

Correlated Equilibrium in Games with Incomplete Information*

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First Version: October 2010

Current Version: May 24, 2011

Abstract

We define a notion of correlated equilibrium for games with incomplete information in a general setting with finite players, finite actions, and finite states. We refer to this solution concept as Bayes correlated equilibrium.

For a given common prior over the payoff relevant states and types, we show that the set of Bayes correlated equilibrium probability distributions equals the set of probability distributions over actions, states and types that might arise in any Bayes Nash equilibrium consistent with the given common prior over states and types.

We define a game of incomplete information in terms of a payoff environment, or the “basic game”, and a belief environment, or the “information structure”. We show how the information structure affects the set of predictions that can be made about the Bayes correlated equilibrium distribution. We show that a more informed information structure reduces the set of Bayes equilibrium distributions as it imposes additional incentive constraints.

KEYWORDS: Correlated equilibrium, incomplete information, robust predictions, information structure.

JEL CLASSIFICATION: C72, D82, D83.

*We acknowledge financial support through NSF Grant SES 0851200. We have benefited from comments from Francoise Forges, Olivier Gossner, Eran Shmaya and seminar participants at the Paris School of Economics.

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1 Introduction

We present a notion of correlated equilibrium in games with incomplete information. Aumann (1974), (1987) introduced the notion of correlated equilibrium in games with complete information. A number of definitions of correlated equilibrium in games with incomplete information have been suggested, notably in Forges (1993). Our definition is driven by a different motivation from the earlier literature; we seek the solution concept which characterizes the set of Bayes Nash equilibria which can be sustained by some information structure in a fixed economic setting. This leads us to suggest an equilibrium notion that we shall call Bayes correlated equilibrium, which is a (weaker) of the weakest definition of incomplete information correlated equilibrium (Bayesian solution) in Forges (1993).

We distinguish the "payoff environment" and the "belief environment" in the definition of the game. By payoff environment, we refer to the set of actions, the set of payoff relevant states, the utility functions of the agents, and the common prior over the payoff relevant states. By belief environment, we refer to the information structure, the type space of the game, which is generated by a mapping from the payoff relevant states to a probability distribution over types. The separation between payoff and belief environment enables us to ask how changes in the belief environment affect the equilibrium set for a given and fixed payoff environment. By contrast with the earlier literature, we allow information not known to any of the players to be reflected in the equilibrium distribution.

In games of complete information, the notion of correlated equilibrium was meant to describe the set of possible equilibrium outcomes which can be achieved when the agents may have access to some, unobserved and unmodelled correlation opportunities. A correlated equilibrium was simply defined by joint distribution over the actions of the agents. In a game with incomplete information, the unobserved and unmodelled correlation opportunities still exist, but the existence of the private information of the agents means that there are some constraints on the correlation opportunities as the actions have to be consistent with the private information of the agents. From this perspective, the private information of the agents imposes restrictions on the correlation that can arise in the Bayes correlated equilibrium.

An important special case is then given by the "null" information system in which the type space of each agent consists of a singleton type for every agent. This "null" information system then imposes "null" restrictions on the joint correlation of the agents' actions and payoff state over and above the common prior of the payoff relevant states. The set of Bayes correlated equilibria for a given payoff environment is then largest under the null information system as the lack of private information means that there are no constraints imposed beyond a consistency requirement which asks that the marginal of the equilibrium distribution over the states equals the common prior of the states. Subsequently, we ask how the presence of private information restricts the correlation opportunities of the agents' actions. In particular, we compare

information structures and ask which information structure contains more information for the agents, and hence imposes more restrictions on the set Bayes correlated equilibria. We present a definition as to when one information structure is *more informed* than another information structure. The criterion of “more informed” represents an extension of the “garbling” condition by Blackwell (1953) to an environment with many agents. We establish that an information structure is more informed than another information structure if and only if it supports a smaller set of Bayes correlated equilibria.

The present definition of Bayes correlated equilibrium is used prominently in the analysis of our companion paper, "Robust Predictions in Games with Incomplete Information", (Bergemann and Morris (2011)). In the companion paper, we analyze how much can be said about the joint distribution of actions and states on the basis of the knowledge of the payoff environment alone. There we refer to “robust predictions” as those predictions which can be made with the knowledge of the payoff environment alone, and without any assumption about the belief environment. In the companion paper, the analysis was confined to an environment with quadratic and symmetric payoff functions, a continuum of agents and normally distributed uncertainty about the common payoff relevant state. But this tractable class of models enabled us to offer robust predictions in terms of restrictions on the first and second moments of the joint distribution over actions and state. By contrast, here we present the definition of the Bayes correlated equilibrium in a canonical game theoretic framework with a finite number of agents, a finite set of pure action and a finite set of payoff relevant states. After we introduce the relevant notions, we show towards the end of this paper how the present results translate into the setting with a continuum of anonymous agents that we considered in (Bergemann and Morris (2011)).

A number of papers have considered alternative definitions of correlated equilibrium in games with incomplete information, most notably Forges (1993) and Forges (2006). In this paper, we document the relationship between our version of correlated equilibrium and the various definitions in the literature. In the discussion of the various definitions of correlated equilibrium, we will find it is useful to divide restrictions that the various solution concepts impose on the joint distribution over actions, states and types into two classes: *feasibility* conditions on the distributions of action type state profiles, which are required to hold independently of the payoff functions, and *incentive compatibility* conditions which are rationality constraints on players’ action choices. The only feasibility condition that we impose in defining the Bayes correlated equilibrium is a consistency requirement that demands that the action type state distribution of the equilibrium implies the distribution on the exogenous variables, namely the common prior on the payoff relevant states and types. In contrast, in many of the existing solution concepts, the feasibility conditions are intended to capture the outcome of some form of communication among the agents with an *uninformed mediator*. It is then natural to impose additional restriction on the action state type

distribution in equilibrium which have hold *conditional* on the agents' types. For example, the "Bayesian solution", the weakest of Forges' five definitions, imposes the restriction, referred to here as join feasibility, that the distribution over states conditional on agents' types is not changed conditional on the mediator's recommendations. Our notion of Bayes correlated equilibrium is closest to the "Bayesian solution" but is strictly weaker than the Bayesian solution, because we do not insist on join feasibility.

A number of papers - notably Gossner (2000), Lehrer, Rosenberg, and Shmaya (2010) and Lehrer, Rosenberg, and Shmaya (2011) - have examined comparative statics of how changing the information structure effects the set of predictions that can be made about players' actions, under Bayes Nash equilibrium or alternative solution concepts. We review these results and report a new result. We discussed above that as the agents become more informed, where information is encoded in their type, the set of possible predictions must be reduced. As the agents have more private information, the incentive constraints, here referred to as obedience constraints, will become tighter. The role of the private information in refining the equilibrium prediction is important in our "Robust Prediction" agenda. We will formalize this result this result here in the general framework of the current paper rather than in the specific environment of quadratic payoff functions and normally distributed uncertainty of (Bergemann and Morris (2011)). We illustrate the notion of Bayes correlated equilibrium and the resulting robust predictions in variety of examples, among them a first price auction with private values and a sender-receiver game, which is closely related to a problem studied by Gentzkow and Kaminca (2010), where senders are allowed to commit to a communication strategy.

In Bergemann and Morris (2005) and later work, we studied a mechanism design environments and defined the notion of robust mechanism. In this earlier setting, the agents knew their own "payoff types", and while there was common knowledge of how utilities depended on the profile of payoff types, the agents were allowed to have *any* beliefs and higher order beliefs about others' payoff types. We then defined a mechanism to be *robust* if the social choice function or correspondence could be truthfully implemented in the direct mechanism as a Bayes Nash equilibrium for *any* beliefs and higher order beliefs about others' payoff types. In Bergemann and Morris (2007), we discussed the game theoretic framework underlying the analysis in the mechanism design environment. The notion of Bayes correlated equilibrium is motivated by the same concern for robustness but it encodes a less demanding notion of robustness. The Bayes correlated equilibrium insists that the common prior over the state and type distribution is preserved, and in the case of the "null information structure" that the common prior over the state alone is preserved, but all additional correlation due to unobserved communication or information among the agents is permitted.

