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Intertemporal Equilibria with Knightian Uncertainty

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Abstract

We study a dynamic and infinite–dimensional model with Knightian uncertainty modeled by incomplete multiple prior preferences. In interior efficient allocations, agents share a common risk–adjusted prior and use the same subjective interest rate. Interior efficient allocations and equilibria coincide with those of economies with subjective expected utility and priors from the agents' multiple prior sets. We show that the set of equilibria with inertia contains the equilibria of the economy with variational preferences anchored at the initial endowments. A case study in an economy without aggregate uncertainty shows that risk is fully insured, while uncertainty can remain fully uninsured. Pessimistic agents with Gilboa–Schmeidler's max-min preferences would fully insure risk and uncertainty.

Key words and phrases: Knightian Uncertainty, Ambiguity, Incomplete Preferences, General Equilibrium Theory, No Trade

JEL subject classification: D51, D81, D91

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1 Introduction

Individuals have to deal with different sorts of risky or uncertain events. For some of them, mortality risk, health care costs, car accidents among others, the probabilities are relatively stable and we have robust and safe ways to estimate them. For others — the event that a firm ranked "B" goes bankrupt in the next year, to give a current example — the probabilities are unknown or at least difficult to estimate. The economic literature distinguishes between risk – where the outcomes are unknown, but the probabilities are known — and uncertainty — where not even the distribution of outcomes is known exactly. Many models of decision under uncertainty have been proposed within the past ten years, most assume complete preferences (see Rigotti, Shannon, and Strzalecki (2008) for an overview and the references therein). Models with incomplete preferences are scarcer but have raised recent interest (see, e.g., Bewley (2002), Rigotti and Shannon (2005), Ok, Ortoleva, and Riella (2008), Nehring (2009), Faro (2010), Nascimento and Riella (2010)).

Gilboa, Maccheroni, Marinacci, and Schmeidler (2010) show that one can justify Bewley's model as "objective" rationality whereas the maxmin–model by Gilboa and Schmeidler corresponds to "subjective" rationality. Under objective rationality, an agent prefers to be silent if she does not have enough information. This leads to incomplete preferences. In market situations, however, an agent has to decide what to buy even if she cannot compare between all consumption plans. In particular, she might move away from her endowment to a plan that she cannot compare to it. Bewley (2002) suggests to exclude such unmotivated betting by an inertia principle: agents never trade to a plan whose expected utility is not strictly higher under every prior than their status quo. The market implications of the Bewley approach in the static setting have been worked out in Rigotti and Shannon (2005).

In this paper, we extend the analysis of Bewley's preferences to a dynamic and infinite–dimensional setting. The first contribution of the paper is the characterization of efficient allocations and equilibria without the restriction of inertia. In dynamic models, the marginal rate of substitution between date 0 and some future date and state of the world determines both the risk–adjusted prior and the subjective interest rate of an agent. In efficient allocations, agents have to agree on both objects. In the multiple prior model with incomplete preferences, it is enough that agents share at least one common model.

We also show that an interior allocation is efficient if and only if for some selection of priors, the allocation is efficient in the corresponding *S–economy* (where the S(avage)–economy has complete preferences and every agent has one single subjective prior). These results imply that the Pareto set is much thicker than for complete preferences and that there are a plethora of equilibria: every interior equilibrium of some S-economy is a Bewley equilibrium, and vice versa. This implies that we have indeterminacy.

We then go on to add the restriction of inertia which as noted already by Rigotti and Shannon (2005) plays the role of a natural equilibrium refinement. Although we do not obtain a characterization of equilibria with inertia, except in the case of no-trade, we nonetheless provide a new, shorter proof for the existence of equilibria with inertia by using a utility in the class of variational preferences that have been axiomatized by Maccheroni, Marinacci, and Rustichini (2006a). If $\mathcal{P}^i$ denotes an agent's set of priors and ω^i her endowment and U^i the utility index, we introduce what one might call variational preferences *anchored*[1] *at* ω^i

$$\min_{P \in \mathcal{P}^i} E^P \left(U^i(x) - U^i(\omega^i) \right) .$$

It is easy to show that an equilibrium in the economy with the above variational preferences is an equilibrium with inertia in the Bewley economy. We establish the existence of such equilibria by standard arguments, Mackey–continuity of the utilities and Bewley's existence theorem for L^∞, see Bewley (1972). In the case of no trade, we are able to establish uniqueness.

Rigotti and Shannon (2005) have examined the implications of Bewley's in general equilibrium. In Section 3, we present here a new and, as we believe, particularly interesting case study. It shows that albeit inertia and pessimism are both plausible reactions to uncertainty on the individual level, they have completely different implications on the market level.

We consider an economy whose individual endowments are subject to both a risky and an uncertain source. To focus on the insurance aspect, we assume that there is no aggregate uncertainty. As is well known, with expected utility and homogenous priors, all risk and uncertainty is fully insured. The same holds true for Gilboa–Schmeidler pessimistic multiple prior preferences if agents share at least one prior (the argument of Billot, Chateauneuf, Gilboa, and Tallon (2000) extends easily to a dynamic setting).

In contrast, with incomplete preferences and inertia, risk and uncertainty get a very different treatment. While risk is completely insured in every

[1] Following in spirit Sagi (2006)

equilibrium, the *full* insurance allocation — where also uncertainty is fully hedged — is not an equilibrium with inertia if ambiguity is sufficiently large. The reason for this is that each agent has some very optimistic prior in his belief set; under this prior, the agent prefers the uncertain endowment over the full insurance allocation because the expected payoff from the uncertain source is quite higher compared to the full insurance plan. The inertia constraint – which requires that the agent trades only if the new consumption plan is preferred to endowment for all priors — then inhibits trade.

As a consequence, in equilibria with inertia, risk is insured and (at least) uncertainty remains. We present a particular type of equilibrium where the market for uncertain contingent claims breaks down completely. Agents remain fully uninsured against any uncertainty.

The paper is organized as follows. The next section sets up the model. Section 2 discusses the dynamic market with incomplete preferences and inertia in general. Section 3 contains our case study. Proofs for Section 2 are collected in the first of two appendices; the other one studies Mackey–continuity of variational preferences anchored at endowment.

2 Allocations and Equilibria in Dynamic Multiple Prior Economies with Incomplete Preferences

This section contains our analysis of dynamic multiple prior economies with incomplete expected utility preferences. After setting up the model, we first characterize efficient allocations before we move on to equilibria with and without inertia. As we explain most of our results in words, we have postponed all proofs to Section A.1.

2.1 Model

We consider a pure exchange economy that consists of I agents that live from time 0 to time T and face uncertainty. For simplicity, there is one good in each state of the world. Agents make contingent consumption plans. Information is described by a filtration $(\mathcal{F}_t)_{t=0,\dots,T}$ on a probability space $(\Omega, \mathcal{F}, P_0)$. The commodity space is given by the set of essentially bounded, adapted processes $\mathcal{X} = L^\infty(\Omega \times \{0, \dots, T\}, \mathcal{A}, P_0 \otimes \zeta)$ and consumption plans are its nonnegative elements in $\mathcal{X}_+ = L^\infty_+(\Omega \times \{0, \dots, T\}, \mathcal{A}, P_0 \otimes \zeta)$. Here, $\mathcal{A}$ is the σ–field on $\Omega \times \{0, \dots, T\}$ generated by all adapted processes and ζ is the uniform probability measure on $\{0, \dots, T\}$. Each agent comes with an endowment $\omega^i = (\omega^i_t)_{t=0,\dots,T} \in \mathcal{X}_+$ that is bounded away from 0. $\omega := \sum_{i=1}^I \omega^i$ is aggregate endowment.

