Bilateral Trading and Renegotiation

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Abstract

For a bilateral trade model with a privately informed buyer we characterize trading rules which are implementable via a mechanism with ex post renegotiation. Let $R(v)$ be the seller's ex ante expected payoff if she could commit to a price of v, and let $\tilde{R}$ be the least concave majorant of R. We show that any non-decreasing trading rule can be implemented subject to two constraints. First, on any interval on which $R(v) < \tilde{R}(v)$ the rule must be constant. Second, it must be equal to 1 above p^*, the highest maximizer of R. These rules are implemented by a direct revelation mechanism in which the buyer systematically understates his value.

Keywords: Bilateral Trade, Renegotiation, Mechanism Design

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1 Introduction

Suppose that two or more parties, some of whom have private information, use a mechanism, or contract, to govern their relationship. It may be that, after the mechanism has delivered an outcome, the parties would each, given the information revealed to them by the play of the mechanism, prefer to have a different outcome. That is, the mechanism may not be renegotiation-proof. It has been argued by many authors that renegotiation-proofness is a robustness property that it is desirable for a mechanism to possess. Various notions of renegotiation-proofness have been proposed. In the incomplete information case, much of the literature concerns interim renegotiation, i.e., the parties have an opportunity to renegotiate before they play the mechanism. For example, Holmström and Myerson (1983) define a decision rule (or mechanism) M as *durable* if, given any type profile, and any alternative mechanism $\tilde{M}$, the players would not vote unanimously to replace M by $\tilde{M}$ if a neutral third party were to propose it to them (see also Crawford (1985), Palfrey and Srivastava (1991) and Lagunoff (1995)). Ex post renegotiation has been studied by Green and Laffont (1987), Forges (1994), and Neeman and Pavlov (2007). In these contributions the concepts employed are variations on the principle that a mechanism is (ex post) renegotiation-proof if, for any outcome x of the mechanism and any alternative outcome y, the players would not vote unanimously for y in preference to x if a neutral third party were to propose it to them. Such definitions of renegotiation-proofness have the merit that, if a given mechanism satisfies it, the mechanism is robust against all possible alternative outcomes. However, it also has the drawback that the implied renegotiation process does not have a non-cooperative character. Under an alternative modeling of this process, a renegotiation proposal would be made by one of the parties to the mechanism, or, more generally, the players would play an exogenously given non-cooperative bargaining game after the mechanism is completed.

The latter notion of renegotiation is closer to the one used for the complete information case (Maskin and Moore (1999), Segal and Whinston (2002)), in which, for any inefficient outcome of the mechanism, there is a single renegotiation outcome, which can be predicted by the players. It also corresponds to the approach used in the literature on contract renegotiation (e.g. Dewatripont and Maskin (1990)) in which a trading opportunity is repeated a number of times and the focus is on comparing the outcomes of long-term contracts, sequences of short-term contracts, and long-term contracts which can be renegotiated (i.e., the parties are committed for one period, but in the second period there is an opportunity to change the contract).

In this paper we study mechanism design with renegotiation as modeled in the above fashion, applied to a single period model with incomplete information. Specifically, we consider a bilateral trading model with one-sided asymmetric information (the seller's cost is common knowledge, but the buyer's value is private information). First they play a mechanism in which the buyer sends a message to the seller, resulting in some trade and payment. Then, if some of the good remains unsold, the seller makes a take-it-or-leave-it price demand to the buyer for the remaining stock. Of course, the seller's demand at the renegotiation stage will depend on what she has learned from the mechanism, so we cannot confine attention to equilibria of direct revelation mechanisms in which the buyer tells the truth, since the seller, knowing the truth, would subsequently extract all the remaining surplus, and this in turn would give the buyer an incentive to understate his value.

We provide a characterization of the implementable outcome functions (that is, functions mapping the buyer's type to expected quantity of trade, taking renegotiation into account). Let p^* be the price which the seller would demand in the absence of any mechanism. Then, for any mechanism and any equilibrium, types above p^* must have expected quantity of trade equal to 1 (i.e. the maximum possible). Secondly, for any mechanism, in any pure strategy equilibrium, the outcome function can take at most two values, i.e., low types trade $q \leq 1$ in expectation, while high types trade 1 (note that, as usual, outcome functions must be monotonic).

On the other hand, it is natural to expect randomization since the buyer will want to hide his type from the seller, and it turns out that much more can be achieved using mixed strategy equilibria. Define a function R on the buyer's types by letting $R(p)$ be the seller's expected payoff if she can commit (before learning anything about the buyer's type) to a take-it-or-leave-it price of p. Let $\tilde{R}$ be the least concave function lying above R, i.e. $\tilde{R}$ is the least concave majorant of R. Then on any interval on which $\tilde{R} > R$, bunching must take place. On the other hand, any non-decreasing quantity schedule which satisfies this condition and is equal to 1 above p^* can be achieved.

For any right-continuous quantity schedule Q which satisfies these conditions there is a simple mechanism which implements it. Note first that Q has the properties of a cumulative distribution function of a random variable taking values in the type set. In the mechanism the buyer announces his type. The contracted quantity after message $\tilde{v}$ is simply $Q(\tilde{v})$, and the contracted per-unit price is $E(v|v \leq \tilde{v})$, where the expectation is taken with respect to Q (not the measure which describes the seller's belief about the buyer's type). In equilibrium a type $\tilde{v}$ randomizes over all types up to $\tilde{v}$, so that the seller, given announcement $\tilde{v}$, has a posterior belief distributed over types $\tilde{v}$ and above. The distribution of the seller's renegotiation demands is given by Q conditional on $v \geq \tilde{v}$.

Section 2 sets out the model. Section 3 analyzes a version of the model with finitely many types. Section 4 extends the results for the finite model to the continuum case, and section 5 concludes. The majority of proofs are in the Appendix.

2 The Model

There are two players, a buyer (B) and a seller (S). They may trade up to one unit of a divisible good. The seller's cost of production, $c \geq 0$, is commonly known, but the buyer's value v is privately known to the buyer, and distributed on the interval $V = [\underline{v}, \overline{v}]$, where $c < \underline{v}$, according to a distribution F. Both players are risk-neutral and have quasi-linear utility for money. If S produces and trades a quantity $q \leq 1$ of the good for price p, then S's payoff is $p - cq$ and B's payoff is $vq - p$. The seller's expected profit function is denoted by $R(p) = (p - c)(1 - F(p))$. This is S's expected payoff, given F, if she commits to price p.

The players use a mechanism to determine how much to trade and at what price.

A mechanism is a set of messages M and a pair of functions $q : M \to \wp([0,1])$ and $t : M \to \wp(\Re)$, where, for any set X, $\wp(X)$ is the set of random variables taking values in X. B chooses a message $m \in M$. When message m is sent, $q(m)$ is the contracted quantity to be supplied by S and $t(m)$ is the contracted payment to be paid by B to S. This allows the possibility that quantity and payment are random after a given message.

Renegotiation

We assume, however, that the two parties are not able to commit to the outcome of the mechanism. Specifically, if the outcome of the mechanism gives quantity $q < 1$, it is common knowledge that this outcome is inefficient. In such a situation, we would expect the parties, if they can, to try to negotiate a Pareto-superior outcome. We consider perhaps the simplest bargaining game, in which S makes a single take-it-or-leave it demand. More specifically, the following sequence of events takes place. S observes the outcome of the mechanism (M, q, t) and the message m which B sent. (That is, we restrict attention to direct, as opposed to mediated, communication). If $q = 1$ then the game is over. If $q < 1$ then S demands a new pair (q', t') and B either accepts or rejects it. If B accepts, then they trade q' and B pays t' to S (and they ignore the mechanism outcome). If B rejects, then the mechanism outcome is implemented, i.e. they trade q and B pays t.

We will assume that B accepts the renegotiation demand if he is indifferent. It is easy to see that S must demand $q' = 1$ and that the demand will be accepted by all types $v \geq v'$ for some threshold value v'. Equivalently, we can assume that S produces q and transfers it for payment t (as stipulated by the outcome of the mechanism), but then makes a take-it-or-leave-it unit price demand p for the remaining quantity $1 - q$, which will be accepted by all types $v \geq p$. S's payoff is then $t - cq$ if the demand is rejected and $t + (1 - q)p - c$ if it is accepted, while type v of B gets payoff $vq - t$ if $v < p$ and $v - t - (1 - q)p$ if $v \geq p$. If, as will turn out to be the case, the buyer's type is not fully revealed by the play of the mechanism, then the outcome of the post-mechanism bargaining need not be efficient since the bargaining game is one of incomplete information. This corresponds to a situation in which, whatever mechanism the parties play, there will always be time to renegotiate after it ends, but from that point there is only a finite amount of time in which to reach agreement (hence there may be unrealized gains).

Equilibrium

The mechanism and the post-mechanism bargaining together define a game of incomplete information. In this game buyer type $v \in V$ has strategy μ_v, which is a distribution over M (as noted above, we assume that he accepts a renegotiation demand p if and only if $p \leq v$). A strategy for the buyer is a function $\mu = \{\mu_v\}_{v \in V}$. The seller's strategy is a function $r : M \to \wp(V)$. That is, $r(m)$ is a mixed strategy over renegotiation demands p in V. This assumes that the mixed strategy depends only on the message, not on the realized values of q and t. A strategy profile is therefore a pair (μ, r).