We proceed as follows. In Section 2, we describe a general incomplete information game and compare Bayes Nash equilibrium with a solution concept which we call Bayes correlated equilibrium. In Section 3,

we describe our robust predictions approach and explain the key role played an "epistemic" result: the set of Bayes correlated equilibrium probability distributions over actions, types and payoff-relevant variables equals the set of probability distributions of actions, types and payoff-relevant variables that might arise in a Bayes Nash equilibrium if players were able to observe additional information signals beyond their original types.

In Section 4, we explain how the solution concept we dub "Bayes Correlated Equilibrium" relates to the literature, in particular Forges (1993) and Forges (2006). In Section 5, we report results on comparing information structures. In Section 6, we review special cases in order to illustrate the robust predictions agenda more broadly. In Section 7, we describe analogues of our results for continuum anonymous player games, which apply to our work in "Robust Predictions in Games with Incomplete Information". Section 8 concludes and contains a discussion of the relation to the signed covariance result of Chwe (2006) and the "payoff types" environments of Bergemann and Morris (2007).

2 Bayes Nash and Bayes Correlated Equilibrium

Throughout the paper, we will fix a finite set of players and a finite set of payoff relevant states of the world. There are I players, $1, 2, \dots, I$, and we write i for a typical player. We write Θ for the payoff relevant states of the world and θ for a typical element of Θ .

A "basic game" G consists of (1) for each player i , a finite set of action A_i and a utility function $u_i : A \times \Theta \rightarrow \mathbb{R}$; and (2) a full support prior $\psi \in \Delta(\Theta)$, where we write $A = A_1 \times \dots \times A_I$. Thus $G = \left((A_i, u_i)_{i=1}^I, \psi \right)$. An "information structure" S consists of (1) for each player i , a finite set of types or "signals" T_i ; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(T)$, where we write $T = T_1 \times \dots \times T_I$. Thus $S = \left((T_i)_{i=1}^I, \pi \right)$.

Together, the "payoff environment" or "basic game" G and the "belief environment" or "information structure" S define a standard "incomplete information game". While we use different notation, this division of an incomplete information game into the "basic game" and the "information structure" is a standard one in the literature, see, for example, Lehrer, Rosenberg, and Shmaya (2010).

A (behavioral) strategy for player i in the incomplete information game (G, S) is $b_i : T_i \rightarrow \Delta(A_i)$. Write B_i for the set of strategies of player i in the game (G, S) . The following is the standard definition of Bayes Nash Equilibrium in this setting.

Definition 1 *A strategy profile b is a Bayes Nash Equilibrium (BNE) of (G, S) if for each $i = 1, 2, \dots, I$,*

$t_i \in T_i$ and $a_i \in A_i$ with $b_i(a_i|t_i) > 0$, we have

$$\begin{aligned} & \sum_{t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a_i, b_{-i}(t_i)), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \\ & \geq \sum_{t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, b_{-i}(t_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta). \end{aligned}$$

for each $a'_i \in A_i$.

The relevant space of uncertainty in the incomplete information game (G, S) is $A \times T \times \Theta$, and we will write ν for a typical element of $\Delta(A \times T \times \Theta)$. There are two kinds of constraints imposed in defining alternative versions of incomplete information correlated equilibrium: "feasibility" constraints and "incentive compatibility" conditions. Our preferred definition will impose one feasibility condition:

Definition 2 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is consistent for (G, S) if, for all $t \in T$ and $\theta \in \Theta$, we have*

$$\sum_{a \in A} \nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \quad (1)$$

This simply says the marginal of distribution ν on the exogenous variables T and Θ is consistent with the description of the game (G, S) . We will also impose the weakest natural incentive compatibility condition, "obedience", that says that a player i who knows his type t_i , his recommended action a_i and the distribution ν only has an incentive to follow that recommendation.

Definition 3 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is obedient for (G, S) if, for each $i = 1, \dots, I$, $t_i \in T_i$ and $a_i \in A_i$, we have*

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \\ & \geq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta); \end{aligned} \quad (2)$$

for all $a'_i \in A_i$.

Now our leading definition of correlated equilibrium for incomplete information games will be:

Definition 4 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a Bayes Correlated Equilibrium (BCE) of (G, S) if it is consistent and obedient.*

As will discuss in detail below in Section 4, this is essentially the definition of "Bayesian solution" in Forges (1993), with the difference that we work with a incomplete information game description that does not integrate out payoff relevant states and thus allows the mediator to make action recommendations that depend on a payoff-relevant state that is observed by nobody. We will discuss in the next section why this definition is interesting for our robust predictions agenda.

A Bayes Nash Equilibrium b is a strategy profile in B . A Bayes Correlated Equilibrium ν is an element of $\Delta(A \times T \times \Theta)$ and thus a distribution over action type state profiles. To compare the two solution concepts, we would like to discuss the distribution of action type state profiles generated by a BNE.

Definition 5 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is induced by strategy profile $b \in B$ if, for each $a \in A$, $t \in T$ and $\theta \in \Theta$, we have*

$$\nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \left(\prod_{i=1}^I b_i(a_i|t_i) \right).$$

Distribution $\nu \in \Delta(A \times T \times \Theta)$ is Bayes Nash action type state distribution of (G, S) if there exists a Bayesian Nash Equilibrium b of (G, S) that induces it.

We also have the following straightforward observation:

Lemma 1 *Every Bayes Nash action type state distribution of (G, S) is a Bayes Correlated Equilibrium of (G, S) .*

We will also sometimes be interested in the induced action state distributions, i.e., what we can say if types are not observed.

Definition 6 *Action state distribution $\mu \in \Delta(A \times \Theta)$ is induced by $\nu \in \Delta(A \times T \times \Theta)$ if it is the marginal of ν on $A \times \Theta$. Action state distribution $\mu \in \Delta(A \times \Theta)$ is a BNE action state distribution of (G, S) if it is induced by a Bayes Nash action type state distribution of (G, S) . Action state distribution $\mu \in \Delta(A \times \Theta)$ is a BCE action state distribution of (G, S) if it is induced by a Bayes Correlated Equilibrium of (G, S) .*

An important special case is when the information system is "null" with the players knowing nothing about the states. Formally, the null information system $S_0 = \left(\left(\{t_i^0\} \right)_{i=1}^I, \pi^0 \right)$, where t_i^0 is the singleton type of player i and $\pi^0(t^0|\theta) = 1$ for each $\theta \in \Theta$. We will abbreviate the (degenerate) incomplete information game (G, S_0) to G . Observe that in the special case of a null information system, the space $A \times T \times \Theta$ reduces to $A \times \Theta$ and the consistency condition (1) on $\mu \in \Delta(A \times \Theta)$ becomes

$$\sum_{a \in A} \mu(a, \theta) = \psi(\theta) \tag{3}$$

for all $\theta \in \Theta$; and the obedience constraint (2) reduces to

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, \theta \in \Theta} u_i((a_i, a_{-i}), \theta) \mu((a_i, a_{-i}), \theta) \\ & \geq \sum_{a_{-i} \in A_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \mu((a_i, a_{-i}), \theta); \end{aligned} \quad (4)$$

for each $i = 1, \dots, I$, $a_i \in A_i$ and $a'_i \in A_i$.

Now we have:

Lemma 2 *If $\mu \in \Delta(A \times \Theta)$ is induced by a BCE action type state distribution $\nu \in \Delta(A \times T \times \Theta)$, then μ is a BCE of G .*

As we will discuss in detail below, this result is in the spirit of Proposition 4 of Forges (1993), which shows that "any" correlated equilibrium solution concept of (G, S) generates an equilibrium of the basic game G .