Agents have incomplete expected utility preferences that are induced by a set of priors $\mathcal{P}^i$ on $(\Omega, \mathcal{F}, P_0)$. Agent i prefers consumption plan c to consumption plan d, or $c \succeq^i d$, if and only if for all priors $Q \in \mathcal{P}^i$

$$E^Q U^i(c) \geq E^Q U^i(d)\,,$$

where the intertemporal preferences of agents are described by an additively separable utility function of the form

$$U^i(c) = \sum_{t=0}^{T} u^i(t, c_t)$$

for some continuous function $u^i : \{0, \dots, T\} \times \mathbb{R}_+$ that is strictly increasing, strictly concave and continuously differentiable in its second variable.
The derived strict preference relation $\succ^i$ satisfies $c \succ^i d$ if and only if $c \succeq^i d$ and for some prior $Q \in \mathcal{P}^i$, we have

$$E^Q U^i(c) > E^Q U^i(d)\,.$$

In order to have a short name, we call these preferences *B-preferences* (B for Bewley) and the corresponding economy a B-economy. We frequently compare this economy with incomplete preferences to some economy with heterogeneous priors $Q^i \in \mathcal{P}^i$, but complete preferences. We call that economy the S-economy with priors[2] $Q = (Q^1, \dots, Q^I)$.

As the priors $\mathcal{P}^i$ are defined on the probability space $(\Omega, \mathcal{F}, P_0)$, all priors $Q \in \mathcal{P}^i$ are absolutely continuous to the reference probability[3] P_0. We assume throughout that they are even equivalent to P_0 and satisfy the following property :

[2]We prefer the name *S-economy* (for Savage economy) to the name *risk economy* that has been used elsewhere because we interpret the economy with priors $Q = (Q^1, \dots, Q^I)$ as an economy under uncertainty where agents' preferences conform to Savage's axioms that allow to derive a unique prior Q^i. In a risk economy, the probabilities would be objectively given, and all agents should use the same prior.

[3]The assumption of absolute continuity is discussed in Epstein and Marinacci (2006).

Assumption 2.1 *The family of densities*

$$\mathcal{D} = \left\{ \left. \frac{dP}{dP_0} \right|_{\mathcal{F}_T} \mid P \in \mathcal{Q} \right\}$$

is convex and $\sigma(L^1(\Omega, \mathcal{F}, P_0), L^\infty(\Omega, \mathcal{F}, P_0))$*–compact.*

From the Dunford-Pettis theorem, this assumption is satisfied for closed convex sets for example if the densities in $\mathcal{D}$ are bounded by a P_0–integrable random variable. In particular, the assumption is satisfied whenever the state space Ω is finite and the set of priors is closed and convex.

2.2 Efficient Allocations

We come now to efficient allocations in B–economies. Let us start by fixing the concepts.

An allocation $x \in (\mathcal{X}_+)^I$ is a family of I contingent consumption processes. A full insurance allocation $x \in (\mathcal{X}_+)^I$ is a family of I deterministic consumption processes (that may depend on time, but not on states of the world). The allocation $x = (x^i)_{i=1,\dots,I}$ is feasible if $\sum x^i = \omega$. It is B-efficient if it is feasible and there is no other feasible allocation $y = (y^i)_{i=1,\dots,I}$ such that $y^i \succeq^i x^i$ for all agents $i = 1, \dots, I$ and $y^i \succ^i x^i$ for some i. It is often more convenient to work with the weak notion of B-efficiency. An allocation $x = (x^i)_{i=1,\dots,I}$ is *weakly* B-efficient if it is feasible and there is no other feasible allocation $y = (y^i)_{i=1,\dots,I}$ such that $y^i \succ^i x^i$ for all agents $i = 1, \dots, I$. The following technical result will turn out to be useful. Our concept of strict preference $y \succ^i x$ allows for the possibility that the expected utility of y is the same than that of x under some priors. With our assumptions on utility functions, it is enough to check for strict inequalities here.

Lemma 2.2 *An allocation* $x = (x^i)_{i=1,\dots,I}$ *is* weakly *B-efficient if it is feasible and there is no other feasible allocation* $y = (y^i)_{i=1,\dots,I}$ *such that* $E^Q U^i(y^i) > E^Q U^i(x^i)$ *for all priors* $Q \in \mathcal{P}^i$ *for all agents* $i = 1, \dots, I$*.*

With a compact set of priors and strictly concave Bernoulli utility functions U^i, *interior* weakly B-efficient allocations are also B-efficient. The argument follows the usual lines, with some technical twinkles as we are in an infinite–dimensional context.

Lemma 2.3 *B-efficient allocations are weakly B-efficient. Interior, weakly B-efficient allocations are B-efficient.*

We recall that in our context, an allocation is interior if it is uniformly bounded away from zero. Our main theorem will show that interior allocations are B-efficient if and only if they are S-efficient in some S-economy.

To this end, we start with a general observation which holds true for all sorts of incomplete preferences that are defined by a family of complete preferences. If an allocation is efficient in some economy with complete preferences in the family, then it is efficient in the economy with incomplete preferences.

Lemma 2.4 *If there exist priors $Q^i \in \mathcal{P}^i$ such that x^* is efficient in the S-economy with priors $Q = (Q^1, \ldots, Q^I)$, then x^* is B-efficient .*

We discuss now the marginal rates of substitution and related probability measures (risk–adjusted priors, or, in a finance context, equivalent martingale measures) that support efficient allocations. Fix a prior $Q^i \in \mathcal{P}^i$ for every agent i. We denote by (q_t^i) the density process of Q with respect to P_0. We can rewrite the utility with prior Q^i as a state–dependent expected utility function with respect to P_0:

$$E^{Q^i} U^i(c^i) = E^{Q^i} \sum_{t=0}^{T} u^i(t, c_t) = E^{P_0} \sum_{t=0}^{T} u^i(t, c_t) q_t^i \,.$$

The marginal rate of substitution between date 0 and date t is given by

$$MRS_t^i(c^i, Q^i) = u_c^i(t, c_t^i) q_t^i / u_c^i(0, c_0^i) \,.$$

We denote the set of all (processes of) marginal rates of substitution at some consumption plan c^i by

$$\Psi^i(c^i) = \left\{ MRS^i(c^i, Q^i) | \, Q \in \mathcal{P}^i \right\} \,. \tag{1}$$

As is well known, marginal rates of substitution coincide at interior S-efficient allocations.

Lemma 2.5 *An interior allocation c is efficient in the Savage economy with priors $Q = (Q^1, \ldots, Q^I)$ if and only if the marginal rates of substitution coincide for all agents,*

$$MRS_t^i(c^i, Q^i) = MRS_t^j(c^j, Q^j), \qquad (t = 0, \ldots, T, i, j = 1, \ldots, I) \,.$$

An immediate corollary of the previous two lemmata is the following. When marginal rates coincide for some priors, we have efficiency in the S–economy. But by Lemma 2.4, we then also have B–efficiency.

Corollary 2.6 *Let c be a feasible, interior allocation. If*

$$\bigcap_{i=1}^{I} \Psi^i(c^i) \neq \emptyset,$$

then c is B-efficient.

We begin now a deeper study of the structure of efficient allocations. One can use the marginal rates of substitution to define a pricing probability $Q^i(c^i)$ — the so–called *risk–adjusted probability* — for agent i and an interest rate process r_t^i that defines a price functional supporting efficient allocations. They are defined in such a way that we have

$$E^{Q^i} \sum MRS_t^i(c^i, Q^i) x_t = E^{Q^i(c^i)} \sum \exp(-\sum_{u=1}^{t} r_s^i) x_t \tag{2}$$

for all $x \in \mathcal{X}$.

Lemma 2.7 *Let (c_t^i) be a feasible, interior allocation. Then there exist predictable* individual interest rate processes $(r_t^i)_{t=1,\dots,T,i=1,\dots,I}$ *and strictly positive P_0–martingales (M_t^i) with expectation 1 such that*

$$MRS_t^i(c^i, Q^i) = M_t^i \exp\left(-\sum_{s=1}^{t} r_s^i\right).$$

r^i and M^i are uniquely determined.