An *equilibrium* of a mechanism (M, q, t) is a perfect Bayesian equilibrium of the game defined by the mechanism and the succeeding bargaining stage. For each $m \in M$, a belief for S is a function $G^m : V \to [0, 1]$, where $G^m(\hat{v})$ is the probability that S attaches, after message m, to the event $v \leq \hat{v}$. An equilibrium then consists of a strategy profile (μ, r) and a system of beliefs $G = \{G^m\}_{m \in M}$ such that (i) where appropriate, the beliefs are updated from F using Bayes' rule given μ, (ii) for each $v \in V$, message strategy μ_v is optimal for buyer type v if the seller uses strategy r, and (iii) for every $m \in M$, renegotiation strategy $r(m)$ is optimal for the seller given belief G^m.

Given message $m \in M$, $v \in V$ and strategy profile (μ, r), let $\bar{q}(v, m, r)$ be the expected quantity traded if S uses strategy r, message m is sent and renegotiation demands less than or equal to v are accepted. That is, letting $I(r(m), v)$ be a random variable which is equal to 1 if $r(m) \leq v$ and equal to zero if $r(m) > v$, $\bar{q}(v, m, r)$ is the expectation of $q(m) + [1 - q(m)]I(r(m), v)$. Let $\bar{q}(v, \mu, r)$ be the expected quantity for type v. So $\bar{q}(v, \mu, r) = E\bar{q}(v, m, r)$, where the expectation is taken over $m \in M$ using μ_v.

We will be interested primarily in the ways that expected trade quantity may vary with the buyer's type.

Definition 1: A function $Q : V \to [0, 1]$ is an *implementable quantity schedule given* F if there exists a mechanism (M, q, t) and an equilibrium (μ, r, G) of the mechanism such that, for all $v \in V$, $Q(v) = \bar{q}(v, \mu, r)$.

The profit-maximizing mechanism for the seller is simply a take-it-or-leave-it price offer p^* (there might be more than one). This offer implements the following quantity schedule: $Q(v) = 0$ for $v < p^*$, and $Q(v) = 1$ for $v \geq p^*$. But there are several reasons why we may be interested in implementing other quantity schedules. Firstly, as in the hold-up literature, there may be a prior investment stage. Suppose, for example, that the buyer first chooses a level of costly unverifiable investment and that higher investment will lead, on average, to a higher value for the buyer. It can be shown that in that case the optimal quantity schedule for the seller (taking into account the need to give investment incentives to the buyer) can be strictly increasing over a range of type values. Secondly, consider a case in which the seller is a division of a bigger firm. The division wants to maximize profits but the headquarters imposes constraints; in particular, a constraint that everybody must be served with some probability, perhaps increasing in willingness-to-pay. It may be, for example, that if consumers are shut out of the market then they are less likely to buy in other markets served by the firm, or in the same market in the future. Alternatively, it may be that there are learning effects: the firm may want to serve low-value customers because they may discover a taste for the good and then become high-value types in the future. If consumers and firm divisions have a one-period perspective but the headquarters takes long-term considerations into account then the headquarters may want to impose an increasing quantity schedule on the division.

3 A Model with Finitely Many Types

In the following we will characterize the implementable quantity schedules in a sequence of finite approximations of the model set out above. We will then use those finite quantity schedules to approximate the implementable quantity schedule in the continuum model.

For $n = 1, 2...$, let $V_n = \{v_0 = \underline{v}, v_1, ..., v_n = \overline{v}\} \subset V$ be a finite type set and F_n be a probability distribution on V_n, i.e. $F_n(v_i) = pr[v < v_i]$. Let the associated probability function be f_n, i.e., $f_n(v_i) = F_n(v_{i+1}) - F_n(v_i)$. We denote by R_n the seller's expected profit function given F_n and by v_{n^*} the highest price demand that maximizes R_n. So, $R_n(v_i) = (v_i - c)(1 - F_n(v_i))$ and $v_{n^*} = \max\{v \in V_n \mid R_n(v) \geq R_n(v') \ \forall \ v' \in V_n\}$.

If renegotiation were not possible, we could apply the Revelation Principle and restrict attention to truthful equilibria of direct revelation mechanisms. We cannot do that in the current model: the buyer will not truthfully reveal his type because the seller would then extract all the surplus at the renegotiation stage.

Fix V_n and F_n and take an arbitrary equilibrium (μ, r, G) of an arbitrary mechanism (M, q, t). (To ease notation, we will refer to a strategy of type v_i as μ_i rather than μ_{v_i}). After any message m renegotiation demands which are not equal to any type's value (i.e. not in V_n) are clearly sub-optimal for S. Therefore, without loss of generality, we assume that $r(m)$ is a distribution on V_n.

Our first Lemma shows that, as in the no-renegotiation case, expected quantity must be monotonically non-decreasing in type.

Lemma 1 (Monotonicity) (i) Given any message m_i which is optimal for v_i and any message m_j which is optimal for $v_j > v_i$, (a) $\bar{q}(v_i, m_i, r) \leq \bar{q}(v_j, m_j, r)$; (b) $\bar{q}(v_i, m_i, r) \leq \bar{q}(v_{j-1}, m_j, r)$.
(ii) If $v_j > v_i$, $\bar{q}(v_j, \mu, r) \geq \bar{q}(v_i, \mu, r)$.

Lemma 1(i)(b) follows from the fact that if type v_j gets a renegotiation offer of v_j he is indifferent between accepting and rejecting, so accepting only offers of v_{j-1} or less is an optimal strategy for him.

Let $\hat{M}$ be the set of messages which have strictly positive probability under μ. Suppose that message $m \in \hat{M}$ has been sent and the resulting quantity is less than 1. An optimal renegotiation demand p for the seller must maximize $(p - c)[1 - G^m(p)]$.

She can obtain a strictly positive value for this by demanding $\underline{v}$, so any v_i which she demands with strictly positive probability must satisfy $1-G^m(v_i) > 0$, i.e. some types v_i or higher must send m. If v_i does not send m in equilibrium then S's posterior probability of v_i, $g^m(v_i)$, is zero, so S would do strictly better by making a higher demand than v_i. This establishes

Lemma 2 *For any $m \in \hat{M}$, and any $v_i \in V_n$, if prob$[r(m) = v_i] > 0$ then $\mu_i(m) > 0$.*

Suppose that, for some v_i and $m \in \hat{M}$, $\bar{q}(v_i, m, r) < 1$ and that no types above v_i send m. But then, by Lemma 2, the renegotiation demand after m must be v_i or less, so v_i will trade for sure if he sends m, which contradicts $\bar{q}(v_i, m, r) < 1$. Hence

Lemma 3 *For $m \in \hat{M}$, if $\bar{q}(v_i, m, r) < 1$ then there exists $k > i$ such that $\mu_k(m) > 0$.*

In equilibrium a given type v_i may randomize over several messages, but each message, taking renegotiation into account, must lead to the same expected trade quantity. To see this, assume that $v_i < v_n$, that v_i sends both m_i and m_i' with strictly positive probability, and that $\bar{q}(v_i, m_i, r) > \bar{q}(v_i, m_i', r)$. (If $v_i = v_n$ then clearly trade quantity equals 1 for sure after any message). In that case, by Lemma 3, there is a higher type who also sends m_i'. Let v_k be the lowest such type. If v_k were to send m_i' and then accept only renegotiation demands *strictly* less than v_k, he would get the same expected quantity as v_i gets after m_i', since, by Lemma 2, there are no renegotiation demands between v_i and v_k after m_i'. That is, $\bar{q}(v_{k-1}, m_i', r) = \bar{q}(v_i, m_i', r)$, so $\bar{q}(v_{k-1}, m_i', r) < \bar{q}(v_i, m_i, r)$, which, since $v_k > v_i$, contradicts monotonicity (Lemma 1(i)(b)). Hence

Lemma 4 *$\bar{q}(v_i, m_i, r) = \bar{q}(v_i, m_i', r)$ for all m_i, m_i' such that $\mu_i(m_i) > 0$ and $\mu_i(m_i') > 0$.*

The next Lemma is the key to characterizing the implementable outcome functions. Suppose that v_i is a point at which the expected quantity function is strictly increasing, i.e. equilibrium expected quantity for v_i is strictly greater than for v_{i-1}. Then, for any type v_j below v_i, and any message m_j which is sent in equilibrium by v_j, S must demand v_i with strictly positive probability after message m_j and, therefore, type v_i must send m_j with positive probability. That is, type v_i must send *all* messages sent by lower types.

To see this, take $v_j < v_i$ and m_j such that $\mu_j(m_j) > 0$ and assume that S does not demand v_i after m_j. Suppose first that, after m_j, S makes no demand above v_{i-1} and let $v_k < v_i$ be the highest value v such that $pr[r(m_j) = v] > 0$. Then, by Lemma 2, $\mu_k(m_j) > 0$, i.e. v_k sends m_j. After sending m_j, v_k will certainly accept the renegotiation demand, so $\bar{q}(v_k, m_j, r) = 1$. However, since type $v_{i-1} \geq v_k$ and v_{i-1} has expected quantity strictly below 1, this either violates monotonicity (Lemma 1) or else violates Lemma 4.

Therefore, S must make some demand above v_{i-1} (and, hence, above v_i) with strictly positive probability after m_j. Let $v_{k'}$ be the lowest such demand, and let v_l be the highest type strictly below v_i which sends m_j. Then, after m_j, there are no renegotiation demands between v_l and $v_{k'}$, so, if $v_{k'}$ sends m_j and accepts renegotiation demands only if strictly below $v_{k'}$ (an optimal strategy for him), his expected quantity is the same as v_l would get, i.e. $\bar{q}(v_{k'-1}, m_j, r) = \bar{q}(v_l, m_j, r)$. But $v_l \leq v_{i-1} < v_i < v_{k'}$ and $\bar{q}(v_{i-1}, \mu, r) < \bar{q}(v_i, \mu, r)$, so monotonicity is violated. Thus, we have

Lemma 5 Suppose $\bar{q}(v_i, \mu, r) > \bar{q}(v_{i-1}, \mu, r)$. Then, for all $j < i$ and all m_j such that $\mu_j(m_j) > 0$, $prob[r(m_j) = v_i] > 0$ and $\mu_i(m_j) > 0$.