3 Robust Predictions

Consider an analyst who knows that

1. G describes actions, payoff functions depending on fundamental states, and a prior distribution on fundamental states.
2. Players have observed at least information system S .
3. The full, common prior, information system is common certainty among the players.
4. The players' actions follow a Bayes Nash Equilibrium.

What can she deduce about the joint distribution of actions, types in the "information structure" S and states? In this section, we will formalize this question and show that all she can deduce is that the distribution will be a BCE distribution of (G, S) .

To formalize this, let $\tilde{S} = ((Z_i)_{i=1}^I, \phi)$ be a supplementary information system, over and above S , and suppose each agent i observes a supplementary signal $z_i \in Z_i$, where $\phi : \Theta \times T \rightarrow Z$ describes the distribution of supplementary signals. Now (G, S, \tilde{S}) is an "augmented incomplete information game". Write $\tau_i : T_i \times Z_i \rightarrow \Delta(A_i)$ for a behavior strategy of player i in the augmented incomplete information game.

Definition 7 A strategy profile τ is a Bayes Nash Equilibrium of the augmented game (G, S, \tilde{S}) if, for each $i = 1, 2, \dots, I$, $t_i \in T_i$, $z_i \in Z_i$ and $a_i \in A_i$ with $\sigma_i(a_i | (t_i, z_i)) > 0$, we have

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i}) | \theta) \phi((z_i, z_{-i}) | (t_i, t_{-i}), \theta) \\ & \geq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a'_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i}) | \theta) \phi((z_i, z_{-i}) | (t_i, t_{-i}), \theta). \end{aligned}$$

for each $a'_i \in A_i$.

Write ν_τ for the probability distribution over $A \times T \times \Theta$ generated by strategy profile τ , so

$$\nu_\tau(a, t, \theta) = \psi(\theta) \pi(t | \theta) \sum_{z \in Z} \phi(z | t, \theta) \left(\prod_{i=1}^I \tau_i(a_i | t_i, z_i) \right)$$

Definition 8 A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a BNE action type state distribution of (G, S, \tilde{S}) if there exists a BNE τ of (G, S, \tilde{S}) such that $\nu = \nu_\tau$.

Proposition 1 A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a Bayes Correlated Equilibrium of (G, S) if and only if it is a BNE action type distribution of (u, ψ, S, \tilde{S}) for some augmented information system \tilde{S} .

Proof. Suppose that ν is a correlated equilibrium of (u, ψ, S) . Thus

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \\ & \geq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta); \end{aligned}$$

for each i , $t_i \in T_i$, $a_i \in A_i$ and $a'_i \in A_i$; and

$$\sum_{a \in A} \nu(a, t, \theta) = \psi(\theta) \pi(t | \theta)$$

for all $t \in T$ and $\theta \in \Theta$. Construct an augmented information system $\tilde{S} = ((Z_i)_{i=1}^I, \phi)$ with each $Z_i = A_i$ and

$$\phi(a | t, \theta) = \nu(a | \theta, t).$$

Now in the augmented incomplete information game (G, S, \tilde{S}) , consider the "truthful" strategy profile σ with $\sigma_i(a_i|t_i, a_i) = 1$ for all i, t_i and a_i . Clearly, we have $\nu_\sigma = \nu$. Now

$$\begin{aligned} & \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a'_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta) \\ = & \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \end{aligned}$$

and thus Nash equilibrium conditions are implied by the correlated equilibrium conditions on ν .

Conversely, suppose that τ is a Nash equilibrium of (G, S, \tilde{S}) . Now $\sigma_i(a_i|(t_i, z_i)) > 0$ implies

$$\begin{aligned} & \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta) \\ \geq & \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a'_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta). \end{aligned}$$

for each $a'_i \in A_i$. Thus

$$\begin{aligned} & \sum_{z_i \in Z_i} \sigma_i(a_i|(t_i, z_i)) \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta) \\ \geq & \sum_{z_i \in Z_i} \sigma_i(a_i|(t_i, z_i)) \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a'_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta). \end{aligned}$$

But

$$\begin{aligned} & \sum_{z_i \in Z_i} \sigma_i(a_i|(t_i, z_i)) \sum_{t_{-i} \in T_{-i}, z_{-i} \in Z_{-i}, \theta \in \Theta} u_i((a'_i, \sigma_{-i}(t_{-i}, z_{-i})), \theta) \psi(\theta) \pi((t_i, t_{-i})|\theta) \phi((z_i, z_{-i})|(t_i, t_{-i}), \theta) \\ = & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu_\tau((a_i, a_{-i}), (t_i, t_{-i}), \theta) \end{aligned}$$

and thus BNE conditions imply that μ is a BCE. ■

An alternative formulation of this result would be to say that BCE captures the implications of common certainty of rationality (and the common prior assumption) in the game (G, S) , since requiring BNE in some game with augmented information is equivalent to describing a belief closed subset where the game (G, S) is being played and there is common certainty of rationality. Thus this is an incomplete information analogue of the Aumann (1987) characterization of correlated equilibrium for complete information games and thus - as described in more detail in the next section - corresponds to the "partial Bayesian approach" of Forges (1993), with the difference that she works with the reduced game - integrating out the payoff states Θ .

4 A Number of Legitimate Definitions of Correlated Equilibrium in Incomplete Information Games

Forges (1993) is titled and identifies "five legitimate definitions of correlated equilibrium in games with incomplete information." Forges (2006) describes a mistake in Forges (1993) that leads to a sixth definition. Our purpose in this section is to review these six definitions and understand their relation to the solution concept we dub "Bayes correlated equilibrium." Let us highlight a few differences between our formulation of games and solution concepts to bear in mind as we describe the relation:

1. While we directly define solution concepts for (G, S) as subsets of action type state distributions $\Delta(A \times T \times \Theta)$, she characterizes the set of equilibrium payoffs satisfying a set of restrictions which implicitly define the solution concept.
2. While we work with a "basic game", $G = ((A_i, u_i)_{i=1}^I, \psi)$, describing prior and payoffs and an "information structure" $S = ((T_i)_{i=1}^I, \pi)$, she distinguishes between the "decision problem with incomplete information," $(A_i, u_i)_{i=1}^I$ and includes the prior on payoff relevant states in her description of the "information scheme".
3. While we include the distribution of payoff relevant states Θ in our solution concept, she integrates out payoff relevant states.
4. While we and Forges (2006) allow for any finite number of players, Forges (1993) focussed on the two player case for simplicity.

We start with five definitions of correlated equilibrium for a incomplete information game (G, S) . It is useful to divide restrictions into two classes: *feasibility* conditions on the distributions of action type state profiles, which are required to hold independent of the payoff functions, and *incentive compatibility* conditions which are rationality constraints on players' action choices. The closest solutions rely only on additional feasibility constraints, maintaining obedience as the only incentive compatibility constraint.

Recall that the only feasibility condition we imposed in defining Bayes Correlated Equilibrium was the consistency requirement (Definition 2) that the action type state distribution implied the distribution on exogenous variables (types and states) was that of the game (G, S) . If the solution concept is intended to capture the outcome of communication among the players perhaps allowing for an uninformed mediator, it is natural to impose the additional restriction that the distribution over states conditional on agents' types is not changed conditional on the mediator's recommendations:

Definition 9 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is join feasible for (G, S) if, for all $a \in A$ and $t \in T$ such that*

$$\sum_{\theta \in \Theta} \nu(a, t, \theta) > 0,$$

we have

$$\frac{\nu(a, t, \theta)}{\sum_{\theta' \in \Theta} \nu(a, t, \theta')} = \frac{\psi(\theta) \pi(t|\theta)}{\sum_{\theta' \in \Theta} \psi(\theta') \pi(t|\theta')} \quad (5)$$

for all $\theta \in \Theta$.

This assumption is (implicitly) maintained in all Forges' solution concepts for (G, S) and is made explicit in Lehrer, Rosenberg, and Shmaya (2011) and Lehrer, Rosenberg, and Shmaya (2010) (e.g., condition 4 on page 676 in Lehrer, Rosenberg, and Shmaya (2010)).

