma_1 & \gamma_1 & \beta_1 & 0 \\ \gamma_1 & 2\beta_2+\gamma_1 & 0 & \beta_2 \\ \beta_1 & 0 & 2\beta_1+\gamma_2 & \gamma_2 \\ 0 & \beta_2 & \gamma_2 & 2\beta_2+\gamma_2 \end{pmatrix}$$

We write $R$ as

$$R = \begin{bmatrix} \sqrt{\beta_1} & 0 & 0 & 0 \\ 0 & \sqrt{\beta_2} & 0 & 0 \\ 0 & 0 & \sqrt{\beta_1} & 0 \\ 0 & 0 & 0 & \sqrt{\beta_2} \\ \sqrt{\gamma_1} & \sqrt{\gamma_1} & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_2} & \sqrt{\gamma_2} \\ \sqrt{\beta_1} & 0 & \sqrt{\beta_1} & 0 \\ 0 & \sqrt{\beta_2} & 0 & \sqrt{\beta_2} \end{bmatrix}$$

Then clearly $D_{g_1} = R^T R$. Given a graph $g$, the same technique can be used to show that $D_g$ is positive definite. For the detailed demonstration we refer the reader to the proof of Proposition 7 in İlkılıç (2010). Hence, for any $g$ and any $\boldsymbol{\alpha}$, $LCP(-\boldsymbol{\alpha}; D_g)$ has a unique solution.

Now, let's check that the second order conditions are satisfied. For firm $f_k$ with $n$ connections we first label the connections from 1 to $n$. Hence, $N_g(f_k) = \{v_1, ..., v_n\}$. Then the Hessian of the profit function $\pi_k$ is $H = [h_{ij}]_{n\times n}$ where

$$h_{ij} = \begin{cases} -2\beta_i - \gamma_k, & \text{if } i = j \\ -\gamma_k, & \text{otherwise} \end{cases}$$

Let $H' = -H$. We can use the same technique applied for $D_g$ to show that $H'$ is positive definite. Hence, $H$ is negative definite. The solution of $LCP(-\boldsymbol{\alpha}; D_g)$ is the equilibrium of the Cournot game. ■

[6] This is equivalent to checking that $D$ is positive definite. For other characterizations of positive definiteness see Strang (1988).

**Proof of Theorem 2** Assume $Q^*_{g-Z(Q_g)}, Q^*_{g-Z(Q''_g)}$ are equilibria of the game at $g$ and $g'$, respectively. Let

$$g - Z(Q^*_g) = g' - Z(Q^*_{g'})$$

Then,

$$D_{g-Z(Q^*_g)}.Q^*_{g-Z(Q^*_g)} = \alpha.\mathbf{1}{=}D_{g'-Z(Q^*_{g'})}.Q^*_{g'-Z(Q^*_{g'})} = D_{g-Z(Q^*_g)}.Q^*_{g'-Z(Q^*_{g'})}$$

As we showed in proposition 6 $D_{g-Z(Q^*_g)}$ is positive definite, hence invertible.

$$Q^*_{g-Z(Q_g)} = Q^*_{g-Z(Q''_g)}$$

■

**Proof of Theorem 3**

$$\begin{aligned} D_{g-Z(Q^*_g)}.Q^*_{g-Z(Q^*_g)} &= \left[A^{-1} + G^*\right].Q^*_{g-Z(Q^*_g)} \\ &= A^{-1}\left[I + AG^*\right].Q^*_{g-Z(Q^*_g)} \end{aligned}$$

Remember that $Q^*_g$ is the solution to $LCP(-\alpha\mathbf{1}_r; D_g)$. Then, when we invert $D_{g-Z(Q^*_g)}$, the matrix multiplication $\left[D_{g-Z(Q^*_g)}\right]^{-1}\boldsymbol{\alpha}$ will give us a strictly positive vector.

$$\begin{aligned} \left[I + AG^*\right] &= \left[I - AG^*\right]^{-1}\left[I - (AG^*)^2\right] \\ \left[I + AG^*\right]^{-1} &= \left[I - (AG^*)^2\right]^{-1}\left[I - AG^*\right] \\ &\text{and} \\ \left[I - (AG^*)^2\right]^{-1} &= \sum_{k=0}^{\infty}(AG^*)^{2k} \end{aligned}$$