Note that the interest rates r^i and martingales M^i depend both on Q^i and c^i, so we write $r^i(Q^i, c^i)$ or $M^i(Q^i, c^i)$ to emphasize this dependence when necessary. The martingales M^i identified in the previous lemma define a probability measure $Q^i(c^i)$ that is equivalent to P_0 and satisfies (2). In the financial tradition, we call these probabilities *risk–adjusted priors*, and denote agent i's set of risk–adjusted priors at consumption c^i by $\Pi^i(c^i)$.

We have now collected the relevant tools to state our main theorem on interior B-efficient allocations. At a B-efficient allocation, agents share a risk–adjusted prior. In contrast to the static framework of Rigotti and Shannon (2005), this condition is only necessary, but not sufficient for B-efficiency. Only if the agents also agree on the interest rate used to discount future consumption, the allocation is B-efficient. We also show that B-efficiency with incomplete preferences is equivalent to having efficiency in some S-economy for some choice of priors $Q^i \in \mathcal{P}^i$.

Theorem 2.8 *The following assertions are equivalent for an interior allocation x:*

1. *x is B-efficient,*
2. *the agents' sets of marginal rates of substitution intersect,*

$$\bigcap_{i=1}^{I} \Psi^i(x^i) \neq \emptyset$$

3. *the agents share a risk–adjusted prior*

$$\bigcap_{i=1}^{I} \Pi^i(x^i) \neq \emptyset$$

and for a common risk–adjusted prior $Q \in \bigcap_{i=1}^{I} \Pi^i(x^i)$ all individual interest rates are equal, i.e.

$$r^i(Q, x^i)_t = r^j(Q, x^j)_t$$

for all $i, j = 1, \ldots, I$ and $t = 0, \ldots, T$,

4. *for some selection of priors $Q^i \in \mathcal{Q}^i, i = 1, \ldots, I$, x is S-efficient in the economy with priors $Q = (Q^1, \ldots, Q^I)$.*

Remark 2.9 *Suppose that instead of considering Bewley's unanimity rule, we consider the Gilboa Schmeidler's max-min rule with utility representation for agent i*

$$\tilde{V}^i(x) = \min_{P \in \mathcal{P}^i} E^P \left(U^i(x)\right)$$

We claim that any efficient allocation $x = (x^i)_{i=1,\ldots,I}$ of the economy with preferences $(\tilde{V}^i)$ is B-efficient. If not, there would exists another feasible allocation $y = (y^i)_{i=1,\ldots,I}$ such that $E^Q U^i(y^i) > E^Q U^i(x^i)$ for all priors $Q \in \mathcal{P}^i$ and all agents $i = 1, \ldots, I$ implying $\tilde{V}^i(y^i) > \tilde{V}^i(x^i)$ for every $i = 1, \ldots, I$, contradicting the $\tilde{V}$–efficiency of x.

As in Rigotti and Shannon (2005), we obtain as a corollary of theorem 2.8.

Corollary 2.10 *Assume that there exists no aggregate uncertainty and that*

$$\bigcap_{i=1}^{I} \mathcal{P}^i \neq \emptyset$$

then any full insurance feasible allocation is efficient.

Remark 2.11 *In the case of no aggregate uncertainty and common prior, any efficient allocation for Gilboa–Schmeidler's max-min rule must be a full insurance allocation. This follows from an easy dynamic extension of Billot, Chateauneuf, Gilboa, and Tallon (2000) or Dana (2002). For Bewley's unanimity rule, full insurance allocations are efficient, but there are lots of other efficient allocations since the equilibria of an expected utility economy with heterogeneous beliefs and no aggregate risk are not full insurance allocations.*

2.3 Equilibria with and without Inertia

We study now equilibrium allocations and prices for B-economies. We show that without further restriction, there are usually infinitely many equilibria as any S-economy equilibrium leads to an equilibrium in the B-economy. We then go on to study equilibria that satisfy Bewley's inertia criterion.

A price for our economy is given by an adapted, integrable, positive process $(p_t) \in L^1(\Omega \times \{0, \ldots, T\}, \mathcal{A}, P_0 \otimes \zeta)$. Let $\mathbb{P}$ be the set of price processes. We denote by

$$p.c := E^{P_0} \sum_{t=0}^{T} p_t c_t$$

the linear functional associated with the price (p_t).

A feasible allocation $x^* \in (\mathcal{X})^I$ and a price $p^* \in \mathbb{P}$ form a B-equilibrium if, for every i, $p^*.x^{*i} = p^*.\omega^i$ and if $x^i \succ x^{*i}$ implies $p^*.x^i > p^*.\omega^i$ (there is no budget feasible consumption plan that strictly dominates x^{*i}). A feasible allocation $x^* \in (\mathcal{X})^I$ and a price $p^* \in \mathbb{P}$ is a B-equilibrium with transfer payments if $x^i \succ x^{*i}$ implies $p^*.x^i > p^*.x^{*i}$ for all i.

Before we start our analysis, let us remark that the first welfare theorem trivially holds true for a B-economy. Indeed let (x^*, p^*) be a B-equilibrium. If x^* is not B-efficient, then from lemma 2.3, there exist y feasible such that $y^i \succ^i x^{*i}$ for all i. But then we must have $p^*.y^i > p^*.x^{*i}$ for all i, contradicting Walras law. A weak form of the second welfare theorem (weak since it requires interior allocations) follows from the next proposition. This proposition also gives also an easy proof of existence of B-equilibria. No abstract general result on existence of equilibria in infinite dimension is used.

Proposition 2.12 *1. Let (x^*, p^*) be an equilibrium for an S-economy with priors $Q^i \in \mathcal{P}^i, i = 1, \ldots, I$. Then (x^*, p^*) is a B-equilibrium. Similarly, any B-equilibrium with transfer payments for an S-economy with priors $Q^i \in \mathcal{P}^i, i = 1, \ldots, I$ is a B-equilibrium with transfer payments.*

2. Any interior B-efficient allocation (x^*) is the allocation of a B-equilibrium with transfer payments.

3. Any interior B-equilibrium (x^*, p^*) is an interior equilibrium for a S-economy with priors $Q^i \in \mathcal{P}^i, i = 1, \ldots, I$.

Let us apply the above result to economies without aggregate uncertainty (where the aggregate endowment is a deterministic process). It is well known that in economies with homogenous priors, agents fully insure in equilibrium. This yields an existence[4] result for full insurance equilibria in our case. If agents have one common prior, full insurance is an equilibrium allocation.

Corollary 2.13 *Assume that there exists no aggregate uncertainty and that agents share at least one prior,*

$$\bigcap_{i=1}^{I} \mathcal{P}^i \neq \emptyset.$$

Then there exists an equilibrium allocation with full insurance.

[4] not a uniqueness result, though, compare Remark 2.11.

It also follows from proposition 2.12 that the set of B-equilibria is monotone with respect to the set of priors: the larger the set of priors, the larger is the set of B-equilibria.

The preceding theorem shows that one usually has a plethora of B-equilibria as every equilibrium of some S-economy is a B-equilibrium. It is plausible hat this leads typically to a continuum of equilibria when the sets of priors are not singletons. Therefore, indeterminacy of equilibrium allocations and prices is the rule, not the exception for B-economies[5].

On the other hand, many of these equilibria lead to consumption plans that the agents cannot compare with their endowment because they have incomplete preferences. In this case, one might well ask why these agents should decide to take these plans in the first place. We are thus led to impose the additional condition of inertia: agents trade only if their new consumption can be compared to their original endowment and is better. An equilibrium (x^*, p^*) satisfies the inertia condition if for all agents i with $x^{*i} \neq \omega^i$, we have $x^{*i} \succ^i \omega^i$.

We provide here a simple proof for existence of equilibria with inertia based on an auxiliary economy with complete static variational preferences as axiomatized in Maccheroni, Marinacci, and Rustichini (2006b). We construct uncertainty–averse preferences such that any equilibrium of the auxiliary economy with those preferences is a B-equilibrium with inertia. The utility function that we use appears here for the first time in the literature. We thus devote a definition to it.