Definition 10 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a Bayesian solution of (G, S) if it is consistent, join feasible and obedient.*

This is the solution concept discussed in Section 4.4 of Forges (1993) and one of the two discussed in section 2.5 of Forges (2006). Lehrer, Rosenberg, and Shmaya (2011) refer to this as a "global equilibrium." It also corresponds to the set of jointly coherent outcomes in Nau (1992), justified from no arbitrage conditions. Forges and Koessler (2005) provide a justification if players are able to certify their types to the mediator.

This solution concept allows players to learn about other players' types from the mediator's recommendation. The following condition removes this possibility:

Definition 11 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is belief invariant for (G, S) if, for all $t_i \in T_i$ and $a_i \in A_i$ such that*

$$\sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) > 0,$$

we have

$$\frac{\sum_{a_{-i} \in A_{-i}, \theta \in \Theta} \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta)}{\sum_{a_{-i} \in A_{-i}, t'_{-i} \in T'_{-i}, \theta \in \Theta} \nu((a_i, a_{-i}), (t_i, t'_{-i}), \theta)} = \frac{\sum_{\theta \in \Theta} \psi(\theta) \pi((t_i, t_{-i})|\theta)}{\sum_{t'_{-i} \in T'_{-i}, \theta \in \Theta} \psi(\theta) \pi((t_i, t'_{-i})|\theta)} \quad (6)$$

for each $t_{-i} \in T_{-i}$.

This is condition 3 on page 676 in Lehrer, Rosenberg, and Shmaya (2010). As Forges (2006) puts it, "the omniscient mediator can use his knowledge of the types to make his recommendations but the players

should not be able to infer anything on the others' types from their recommendations." This restriction is added to give the second solution concept:

Definition 12 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a belief invariant Bayesian solution of (G, S) if it is consistent, join feasible, belief invariant and obedient.*

This is the second solution concept discussed in Section 2.5 of Forges (2006); it was discussed informally in Section 4.4 of Forges (1993) but it was then mistakenly claimed that it was equivalent to agent normal form correlated equilibrium. This solution concept is also used in Lehrer, Rosenberg, and Shmaya (2011) and Lehrer, Rosenberg, and Shmaya (2010). Because they do not work with the reduced game, i.e., they explicitly discussed payoff relevant states like Θ , and they must explicitly impose a join feasibility restriction.

The belief invariant Bayesian solution allows the mediator to use information about players' types to make a recommendation to players. Suppose that the mediator has no information about the players' types when deciding what strategy to recommend as a function of the players' types. This is reflected in the next feasibility restriction. A pure strategy in the incomplete information game is function $\sigma_i : T_i \rightarrow A_i$. Write Σ_i for the set of pure strategies of agent i and Σ for the set of pure strategy profiles, $\Sigma = \Sigma_1 \times \dots \times \Sigma_I$.

Definition 13 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is agent normal form feasible for (G, S) if there exists $q \in \Delta(\Sigma)$ such that*

$$\nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \sum_{\{\sigma \in \Sigma | \sigma(t) = a\}} q(\sigma) \quad (7)$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$.

One can show that agent normal form feasibility implies belief invariance. This restriction is added to give the third solution concept:

Definition 14 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is an agent normal form correlated equilibrium of (G, S) if it is consistent, uninformed mediator feasible, agent normal form feasible (and thus belief invariant) and obedient.*

This is the solution concept discussed in Section 4.2 of Forges (1993) and Section 2.3 of Forges (2006). It corresponds to applying the complete information definition of correlated equilibrium to the agent normal form of the reduced incomplete information game. It was also studied by Samuelson and Zhang (1989) and Cotter (1994). The solution concept only makes sense on the understanding that the players receive a recommendation for each type but do not learn what recommendation they would have received if they had been different types. If they did learn the whole strategy that the mediator choose for them in the strategic form game, then an extra incentive compatibility condition would be required:

Definition 15 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is strategic form incentive compatible for (G, S) if there exists $q \in \Delta(\Sigma)$ such that*

$$\nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \sum_{\{\sigma \in \Sigma | \sigma(t)=a\}} q(\sigma) \quad (8)$$

for each $a \in A$, $t \in T$ and $\theta \in \Theta$; and, for each $i = 1, \dots, I$, $t_i \in T_i$, $a_i \in A_i$ and $\sigma_i \in \Sigma_i$ such that $\sigma_i(t_i) = a_i$, we have

$$\begin{aligned} & \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta) \pi(t|\theta) \left(\sum_{\{\sigma_{-i} \in \Sigma_{-i} | \sigma_{-i}(t_{-i})=a_{-i}\}} q(\sigma_i, \sigma_{-i}) \right) u_i((a_i, a_{-i}), \theta) \\ & \geq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta) \pi(t|\theta) \left(\sum_{\{\sigma_{-i} \in \Sigma_{-i} | \sigma_{-i}(t_{-i})=a_{-i}\}} q(\sigma_i, \sigma_{-i}) \right) u_i((a'_i, a_{-i}), \theta) \end{aligned} \quad (9)$$

for all $a'_i \in A_i$.

Note that this condition implies both agent normal form feasibility and obedience. This restriction gives the fourth solution concept:

Definition 16 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a strategic form correlated equilibrium of (G, S) if it is consistent, uninformed mediator feasible and strategic form incentive compatible (and thus agent normal form feasible, belief invariant and obedient).*

This is the solution concept discussed in Section 4.1 of Forges (1993) and Section 2.2 of Forges (2006). This solution concept was studied by Cotter (1991).

Thus far we have simply been adding restrictions, so that the solution concept have become stronger as we go from Bayesian solution, to belief invariant Bayesian solution, to agent normal form correlated equilibrium, to strategic form correlated equilibrium. For the Bayesian solution, an omniscient mediator who observes players' types for free is assumed. For agent normal form and strategic form correlated equilibrium, the players' types cannot play a role in the selection of recommendations to the players. An intermediate assumption is that the players can report their types to the mediator, but will do so truthfully only if it is incentive compatible to do so. Write $\xi_\nu : T \times \Theta \rightarrow A$ for the mediator's recommendation strategy implied by $\nu \in \Delta(A \times T \times \Theta)$, so that, for each $t \in T$ and $\theta \in \Theta$ with $\sum_{a' \in A} \nu(a', t, \theta) > 0$,

$$\xi_\nu(a|t, \theta) = \frac{\nu(a, t, \theta)}{\sum_{a' \in A} \nu(a', t, \theta)}$$

for each $a \in A$.

Definition 17 *Distribution $\nu \in \Delta(A \times T \times \Theta)$ is truth telling for (G, S) if, for each $i = 1, \dots, I$ and $t_i \in T_i$, we have*

$$\begin{aligned} & \sum_{a \in A, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta) \pi((t_i, t_{-i}) | \theta) \xi_\nu((a_i, a_{-i}) | (t_i, t_{-i}), \theta) \\ & \geq \sum_{a \in A, t_{-i} \in T_{-i}, \theta \in \Theta} \psi(\theta) \pi((t_i, t_{-i}) | \theta) \xi_\nu((\delta_i(a_i), a_{-i}) | (t'_i, t_{-i}), \theta); \end{aligned} \quad (10)$$

for all $t'_i \in T_i$ and $\delta_i : A_i \rightarrow A_i$.

Note that this condition implies obedience (Definition 2). One can show that this condition is implied by strategic form incentive compatibility. Now we have the fifth solution concept:

Definition 18 *A probability distribution $\nu \in \Delta(A \times T \times \Theta)$ is a communication equilibrium of (G, S) if it is consistent, join feasible and incentive compatible (and thus obedient).*

This is the solution concept discussed in Section 4.3 of Forges (1993) and Section 2.4 of Forges (2006), and developed earlier in the work of Myerson (1982) and Forges (1986).