Substituting this into $D_{g-Z(Q^*_g)}.Q^*_{g-Z(Q^*_g)} = \boldsymbol{\alpha}$,

$$\begin{aligned} Q^*_{g-Z(Q^*_g)} &= \left[I - (AG^*)^2\right]^{-1}\left[I - AG^*\right]A\boldsymbol{\alpha} \\ &= \sum_{k=0}^{\infty}(AG^*)^{2k}\left[I - AG^*\right]A\boldsymbol{\alpha} \\ &= \left[\sum_{k=0}^{\infty}(AG^*)^{2k} - \sum_{k=0}^{\infty}(AG^*)^{2k+1}\right]A\boldsymbol{\alpha} \\ &= \left[M((AG^*)^2, 1) - M((AG^*)^2, 1).(AG^*)\right]A\boldsymbol{\alpha} \end{aligned}$$

■

**Proof of Proposition 4** *i*)If the firms do not share any markets, then the best response function of each firm is a function of the non-cartel firms' supplies. Hence the optimal supply is the same as if there was no merger.

*ii*) Suppose firms $f_j$ and $f_k$ share a market $m_i$. Let $Q_g^*$ be the pre-merger Cournot equilibrium. Then, the marginal profit of $f_j$ from supplying to market $m_i$ before the merger was

$$\frac{d\pi_j}{dq_{ij}}|_{Q_g^*} = \alpha_i - \gamma_j \sum_{m_l \in N_g(f_j)} q_{lj}^* - \beta_i \sum_{f_k \in N_g(m_i)} q_{ik}^* - \beta_i q_{ij}^* = 0$$

After the merger, the new marginal profit, calculated at the pre-merger equilibrium is

$$\frac{d\Pi_{jk}}{dq_{ij}}|_{Q_g^*} = \alpha_i - \gamma_j \sum_{m_l \in N_g(f_j)} q_{lj}^* - \beta_i \sum_{f_k \in N_g(m_i)} q_{ik}^* - \beta_i q_{ij}^* - \beta_i q*ik = -\beta_i q_{ik}^* < 0 \qquad (7)$$

Hence post-merger marginal profits from supplies to the shared markets from both of the firms are strictly negative at the pre-merger Cournot equilibrium. Let market $m_i = argmax_{m_l \in N_g(f_j) \cap N_g(f_k)} max \beta_l q_{lj}^*, \beta_l q_{lk}^*$ and w.l.o.g. let this maximum be $\beta_i q_{ij}^*$. Then firm $f_j$ will decrease its supply to market $m_i$ after the merger. Although firm $f_k$ might increase its supply to market $m_i$, the total merger supply to $m_i$ is lower than the pre-merger levels follows from 7.

Suppose firm $f_j$ decreases its supply to $m_i$ by an infinitesimal amount $\Delta$. If there exists a market $m_t \in N_g(f_j)$ and $m_t \notin N_g(f_k)$. Then, the marginal profit of $f_j$ from supplying to market $m_t$ before the merger was

$$\frac{d\pi_j}{dq_{tj}}|_{Q_g^*} = \alpha_t - \gamma_j \sum_{m_l \in N_g(f_j)} q_{lj}^* - \beta_t \sum_{f_k \in N_g(m_t)} q_{tk}^* - \beta_t q_{tj}^* = 0$$

After the $\Delta$ decrease in firm $f_j$'s supply to $m_i$, the marginal profit from supplying to market $m_t$ becomes $\Delta\gamma_j > 0$. Hence, firm $f_j$ will have incentives to increase its supply to $m_t$.

**Proof of Proposition 5** Given a graph $g$, at any the cartel supplies cannot be negative

$$\tilde{Q}_g \geq 0 \qquad (8)$$

For each link $(i,j) \in g$, at the profit maximizing supply $\frac{\partial \Pi}{\partial \tilde{q}_{ij}}|_{\tilde{q}_{ij}} \leq 0$. More explicitly

$$\frac{\partial \Pi}{\partial q_{ij}}|_{\tilde{q}_{ij}} = \alpha_i - \gamma_j \sum_{m_k \in N_g(f_j)} \tilde{q}_{kj} - 2\beta_i \sum_{f_k \in N_g(m_i)} \tilde{q}_{ik} \leq 0$$

These set of equations can be written in matrix form

$$-\boldsymbol{\alpha} + B_g \tilde{Q}_g \geq 0 \tag{9}$$

where $\boldsymbol{\alpha} = [\alpha_t]_r$ such that for $t = \tau(i,j)$, $\alpha_t = \alpha_i$ and $B_g = [b_{tz}]_{r \times r}$ such that

$$b_{tz} = \begin{cases} 2\beta_i + \gamma_j, & \text{if } t = z = \tau(i,j) \text{ for some } m_i \in M, f_j \in F \\ \gamma_j & , \text{ if } t \neq z, t = \tau(i,j), z = \tau(k,j) \text{ for some } m_i, m_k \in M, f_j \in F \\ 2\beta_i & , \text{ if } t \neq z, t = \tau(i,j), z = \tau(i,k) \text{ for some } m_i \in M, f_j, f_k \in F \\ 0 & , \text{ otherwise} \end{cases}$$