Definition 2.14 *We call a utility function of the form*

$$V^i(x) = \min_{Q \in \mathcal{P}^i} E^Q \left((U^i(x) - U^i(\omega^i)) \right)$$

variational utility anchored at ω^i*.*

An agent of the above type compares the expected gain from moving away from her endowment ω^i to the new consumption plan x^i; her concern is thus more of a relative nature, as documented in many empirical and psychological

[5] A detailed analysis of indeterminacy in the static setting is in Rigotti and Shannon (2005). Their arguments are valid here, too.

findings. She is pessimistic and computes the worst case outcome. Variational utility functions anchored at ω^i belong to the class of variational preferences that generalize the Gilboa–Schmeidler preferences; they have been axiomatized in Maccheroni, Marinacci, and Rustichini (2006a).

In our case, these (complete) preferences are useful because equilibria in economies with such agents are also equilibria *with inertia* in our B–economy.

Theorem 2.15 *Any equilibrium of an economy with variational utilities anchored at endowments* $V^i(x) = \min_{Q \in \mathcal{P}^i} E^Q \left((U^i(x) - U^i(\omega^i)) \right)$ *is a B-equilibrium with inertia. In particular, B-equilibria with inertia exist.*

We call V–equilibrium an equilibrium of the economy with complete preferences (V^i). Inertia is a very strict requirement, and can lead to market breakdown in the sense that the initial endowment is the unique equilibrium allocation. Our above argument leads easily to a characterization of such no trade situations.

Corollary 2.16 (ω^i) *is the unique B-equilibrium allocation with inertia if and only if* (ω^i) *is a no trade V-equilibrium allocation.*

3 Pessimism versus Inertia: A Case Study

Model uncertainty or ambiguity about subjective probabilities naturally lead to multiple prior models. An individual can react to such multiplicity in several plausible ways. The more cautious approach would be to use incomplete preferences as in the present paper; in this case, the agent might use the inertia rule when she is faced with alternatives that she cannot compare. A plausible alternative that has been studied in detail is to perform a worst case analysis leading to the pessimistic multiple prior preferences axiomatized by Gilboa and Schmeidler (1989) in the static, and Epstein and Schneider (2003) in the dynamic context.

While both inertia and pessimism are plausible on the individual level, they lead to quite different outcomes on the market level, as we show now by a case study. Pessimistic agents insure themselves against risk and uncertainty, whereas inertia may leave individuals uninsured against uncertainty.

We consider an economy whose individual endowments are subject to both a risky and an uncertain source. To focus on the insurance aspect, we assume that there is no aggregate uncertainty. As is well known, with expected utility and homogenous priors, all risk and uncertainty is fully insured. The same holds true for Gilboa–Schmeidler pessimistic multiple prior preferences if agents share at least one prior (Billot, Chateauneuf, Gilboa, and Tallon (2000)). In contrast, we show that in the B–economy with sufficiently large ambiguity

- risk is completely insured,
- the full insurance allocation (where also the uncertain part of the endowment is fully insured) is *not* an equilibrium with inertia,
- and there is an equilibrium with inertia where subjective uncertainty is not insured at all.

We now introduce the formal model of our case study. The basic building block is a probability space $(\Omega, \mathcal{F}, P_0)$. Whenever we write the expectation operator E below, we mean the expectation under our reference measure P_0.

On $(\Omega, \mathcal{F}, P_0)$, there are two independent random walks

$$R_t = \sum_{s=1}^{t} \rho_s \quad \text{and} \quad U_t = \sum_{s=1}^{t} \nu_s \tag{3}$$

for $t \geq 1$ and $R_0 = U_0 = 0$. Under our reference measure P_0, the (ρ_t) and (ν_t) are independent and identically distributed with common distribution $F = N(0,1)$, the standard normal distribution.

The information filtration is the natural filtration generated by the two processes R and U, i.e.

$$\mathcal{F}_t = \sigma\left(R_1, \ldots, R_t, U_1, \ldots, U_t\right) .$$

We also introduce the information generated by U alone,

$$\mathcal{F}_t^U = \sigma\left(U_1, \ldots, U_t\right) .$$

Let us assume that there are two agents who use the same class of priors $\mathcal{P}$. The priors $Q \in \mathcal{P}$ are such that all agents agree that the (ρ_t) are standard normal and independent of U.

In contrast, agents are uncertain about the distribution of the ν_t that generate the random walk U. For the sources of uncertainty ν_t, we model the idea that they come from identical experiments, but we use multiple priors. One way to do this in such a way that we can later compare the results with ambiguity–averse decision makers is to use the following model of "independent experiments with identical ambiguity"[6].

The family of priors is defined by their density processes with respect to P_0; they are of the form

$$q_t = \exp\left(\sum_{s=1}^{t}\left(\alpha_s\nu_s - \frac{1}{2}\alpha_s^2\right)\right)$$

for some $\mathcal{F}^U$–predictable process (α_t) with values in the interval $[-\kappa, \kappa]$ for some $\kappa > 0$[7]. Note that α_t depends only on the values of $U_1, \ldots, U_{t-1}$ as it is $\mathcal{F}^U$–predictable. Note also that (ρ_t) are standard normal under all priors with such density processes (as the density depends only on U) and independent of U. It is also worthwhile to remark that the set of densities that we use here is not convex. This is not needed for our current results.

We assume that the two agents have period utilities

$$u^i(t, c) := \exp(-\rho t)v^i(c) := -\exp(-\rho t - c)$$

for some subjective discount rate[8] ρ. Agents thus have constant absolute risk aversion 1.

Let endowments be

$$\omega_t^1 = R_t + U_t \quad \text{and} \quad \omega_t^2 = -\omega_t^1.$$

Aggregate endowment is thus zero[9].

[6]This model is discussed at length in Epstein and Schneider (2003) where it is called "independent and indistinguishably distributed". We use the name coined by Riedel (2009). For Gilboa–Schmeidler preferences, time–consistency is an issue. Indeed, not every convex set of priors leads to time–consistent preferences. Epstein and Schenider have identified the property of rectangularity, or as it has been called elsewhere, stability under pasting, as necessary and sufficient for time–consistency. Our set of priors is rectangular.

[7]Readers familiar with the continuous–time literature recognize here the usual Girsanov transform for a change of measure; our model is thus the direct discrete–time counterpart of κ–ambiguity as in Chen and Epstein (2002).

[8]The case of different discount rates leads to the same results. Here, a (deterministic) trade pattern appears to the different degrees of time preferences. As we are interested in the insurance aspect, we take homogenous discount rates.

[9]We allow here for negative consumption and take the commodity space to be the space of all square–integrable adapted processes. Even though this does not fit into our general framework of the previous subsection, the basic results carry over. We give details below when needed.

Initially, both agents are affected by risk and uncertainty. As there is no aggregate risk (nor uncertainty), one might expect rational agents to insure perfectly in equilibrium. This is indeed the case for Gilboa–Schmeidler preferences; we recall here the result of Billot, Chateauneuf, Gilboa, and Tallon (2000) (see also Chateauneuf, Dana, and Tallon (2000) and Dana (2002)) that translates easily to the dynamic setting.

Lemma 3.1 *In every equilibrium of the Gilboa–Schmeidler economy, there is full insurance.*

Let us now come to the B–economy. We know from our above analysis that every equilibrium of an S–economy for some selection of priors is also an equilibrium of the B–economy. In particular, if we choose the same prior for both agents, we get that the full insurance allocation $x^1 = x^2 = 0$ is an equilibrium of the B–economy. However, it is not an equilibrium with inertia if ambiguity is sufficiently large.