Lemma 5 implies that there must be some bunching of types; that is, some types must, taking renegotiation into account, trade the same amount as other types. This is immediate in the case in which the seller uses a pure strategy - in fact, in that case, the types must be grouped into at most two intervals. Suppose that, for some equilibrium (μ, r, G) in which r is pure, the function $\bar{q}(v, \mu, r)$ has a jump at $v_i > \underline{v}$ and another jump at $v_j > v_i$. Then, after any message m sent by $\underline{v}$, Lemma 5 implies that $prob[r(m) = v_i] > 0$ and $prob[r(m) = v_j] > 0$, which contradicts the fact that r

is pure. Therefore, the quantity schedule must be a step function with at most two values: quantity is equal to $\hat{q} \leq 1$ up to some critical type, and equal to 1 for all higher types.

Theorem 1 *In any equilibrium* (μ, r, G) *of any mechanism* (M, q, t) *for the finite type set* V_n *with prior* F_n*, if* r *is a pure strategy then, for some* $\hat{q} \leq 1$ *and* $v_k \leq v_{n^*}$*,* $\bar{q}(v, \mu, r)$ *takes the form* $\bar{q}(v_i, \mu, r) = \hat{q}$ *for* $v_i < v_k$*;* $\bar{q}(v_i, \mu, r) = 1$ *for* $v_i \geq v_k$*.*

Our aim is to characterize the implementable schedules in the case in which S may use a mixed strategy. It turns out that in this case too, bunching must take place. We show below that all types above v_{n^*} (the highest optimal renegotiation demand when the belief is F_n) must trade quantity 1. But for lower types too there may be bunching, depending on the shape of the profit function R_n.

To show this, we take a v_i which is a jump point of the quantity schedule, i.e. $\bar{q}(v_i, \mu, r) > \bar{q}(v_{i-1}, \mu, r)$, and we consider the renegotiation demand decision of S after message m in the support of the strategy of some arbitrary type $v_j < v_i$. By Lemma 5, it is optimal for S to demand v_i rather than v_{i-k}, for any k such that $1 \leq k \leq i$, and it is also optimal for S to demand v_i rather than v_{i+l}, for any l such that $1 \leq l \leq n-i$. Fix such a pair (k, l). Then we obtain the following inequalities.

$$(v_i - c)[1 - G^m(v_i)] \geq (v_{i-k} - c)[1 - G^m(v_{i-k})],$$

and

$$(v_i - c)[1 - G^m(v_i)] \geq (v_{i+l} - c)[1 - G^m(v_{i+l})].$$

Rearranging we get

$$(v_i - v_{i-k})[1 - G^m(v_i)] \geq (v_{i-k} - c)[G^m(v_i) - G^m(v_{i-k})] \tag{1}$$

and

$$(v_i - c)[G^m(v_{i+l}) - G^m(v_i)] \geq (v_{i+l} - v_i)[1 - G^m(v_{i+l})] \tag{2}$$

Using (1) we can write

$$
\begin{aligned}
1 - G^m(v_{i+l}) &= 1 - G^m(v_i) - [G^m(v_{i+l}) - G^m(v_i)] \\
&\geq \left(\frac{v_{i-k} - c}{v_i - v_{i-k}}\right)[G^m(v_i) - G^m(v_{i-k})] - [G^m(v_{i+l}) - G^m(v_i)].
\end{aligned}
$$

Then, replacing $1 - G^m(v_{i+l})$ in (2), we obtain

$$
\begin{aligned}
(v_i - c)[G^m(v_{i+l}) - G^m(v_i)] \geq \; & \frac{(v_{i-k} - c)(v_{i+l} - v_i)}{v_i - v_{i-k}}[G^m(v_i) - G^m(v_{i-k})] \\
& -(v_{i+l} - v_i)[G^m(v_{i+l}) - G^m(v_i)].
\end{aligned}
$$

This expression simplifies to

$$
G^m(v_{i+l}) - G^m(v_i) \geq \frac{(v_{i-k} - c)(v_{i+l} - v_i)}{(v_{i+l} - c)(v_i - v_{i-k})}\left[G^m(v_i) - G^m(v_{i-k})\right],
$$

which, together with Bayes' Rule, implies that

$$
\frac{F_n(v_i) - F_n(v_{i-k})}{F_n(v_{i+l}) - F_n(v_i)} \leq \frac{pr(m|v_i \leq v < v_{i+l})}{pr(m|v_{i-k} \leq v < v_i)} \frac{(v_{i+l} - c)(v_i - v_{i-k})}{(v_{i-k} - c)(v_{i+l} - v_i)}. \tag{3}
$$

Denote by $\Gamma(i-k, i-1)$ the set of all messages sent with strictly positive probability by types v_j, with $i - k \leq j \leq i - 1$. There must exist at least one message (say m') in $\Gamma(i-k, i-1)$ such that

$$
pr(m'|v_i \leq v < v_{i+l}) \leq pr(m'|v_{i-k} \leq v < v_i).
$$

Otherwise, by summing over all messages in $\Gamma(i-k, i-1)$ we obtain

$$
\sum_{m \in \Gamma(i-k,i-1)} pr(m|v_i \leq v < v_{i+l}) > \sum_{m \in \Gamma(i-k,i-1)} pr(m|v_{i-k} \leq v < v_i) = 1,
$$

a contradiction. Hence, since (3) is true for any message in $\Gamma(i-k, i-1)$, and therefore for m', we have

Lemma 6   Take any v_i which is a jump point of $\bar{q}(v, \mu, r)$ for some equilibrium

(μ, r) of some mechanism. Then, for all k, l such that $1 \leq k \leq i, 1 \leq l \leq n-i$,

$$\frac{F_n(v_{i+l}) - F_n(v_i)}{v_{i+l} - v_i} \geq \frac{(v_{i-k} - c)}{(v_{i+l} - c)} \frac{F_n(v_i) - F_n(v_{i-k})}{(v_i - v_{i-k})} \tag{4}$$

or, equivalently,

$$\frac{R_n(v_{i+l}) - R_n(v_i)}{v_{i+l} - v_i} \leq \frac{R_n(v_i) - R_n(v_{i-k})}{v_i - v_{i-k}}. \tag{5}$$

Furthermore, if v_i is an optimal renegotiation demand after each message sent with strictly positive probability by an arbitrary type $v_j < v_i$, then (4) and (5) hold.

Lemma 6 places restrictions on the set of points at which the quantity schedule can jump. Note that if (5) is satisfied for v_i, then the following inequality must be satisfied for all k, l such that $1 \leq k \leq i, 1 \leq l \leq n-i$

$$R_n(v_i) \geq \frac{v_{i+l} - v_i}{v_{i+l} - v_{i-k}} R_n(v_{i-k}) + \frac{v_i - v_{i-k}}{v_{i+l} - v_{i-k}} R_n(v_{i+l}), \tag{6}$$

i.e., for any pair (v', v'') of points in V_n such that $v' < v_i < v''$, the chord between $(v', R_n(v'))$ and $(v'', R_n(v''))$ must lie below the graph of R_n at v_i. It follows that we can characterize the points at which the schedule *cannot* jump in terms of the least concave majorant of R_n, i.e., the least concave function that lies above R_n. Define $C(R_n)$ as the set of all concave functions $f : V_n \to \Re$ such that $f \geq R_n$. Define $\tilde{R}_n$ by $\tilde{R}_n(v_i) := \inf\{f(v_i) \mid f \in C(R_n)\}$. Clearly $\tilde{R}_n \geq R_n$ and it is straightforward to show that $\tilde{R}_n$ is concave, hence the least concave majorant of R_n. The set of values of v_i such that $\tilde{R}_n(v_i) > R_n(v_i)$, if non-empty, consists of a collection of intervals $\{v_{a(k)}, v_{a(k)+1}, ..., v_{b(k)}\}_{k=1}^{k'}$. The next Lemma shows that the quantity schedule must be flat on any such interval.

Lemma 7 (i) If $\tilde{R}_n(v_i) > R_n(v_i)$, then v_i cannot be a jump point of the quantity schedule $\bar{q}(v, \mu, r)$; that is, $\bar{q}(v, \mu, r)$ is constant on each interval $\{v_{a(k)-1}, v_{a(k)}, ..., v_{b(k)}\}$.
(ii) $\bar{q}(v, \mu, r) = 1$ for $v \geq v_{n^*}$.

The idea of the proof that equilibrium quantity equals 1 for types v_{n^*} and higher is the following. Suppose that in some equilibrium (μ, r) of some mechanism the quantity jumps to 1 at $v_k > v_{n^*}$. Then, after any message sent by a type lower than v_k, one optimal demand is v_k, by Lemma 5. Suppose we change the strategies of types v_k and above so that they are all the same (essentially, the 'average' strategy of these types under μ). Then, conditional on $v \geq v_k$, S's belief under the new profile $\tilde{\mu}$ is a truncation of F_n, which implies that v_k is the optimal demand (the least concave majorant $\tilde{R}_n$ must be decreasing above v_{n^*}, and if v_k is a jump point, it must be that $R_n = \tilde{R}_n$ at v_k, so $R_n(v_k) \geq R_n(r)$ for all $r \geq v_k$). v_k remains optimal, under $\tilde{\mu}$, after messages sent by types below v_k, since the change does not affect the probability of acceptance of any demand $r \leq v_k$, while demands $r \geq v_k$ will only be accepted by types v_k and higher and, conditional on these types, v_k is optimal, as just argued. Therefore v_k is optimal regardless of the message sent, hence optimal ex ante. But this contradicts the assumption that v_{n^*} is the maximal ex ante optimal demand, so the quantity schedule must reach 1 at v_{n^*} or lower.