Lastly, for each link $(i,j) \in g$, at equilibrium $\frac{\partial \Pi}{\partial q_{ij}}|_{\tilde{q}_{ij}} \tilde{q}_{ij} < 0$. In matrix form

$$(\tilde{Q}_g)^T(-\boldsymbol{\alpha} + B_g \tilde{Q}_g) \geq 0 \tag{10}$$

The first order profit maximizing conditions 8,9 and 10 for the cartel constitute a $LCP(-\boldsymbol{\alpha}; F_g)$. We will show that the matrix, $B_g$ is positive semi-definite. Hence, $LCP(-\alpha\boldsymbol{\alpha}; B_g)$ has a solution, though not necessarily unique.

We show that for any matrix $B_g$ we can find a matrix $R$ such that $B_g = R^T R$. [7]

For example for graph $g_1$,

$$B_{g_1} = \begin{pmatrix} 2\beta_1 + \gamma_1 & \gamma_1 & 2\beta_1 & 0 \\ \gamma_1 & 2\beta_2 + \gamma_1 & 0 & 2\beta_2 \\ 2\beta_1 & 0 & 2\beta_1 + \gamma_2 & \gamma_2 \\ 0 & 2\beta_2 & \gamma_2 & 2\beta_2 + \gamma_2 \end{pmatrix}$$

We write $R$ as

$$R = \begin{bmatrix} \sqrt{\gamma_1} & \sqrt{\gamma_1} & 0 & 0 \\ 0 & 0 & \sqrt{\gamma_2} & \sqrt{\gamma_2} \\ \sqrt{2\beta_1} & 0 & \sqrt{2\beta_1} & 0 \\ 0 & \sqrt{2\beta_2} & 0 & \sqrt{2\beta_2} \end{bmatrix}$$

[7] This is equivalent to checking that $B_g$ is positive semi-definite.

Then clearly $B_{g_1} = R^T R$. Given a graph $g$, the same technique can be used to show that $B_g$ is positive semi-definite. For the detailed demonstration we refer the reader to the proof of Proposition 8 in İlkılıç (2010). Hence, for any $g$ and any $\boldsymbol{\alpha}$, $LCP(-\boldsymbol{\alpha}; B_g)$ has a solution.

The Hessian matrix of $\Pi$ is $H_\Pi = -B_g$. Since $B_g$ is positive semi-definite, $H_\Pi$ is negative semi-definite. Meaning that any $\tilde{Q}_g$ maximizes $\Pi$. ■

**Proof of Proposition 6** We know that the supply of $\tilde{s}_0$ by each firm and the consumption of $\tilde{c}_0$ satisfies the first order conditions in $\overleftrightarrow{g_0}$. Since $g_0$ and $\overleftrightarrow{g_0}$ have the same set of nodes, they also satisfy the conditions in $g_0$. ■

**Proof of Proposition 7** By assumption, $g_0$ has no least inclusive subgraphs.

Take a firm $f_j$ in $g_0$. Let $f_j$ supply a total of $\tilde{q}_0$, such that none of the markets consume more than $\tilde{c}_0$. $\tilde{s}_0$ and $\tilde{c}_0$ are functions of the market/firm ratio. If $f_j$ is not linked to enough markets to achieve such a supply, then firm $f_j$ and the markets $N_g(f_j)$ form a least inclusive subgraph in $g_0$, which is a contradiction with $g_0$ having no least inclusive subgraphs.

Now, we are going to show by induction that $s_0$ supply by a firm in $g_0$ such that no market consumes more than $\tilde{c}_0$ is possible in any inclusive subgraph of $g_0$ that contains $f_j$. As $g_0$ is an inclusive subgraph of itself, this will imply that such levels of supply are possible in $g_0$.

We know that it is possible for the inclusive subgraph with firm $f_j$ and the markets $N_g(f_j)$. Take an inclusive subgraph $g_{k-1}$ of $g_0$ that contains $k-1$ firms including $f_j$. Suppose that such levels of supply are possible in $g_{k-1}$. Denote by $Q_{g_{k-1}}$ such a possible amount of flows in $g_{k-1}$.

Now take an inclusive subgraph $g_k$ of $g_0$ that contains $k$ firms, $k-1$ which were in $g_{k-1}$ and a fixed firm $f_k$ which was not in $g_{k-1}$.