Lemma 3.2 *If ambiguity is sufficiently large, $\kappa > 1$, the full insurance allocation $x^1 = x^2 = 0$ is not an equilibrium allocation with inertia.*

PROOF: We verify that the inertia condition is not satisfied. Recall that for a standard normally distributed random variable X, the Laplace transform is $Ee^{mX} = e^{\frac{1}{2}m^2}$ for $m \in \mathbb{R}$. Consider the prior with density

$$q_t = \exp\left(\kappa U_t - \frac{\kappa^2}{2}t\right) .$$

Expected utility under this prior for agent 1 is then

$$\begin{aligned}
&-\sum_{t=0}^{T} E\exp\left(-R_t - U_t + \kappa U_t - \frac{\kappa^2}{2}t - \rho t\right) \\
&= -\sum_{t=0}^{T} E\exp(-R_t)E\exp((\kappa-1)U_t)\exp(-(\frac{\kappa^2}{2}+\rho)t) \\
&= -\sum_{t=0}^{T} \exp\left(\frac{1}{2}t + \frac{1}{2}(\kappa-1)^2 t - (\frac{\kappa^2}{2}+\rho)t\right) \\
&= -\sum_{t=0}^{T} \exp\left((1-\kappa-\rho)\,t\right) > -\sum_{t=0}^{T} \exp\left(-\rho t\right) .
\end{aligned}$$

Hence, agent 1's expected utility under this prior is higher than for the plan $x^1 = 0$. □

In order to understand this result, think about the inertia condition. Agents move away from their endowment only if they prefer the full insurance allocation under *all* priors. With large ambiguity, they have a very optimistic prior for the uncertain part U in their belief set. Under this prior, the average value of U_t is so high that they would like to keep it; this can be even so high that it offsets their sufferings from keeping the risky part R_t (which they would want to trade in all cases).

Having said this, let us record a fact about insurance of the risky part.

Lemma 3.3 *In every equilibrium (x, p) of the B–economy, risk is fully insured in the sense that the equilibrium allocation x depends only on U not on R.*

PROOF: See the following text. □

The fact that risk is completely insured is due to the fact that agents share a common prior for the distribution of the risky random walk R. Whenever there is a feasible allocation $x = (x^1, x^2)$ that contains R in a nontrivial way, agents can pass to the corresponding (conditional) expectation $y_t^1 = E[x_t^1 | U_1, \ldots, U_t], y_t^2 = E[x_t^2 | U_1, \ldots, U_t]$. Note that we do not use different priors here because the distribution of R is the same under every prior. As there is no aggregate uncertainty, the allocation $y = (y^1, y^2)$ is still feasible, and by risk aversion, both agents are better off.

Theorem 3.4 *If ambiguity is sufficiently large, $\kappa \geq 1$, the B–economy has an equilibrium with inertia (x, p) with allocation*

$$x_t^1 = U_t = -x_t^2$$

and equilibrium price

$$p_t = \exp\left(-(\rho + 1/2)t\right) .$$

PROOF: We first show that the inertia condition is satisfied. For every prior $Q \in \mathcal{P}$ we have by the usual risk aversion (concavity) argument that $E^Q v^i(U_t + R_t) = -E^Q\left[\exp(-U_t) E^Q\left[\exp(-R_t)\right]\right] < -E^Q\left[\exp(-U_t)\right]$ because R_t is independent of U under all priors and normally distributed with mean 0 under all priors $Q \in \mathcal{P}$.

We show next that (x, p) is an equilibrium. As in the previous section, it is enough to show that (x, p) is an equilibrium in some S–economy with priors Q^1 and Q^2.

By definition of x, markets clear. The budget constraint is also satisfied as we have

$$E \sum_{s=1}^{T} p_t \left(\omega_t^1 - x_t^1\right) = \sum_{s=1}^{T} \exp(-(\rho + 1/2)s)\, ER_t = 0\,.$$

By Walras' law, the budget constraint also holds for agent 2.

Now we take the densities $q_t^1 = \exp(U_t - \frac{1}{2}t)$ and $q_t^2 = \exp(-U_t - \frac{1}{2}t)$. Then the marginal rate of substitution between time 0 and time t for agent 1 is

$$\exp(-\rho t - U_t) q_t^1 = \exp(-\rho t - \frac{1}{2}t) = p_t\,.$$

The marginal rate of substitution of agent 2 at x^2 is

$$\exp(-\rho t + U_t) q_t^2 = \exp(-\rho t - \frac{1}{2}t) = p_t\,.$$

Hence, the first order conditions of utility maximization in the S–economy with priors Q^1 and Q^2 are satisfied. We thus have an equilibrium in that S–economy. □

We conclude that the market for insurance of uncertainty can break down if agent use the inertia criterion in combination with incomplete expected utility preferences[10].

4 Conclusion

The two prevalent ways to interpret Knightian uncertainty, Bewley's incomplete expected utility combined with an inertia principle and Gilboa–Schmeidler's pessimistic, worst–case approach, have quite different implications for market equilibria. Inertia combined with a unanimity rule easily leads to no–trade situations. This may partly explain the liquidity crisis that financial markets recently experienced[11].

[10]The equilibrium allocation is not unique, by the way, as one can find other equilibria with inertia where the equilibrium allocation depends in a more complicated way on U. We do not expand on this here.

[11]This point is also made, in a simple static model, in Easley and O'Hara (2010).

From a different, but equally important perspective, our analysis sheds some light on possible regulatory approaches to financial markets. Our analysis supports the claim that the worst–case approach as used for coherent monetary risk measures[12] leads to a better regulation than an approach based on stress–testing — where we interpret a regulation based on stress testing as incomplete preferences plus inertia: a bank is allowed to perform a trade only if it leads to a better outcome under all stress tests. Our case study indicates that such a regulation might easily dry up markets. Of course, more research in this direction is needed.

A Appendix

A.1 Proofs for Section 2

Proof of lemma 2.2

PROOF: Clearly if x is weakly B-efficient, there is no other feasible allocation $y = (y^i)_{i=1,\dots,I}$ such that $E^Q U^i(y^i) > E^Q U^i(x^i)$ for all priors $Q \in \mathcal{P}^i$ for all agents $i = 1, \dots, I$. Conversely, assume that there is no other feasible allocation $y = (y^i)_{i=1,\dots,I}$ such that $E^Q U^i(y^i) > E^Q U^i(x^i)$ for all priors $Q \in \mathcal{P}^i$ for all agents $i = 1, \dots, I$ and there exists another feasible allocation $y = (y^i)_{i=1,\dots,I}$ such that $y^i \succ x^i$ for all agents $i = 1, \dots, I$. Since $y^i \neq x^i$ and U^i is strictly concave and priors are equivalent to P_0, $E^Q U^i(\frac{y^i+x^i}{2}) > E^Q U^i(x^i)$ for all priors $Q \in \mathcal{P}^i$ and agents $i = 1, \dots, I$. As the allocation $\frac{x+y}{2}$ is feasible, we obtain a contradiction to the weak efficiency of x. □

Proof of lemma 2.3

PROOF: Clearly if x is efficient, it is weakly efficient. Conversely, assume that x is an interior weakly efficient allocation and w.l.o.g. assume that there exists a feasible allocation y such that $y^1 \succ x^1$ and $y^i \succeq x^i$, $i \neq 1$. As x^1 is interior, it is uniformly bounded below. By considering $\frac{(y^1+x^1)}{2}$ instead of y^1, we may also assume that y^1 is uniformly bounded below.