Thus we have Forges' five solution concepts for the incomplete information game (G, S) :

1. Bayesian solution (Definition 10);
2. Belief invariant Bayesian solution (Definition 12);
3. Agent normal form correlated equilibrium (Definition 14);
4. Strategic form correlated equilibrium (Definition 16); and
5. Communication equilibrium (Definition 18).

As documented by Forges (1993) and Forges (2006) and implied by the above definitions, we have that the Bayesian solution [1] is weaker than the belief invariant Bayesian equilibrium solution [2], which is weaker than the agent normal form correlated equilibrium [3], which is weaker than the strategic form correlated equilibrium [4]; and also the Bayesian solution [1] is weaker than communication equilibrium [5] which is weaker than strategic form correlated equilibrium [4]. Examples reported in Forges (1993) and Forges (2006) that each weak inclusion is strict and that the belief invariant Bayesian solution [2] and agent normal form correlated equilibrium [3] cannot be ranked relative to communication equilibrium [5]. Our definition of Bayes Correlated Equilibrium is weaker than the Bayesian solution, the weakest of Forges' five, because we do not maintain join feasibility.

The following is a trivial (one player) example showing that Bayes Correlated Equilibrium is a more permissive solution concept than any of Forges' five solution concepts for (G, S) . Suppose there is one player, $I = 1$, and two states, $\Theta = \{\theta, \theta'\}$. Let the basic game $G = (A_1, u_1, \psi)$ be defined by $A_1 = \{a_1, a'_1\}$, $u_1(a_1, \theta) = 2$, $u_1(a_1, \theta') = -1$ and $u_1(a'_1, \theta) = u_1(a'_1, \theta') = 0$, and $\psi(\theta) = \psi(\theta') = \frac{1}{2}$. And consider the null information system S_0 . Consistency (3), obedience (4) and join feasibility (5) together imply that

$$\mu(a_1, \theta) = \mu(a_1, \theta') = \frac{1}{2} \text{ and } \mu(a'_1, \theta) = \mu(a'_1, \theta') = 0.$$

This is thus the unique Bayesian solution, belief invariant Bayesian solution, agent normal form correlated equilibrium, strategic form correlated equilibrium and communication equilibrium. However, consistency (3) implies only that

$$\begin{aligned} \mu(a_1, \theta) + \mu(a'_1, \theta) &= \frac{1}{2} \\ \mu(a_1, \theta') + \mu(a'_1, \theta') &= \frac{1}{2} \end{aligned}$$

and obedience (4) implies only that

$$\begin{aligned} 2\mu(a_1, \theta) - \mu(a_1, \theta') &\geq 0 \\ 2\mu(a'_1, \theta) - \mu(a'_1, \theta') &\leq 0. \end{aligned}$$

There are many Bayesian Correlated Equilibria satisfying the above constraints. The one maximizing the player's utility has

$$\mu(a_1, \theta) = \mu(a'_1, \theta') = \frac{1}{2} \text{ and } \mu(a_1, \theta') = \mu(a'_1, \theta) = 0.$$

In Section 4.5, Forges (1993) discusses how more solution concepts are conceivable, including by dropping join feasibility, and gives an example like the above illustrating this point.

In Section 6, Forges (1993) considers a "universal Bayesian approach" in which a prior "information scheme" (in our language, prior on Θ and information system) is not taken as given. Thus her "universal Bayesian solution" is defined for $(A_i, u_i)_{i=1}^I$. Expressing her ideas in the language of action state distributions, she studies the following solution concept.

Definition 19 *A probability distribution $\mu \in \Delta(A \times \Theta)$ is a universal Bayesian solution of $(A_i, u_i)_{i=1}^I$ if it satisfies (4).*

Thus a probability distribution $\mu \in \Delta(A \times \Theta)$ is Bayes Correlated Equilibrium of $G = ((A_i, u_i)_{i=1}^I, \psi)$ if and only if it is a universal Bayesian solution and satisfies (3). Note that applying Proposition 1 to the special case of the null information system, we have that $\mu \in \Delta(A \times \Theta)$ is a Bayes Correlated

Equilibrium of G if and only if there exists an information system S and a Bayes Correlated Equilibrium $\nu \in \Delta(A \times T \times \Theta)$ of (G, S) which induces $\mu \in \Delta(A \times \Theta)$. This then corresponds to Forges' Proposition 4 when applied to the solution concept of Nash equilibrium (although she states the results in terms of equilibrium payoffs rather than distributions). As she notes, her Proposition 4 is a natural incomplete information generalization of Aumann (1987) and our Proposition 1 is an example of such a generalization stated in a different language.

5 Comparing Information Systems

An important result for our robust predictions agenda is that as players become more informed, the set of possible predictions must be reduced, since obedience constraints will become tighter. We will formalize this result in the next sub-section. First, we review some existing results on comparing information systems.

5.1 The Existing Literature

The following useful terminology is used in Lehrer, Rosenberg, and Shmaya (2011) and Lehrer, Rosenberg, and Shmaya (2010).

Definition 20 *Information system S' is a garbling of S if there exists $\phi : T \rightarrow \Delta(T')$ and satisfying*

$$\pi'(t'|\theta) = \sum_{t \in T} \pi(t|\theta) \phi(t'|t)$$

for each $t' \in T'$ and $\theta \in \Theta$. The map ϕ is called a garbling that transforms S to S' .

This says that the join of the information in S' is a garbling in the sense of Blackwell (1951) of the join of the information in S . Garbling ϕ is *non-communicating* if, for each $i = 1, \dots, I$, $t_i \in T_i$, $t'_i \in T_i$,

$$\sum_{t'_{-i} \in T'_{-i}} \phi((t'_i, t'_{-i}) | (t_i, t_{-i})) = \sum_{t'_{-i} \in T'_{-i}} \phi((t'_i, t'_{-i}) | (t_i, \tilde{t}_{-i}))$$

for all $t_{-i}, \tilde{t}_{-i} \in T_{-i}$.

Definition 21 *Information system S' is a non-communicating garbling of S if there exists a non-communicating garbling ϕ that transforms S into S' .*

This condition requires that each player's information in S' is a Blackwell garbling of his information in S . If garbling ϕ is a non-communicating garbling, we write $\phi_i(t'_i|t_i)$ for the (t_{-i} independent) probability

of t'_i conditional on t_i , i.e.,

$$\phi_i(t'_i|t_i) \equiv \sum_{t'_{-i} \in T_i} \phi((t'_i, t'_{-i}) | (t_i, t_{-i}))$$

Garbling ϕ is *coordinated* if there exist $\lambda \in \Delta(\{1, \dots, K\})$ and, for each i , $\phi_i : T_i \times \{1, \dots, K\} \rightarrow \Delta(T_i)$ such that

$$\phi(t'|t) = \sum_{k=1}^K \lambda(k) \prod_{i=1}^I \phi_i(t'_i|t_i, k)$$

for each $t \in T$ and $t' \in T'$.

Definition 22 *Information system S' is a coordinated garbling of S if there exists a coordinated garbling ϕ that transforms S into S' .*

A garbling is independent if it is coordinated with $K = 1$, so that there exists, for each i , $\phi_i : T_i \rightarrow \Delta(T_i)$ such that

$$\phi(t'|t) = \sum_{k=1}^K \lambda(k) \prod_{i=1}^I \phi_i(t'_i|t_i, k)$$

for each $t \in T$ and $t' \in T'$.

Definition 23 *Information system S' is an independent garbling of S if there exists an independent garbling ϕ that transforms S into S' .*

Lehrer, Rosenberg, and Shmaya (2010) and Lehrer, Rosenberg, and Shmaya (2011) note that, by definition, an independent garbling is a coordinated garbling, a coordinated garbling is a non-communicating garbling and a non-communicating garbling is a garbling, and present elegant examples showing that none of the reverse implications is true.

Say that an information system S is larger than S' under a given equilibrium concept if, for every game G , every action state distribution induced by an equilibrium of (G, S') is also induced by an equilibrium of (G, S) . Information system S is equivalent to S' under a given equilibrium concept if S is larger than S' and S' is larger than S under that equilibrium.