Lemmas 1 and 7 show that the quantity schedule $\overline{q}$ must be weakly increasing and also flat on certain intervals. Can any function satisfying these conditions be implemented? We show that it can.

Let $I_n \subseteq \{1, 2, .., n^* - 1\}$ be the set of indices of types v_i between v_1 and v_{n^*-1} inclusive for which $\tilde{R}_n(v_i) > R_n(v_i)$, i.e., the indices $\{a(k), .., b(k)\}_{k=1}^{k'}$. Take any direct revelation mechanism. Without loss of generality, suppose that the buyer announces the index of his type rather than the type itself, i.e. we take the message set to be $\{0, 1, 2, ..., n\}$ rather than V_n. We show first that, for such a mechanism, it is possible for the agent to randomize over announcements in such a way that, after receiving message $i < n^*$, the seller is indifferent between all renegotiation demands between i and n^* inclusive, excluding those in I_n.

Lemma 8 Given the finite type set V_n with prior F_n, if the message set is $\{0, 1, 2, ..., n\}$, there exists a strategy profile μ and associated belief system such that, after any message $i \leq n^*$, the optimal demands for S are $\{v_j \mid n^* \geq j \geq i, j \notin I_n\}$.

The details of the proof are in the Appendix, but we explain the main ideas here.

For simplicity, take $c = 0$. Suppose first that I_n is empty, so (6) is satisfied for all $v_i < v_{n^*}$. We define strategies in such a way that each type $v_i \leq v_{n^*}$ randomizes over all messages i and lower, and types above v_{n^*} have the same strategy as v_{n^*}. This means that, on receiving message i, S has a belief G^i which is distributed on $\{v_i, v_{i+1}, ..., v_n\}$. We proceed in a recursive fashion. Let $\mu_0(0) = 1$, i.e. the lowest type tells the truth. In order for S to be indifferent, after getting message 0, between demanding v_0 and demanding v_j such that $v_0 < v_j \leq v_{n^*}$, we need

$$G^0(v_j) = \frac{v_j - v_0}{v_j}$$

for $j = 1, 2, .., n^*$, which pins down $g^0(v_j)$ for all $v_j \leq v_{n^*-1}$. Accordingly, using Bayes' Rule:

$$\frac{g^i(v_j)}{g^i(v_k)} = \frac{f_n(v_j)\mu_j(i)}{f_n(v_k)\mu_k(i)},$$

we define $\mu_j(0)$ for $0 < j < n^*$ in such a way that the relative conditional probabilities, $g^0(v_j)/g^0(v_0)$, of these types are as required, and we define $\mu_{n^*}(0)$ (which is also equal to $\mu_j(0)$ for all j higher than n^*) so as to ensure that the total conditional probabilities of types n^* and higher are as required. This defines $\mu_i(0)$ for all i.

Next we define μ_1 by setting $\mu_1(1) = 1 - \mu_1(0)$, so that type 1 randomizes over two messages - his true type and type 0. It turns out that the fact that v_1 satisfies (4) for $k = l = 1$ ensures that $\mu_1(0) \leq \mu_0(0) = 1$, so μ_1 is a well-defined mixed strategy. Define $\mu_i(1)$ for $i > 1$ in an analogous fashion to the above, so that after message i, S is indifferent between all demands $v_i, v_{i+1}, ..., v_{n^*}$. The fact that v_2 satisfies (4) for $k = l = 1$ then implies that $\mu_2(0) \leq \mu_1(0)$ and $\mu_2(1) \leq \mu_1(1)$, so we can define $\mu_2(2) = 1 - \mu_2(0) - \mu_2(1)$. Proceeding in this way, we can define μ_i for all $i < n^*$. Finally, let $\mu_{n^*}(n^*) = 1 - \sum_{i<n^*} \mu_{n^*}(i)$. The fact that $\mu_{n^*}(n^*) \geq 0$ follows, as shown in the Appendix, from the fact that v_{n^*} is an ex ante optimal demand for S.

In the case in which I_n is non-empty, we first consider the associated model in which every type v_i with $i \in I_n$ is dropped and its probability re-assigned to the nearest type v_j below such that $j \notin I_n$. This model satisfies (4) for all $v_i < v_{n^*}$, so the argument just given implies that there exists a strategy profile $\hat{\mu}$ for this model under which, after message $i < n^*$, S is indifferent between all demands v_j for $j \in \{i, i{+}1, .., n^*\}/I_n$. Define μ for the original model so that types in $V_n/\{v_j \mid j \in I_n\}$ play according to $\hat{\mu}$ and, for each $i \in I_n$, type v_i plays strategy $\hat{\mu}_{\tau(i)}$, where $v_{\tau(i)}$ is the nearest type below v_i with $\tau(i) \notin I_n$, i.e. $v_{\tau(i)} := \max_{j \notin I_n}\{v_j \mid v_j < v_i\}$. Then it remains true that, after message $i < n^*$, S is indifferent between all demands v_j for $j \in \{i, i{+}1, .., n^*\}/I_n$ since the probabilities of acceptance of these demands have not altered. Moreover, Lemma 6 implies that demand v_j for $j \in I_n$ cannot be optimal.

Implementable Quantity Schedules

Suppose that we want to implement a quantity schedule Q which satisfies the conditions given in Lemma 7. In particular Q is non-decreasing and takes the value 1 for types v_{n^*} and higher. We show in Theorem 2 that there is a simple mechanism which implements Q. Note first that Q has the properties of a cumulative distribution function of a random variable taking values in V_n. Denote by γ_Q the associated measure on V_n, that is $\gamma_Q(v_i) = Q(v_i) - Q(v_{i-1})$, and let $\gamma_{Q|A}$ denote the measure conditional on $A \subseteq V_n$. In the mechanism the buyer announces his type. The contracted quantity after message v_i is simply $Q(v_i)$, and the contracted per-unit price is $E(v|v \leq v_i)$, where the expectation is taken with respect to the measure γ_Q. In the equilibrium the distribution of the seller's renegotiation demands is $\gamma_{Q|\{v>v_i\}}$.

Theorem 2 *Take a finite distribution F_n on V_n. A function $Q : V_n \to [0, 1]$ is an implementable quantity schedule for F_n if and only if it is non-decreasing, constant on each interval $\{v_{a(k)-1}, v_{a(k)}, ..., v_{b(k)}\}$ and equal to 1 for $v_i \geq v_{n^*}$.*

Proof The 'only if' part has been established by Lemma 1 and Lemma 7. Suppose first that I_n is empty, so that (4) is satisfied by all v_i with $i < n^*$. Take a function Q with the stated properties. We define a mechanism (M_n, q, t) which implements it. Let $M_n = \{0, 1, , 2, .., n\}$ (this is equivalent to a revelation mechanism). For all

$i \in M_n$, let $q(i) = Q(v_i)$ and let

$$t(i) = \sum_{j=0}^{i} \gamma_Q(v_j) v_j,$$

where $\gamma_Q(v_0) = Q(v_0)$. This defines the mechanism.

Let the buyer's strategy μ and associated beliefs G^i be as given by Lemma 8. Define the seller's strategy r as follows. For any $i \in M_n$ such that $i < n^*$,

$$prob[r(i) \leq v_i] = prob[r(i) > v_{n^*}] = 0$$

and, for $i < j \leq n^*$,

$$prob[r(i) = v_j] = \frac{\gamma_Q(v_j)}{1 - Q(v_i)}.$$

For $i \geq n^*$, $r(i)$ can be arbitrary since renegotiation is unnecessary after such messages.

To check that the strategy profile (μ, r) gives expected quantity $Q(v_j)$ for each type v_j, note that if type $v_j \leq v_n$ sends message $i \leq j$ her expected volume of trade is

$$\begin{aligned} & q(i) + [1 - q(i)] prob[r(i) \leq v_j] \\ = {} & Q(v_i) + [1 - Q(v_i)] \frac{\gamma_Q(v_{i+1}) + \gamma_Q(v_{i+2}) + ... + \gamma_Q(v_j)}{(1 - Q(v_i))} \\ = {} & Q(v_j). \end{aligned}$$

By Lemma 8, r is optimal given μ, since S, after message i, is indifferent between all demands in $\{v_i, v_{i+1}, .., v_{n^*}\}$. It remains to show that μ_j is optimal for each type v_j given r. Suppose that $v_j < v_{n^*}$ sends message $i > j$. Then her expected payoff is $v_j Q(v_i) - t(i)$. If she sends message j her payoff is $v_j Q(v_j) - t(j)$. The latter payoff is higher if $t(i) - t(j) \geq v_j(Q(v_i) - Q(v_j))$, i.e., if

$$\sum_{l=j+1}^{i} \gamma_Q(v_l) v_l \geq \sum_{l=j+1}^{i} \gamma_Q(v_l) v_j,$$

which is true since $v_l \geq v_j$ for all $l \geq j$. Therefore sending a message higher than one's type cannot be a profitable deviation. To show that, for any $j \in M_n$ with $j \leq n^*$, type v_j is indifferent between all messages in $\{0, 1, 2, ..., j\}$, note that all these messages give the same expected volume of trade for v_j. Therefore we only need to check that the expected payments are the same. The expected payment given message i is

$$t(i) + [1 - Q(v_i)](prob[r(i) = v_{i+1}]v_{i+1} + ... + prob[r(i) = v_j]v_j)$$

$$= t(i) + \gamma_Q(v_{i+1})v_{i+1} + ... + \gamma_Q(v_j)v_j = t(j)$$

This shows that v_j is indifferent between j and all lower messages. Finally, any type $v_j > v_{n^*}$ gets the same payment and trade volume from any message m as type v_{n^*} does (since $r(m) \leq v_{n^*}$ for all $m \in M_n$), so these types are also indifferent between all messages. Hence μ is an optimal strategy.