For each $Q \in \mathcal{P}$, there exists ε_Q such that $E^Q U^1(y^1 - \varepsilon_Q) > E^Q U^1(x^1)$

[12] Artzner, Delbaen, Eber, and Heath (1999) have introduced coherent risk measures. From a technical point of view, they are equivalent to Gilboa–Schmeidler preferences with linear Bernoulli utility. They were later generalized to convex risk measures by Föllmer and Schied (2002); convex risk measures are a precursor of variational preferences.

as the map $x \to E^Q U^1(x)$ is norm-L^∞–continuous. For a given $\varepsilon > 0$, let

$$\mathcal{V}_\varepsilon = \left\{ Q \in \mathcal{P} \mid E^Q U^1(y^1 - \varepsilon) > E^Q U^1(x^1) \right\}$$

As $Q \to E^Q U^1(y^1 - \varepsilon) - E^Q U^1(x^1)$ is linear and L^1–continuous, $\mathcal{V}_\varepsilon$ is $\sigma(L^1, L^\infty)$ (relatively) open and from the previous argument $\cup_\varepsilon \mathcal{V}_\varepsilon = \mathcal{P}$. Since $\mathcal{P}$ is compact, there exists a finite subcovering of $\mathcal{P}$ by $(\mathcal{V}_{\varepsilon_i})$. Let $\varepsilon = \min_i \varepsilon_i$ and $\varepsilon' = \frac{\varepsilon}{I-1}$. We then have

$$y^1 - \varepsilon \succ x^1 \text{ and } \; y^i + \varepsilon' \succ x^i$$

contradicting the weak efficiency of x. □

Proof of lemma 2.4

PROOF: Suppose that x^* is efficient in the S-economy with priors $Q = (Q^1, \ldots, Q^I)$. From lemmas 2.3 and 2.2, to show that x^* is efficient, it suffices to show that there exists no allocation y such that

$$E^{Q^i} U^i(y^i) > E^{Q^i} U^i(x^i) \text{ for all } i \text{ and all } Q^i \in \mathcal{P}^i \tag{4}$$

This is obvious since 4 contradicts x being efficient in the S-economy with priors Q. □

Proof of corollary 2.6

PROOF: If

$$\bigcap_{i=1}^{I} \Psi^i(c^i) \neq \emptyset,$$

then agents' marginal rates of substitution coincide for some priors $Q^i \in \mathcal{P}^i$. By Lemma 2.5, c is efficient in the Savage economy with priors (Q^i). Lemma 2.4 yields that c is efficient. □

Proof of Lemma 2.7

PROOF: This lemma is a version of the multiplicative Doob decomposition. Let (c^i) be a feasible, interior allocation, take some c^i and write $Z_t = MRS^i_t(c^i, Q^i)$. Note that Z is strictly positive, and bounded, because c^i is bounded away from zero. Moreover, $Z_0 = 1$.

If we have a decomposition $Z_t = M_t \exp\left(-\sum_{s=1}^t r_s\right)$ with a strictly positive martingale M and a predictable process r, then we must have for $t = 0, \ldots, T-1$

$$1 = E^{P_0}\left[\left. \frac{M_{t+1}}{M_t} \right| \mathcal{F}_t \right] = E^{P_0}\left[\left. \frac{Z_{t+1}}{Z_t} \right| \mathcal{F}_t \right] \exp(r_{t+1}),$$

because r_{t+1} is $\mathcal{F}_t$–measurable. So the only possible choice is

$$r_{t+1} = -\log E^{P_0}\left[\left.\frac{Z_{t+1}}{Z_t}\right| \mathcal{F}_t\right]$$

and

$$M_t = Z_t \exp\left(\sum_{s=1}^{t} r_s\right) .$$

A straightforward calculation shows that M is a martingale with $M_0 = Z_0 = 1$. □

Proof of theorem 2.8

PROOF: 3. is equivalent to 2. Let us first show that 3 implies 2. Let $Q \in \bigcap_{i=1}^{I} \Pi^i(c^i)$ and denote by (q_t) the corresponding density process with respect to P_0. Let $r_t = r^i(Q, c^i)_t$ be the common interest rate. Then we have for all i

$$Q = Q^i(c^i)$$

for some $Q^i \in \mathcal{P}^i$ and hence

$$MRS_t^i(c^i, Q^i) = MRS_t^j(c^j, Q^j)$$

or

$$\bigcap_{i=1}^{I} \Psi^i(x^i) \neq \emptyset\,.$$

Conversely if $\bigcap_{i=1}^{I} \Psi^i(x^i) \neq \emptyset$ or equivalently if $MRS_t^i(c^i, Q^i) = MRS_t^j(c^j, Q^j)$ for all $i, j = 1, \ldots, I$ and $t = 0, \ldots, T$ from lemma 2.7, the martingales and interest rates r^i coincide. Hence the agents share a risk–adjusted prior $\bigcap_{i=1}^{I} \Pi^i(x^i) \neq \emptyset$ and $r^i(Q, c^i)_t = r^j(Q, c^j)_t$ for all $i, j = 1, \ldots, I$ and $t = 0, \ldots, T$

2. implies 4. follows from Lemma 2.5.

4. implies 1. follows from Lemma 2.4.

Let us now show that 1. implies 2.

We will work on the product probability space $(S, \mathcal{S}, \nu)$ given by

$$S = \Omega \times \{0, \ldots, T\}, \mathcal{S} = \mathcal{A}, \nu = P_0 \otimes \zeta\,,$$

where we recall that $\mathcal{A}$ is the σ–field generated by all adapted processes and ζ the uniform probability measure on $\{0, \ldots, T\}$.

Take an interior efficient allocation (c^i) and form the sets

$$H^i := \Phi^i(c^i) := \left\{ \left(\frac{MRS_t^i(c^i, Q^i)}{\int_S MRS^i(c^i, Q^i) d\nu} \right)_{t=0,\ldots,T} \mid Q^i \in \mathcal{P}^i \right\}.$$

If we treat the product space S as our basic state space, these sets are the risk–adjusted priors as in Rigotti and Shannon (2005). Note that the ratios are well–defined because (c^i) is an interior allocation. The same argument as in Rigotti and Shannon (2005), Lemma 3, Appendix shows that H^i is convex.

H^i is $\sigma(L^1(S, \mathcal{S}, \nu), L^\infty(S, \mathcal{S}, \nu))$–compact because the marginal utilities are bounded above and below and $\mathcal{P}^i$ is $\sigma(L^1(S, \mathcal{S}, \nu), L^\infty(S, \mathcal{S}, \nu))$–compact.

Now suppose $\bigcap_i H^i = \emptyset$. Samet's Separation Theorem (see Lemma A.1 below for our infinite–dimensional context) then implies that there exist $(g^i)_{i=1}^n \in (L^\infty(S, \mathcal{S}, \nu))^n$ with $\sum_i g^i = 0$ such that $\int_S h^i g^i \, d\nu > 0$ for all $h^i \in H^i$ and all i. Let $d^i = c^i + \lambda g^i$ wth $\lambda > 0$. For λ small enough, the allocation (d^i) is feasible and Pareto–dominates (c^i). Indeed, for any $Q^i \in \mathcal{P}^i$,

$$E^{Q^i}(U_i(d^i) - U_i(c^i)) \geq \lambda \int_S MRS_u^i(d^i, Q^i)\nu(du) \frac{\int_S MRS_u^i(d^i, Q^i) g^i \nu(du)}{\int_S MRS_u^i(d^i, Q^i)\nu(du)} > 0$$

for λ small enough. This is a contradiction to c being efficient. Therefore, we have $\bigcap_i H^i \neq \emptyset$, and we can find priors $Q^i \in \mathcal{P}^i$ such that

$$\frac{MRS_t^i(c^i, Q^i)}{\int_S MRS_u^i(c^i, Q^i)\nu(du)} = \frac{MRS_t^j(c^j, Q^j)}{\int_S MRS_u^j(c^j, Q^j)\nu(du)}$$

for all $i, j = 1, \ldots, I$ and all $t = 0, \ldots, T$ a.s. For $t = 0$, the marginal rates of substitution are 1, so we get

$$\int_S MRS_u^i(c^i, Q^i)\nu(du) = \int_S MRS_u^j(c^j, Q^j)\nu(du)$$

for all agents i, j. This implies

$$MRS_t^i(c^i, Q^i) = MRS_t^j(c^j, Q^j)$$

and hence

$$\bigcap_{i=1}^I \Psi^i(x^i) \neq \emptyset.$$

□

Let us prove here the version of Samet's Theorem for infinite–dimensional spaces that we used above [13].