Assume that in $g_k$, $\frac{|\hat{M}_k|}{|\hat{F}_k|} < \frac{|\hat{M}|}{|\hat{F}|}$. Then $g_k$ is a least inclusive subgraph of $g_0$, which is a contradiction.

Then, $\frac{|\hat{M}_k|}{|\hat{F}_k|} \geq \frac{|\hat{M}|}{|\hat{F}|}$. Take $Q_{g_{k-1}}$ such that each firm supplies $\tilde{s}_0$in $g_{k-1}$. As $g_k$ contains $g_{k-1}$ the firms in $g_{k-1}$ can supply $\tilde{s}_0$ without exceeding $\tilde{c}_0$ in any market. Now let $f_k$ supply through its links such that the consumption at each market in $N_g(f_k)$ is $\tilde{c}_0$. If the total supply of $f_k$ is al least $\tilde{s}_0$, then we are done.

If not, denote by $Q^1$ the flow vector for $g_k$ such that flows for the links which were already in $g_{k-1}$ equals to $Q_{g_{k-1}}$, and the flows for the links which were not in $g_{k-1}$ equals to 0. Now, given that $f_k \notin F_{k-1}$, let[8] $Q^2$ be the flow vector for $g_k$ such that

$$\begin{aligned} q^2_{jk} &= \tilde{c}_0 - q^1_i, \text{ for } m_j \in N_g(f_k) \\ q^2_{jl} &= q^1_{jl}, \text{ for } l \neq k \end{aligned}$$

Since $\frac{|\hat{M}_k|}{|\hat{C}_F|} \geq \frac{|\hat{M}|}{|\hat{F}|}$, there must be a market $m_i$ in $g_k$ not connected to $f_k$, such that its consumption in $Q^2$ is strictly less than $\tilde{c}_0$. Let $M^-_k$ be the set of markets in $g_k$ which are not connected to $f_k$ and which have consumption in $Q^2$ strictly less than $\tilde{c}_0$.

$$M^-_k = \{m_i \in M_k : m_i \notin N_g(f_k) \text{ and } q^2_i < \tilde{c}_0\}$$

Suppose that for any market $m_i \in M^-_k$ and for all paths

$$P = \{(m_i, f_1), (f_1, m_1), ..., (f_t, m_t), (m_t, f_k)\}$$

that connects $m_i$ with $f_k$, there exists $(f_j, m_j) \in P$ such that $q^2_{jj} = 0$. Given such a path $P$, let $m_P$ denote the market $m_l$ such that $(f_l, m_l) \in P$, $q^2_{ll} = 0$ and there exists no other market $m_j$ in $P$, closer to $f_k$ than $m_l$ such that $(f_j, m_j) \in P$ and $q^2_{jj} = 0$. Let $\overline{F}_k = \{f_j \in F_k :$ there exists a path $P$ from $m_i$ to $f_k$ for some $m_i \in M^-_k$ and in $P$, $f_j$ is between $m_P$ and $f_k\}$. Then the inclusive subgraph with firms $\overline{F}_k \cup f_k$is least inclusive in $g_k$, which is a contradiction.

Then there exists a market $m_i \in M^-_k$ such that there exists a path

$$P = \{(m_i, f_1), (f_1, m_1), ..., (f_t, m_t), (m_t, f_k)\}$$

that connects $m_i$ with $f_k$ and $\min_{(f_j,m_j)\in P} q^2_{jj} \neq 0$. Let

$$d = \min_{(f_j,m_j)\in P} \{q^2_{jj}, q^2_i\}$$

Now, given such a path $P$, let $Q^3$ be the flow vector for $g_k$ such that

$$\begin{aligned} q^3_{i1} &= q^2_{i1} + d, \\ q^3_{jj} &= q^2_{jj} - d, \\ q^3_{j(j+1)} &= q^2_{j(j+1)} + d \\ q^3_{tk} &= q^2_{tk} + d \\ q^3_{ll'} &= q^2_{ll'}, \text{ for all other links } (l, l') \end{aligned}$$

[8]The subscripts will be used as indices. Hence, for market $m_i$, $q^1_i$ will denote its outflow at the vector $Q^1$.

It is possible to make $f_k$ supply at least $\tilde{s}_0$ by finding such paths from markets in $\hat{M}_k^-$ to $f_k$ and changing the flows as explained above for each path from a market in $\hat{M}_k^-$ to $f_k$. If after using all such paths, $f_k$ could still not supply $\tilde{s}_0$, then $g_k$ is a least inclusive subgraph in $g_0$, a contradiction.

Then the desired levels of supply are possible in $g_0$. ■