Lehrer, Rosenberg, and Shmaya (2011) show that (in Theorem 2.8) that

1. Two information systems are equivalent under Bayes Nash Equilibrium if and only if they are independent garblings of each other.
2. Two information systems are equivalent under Agent Normal Form Correlated Equilibrium (Definition 14) if and only if they are coordinated garblings of each other.

3. Two information systems are equivalent under the Belief Invariant Bayesian Solution (Definition 12) if and only if they are non-communicating garblings of each other.

They do not report an analogous result for Bayes Correlated Equilibrium. They do not report results for the "larger than" relation.

Lehrer, Rosenberg, and Shmaya (2010) consider common interest games. Say that information system is S better than S' under a given solution concept if, for every common interest game G , the maximum (common) equilibrium payoff is higher in (G, S) than (G, S') . They show

1. (Theorem 3.5) Information system S is better than S' under Bayes Nash Equilibrium if and only if S' is a coordinated garbling of S .
2. (Theorem 4.2) Information system S is better than S' under Agent Normal Form Correlated Equilibrium (Definition 14) if and only if S' is a coordinated garbling of S .
3. (Theorem 4.2) Information system S is better than S' under Strategic Form Correlated Equilibrium (Definition 16) if and only if S' is a coordinated garbling of S .
4. (Theorem 4.5) Information system S is better than S' under the Belief Invariant Bayesian Solution (Definition 12) if and only if S' is a non-communicating garbling of S .
5. (Theorem 4.6) Information system S is better than S' under Communication Equilibrium (Definition 18) if and only if S' is a garbling of S .

Gossner (2000) studies Bayes Nash equilibrium only as a solution concept. His focus is on complete information games but also reports results for incomplete information games. The idea of his results is that more correlation possibilities are better for the set of BNE that can be supported. To state Gossner's result, write $BNE(G, S)$ for the set of BNE action state distributions of (G, S) (see Definition 6), i.e., the set of distributions on $A \times \Theta$ that can be induced by a BNE of (G, S) .

Definition 24 *Information system S is BNE-larger than information system S' if $BNE(G, S') \subseteq BNE(G, S)$ for all basic games G .*

An independent garbling ϕ is *faithful* if whenever for each i , $t_i \in T_i$ and $t'_i \in T'_i$ with $\phi_i(t'_i|t_i) > 0$, we have

$$\frac{\psi(\theta) \pi'((t'_i, t'_{-i})|\theta)}{\sum_{\tilde{t}'_{-i} \in T'_{-i}, \tilde{\theta} \in \Theta} \psi(\tilde{\theta}) \pi'((t'_i, \tilde{t}'_{-i})|\tilde{\theta})} = \frac{\psi(\theta) \sum_{t_{-i} \in T_{-i}} \pi((t_i, t_{-i})|\theta) \left(\prod_{j \neq i} \phi_j(t'_j|t_j) \right)}{\sum_{t_{-i} \in T_{-i}, \tilde{\theta} \in \Theta} \psi(\tilde{\theta}) \pi((t_i, t_{-i})|\tilde{\theta})}$$

for all $t'_{-i} \in T'_{-i}$ and $\theta \in \Theta$.

Definition 25 *Information system S' is a faithful independent garbling of S if there exists a faithful independent garbling ϕ that transforms S into S' .*

Intuitively, this states that information system S allows more correlation possibilities than S' but does not give more information about beliefs and higher order beliefs about payoff relevant states. Now we have:

Proposition 2 *Information system S is BNE-larger than S' if and only if S' is a faithful independent garbling of S .*

This is Theorem 19 in Gossner (2000). [In the briefly described (Section 6) statement of Gossner's result, his definition of BNE-larger ("richer" in his language) refers only to distributions over action profiles, and not over action profiles and Θ ; however his arguments would apply the above result.] An interesting special case is when S' is uninformative, i.e., contains neither information about Θ nor correlation opportunities, so that there exist, for each i , $\lambda_i \in \Delta(T'_i)$ such that

$$\pi'(t'|\theta) = \prod_{i=1}^I \lambda_i(t'_i)$$

for all $t' \in T'$ and $\theta \in \Theta$. In this case, $BNE(G, S')$ is just equal to the independent distributions over actions generated by Nash equilibria in the basic game G . This S' is a faithful independent garbling of S for *any* S which is not informative about Θ : simply set

$$\phi(t'|t) = \prod_{i=1}^I \lambda_i(t'_i)$$

for all $t' \in T'$ and $\theta \in \Theta$. Now $BNE(G, S)$ contains $BNE(G, S')$ because there are weakly more correlation possibilities in S .

5.2 More Information Reduces the set of Bayes Correlated Equilibria

We present a new result showing that more information reduces the set of Bayes Correlated Equilibria. Note that this result seems to go in the opposite direction to Gossner's result, as we are seeing that more information rules out more outcomes. The explanation for this apparent contradiction is that by using BCE as a solution concept, the players can correlate their behavior for free in equilibria and the only impact of more information is to impose more incentive constraints.

The relevant formalization of less information is as follows:

Definition 26 *Information system S' is less informed than S if there exist $\sigma : T \times \Theta \rightarrow \Delta(T')$ and, for each i , $\phi_i : T_i \rightarrow \Delta(T'_i)$, such that*

$$\pi'(t'|\theta) = \sum_{t \in T} \sigma(t'|t, \theta) \pi(t|\theta)$$

for each $t' \in T'$ and $\theta \in \Theta$, satisfying also that for each $i = 1, \dots, I$, $t_i \in T_i$, $t'_i \in T'_i$,

$$\sum_{t'_{-i} \in T'_{-i}} \sigma((t'_i, t'_{-i}) | (t_i, t_{-i}), \theta) = \phi_i(t'_i | t_i)$$

for all $t_{-i} \in T_{-i}$ and $\theta \in \Theta$.

Note that if S' is a non-communicating garbling of S exactly if the above definition is satisfied but with the function σ not dependent on Θ . Thus if S' is a non-communicating garbling of S , then S' is less informed than S . But a robust example in the Appendix (Section 9.1) shows that the converse is not true. Now we have:

Definition 27 *Information system S' is BCE-larger than information system S if $BCE(G, S) \subseteq BCE(G, S')$ for all games G .*

Theorem 1 *S' is BCE-larger than S if and only if S' is less informed than S .*

A sketch of the proof is the Appendix (Section 9.2). The argument involves relating the higher order beliefs about Θ under the two information systems. See Tang (2010) for more discussion of this issue.

6 The Robust Predictions Agenda

We will report a couple of examples to illustrate the logic of our approach.

6.1 One Player, Binary State, Binary Action Games

Suppose there is one player and two states. There are two states, $\Theta = \{\theta_0, \theta_1\}$. Consider the game G with $A = \{a_0, a_1\}$; $u(a_0, \theta_0) = \kappa$, $u(a_1, \theta_1) = 1 - \kappa$ and $u(a_0, \theta_1) = u(a_1, \theta_0) = 0$; and $\psi(\theta_0) = \xi$ and $\psi(\theta_1) = 1 - \xi$. Consider an arbitrary information system $S = (T, \pi)$ and write $\pi_k(t)$ for the probability of signal t in state θ_k .

We are interested in Bayes Correlated Equilibria of the game (G, S) . In the special case where S is the null information system, this reduces to the problem of finding outcomes of a sender receiver game where we can exogenously choose the sender strategy. This is thus closely related to the problem studied by Gentzkow and Kaminca (2010), where senders are allowed to commit to a communication strategy.