Lemma A.1 (Samet's Separation Theorem for L^∞) *Let $(S, \mathcal{S}, \nu)$ be a probability space and $(H^i)_{i=1}^n$ be nonempty, convex, and $\sigma(L^1(S,\mathcal{S},\nu), L^\infty(S,\mathcal{S},\nu))$–compact subsets of densities in $L^1_+(S,\mathcal{S},\nu)$. Then $\bigcap_{i=1}^n H^i = \emptyset$ if and only if there exists $(g^i)_{i=1}^n \in (L^\infty(S,\mathcal{S},\nu))^n$ with $\sum_i g^i = 0$ such that $\int_S h^i g^i \, d\nu > 0$ for all $h^i \in H^i$ and all i.*

PROOF: We write L^∞ for $L^\infty(S,\mathcal{S},\nu)$ and L^1 for $L^1(S,\mathcal{S},\nu)$ and Eg for $\int_S g d\nu$. Assume that $\bigcap_{i=1}^n H^i = \emptyset$ and let $H = H^1 \times H^2 \ldots \times H^n$ and $L = \{(h, h, \ldots, h),\ h \in L^1\}$. H is $\sigma(L^1((S,\mathcal{S},\nu)^n), L^\infty((S,\mathcal{S},\nu)^n))$–compact and convex as a product of $\sigma(L^1, L^\infty)$–compact and convex sets and L is a norm–closed vector subspace, hence $\sigma(L^1((S,\mathcal{S},\nu)^n), L^\infty((S,\mathcal{S},\nu)^n))$–closed. From the separation theorem for convex sets, there exists $c \in \mathbb{R}$, $(f^i)_{i=1}^n \in (L^\infty)^n$ such that

$$\sum_i E(f^i h^i) > c \geq E(h \sum_i f^i) \text{ for all } h \in L^1\ h^i \in H^i,\ i = 1, \ldots, n$$

From the right hand side of the inequality, we obtain since L is a subspace that $c \geq 0 = E(h \sum_i f^i)$ for all $h \in L^1$. Hence $\sum_i f^i = 0$ a.e. From the left hand side, one obtains that $\sum_i E(f^i h^i) > 0$ for all $h^i \in H^i$, $i = 1, \ldots, n$. Since $h \to E(f^i h)$ is $\sigma(L^1, L^\infty)$–continuous and H^i is $\sigma(L^1, L^\infty)$ –compact, there exists $\bar{h}^i$ minimizing $E(f^i h^i)$ on H^i. Since $\sum_i E(f^i \bar{h}^i) > 0$ for all i, there exists $(m^i)_{i=1}^n \in \mathbb{R}^n$ such that $\sum_i m^i = 0$ and $E(f^i \bar{h}^i) + m^i > 0$ for all i. Let $g^i = f^i + m^i$. We then have $E(g^i \bar{h}^i) = E(f^i \bar{h}^i) + m^i > 0$ and therefore for all $h^i \in H^i$

$$E(g^i h^i) = E(f^i h^i) + m^i \geq E(f^i \bar{h}^i) + m^i > 0$$

and $\sum_i g^i = \sum_i g^i + \sum_i m^i = 0$ proving one direction of the lemma. The reverse direction is trivially true. □

Proof of Proposition 2.12 PROOF: Let (x^*, p^*) be an equilibrium for a S-economy with priors $Q^i \in \mathcal{P}^i, i = 1, \ldots, I$. Suppose that $y^i \succ^i x^{*i}$ and

[13] This version of Samet's theorem may also be obtained as a corollary of theorem 2 in Billot, Chateauneuf, Gilboa, and Tallon (2000), but the proof given here is more direct and much simpler

$p.y^i \leq p.\omega^i$. As already proven, w.l.o.g. we may assume that $E^{Q^i}U^i(y^i) > E^{Q^i}U^i(x^{*i})$ for any $Q^i \in \mathcal{P}^i$. Hence $p.y^i > p.\omega^i$ contradicting the hypothesis that (x^*, p^*) is an equilibrium for the S-economy with priors (Q^i). The proof for equilibria with transfer payments is similar. To prove the second claim, from theorem 2.8, any interior efficient B-allocation (x^*) is an efficient allocation, hence an equilibrium with transfer payments for some S-economy with priors $Q^i \in \mathcal{P}^i, i = 1, \ldots, I$. From assertion one, (x^*) is a B-equilibrium with transfer payments. To prove the third claim, let (x^*, p^*) be an interior B-equilibrium. By the first welfare theorem, x^* is efficient. Assume that $\lambda p^* \notin H^i$ for any $\lambda \geq 0$ where H^i is defined in the proof of 4 implies 1 of theorem 2.8. Since H^i is $\sigma(L^1((S, \mathcal{S}, \nu)), L^\infty((S, \mathcal{S}, \nu))$–compact and convex and $p^*\lambda$ for $\lambda \geq 0$ is a $\sigma(L^1((S, \mathcal{S}, \nu), L^\infty(S, \mathcal{S}, \nu))$–closed convex cone, from the separation theorem for a convex cloed cone and a convex compact set, there exists $f^i \in L^\infty$ such that

$$p^* \dot{f}^i \leq 0 < \min_{H^i} f^i \dot{h}^i$$

Using again the proof of theorem 2.8, $d^i = x^{*i} + \mu f^i$ with $\mu > 0$ small enough is such that $d^i \succ x^{*i}$ while $p^* \dot{d}^i \leq p^* \dot{x}^{*i}$ contradicting the hypothesis that p^* is a B-equilibrium price. Hence $\frac{p^*}{\int_S p^* d\nu} \in H^i$. Since this is true for each agent, $\cap H^i \neq \emptyset$ and $\frac{p^*}{\int_S p^* d\nu} \in \cap H^i$. From the proof of theorem 2.8, we can find priors $Q^i \in \mathcal{P}^i$ such that

$$p^*_t = aMRS^i_t(c^i, Q^i), \ a > 0$$

for all $i = 1, \ldots, I$ and all $t = 0, \ldots, T$ a.s. which implies that (x^*, p^*) is an equilibrium for the S-economy with priors (Q^i). □

Proof of corollary 2.13

Proof: Let $Q \in \bigcap_{i=1}^{I} \mathcal{P}^i$. Then the S-economy with common prior Q and no aggregate risk has a full insurance equilibria. Corollary 2.13 then follows from assertion 3 of Proposition 2.12 □

Proof of theorem 2.15

Proof: In order to prove existence of an equilibrium with inertia, let

$$V^i(x) = \min_{P \in \mathcal{P}^i} E^P \left(U^i(x) - U^i(\omega^i) \right) .$$

Bn Lemma A.3 in next section, these preferences are Mackey–continuous. Hence, the existence theorem of Bewley (1972) may be applied to get an equilibrium with strictly positive price $(\bar{p}, \bar{x})$ for the economy with complete preferences (V^i) which will be refered to as the V-economy. Let us show that $(\bar{p}, \bar{x})$ is a B-equilibrium. Budget constraints are trivially fulfilled. Let $x^i \succ \bar{x}^i$. W.l.o.g, as we already argued, we may assume that $E^Q U^i(x^i) > E^Q U^i(\bar{x}^i)$ for any $Q \in \mathcal{P}^i$. Hence $V^i(x^i) > V^i(\bar{x}^i)$ and therefore $\bar{p}.x^i > \bar{p}.\bar{x}^i$ proving that $(\bar{p}, \bar{x})$ is a B-equilibrium. Let us now show that any V-equilibrium is a B-equilibrium with inertia. We claim that any V-equilibrium allocation $\bar{x}$ verifies $V^i(\bar{x}^i) > V(\omega^i) = 0$ for any i such that $\bar{x}^i \neq \omega^i$. If not, V^i being strictly concave $V^i((\bar{x}^i + \omega^i)/2) > V(\omega^i) = 0$ while $\bar{p}.(\bar{x}^i + \omega^i)/2 = \bar{p}.\bar{x}^i$ a contradiction to $(\bar{p}, \bar{x})$ being an equilibrium of the V-economy. Therefore $E^Q U^i(\bar{x}^i) > E^Q U^i(\omega^i)$ for any i such that $\bar{x}^i \neq \omega^i$ and any $Q \in \mathcal{P}^i$. Thus $\bar{x}^i \succ^i \omega^i$. Hence any equilibrium in the economy with preferences (V^i) is an equilibrium with inertia. □