Suppose now that I_n is non-empty. Take the model in which types v_i with $i \in I_n$ have been dropped and their probability assigned to $v_{\tau(i)} = \max_{j \notin I_n}\{v_j \mid v_j < v_i\}$. The argument above shows that there is a mechanism $(M_n = \{0, 1, .., n^*\}/I_n, q, t)$ and strategy profile (μ, r) which implement Q on the type space $V_n/\{v_j \mid j \in I_n\}$. For the original model (V_n, F_n), take the same mechanism, the same r, and extend μ so that for any $i \in I_n$, $\mu_i = \mu_{\underline{i}}$. Then, as argued in the proof of Lemma 8, r remains optimal. By the argument above, types v_i with $i \notin I_n$ are playing optimally and get expected quantity $Q(v_i)$. Furthermore, for any type v_i with $i \in I_n$ expected quantity is the same for all messages $m \leq \tau(i)$, as is expected payment. Therefore $\mu_{\tau(i)}$ is optimal for v_i and v_i gets expected quantity $Q(v_{\tau(i)}) = Q(v_i)$. This completes the proof of the Theorem.

4 Analysis of the Continuum Model

Now we are in a position to examine the continuum case (V, F), where $V = [\underline{v}, \overline{v}]$ and F is a distribution on V with continuously differentiable density function f. Define by $\tilde{R}$ the least concave majorant of R. Set $I = \{v \in V \mid R(v) < \tilde{R}(v)\}$. I, if non-empty, is then a union of disjoint open intervals $\{(a_k, b_k)\}_k$. We assume that there is a finite number of these intervals. Let p^* be the supremum of the set of maximizers of R. Define a sequence of finite type spaces $V_n = \{v_0^n, v_1^n, ..., p^*, ..., v_{2^n}^n\}$ for $n = 1, 2, ...$ by setting $v_i^n = \underline{v} + \frac{i}{2^n}(\overline{v} - \underline{v})$ for all $i = 0, 1, ..., 2^n$. Then, $V_n \subset V_{n+1}$ for all n. Let the distance between two functions f and g on V_n be given by the supremum norm, i.e. $|f - g| = \sup_{v \in V_n} |f(v) - g(v)|$.

Definition 2 *Given (V, F), a quantity schedule Q on V is defined to be finitely implementable if there exist, for each n, a distribution F_n on V_n , a mechanism (M_n, q_n, t_n), with M_n finite, and a quantity schedule $Q_n : V_n \to [0, 1]$ such that*

(i) *F_n converges to F:* $\lim_n |F_n - F| = 0$,

(ii) *Q_n is implementable by (M_n, q_n, t_n) given F_n,*

(iii) *Q_n converges to Q:* $\lim_n |Q_n - Q| = 0$.

The following theorem is the analogue of Theorem 2 for the continuum model:

Theorem 3 *Take a distribution F on V. A right-continuous function $Q : V \to [0, 1]$ is a finitely implementable quantity schedule for F if and only if it is non-decreasing, constant on each interval (a_k, b_k) and equal to 1 for $v \geq p^*$*

Proof The "if" part of the statement follows from Theorem 2 by setting, for each n, $F_n(v_i^n) = F(v_i^n)$ and $Q_n(v_i^n) = Q(v_i^n)$ for $v_i^n \in V_n$. First, note that for each n, F_n is a well defined distribution on V_n. Also, for all $v \in V_n$, $R_n(v) = R(v)$. Parts (i) and (iii) of the definition are trivially satisfied. By Theorem 2, Q_n is implementable if it is non-decreasing, constant on each interval $\{v_{a(k)}^n, v_{a(k)+1}^n, ..., v_{b(k)}^n\}$ in $I_n = \{v \in V_n \mid R_n(v) < \tilde{R}_n(v)\}$ and equal to 1 above v_n^*. By definition, Q_n is non-decreasing. Since, by construction, $p^* \in V_n$, $v_n^* = p^*$ and so Q_n is equal to 1 above v_n^*. Take $v \in \{v_{a(k)}^n, v_{a(k)+1}^n, ..., v_{b(k)}^n\}$. We show below that $\tilde{R}(v) > R(v)$. V can be partitioned into a finite number of intervals, namely $\{(a_k, b_k)\}_k$, on each of which $\tilde{R} > R$, and a finite number of complementary intervals, on each of which $\tilde{R} = R$.

Take n sufficiently large that V_n contains elements from each of these intervals. Then since $\tilde{R} > R$ on the interval $\{v^n_{a(k)}, v^n_{a(k)+1}, ..., v^n_{b(k)}\}$, $\{v^n_{a(k)}, v^n_{a(k)+1}, ..., v^n_{b(k)}\} \subseteq (a_k, b_k)$ for some k. Therefore Q, and hence Q_n, is constant on this interval. To show that $\tilde{R}(v) > R(v)$, note that, since $v \in \{v^n_{a(k)}, v^n_{a(k)+1}, ..., v^n_{b(k)}\}$, $\tilde{R}_n(v) > R_n(v) = R(v)$. Also, there exist $v', v'' \in V_n$ and $\lambda \in (0,1)$ such that $v = \lambda v' + (1-\lambda)v''$ and

$$\tilde{R}_n(v) = \lambda R_n(v') + (1-\lambda)R_n(v'').$$

Hence, if $\tilde{R}_n(v) > \tilde{R}(v)$,

$$\tilde{R}(v) < \lambda R_n(v') + (1-\lambda)R_n(v'')$$

$$= \lambda R(v') + (1-\lambda)R(v'')$$

$$\leq \lambda \tilde{R}(v') + (1-\lambda)\tilde{R}(v''),$$

which contradicts concavity of $\tilde{R}$. Therefore $\tilde{R}(v) \geq \tilde{R}_n(v) > R_n(v) = R(v)$ and so $\tilde{R}(v) > R(v)$.

For the "only if" part assume that Q is finitely implementable, that is, for each n, there exist a distribution F_n on V_n, a mechanism (M_n, q_n, t_n), and a quantity schedule Q_n on V_n such that conditions (i), (ii) and (iii) of Definition 2 hold.

(a) Q *non-decreasing.* Assume that there exist $a, b \in V$ with $a < b$ and $Q(a) > Q(b)$. Since Q is a right-continuous function we can assume, without loss of generality, that $a, b \in \bigcup_n V_n$. Then, from Theorem 2, $\exists \bar{n}$, such that $a, b \in V_n$ and $Q_n(a) \leq Q_n(b)$ for all $n \geq \bar{n}$. Then (iii) is in contradiction with $Q(a) > Q(b)$, so Q must be non-decreasing.

(b) Q *constant on each interval* $(a_k, b_k) \in I$. Suppose not. Then there exist $v', v'' \in (a_k, b_k)$ such that $v'' > v'$ and $Q(v'') > Q(v')$. For $v \in (a_k, b_k)$, let $L(v) = \lambda R(a_k) + (1-\lambda)R(b_k)$, where $\lambda \in (0,1)$ is defined by $v = \lambda a_k + (1-\lambda)b_k$. (So $L(v) = \tilde{R}(v)$). Then there exists $\varepsilon > 0$ such that $R(v) < L(v) - \varepsilon$ for all $v \in [v', v'']$. This follows because R is upper semicontinuous, hence achieves a maximum on $[v', v'']$, but $R(v) < \tilde{R}(v) = L(v)$ on $[v', v'']$.

Since R is left-continuous and upper semicontinuous and Q is right-continuous, we can assume without loss of generality that $\{a_k, b_k, v', v''\} \subset \bigcup_n V_n$ (if necessary, take points just below a_k and b_k respectively and just above v' and v'' respectively - for simplicity we keep the same notation). Therefore there exists $\bar{n}$ such that for all $n \geq \bar{n}$, $Q_n(v'') > Q_n(v')$ and $R_n(v) < L_n(v)$ for all $v \in [v', v''] \bigcap V_n$, where $L_n(v) = \lambda R_n(a_k) + (1-\lambda) R_n(b_k)$. It follows that for all $v \in [v', v''] \bigcap V_n$, $\tilde{R}_n(v) > R_n(v)$, so that there are no jumps of Q_n in this set. Hence $Q_n(v'') = Q_n(v')$, which gives a contradiction. Therefore Q is constant on (a_k, b_k).

(c) *Q equals 1 for $v \geq p^*$.* Suppose there exists $a > p^*$ such that $Q(a) < 1$. W.l.o.g (by right-continuity) take $a \in \bigcup_n V_n$. Then there exists $\bar{n}$ such that $Q_n(a) < 1$ for all $n \geq \bar{n}$, so $v_n^* \geq a$ for all $n \geq \bar{n}$. Hence $max_{\bar{v} \geq v_n \geq a} R_n(v_n) \geq R_n(p^*)$ for all $n \geq \bar{n}$. Taking limits as $n \to \infty$, $sup_{\bar{v} \geq v \geq a} R(v) \geq R(p^*)$. R is upper semicontinuous, so achieves a maximum in $[a, \bar{v}]$, which implies that $R(v) \geq R(p^*)$ for some $v \geq a > p^*$, which contradicts the definition of p^*. This shows that Q equals 1 for $v \geq p^*$ and completes the proof of the Theorem.