Suppose that the mediator recommends action a_1 if the player observes signal t in state θ_k with probability $\beta_k(t)$ (and thus a_0 with probability $1 - \beta_k(t)$). Thus the mediator's behavior is given by (β_1, β_2) with each $\beta_k : T \rightarrow [0, 1]$. Now if the player observes signal t and is advised to take action a_1 , he attaches probability

$$\frac{\xi \pi_0(t) \beta_0(t)}{\xi \pi_0(t) \beta_0(t) + (1 - \xi) \pi_1(t) \beta_1(t)}$$

to state θ_0 and thus follows the recommendation if

$$(1 - \xi) \pi_1(t) \beta_1(t) (1 - \kappa) \geq \xi \pi_0(t) \beta_0(t) \kappa \quad (11)$$

or

$$\beta_1(t) \geq \left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \beta_0(t). \quad (12)$$

If the player observes signal t and is advised to take action a_0 , he attaches probability

$$\frac{\xi \pi_0(t) (1 - \beta_0(t))}{\xi \pi_0(t) (1 - \beta_0(t)) + (1 - \xi) \pi_1(t) (1 - \beta_1(t))}$$

to state θ_0 and thus follows the recommendation if

$$(1 - \xi) \pi_1(t) (1 - \beta_1(t)) (1 - \kappa) \leq \xi \pi_0(t) (1 - \beta_0(t)) \kappa$$

or

$$(1 - \xi) \pi_1(t) \beta_1(t) (1 - \kappa) \geq \xi \pi_0(t) \beta_0(t) \kappa + (1 - \xi) \pi_1(t) (1 - \kappa) - \xi \pi_0(t) \kappa \quad (13)$$

or

$$\beta_1(t) \geq \left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \beta_0(t) + \left(1 - \left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \right). \quad (14)$$

Now the two obedience constraints (12) and (14) can be combined in the constraint that

$$\beta_1(t) \geq \left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \beta_0(t) + \max \left(0, 1 - \left(\frac{\kappa}{1 - \kappa} \right) \left(\frac{\xi}{1 - \xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \right). \quad (15)$$

Now distribution $\nu \in \Delta(A \times T \times \Theta)$ is a Bayes Correlated Equilibrium if and only if

$$\nu(a, t, \theta) = \begin{cases} (1 - \xi) \pi_1(t) \beta_1(t), & \text{if } (a, \theta) = (a_1, \theta_1) \\ (1 - \xi) \pi_1(t) (1 - \beta_1(t)), & \text{if } (a, \theta) = (a_0, \theta_1) \\ \xi \pi_0(t) \beta_0(t), & \text{if } (a, \theta) = (a_1, \theta_0) \\ \xi \pi_0(t) (1 - \beta_0(t)), & \text{if } (a, \theta) = (a_0, \theta_0) \end{cases}$$

for some (β_1, β_2) satisfying (15).

To understand how the set of BCE vary with different information structures, we can consider some extreme points. Consider the player's ex ante utility

$$\sum_{t \in T} (\xi \kappa \pi_0(t) (1 - \beta_0(t)) + (1 - \xi) (1 - \kappa) \pi_1(t) \beta_1(t)).$$

This is maximized by setting $\beta_0(t) = 0$ and $\beta_1(t) = 1$ for all $t \in T$, giving maximum ex ante utility

$$\bar{U}(S) = \xi\kappa + (1 - \xi)(1 - \kappa).$$

We write this as a function of the information system S , although it turns out to be independent of the information system. Now let's find the BCE minimizing the player's ex ante utility. From (11) and (13), we have that

$$(1 - \xi)\pi_1(t)\beta_1(t)(1 - \kappa) - \xi\pi_0(t)\beta_0(t)\kappa \geq \max\{0, (1 - \xi)\pi_1(t)(1 - \kappa) - \xi\pi_0(t)\kappa\} \quad (16)$$

Thus the utility minimizing BCE is attained by setting $\beta_0(t) = \beta_1(t) = 0$ if

$$\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) \leq 1$$

and $\beta_0(t) = \beta_1(t) = 1$ if

$$\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1$$

This gives minimum ex ante utility

$$\underline{U}(S) = \xi\kappa \Pr\left(\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1\right) + (1 - \xi)(1 - \kappa) \Pr\left(\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) \leq 1\right).$$

Notice that more information will increase the minimum ex ante utility and not change the maximum ex ante utility.

Now consider the probability that action a_1 is chosen,

$$\sum_{t \in T} (\xi\pi_0(t)\beta_0(t) + (1 - \xi)\pi_1(t)\beta_1(t)).$$

This is maximized by setting $\beta_0(t) = \beta_1(t) = 1$ if

$$\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) \leq 1$$

and $\beta_1(t) = 1$ and $\beta_0(t)$ solves

$$\beta_0(t) = \left(\frac{1 - \kappa}{\kappa}\right) \left(\frac{1 - \xi}{\xi}\right) \left(\frac{\pi_1(t)}{\pi_0(t)}\right) \quad (17)$$

otherwise. Thus the maximum probability of action a_1 in a BCE is

$$\bar{\Pi}(S) = 1 - \Pr\left(\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1\right) \left(1 - \left(\frac{1 - \kappa}{\kappa}\right) \left(\frac{1 - \xi}{\xi}\right) \left(\frac{\pi_1(t)}{\pi_0(t)}\right)\right) \left(\frac{\xi\pi_0(t)}{\xi\pi_0(t) + (1 - \xi)\pi_1(t)}\right)$$

This is minimized by setting $\beta_0(t) = \beta_1(t) = 0$ if

$$\left(\frac{\kappa}{1 - \kappa}\right) \left(\frac{\xi}{1 - \xi}\right) \left(\frac{\pi_0(t)}{\pi_1(t)}\right) > 1$$

and $\beta_0(t) = 0$ and $\beta_1(t)$ solves

$$\beta_1(t) = 1 - \left(\frac{\kappa}{1-\kappa} \right) \left(\frac{\xi}{1-\xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \quad (18)$$

otherwise. Thus the minimum probability of action a_1 in a *BCE* is

$$\underline{\Pi}(S) = \Pr \left(\left(\frac{\kappa}{1-\kappa} \right) \left(\frac{\xi}{1-\xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \leq 1 \right) \left(1 - \left(\frac{\kappa}{1-\kappa} \right) \left(\frac{\xi}{1-\xi} \right) \left(\frac{\pi_0(t)}{\pi_1(t)} \right) \right) \left(\frac{(1-\xi)\pi_1(t)}{\xi\pi_0(t) + (1-\xi)\pi_1(t)} \right)$$

6.2 First Price Auctions

We consider a discretized first price private value auction with independent uniform priors. Suppose there are two players and K^2 states, $\Theta = \left\{ \frac{1}{K}, \frac{2}{K}, \dots, \frac{K-1}{K}, 1 \right\}^2$ with typical element $\theta = (\theta_1, \theta_2) \in \Theta$. Consider the game G with $A_1 = A_2 = \left\{ 0, \frac{1}{M}, \frac{2}{M}, \dots, \frac{M-1}{M}, 1 \right\}$; $\psi(\theta) = \frac{1}{K^2}$ for each $\theta \in \Theta$;

$$u_i((a_i, a_j), (\theta_i, \theta_j)) = \begin{cases} \theta_i - a_i, & \text{if } \theta_i > \theta_j \\ \frac{1}{2}(\theta_i - a_i), & \text{if } \theta_i = \theta_j \\ 0, & \text{if } \theta_i < \theta_j \end{cases}.$$

The information structure S has each player i observing his private value θ_i . Formally, we have $S = \left((T_i)_{i=1,2}, \pi \right)$ where $T_1 = T_2 = \left\{ \frac{1}{K}, \frac{2}{K}, \dots, \frac{K-1}{K}, 1 \right\}$ and

$$\pi(t|\theta) = \begin{cases} 1, & \text{if } t = \theta \\ 0, & \text{otherwise} \end{cases}.$$

Now a Bayes Correlated Equilibrium is a distribution $\nu \in \Delta(A \times T \times \Theta)$ consistency

$$\sum_{a \in A} \nu(a, t, \theta) = \begin{cases} \frac{1}{K^2}, & \text{if } t = \theta \\ 0, & \text{otherwise} \end{cases}$$

for all t and θ ; and obedience

$$\sum_{a_j, t_j, \theta_j} \nu((a_i, a_j), (t_i, t_j), (\theta_i, \theta_j)) u_i((a_i, a_j), (\theta_i, \theta_j)) \geq \sum_{a_j, t_j, \theta_j} \nu((a_i, a_j), (t_i, t_j), (\theta_i, \theta_j)) u_i((a'_i, a_j), (\theta_i, \theta_j))$$

for each i , θ_i, t_i, a_i and a'_i .