Proof of corollary 2.16

Proof: If (ω^i) is an equilibrium allocation of the V-economy, then from theorem 2.15, it is a B-equilibrium allocation and there is no trade. Conversely (ω^i) is a B-equilibrium allocation, then it is B-efficient and since (ω^i) is an interior allocation, $\bigcap_i \Phi^i(\omega^i) \neq \emptyset$ (where Φ^i is defined in the proof of theorem 2.8), hence V- efficient since any $Q \in \mathcal{P}^i$ is a minimizing probability for V at ω^i. Since (ω^i) is a B-equilibrium allocation, there exists a price process $p \in \mathbb{P}$ such that $E^Q(U^i(x)) > E^Q(U^i(\omega^i))$ for all $Q \in P^i$ implies $p.x^i > p.\omega^i$. Hence $V^i(x) > V^i(\omega^i) = 0$ implies $p.x > p.\omega^i$ and therefore (ω^i) is a no-trade V-equilibrium allocation. (ω^i) is the unique B-equilibrium allocation with inertia. Let (x) be another equilibrium with inertia and i be such that $x^i \succ \omega^i$. Let p support (ω^i). Then $p.x^i > p.\omega^i$ contradicting (x) being another equilibrium. □

A.2 Mackey–Continuity of Variational Preferences Anchored at Endowments

Lemma A2 which shows Mackey–continuity of a special variational preference introduced in the paper, extends a proof in Dana (2002). The reader may verify that this proof does not extend to general variational preferences with a lower semi-continuous penalty function. The following estimates for utility increments will be useful for lemma A2.

Lemma A.2 *Let $u : \mathbb{R}_+ \to \mathbb{R}$ be strictly convave increasing, C^1 and verify*

$u(0) = 0$. *Then for any* $\eta > 0,\ x,\ y \in \mathbb{R}_+$, *we have*

$$|u(x) - u(y)| \leq \frac{u(\eta)}{\eta}|x - y| + 2u(\eta) \tag{5}$$

Proof: Let $\eta > 0$ be given. If $x < \eta$ and $y > \eta$,

$$\frac{|u(y) - u(x)|}{|y - x|} = \frac{u(y) - u(x)}{y - x} \leq \frac{u(y)}{y} \leq \frac{u(\eta)}{\eta}$$

If $x < \eta$ and $y < \eta$, $|u(x) - u(y)| \leq 2u(\eta)$. Hence if $x < \eta$, we have

$$|u(x) - u(y)| \leq \frac{u(\eta)}{\eta}|x - y| + 2u(\eta)$$

If $x > \eta$ and $y < \eta$, we similarly have $|u(x) - u(y)| \leq \frac{u(\eta)}{\eta}|x - y|$ while if $x > \eta$ and $y > \eta$, $|u(x) - u(y)| \leq u'(\theta)|x - y|$ for some $\theta \in]x, y[$. Hence

$$|u(x) - u(y)| \leq \frac{u(\eta)}{\eta}|x - y|$$

proving (5) □

Let $\tau(L^\infty, L^1)$ denote the Mackey topology on $L^\infty(\Omega \times \{0, \ldots, T\}, \mathcal{A}, P_0 \otimes \zeta)$.

Lemma A.3 *Let* $U : {\mathbb{R}_+}^{\{0,\ldots,T\}} \to \mathbb{R}$ *be defined by* $U(c) = \sum_{t=0}^{T} \rho^t u(c_t)$ *with* u *strictly increasing, strictly concave and* C^1 *and verify* $u(0) = 0$. *Let* $\mathcal{P}$ *fulfill assumption 2.1. Let* $y \in L^\infty(\Omega \times \{0, \ldots, T\}, \mathcal{A}, P_0 \otimes \zeta)$ *be fixed. Then the utility function* $V : \mathcal{X} \to \mathbb{R}$ *defined by*

$$V(x) = \min_{Q \in \mathcal{P}} E^Q \left(U(x) - U(y)\right)$$

is $\tau(L^\infty, L^1)$*–continuous, strictly concave and monotone.*

Proof: For $Q \in \mathcal{P}$, let $q = \frac{dQ}{dP_0}|_{\mathcal{F}_T}$ denote the time-T density with respect to P_0 and let $\mathcal{D}$ be the set of densities. Let $x_\alpha \xrightarrow{\tau} x$ (or equivalently, let $x_{t_\alpha} \xrightarrow{\tau} x_t$ for all t) and q_α be such that

$$V(x_\alpha) = E^{P_0}[q_\alpha(U(x_\alpha) - U(y))].$$

Such an q_α exists since $\mathcal{D}$ is $\sigma(L^1, L^\infty)$ compact and $(U(x_\alpha) - U(y)) \in L^\infty$. Since $\mathcal{D}$ is $\sigma(L^1, L^\infty)$ compact, we may assume w.l.o.g. that $q_\alpha \xrightarrow{\sigma} q$. Let us show that the restriction to $L^\infty_+(\Omega \times \{0, \dots, T\}, \mathcal{A}, P_0 \otimes \zeta) \times \mathcal{D}$ endowed with $\tau(L^\infty, L^1) \times \sigma(L^1, L^\infty)$ of the map $(x, q) \to E(q(U(x) - U(y)))$ is jointly continuous. Let $x_\alpha \xrightarrow{\tau} x$ and $q_\alpha \xrightarrow{\sigma} q$. Let us prove that $E^{P_0}(q_\alpha(U(x_{t_\alpha}) - U(x) + (q_\alpha - q)(U(x_t) - U(y)) \to 0$. Since $q_\alpha \xrightarrow{\sigma} q$, the second term goes to zero. To study the first term, since $x_{t_\alpha} \xrightarrow{\tau} x_t$ and the Mackey topology is locally solid, $|x_{t_\alpha} - x_t| \xrightarrow{\tau} 0$. From lemma A1, we have for any η

$$E^{P_0}|U(x_{t_\alpha} - U(x_t)|q_\alpha \leq U(\eta)[2 + \frac{1}{\eta} E^{P_0}(|x_{t_\alpha} - x_t|q_\alpha)]$$

For any $\varepsilon > 0$, choose $\eta > 0$ such that $U(\eta) \leq \varepsilon$ and α such that $\sup_{q \in \mathcal{D}} E^{P_0}[|x_{t_\alpha} - x_t|q] < \eta$, then the above integral is smaller then 3ε which proves the claimed joint continuity.
Hence

$$V(x_\alpha) = E^{P_0}[q_\alpha(U(x_\alpha) - U(y))] \to E^{P_0}[q(U(x) - U(y))]$$

By definition of q_α

$$E^{P_0}[q_\alpha(U(x_\alpha) - U(y))] \leq E^{P_0}[q(U(x_\alpha) - U(y))] \text{ for all } q \in \mathcal{D}$$

In the limit, we obtain

$$E^{P_0}[q(U(x) - U(y))] \leq \min_{q \in \mathcal{D}} E^{P_0}[q(U(x) - U(y))] = V(x)$$

hence

$$E^{P_0}[q(U(x) - U(y))] = \min_{q \in \mathcal{D}} E^{P_0}[q(U(x) - U(y))] = V(x)$$

proving the continuity of V with respect to the Mackey topology.
Let us prove that V is strictly monotone. From assumption 2.1., any $q \in \mathcal{D}$ is strictly positive. As U is strictly concave, for any $z \in L^\infty_+$, we have

$$V(x + z) - V(x) \geq E^{P_0}[q_{x+z}(U(x + z) - U(x))] > E^{P_0}[q_{x+z}U'(x + z)z] > 0$$

where q_{x+z} is a minimizing density for $V(x + z)$. Therefore V is strictly monotone.
Finally, let us show the strict concavity of V. Let $\lambda \in]0, 1[$ be given.

$$V(\lambda x+(1-\lambda)z) \geq E^{P_0}[q_{\lambda x+(1-\lambda)z}U(\lambda x+(1-\lambda)z) > \lambda E^{P_0}(q_x U(x)+(1-\lambda)q_z U(z))$$

since U is strictly concave and q_x is a minimizing probability for V at x. □

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