5 Conclusion

In this paper we analyzed the impact of ex-post renegotiation on the set of implementable outcomes in a bilateral trade problem. With full commitment, any increasing trading rule can be implemented via a direct revelation mechanism that is designed to elicit the truth from privately informed parties. Without commitment the set of implementable trading rules is restricted as a direct revelation mechanism cannot fully extract all information from parties. Earlier papers on mechanism design with renegotiation have shown that informed parties must use mixed strategies at the revelation stage. In this paper we have shown that parties can gain from designing the mechanism in such a way that the uninformed party also uses a mixed strategy. Namely, if the uniformed party is restricted to the use of pure strategies, only very simple trading rules prescribing a low level of trade for low types and the efficient quantity for high types can be implemented. In contrast, if the uninformed party is allowed to use a mixed strategy more general trading rules can be achieved. Nevertheless, some bunching of types must always occur.

6 Appendix

Proof of Lemma 1. Let $\bar{t}(v, m, r)$ be the expected amount paid, including renegotiation price, if S uses strategy r, message m is sent, and renegotiation demands less than or equal to v are accepted. That is, $\bar{t}(v, m, r)$ is the expectation of $t(m) + [1 - q(m)]r(m)I(r(m), v)$. Let $\bar{t}(v, \mu, r)$ be the expected amount paid by type v, i.e. $\bar{t}(v, \mu, r) = E\bar{t}(v, m, r)$.

(i) If v_i sends m_i and accepts renegotiation demands less than or equal to v_i (which is an optimal strategy for v_i), v_i gets

$$v_i\bar{q}(v_i, m_i, r) - \bar{t}(v_i, m_i, r).$$

If v_i sends m_j and accepts renegotiation demands less than or equal to v_j, v_i gets

$$v_i\bar{q}(v_j, m_j, r) - \bar{t}(v_j, m_j, r).$$

So

$$v_i\bar{q}(v_i, m_i, r) - \bar{t}(v_i, m_i, r) \geq v_i\bar{q}(v_j, m_j, r) - \bar{t}(v_j, m_j, r).$$

Similarly

$$v_j\bar{q}(v_j, m_j, r) - \bar{t}(v_j, m_j, r) \geq v_j\bar{q}(v_i, m_i, r) - \bar{t}(v_i, m_i, r).$$

Hence $\bar{q}(v_i, m_i, r) \leq \bar{q}(v_j, m_j, r)$. This proves (a).

Similarly, since it is also optimal for v_j to send m_j and accept renegotiation demands less than or equal to v_{j-1}, (b) follows.

(ii) follows from (a) since $\bar{q}(v_j, \mu, r) = E\bar{q}(v_j, m, r)$.

Proof of Lemma 7. We first show that $\tilde{R}_n$ is concave. Suppose not. Then, for some $v_k < v_i < v_l$,

$$\tilde{R}_n(v_i) < \frac{v_l - v_i}{v_l - v_k}\tilde{R}_n(v_k) + \frac{v_i - v_k}{v_l - v_k}\tilde{R}_n(v_l).$$

Hence, by the definition of $\tilde{R}_n$, for some concave $f \geq R_n$,

$$f(v_i) < \frac{v_l - v_i}{v_l - v_k}\tilde{R}_n(v_k) + \frac{v_i - v_k}{v_l - v_k}\tilde{R}_n(v_l).$$

But $\tilde{R}_n(v_k) \leq f(v_k)$ and $\tilde{R}_n(v_l) \leq f(v_l)$, so

$$f(v_i) < \frac{v_l - v_i}{v_l - v_k}f(v_k) + \frac{v_i - v_k}{v_l - v_k}f(v_l),$$

which contradicts the concavity of f. Therefore $\tilde{R}_n$ is concave.

Set $v' = \max\{v < v_i \mid \tilde{R}_n(v) = R_n(v)\}$ and $v'' = \min\{v > v_i \mid \tilde{R}_n(v) = R_n(v)\}$. Then

$$\tilde{R}_n(v_i) = \frac{v'' - v_i}{v'' - v'}\tilde{R}_n(v') + \frac{v_i - v'}{v'' - v'}\tilde{R}_n(v''). \tag{7}$$

This follows because, firstly, if $\tilde{R}_n(v_i)$ were strictly smaller than the right-hand side of (7), this would contradict the concavity of $\tilde{R}_n$. If, on the other hand, $\tilde{R}_n(v_i)$ were strictly larger, then there would exist v_j such that $v'' < v_j < v'$ and such that

$$\frac{\tilde{R}_n(v_j) - \tilde{R}_n(v_{j-1})}{v_j - v_{j-1}} > \frac{\tilde{R}_n(v_{j+1}) - \tilde{R}_n(v_j)}{v_{j+1} - v_j}.$$

Let $\tilde{R}_n^{\varepsilon}(v) = \tilde{R}_n(v)$ if $v \neq v_j$ and let $\tilde{R}_n^{\varepsilon}(v_j) = \tilde{R}_n(v_j) - \varepsilon$. Since $\tilde{R}_n(v_j) > R_n(v_j)$, for small enough $\varepsilon > 0$, $\tilde{R}^{\epsilon} \geq \tilde{R}_n$ and $\tilde{R}^{\epsilon}$ is concave, contradicting the assumption that $\tilde{R}_n$ is the least concave majorant of R_n.

Since $\tilde{R}_n(v_i) > R_n(v_i)$, $\tilde{R}_n(v') = R_n(v')$ and $\tilde{R}_n(v'') = R_n(v'')$, (7) implies that

$$R_n(v_i) < \frac{v'' - v_i}{v'' - v'}R_n(v') + \frac{v_i - v'}{v'' - v'}R_n(v'').$$

Hence (6) is violated and v_i cannot be a jump point. This proves (i).

It remains to show that $\bar{q}(v, \mu, r) = 1$ for $v \geq v_{n^*}$. Let $(M_n, 0, 0)$ be the mechanism with message set M_n and $q(m) = t(m) = 0$ for all $m \in M_n$.

Claim 1 Suppose that, in some equilibrium (μ, r) of some mechanism (M_n, q, t), the function $\bar{q}(v, \mu, r)$ jumps to 1 at $v_k > v_{n^*}$, i.e., $1 = \bar{q}(v_k, \mu, r) > \bar{q}(v_{k-1}, \mu, r)$. In that case there is a message set $\tilde{M_n}$ and strategy profile $(\tilde{\mu}, \tilde{r})$ such that (a) $\tilde{r}$ is a best response to $\tilde{\mu}$ in $(\tilde{M_n}, 0, 0)$; and (b) $prob[\tilde{r}(m) = v_k] > 0$ for all $m \in \tilde{M_n}$.

Proof of Claim 1 First, note that r is optimal in $(M_n, 0, 0)$ given μ, since the seller's updated beliefs would be the same as in (M_n, q, t), and the optimal choice of renegotiation demand is not affected by the quantity at stake, or price already paid.

Second, denote by $\Gamma(i)$ the set of all messages in M_n sent with strictly positive probability by type v_i given μ (recall that $\Gamma(j, l)$ is the set of all messages sent with strictly positive probability by types between v_j and v_l). Then, since $\bar{q}(v, \mu, r)$ jumps to 1 at v_k, for all $i < k$, $\bar{q}(v_i, \mu, r) < 1$. By Lemma 4, $\bar{q}(v_i, \mu, r) = \bar{q}(v_i, m, r)$ for all $m \in \Gamma(i)$, which implies $q(m) < 1$ for all $m \in \Gamma(0, k-1)$. By Lemma 5, $\Gamma(0, k-1) \subseteq \Gamma(k)$ and so for all $m \in \Gamma(0, k-1)$ we must have $r(m) \leq v_k$.

Let $\tilde{M_n} = \Gamma(0, k-1) \bigcup \hat{m}$, where $\hat{m}$ is a message not in M_n. Let $\tilde{r}(m) = r(m)$ for all $m \in \Gamma(0, k-1)$ and let $\tilde{r}(\hat{m}) = v_k$ with probability 1.

For any $i < k$, let $\tilde{\mu}_i = \mu_i$. For any $i \geq k$, define $\tilde{\mu}_i$ as follows. For any message $m \in \Gamma(0, k-1)$, let

$$\tilde{\mu}_i(m) = \frac{\sum_{j \geq k} \mu_j(m) f_n(v_j)}{\sum_{j \geq k} f_n(v_j)},$$

and let

$$\tilde{\mu}_i(\hat{m}) = 1 - \sum_{m \in \Gamma(0,k-1)} \tilde{\mu}_i(m).$$

This is a well-defined mixed strategy if $\sum_{m \in \Gamma(0,k-1)} \tilde{\mu}_i(m) \leq 1$, i.e. if

$$\sum_{j \geq k} \sum_{m \in \Gamma(0,k-1)} \mu_j(m) f_n(v_j) \leq \sum_{j \geq k} f_n(v_j),$$

which is true since $\sum_{m \in \Gamma(0,k-1)} \mu_j(m) \leq 1$.

We need to show that $\tilde{r}$ is optimal for the seller given $\tilde{\mu}$. Let

$$\tilde{G}^m(v_i) = prob[v < v_i | \tilde{\mu}, m]$$

be the seller's belief given m if the buyer's strategy is $\tilde{\mu}$. Thus,

$$G^m(v_i) = \frac{\sum_{j<i} \mu_j(m) f_n(v_j)}{\sum_{j<i} \mu_j(m) f_n(v_j) + \sum_{j \geq i} \mu_j(m) f_n(v_j)}$$

and $\tilde{G}^m(v_i)$ is given by the same expression, with $\tilde{\mu}$ replacing μ.

Among possible demands $r \geq v_k$, the optimal demands must solve

$$\max_r (r - c)[1 - \tilde{G}^m(r | v \geq v_k)],$$

where $\tilde{G}^m(.|v \geq v_k)$ is the buyer's belief about v, given $\tilde{\mu}$ and m, conditional on $v \geq v_k$ (since only such types would accept any such demand).

Moreover, for $i \geq k, j \geq k$, $\tilde{\mu}_i = \tilde{\mu}_j$, so, conditional on $v \geq v_k$, the seller's belief over the buyer's types is given by

$$\tilde{G}^m(v_i | v \geq k) = \frac{\sum_{k \leq j<i} \tilde{\mu}_j(m) f_n(v_j)}{\sum_{k \leq j<i} \tilde{\mu}_j(m) f_n(v_j) + \sum_{j \geq i} \tilde{\mu}_j(m) f_n(v_j)}$$

$$= \frac{F_n(v_i) - F_n(v_k)}{1 - F_n(v_k)}.$$

Hence, among $r \geq v_k$, the seller maximizes

$$\frac{(r - c)[1 - F_n(r)]}{1 - F_n(v_k)} := R_n(r | v_k).$$

We next show that $R_n(v_k) \geq R_n(r)$ (and so $R_n(v_k | v_k) \geq R_n(r | v_k)$) for all $r \geq v_k$. To see this, we show that $\tilde{R}_n(v_k) \geq \tilde{R}_n(r)$ for all $r \geq v_k$. The claim then follows from $\tilde{R}_n(v_k) = R_n(v_k)$ (because v_k is a jump point) and $\tilde{R}_n(r) \geq R_n(r)$ for all r. Assume

$\tilde{R}_n(v_k) < \tilde{R}_n(v_j)$ for some $v_j > v_k$. This implies that

$$\frac{\tilde{R}_n(v_j) - \tilde{R}_n(v_k)}{v_j - v_k} > \frac{\tilde{R}_n(v_k) - \tilde{R}_n(v_{n^*})}{v_k - v_{n^*}},$$

which is equivalent to

$$\tilde{R}_n(v_k) < \frac{v_k - v_{n^*}}{v_j - v_{n^*}}\tilde{R}_n(v_j) + \frac{v_j - v_k}{v_j - v_{n^*}}\tilde{R}_n(v_{n^*}).$$

This contradicts the concavity of $\tilde{R}_n$. Hence, given $\tilde{\mu}$, v_k is the optimal renegotiation demand for the seller among all $r \geq v_k$.

After $\hat{m}$, the seller knows that $v \geq v_k$, so, by the preceding argument, v_k is optimal, hence $\tilde{r}(\hat{m})$ is an optimal response to $\tilde{\mu}$ after this message.

Given $i \leq k$, $\tilde{G}^m(v_i) = G^m(v_i)$ since $\tilde{\mu}_j(m) = \mu_j(m)$ for all $j < k$, and since $\sum_{j \geq k} \mu_j(m) f_n(v_j) = \sum_{j \geq k} \tilde{\mu}_j(m) f_n(v_j)$. After message $m \in \Gamma(0, k-1)$, the seller chooses the renegotiation demand r to maximize $(r-c)[1 - G^m(r)]$, given buyer strategy μ, or to maximize $(r-c)[1 - \tilde{G}^m(r)]$ given $\tilde{\mu}$. Therefore, among demands $r \leq v_k$, the optimal set is the same in the two cases.

We argued above that $r(m) \leq v_k$ for all $m \in \Gamma(0, k-1)$, and this shows that $\tilde{r}(m)$ is an optimal response to $\tilde{\mu}$ for all those m, hence for all messages. This proves (a).

For $m \in \Gamma(0, k-1)$, $prob[r(m) = v_k] > 0$ by Lemma 5, since $\bar{q}(v, \mu, r)$ jumps at v_k. Hence $prob[\tilde{r}(m) = v_k] > 0$. $Prob[\tilde{r}(\hat{m}) = v_k] > 0$ by definition. This proves (b) and so proves the Claim.

Therefore, in $(\tilde{M}_n, 0, 0)$, given $\tilde{\mu}$, the seller finds it optimal to demand v_k after every possible message. It follows that in this setting a policy of demanding v_k after receiving any message is weakly better for the seller than a policy of demanding r after any message, for all r. The seller's expected payoff from such a policy is

$$\sum_{m \in \tilde{M}_n} \sum_i f_n(v_i) \tilde{\mu}_i(m)(r-c)[1 - \tilde{G}^m(r)].$$

Since

$$1 - \tilde{G}^m(r) = \frac{\sum_{v_i \geq r} \tilde{\mu}_i(m) f_n(v_i)}{\sum_i \tilde{\mu}_i(m) f_n(v_i)},$$

the seller's expected payoff is

$$\sum_{m \in \tilde{M}_n} (r-c) \sum_{v_i \geq r} \tilde{\mu}_i(m) f_n(v_i) = \sum_{v_i \geq r} (r-c) \sum_{m \in \tilde{M}_n} \tilde{\mu}_i(m) f_n(v_i) = (r-c)[1 - F_n(r)].$$

Therefore $R_n(v_k) = (v_k - c)[1 - F(v_k)] \geq (r-c)[1 - F_n(r)] = R_n(r)$ for all r, in particular for $r = v_{n^*}$, a contradiction.

Proof of Lemma 8. First, consider a prior belief F_n such that I_n is empty.

Claim 2. Suppose that S's belief system G satisfies, for each message $i \leq n^*$,

(a)

$$G^i(v_j) = 0 \text{ if } j \leq i$$

and

$$G^i(v_j) = \frac{(v_j - v_i)}{(v_j - c)} \text{ if } i < j \leq n^*$$

(b) conditional on $v_i \geq v_{n^*}$, $G^i(v)$ is a truncation of F_n.

Then, after any message $i \in \{0, 1, ..., n^*\}$ the set of optimal demands for S is $\{v_i, v_{i+1}, .., v_{n^*}\}$.

Proof of Claim 2. Suppose that message i has been sent. If S makes demand v_j such that $v_i < v_j \leq v_{n^*}$ she gets $(v_j - c)[1 - G^i(v_j)] = v_i - c$, so S is indifferent between all offers from v_i to v_{n^*} inclusive. Demanding $v_j \leq v_i$ gives $(v_j - c)[1 - G^i(v_j)] = v_j - c$, and so any demand strictly below v_i is suboptimal. Since S's belief, conditional on types v_{n^*} and above, is a truncation of F_n, the argument in the proof of Lemma 7 shows that v_{n^*} is uniquely optimal among offers of v_{n^*} and above. This proves the claim.

Claim 3. There exists a strategy μ and corresponding belief system G such that the hypotheses of Claim 2 hold.

Proof of Claim 3. We define strategies μ_i so that for each type $v_i \leq v_{n^*}$, μ_i has support $\{0, 1, 2, ..., i\}$. This implies that message i is sent only by types v_i and above, so $G^i(v_j) = 0$ for $j \leq i$. In addition, we set $\mu_i = \mu_{n^*}$ for all $i > n^*$, so $G^i(v)$ is a truncation of F_n conditional on $v_i \geq v_{n^*}$. Thirdly, we define $\mu_0, \mu_1, .., \mu_{n^*}$ recursively so that the updated beliefs $g^i(v_j)$ satisfy

$$g^i(v_j) = \frac{(v_i - c)(v_{j+1} - v_j)}{(v_j - c)(v_{j+1} - c)} \tag{8}$$

for all i, j such that $0 \leq i \leq n^* - 1$, $j = i, i+1, i+2, ..., n^* - 1$ and message i has strictly positive probability. Since

$$G^i(v_{j+1}) = G^i(v_j) + g^i(v_j)$$

we then have, by induction,

$$G^i(v_j) = \frac{(v_j - v_i)}{(v_j - c)} \tag{9}$$

if $i < j \leq n^*$, as required.

For any $i \in \{0, 1, 2, ..., n^*\}$, let $\mu_j(i) = 0$ for all $j < i$. Assume for the moment that $\mu_i(i)$ has been defined for each $i \in \{0, 1, 2, ..., n^*\}$. Then, for any $i \leq n^* - 1$ and any k such that $i < k < n^*$, we let

$$\mu_k(i) = \frac{\mu_i(i) f_n(v_i)(v_i - c)(v_{i+1} - c)(v_{k+1} - v_k)}{f_n(v_k)(v_k - c)(v_{k+1} - c)(v_{i+1} - v_i)}, \tag{10}$$

and we let

$$\mu_{n^*}(i) = \frac{\mu_i(i) f_n(v_i)(v_i - c)(v_{i+1} - c)}{[1 - F_n(v_{n^*})](v_{n^*} - c)(v_{i+1} - v_i)}. \tag{11}$$

For $j > n^*$ and $i \leq n^*$, we let $\mu_j(i) = \mu_{n^*}(i)$.

We need to show that a strategy with these properties, supposing it exists, satisfies (8). It is enough to show, for any positive-probability message i, (a) that, for j, k

such that $i \leq j, k < n^*$,

$$\frac{g^i(v_j)}{g^i(v_k)} = [\frac{(v_i - c)(v_{j+1} - v_j)}{(v_j - c)(v_{j+1} - c)}][\frac{(v_k - c)(v_{k+1} - c)}{(v_i - c)(v_{k+1} - v_k)}], \tag{12}$$

and (b) that

$$\frac{g^i(v_i)}{1 - G^i(v_{n^*})} = [\frac{(v_i - c)(v_{i+1} - v_i)}{(v_i - c)(v_{i+1} - c)}][\frac{(v_{n^*} - c)}{(v_i - c)}]. \tag{13}$$

(a) would imply that the updated relative probabilities of types $\{v_i, v_{i+1}, .., v_{n^*-1}\}$ are consistent with (8), and (b) would show that the total updated probability of these types is as given by (8) and (9).

By Bayes' rule, for all $i \in \{0, 1, .., n^*\}$ and k such that $g^i(v_k) \neq 0$,

$$\frac{g^i(v_j)}{g^i(v_k)} = \frac{f_n(v_j)\mu_j(i)}{f_n(v_k)\mu_k(i)}. \tag{14}$$

From (10),

$$\frac{\mu_j(i)}{\mu_k(i)} = \frac{f_n(v_k)(v_{j+1} - v_j)(v_k - c)(v_{k+1} - c)}{f_n(v_j)(v_j - c)(v_{j+1} - c)(v_{k+1} - v_k)},$$

for j, k such that $i \leq j, k < n^*$, so that (12) is satisfied.

$$\frac{g^i(v_i)}{1 - G^i(v_{n^*})} = \frac{f_n(v_i)\mu_i(i)}{\sum_{j \geq n^*} f_n(v_j)\mu_j(i)}$$

$$= \frac{f_n(v_i)\mu_i(i)}{\mu_{n^*}(i)[1 - F_n(v_{n^*})]},$$

so (11) implies (13).

It remains to show that it is possible to define $\mu_i(i)$ for each $i \in \{0, 1, .., n^*\}$ in such a way that (10) and (11) give a well-defined mixed strategy profile. We do this by recursion. Let $\mu_0(0) = 1$. Then, by (10) and (11), $\mu_j(0)$ is defined for all j. Let $\mu_1(1) = 1 - \mu_1(0)$. From (10) and the fact that (4) holds for $i = l = k = 1$, $\mu_1(0) \leq \mu_0(0)$. Hence μ_1 is a well-defined mixed strategy over the messages $\{0, 1\}$.

Now suppose that, for all $j = 0, 1, 2, ..., i$ where $i < n^* - 1$, μ_j has been defined. Then $\mu_{i+1}(h)$ has been defined for all $h \leq i$. Let $\mu_{i+1}(i+1) = 1 - \sum_{h \leq i} \mu_{i+1}(h)$. If

$\mu_i(h) = 0$, then, by (10), $\mu_j(h) = 0$ for all $j > i$. From (10), if, for $h \leq i$, $\mu_i(h) \neq 0$,

$$\frac{\mu_{i+1}(h)}{\mu_i(h)} = \frac{f_n(v_i)(v_i - c)(v_{i+2} - v_{i+1})}{f_n(v_{i+1})(v_{i+2} - c)(v_{i+1} - v_i)},$$

so, from (4) for $k = l = 1$, $\mu_{i+1}(h) \leq \mu_i(h)$ for all $h \leq i$. Hence $\mu_{i+1}(i+1) \geq 0$ and so μ_{i+1} is a well-defined mixed strategy. Therefore, by induction, μ_j is defined for all $j \leq n^* - 1$, as is $\mu_{n^*}(j)$ for all $j \leq n^* - 1$. Now let $\mu_{n^*}(n^*) = 1 - \sum_{j \leq n^*-1} \mu_{n^*}(j)$. This is well-defined as long as $\mu_{n^*}(j) \leq \mu_{n^*-1}(j)$ for all $j \leq n^* - 1$. But, if $\mu_{n^*-1}(j) \neq 0$,

$$\frac{\mu_{n^*}(j)}{\mu_{n^*-1}(j)} = \frac{f_n(v_{n^*-1})(v_{n^*-1} - c)}{[1 - F_n(v_{n^*})](v_{n^*} - v_{n^*-1})}.$$

By assumption, $R_n(r) \equiv (r - c)[1 - F_n(r)]$ is maximized at v_{n^*}, so

$$(v_{n^*} - c)[\sum_{k \geq n^*} f_n(v_k)] > (v_{n^*-1} - c)[\sum_{k \geq n^*-1} f_n(v_k)]$$

i.e.,

$$f_n(v_{n^*-1})(v_{n^*-1} - c) < [1 - F_n(v_{n^*})](v_{n^*} - v_{n^*-1}).$$

This shows that μ is a well-defined strategy.

If $\mu_{i-1}(h) = \mu_i(h)$ for all $h \leq i - 1$ (this corresponds to a type v_i for which (4) holds with equality for $k = l = 1$), $\mu_i(i) = 0$. Therefore, by (10), message i has zero probability under μ and we can simply define G^i to have the properties given in Claim 2. This completes the proof of Claim 3.

Now suppose that I_n is not empty. Consider a model $(\hat{V}_n, \hat{F}_n)$ in which each type with an index in I_n is removed and its probability transferred to the highest type below it with an index outside I_n. That is, let $\hat{V}_n = V_n \backslash \{v_j | j \in I_n\}$, let $\tau : I_n \rightarrow \{0, 1, .., n^* - 1\} \backslash I_n$ be defined by $\tau(i) = max\{j | v_j < v_i, j \notin I_n\}$, and define the probability distribution $\hat{f}_n$ (and hence the cumulative $\hat{F}_n$) as follows. For $k \in \tau(I_n)$,

$$\hat{f}_n(v_k) = \sum_{j \in \tau^{-1}(k)} f_n(v_j) + f_n(v_k),$$

and for $k \notin \tau(I_n)$, $\hat{f}_n(v_k) = f_n(v_k)$.

Let $\hat{R}_n(v_k) = (v_k - c)[1 - \hat{F}_n(v_k)]$ for $v_k \in \hat{V}_n$. Then, for all $v_k \in \hat{V}_n$, $\hat{R}_n(v_k) = R_n(v_k)$ since $\hat{F}_n(v_k) = F_n(v_k)$. Therefore $\hat{F}_n$ satisfies (4) since F_n satisfies (4) for all $k \notin I_n$. Hence, by the previous argument, there exists $\hat{\mu}$ and corresponding beliefs $\hat{G}$ such that, after any message $i \in \{0, 1, ..., n^*\} \backslash I_n$, S's optimal set of demands is $\{v_k | i \leq k \leq n^*, k \notin I_n\}$.

Now consider the original distribution F_n on V_n. Let the message space be $\{0, 1, ..., n^*\} \backslash I_n$. Define a mixed strategy μ by

$$\mu_i = \hat{\mu}_i \text{ for all } i \notin I_n,$$

$$\mu_i = \hat{\mu}_{\tau(i)} \text{ for all } i \in I_n.$$

Given message $i \in \{0, 1, ..., n^*\} \backslash I_n$, the probability that renegotiation demand $v_j \in \hat{V}_n$ will be accepted is

$$1 - G^i(v_j) = \frac{\sum_{k \geq j} \mu_k(i) f_n(v_k)}{\sum_{k \geq 0} \mu_k(i) f_n(v_k)}$$

if the mixed strategy is μ and the distribution is F_n, and

$$1 - \hat{G}^i(v_j) = \frac{\sum_{k \geq j, k \notin I_n} \hat{\mu}_k(i) \hat{f}_n(v_k)}{\sum_{k \geq 0, k \notin I_n} \hat{\mu}_k(i) \hat{f}_n(v_k)}$$

if the mixed strategy is $\hat{\mu}$ and the distribution is $\hat{F}_n$. However, for all $j \notin I_n$,

$$\sum_{k \geq j, k \notin I_n} \hat{\mu}_k(i) \hat{f}_n(v_k) = \sum_{k \geq j, k \in \tau(I_n)} \hat{\mu}_k(i) \hat{f}_n(v_k) + \sum_{k \geq j, k \notin \tau(I_n), k \notin I_n} \hat{\mu}_k(i) \hat{f}_n(v_k)$$

$$= \sum_{k \geq j, k \in \tau(I_n)} \hat{\mu}_k(i)[f_n(v_k) + \sum_{l \in \tau^{-1}(k)} f_n(v_l)] + \sum_{k \geq j, k \notin \tau(I_n), k \notin I_n} \mu_k(i) f_n(v_k)$$

$$= \sum_{k \geq j} \mu_k(i) f_n(v_k).$$

It follows that, after any message $i \in \{0, 1, .., n^*\} \backslash I_n$, S's expected profit from any demand $v_j \in \hat{V}_n$ is the same under μ as it would be under $\hat{\mu}$. Therefore, if we can show that there is no demand v_j, with $j \in I_n$, which gives a strictly higher expected profit to S than demands in $\{v_k | i \le k \le n^*, k \notin I_n\}$, the Lemma is proved.

Consider, therefore, $v_j \notin \hat{V}_n$. Under μ, v_j sends all messages sent by $v_{\tau(j)}$. Suppose that, under μ, v_j is an optimal renegotiation demand after some message i. Therefore, for some i such that $\mu_{\tau(j)}(i) > 0$, v_j is at least as good for S as $v_{\tau(j)}$.

Let $v_{\sigma(j)} = min\{v \in \hat{V}_n | v > v_j\}$. Let $G^i(\tau(j), \sigma(j))$ be S's belief after message i, given μ, conditional on $v_{\tau(j)} \le v < v_{\sigma(j)}$. For all i such that $\mu_{\tau(j)}(i) > 0$ and all k, k' such that $v_{\tau(j)} \le v_k < v_{\sigma(j)}$ and $v_{\tau(j)} \le v_{k'} < v_{\sigma(j)}$, $\mu_k(i) = \mu_{k'}(i)$. Therefore $G^i(\tau(j), \sigma(j))$ is the same as F_n conditional on $v_{\tau(j)} \le v < v_{\sigma(j)}$, hence independent of i. Moreover, after any i such that $\mu_{\tau(j)}(i) > 0$, S is indifferent between demands $v_{\tau(j)}$ and $v_{\sigma(j)}$, so, conditional on $v \ge v_{\tau(j)}$, the probability that $v \ge v_{\sigma(j)}$ is independent of i. Hence the probability that demand v_j will be accepted is the same for all such messages. Therefore v_j is optimal after all messages sent by $v_{\tau(j)}$. For F_n, v_j fails (4). Hence, by Lemma 6, it cannot be optimal after all messages sent by $v_{\tau(j)}$. This completes the proof of the lemma.

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