The following simple example illustrates the difference in the bidding behavior in the Bayes Nash Equilibria (BNE) and Bayes Correlated Equilibria (BCE) of the game (G, S) with a payoff type space and a rich type space. Let $K = 3$ and $M = 8$. The unique BNE is given by a symmetric pure strategy profile

displayed in the left matrix below:

$a_i \setminus \theta_i$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$
0	0	0	0
$\frac{1}{8}$	0	0	0
$\frac{2}{8}$	1	0	0
$\frac{3}{8}$	0	1	0
$\frac{4}{8}$	0	0	1

$a_i \setminus \theta_i$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{3}{3}$
0	0.18	0	0
$\frac{1}{8}$	0.56	0	0
$\frac{2}{8}$	0.26	0.50	0
$\frac{3}{8}$	0	0.50	0.70
$\frac{4}{8}$	0	0	0.30

The entry in each cell is the conditional probability, $\sigma_i(a_i | \theta_i)$, that agent i with value θ_i submits a bid a_i . In contrast, the revenue minimizing BCE gives rise to the conditional distributions of bids a_i given values θ_i described the above right matrix. We observe that the average bid in the revenue minimizing BCE is strictly below the bid in the BNE. The revenue in the BCE is given by 0.33 whereas in the BNE it is 0.43. (The examples were computed with programs written for Matlab.)

7 Anonymous Games

In this section, we specialize our analysis to the case of anonymous games, where each player is symmetric in payoffs and information, so that players' labels are assumed to not matter for either the description of the game or their choice of strategy. Once we have an anonymous finite player, finite action, finite state version of Bayes Correlated Equilibrium, it is then possible to present analogue results for continuum player, continuum action, continuum state games. This is the foundation for our quadratic normal modelling in Bergemann and Morris (2011).

7.1 The Finite Case

As before, there are I players and finite state space Θ . A "basic game" G now consists of (1) a common action set A ; (2) a common utility function $u : A \times \Delta_I(A) \times \Theta \rightarrow \mathbb{R}$; , where $u(a, h, \theta)$ is a player's payoff if he chooses action a , the distribution of actions among the I players is $h \in \Delta_I(A)$ and the state is θ . (For any finite set X , we write $\Delta_I(X)$ for the set of probability distributions on X with support on $(0, \frac{1}{I}, \frac{2}{I}, \dots, 1)$); and (3) a full support prior $\psi \in \Delta(\Theta)$. Thus a basic game $G = (A, u, \psi)$. An "information structure" S now consists of (1) a common set of types or "signals" T ; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(\Delta_I(T))$. Now $\pi(\theta) \in \Delta(\Delta_I(T))$ is a probability distribution over the realized distribution of signals in the population. Thus $S = (T, \pi)$. Now (G, S) describes a standard (anonymous) Bayesian game.

If $\xi \in \Delta_I(A \times T)$ is a distribution over action-signal pairs, write $\text{marg}_T \xi \in \Delta_I(T)$ for the marginal distribution over signals, so

$$\text{marg}_T \xi(t) = \sum_{a \in A} \xi(a, t)$$

for each $t \in T$; write $\text{marg}_A \xi \in \Delta_I(A)$ for the marginal distribution over actions, so

$$\text{marg}_A \xi(a) = \sum_{t \in T} \xi(a, t)$$

for each $a \in A$. If $\nu \in \Delta(\Delta_I(A \times T) \times \Theta)$ is a distribution over action-signal pair distributions and states, write $\text{marg}_{\Delta_I(T) \times \Theta} \nu \in \Delta(\Delta_I(T) \times \Theta)$ for the marginal distribution over realized distributions of signals and states, so

$$\text{marg}_{\Delta_I(T) \times \Theta} \nu(g, \theta) = \sum_{\{\xi \in \Delta_I(A \times T) : \text{marg}_T \xi = g\}} \nu(\xi, \theta)$$

for each $g \in \Delta_I(T)$ and $\theta \in \Theta$. Finally, write $\pi \circ \psi$ for the probability distribution on $\Delta_I(T) \times \Theta$ induced by $\psi \in \Delta(\Theta)$ and $\pi : \Theta \rightarrow \Delta(\Delta_I(T))$, so

$$\pi \circ \psi(g, \theta) = \psi(\theta) \pi(g|\theta)$$

for each $g \in \Delta_I(T)$ and $\theta \in \Theta$.

Definition 28 (Bayes Correlated Equilibrium)

A probability distribution $\nu \in \Delta(\Delta_I(A \times T) \times \Theta)$ is a Bayes Correlated Equilibrium (BCE) of (G, S) if

$$\sum_{\xi \in \Delta_I(A \times T), \theta \in \Theta} u(a, \text{marg}_A \xi, \theta) \xi(a, t) \nu(\xi, \theta) \geq \sum_{\xi \in \Delta_I(A \times T), \theta \in \Theta} u(a', \text{marg}_A \xi, \theta) \xi(a, t) \nu(\xi, \theta); \quad (19)$$

for each $t \in T$, $a \in A$ and $a' \in A$; and

$$\text{marg}_{\Delta_I(T) \times \Theta} \nu = \pi \circ \psi. \quad (20)$$

In the special case of a null information system (so there are no signals), then the obedience condition (19) for $\mu \in \Delta(\Delta_I(A) \times \Theta)$ will be

$$\sum_{g \in \Delta_I(A), \theta \in \Theta} u(a, g, \theta) g(a) \mu(g, \theta) \geq \sum_{g \in \Delta_I(A), \theta \in \Theta} u(a', g, \theta) g(a) \mu(g, \theta);$$

for each $a \in A$ and $a' \in A$ while the consistency condition (20) will be

$$\text{marg}_\Theta \mu = \psi. \quad (21)$$

7.2 The Continuum Case

There is a continuum $[0, 1]$ of players and state space Θ . A "basic game" G now consists of (1) a common action set A ; (2) a common utility function $u : A \times \Delta(A) \times \Theta \rightarrow \mathbb{R}$; where $u(a, h, \theta)$ is a player's payoff if he chooses action a , the distribution of actions among the continuum players is $h \in \Delta(A)$ and the state is θ ; and (3) a full support prior $\psi \in \Delta(\Theta)$. Thus $G = (A, u, \psi)$. An "information structure" S now consists of (1) a common set of types or "signals" T ; and (2) a signal distribution $\pi : \Theta \rightarrow \Delta(\Delta(T))$. Now $\pi(\theta) \in \Delta(\Delta(T))$ is a probability distribution over realized distributions of signals in the population. Thus $S = (T, \pi)$. Now (G, S) describes a standard continuum (anonymous) Bayesian game.

Now the definitions for the continuum case are as before, except that distributions are over a continuum population and summations are replaced with integrals. We omit the measurability conditions that will be required in general (they are not an issue for applications using densities).

As before, if $\xi \in \Delta(A \times T)$ is a distribution over action-signal pairs, write $\text{marg}_T \xi(t) \in \Delta_I(T)$ and $\text{marg}_A \xi \in \Delta_I(A)$ for the marginal distributions over signals and actions respectively. If $\nu \in \Delta(\Delta(A \times T) \times \Theta)$, write $\text{marg}_{\Delta(T) \times \Theta} \nu \in \Delta(\Delta(T) \times \Theta)$ for the marginal distribution over realized distributions of signals and states. Write $\pi \circ \psi$ for the probability distribution on $\Delta(T) \times \Theta$ induced by $\psi \in \Delta(\Theta)$ and $\pi : \Theta \rightarrow \Delta(\Delta(T))$.

Definition 29 (Bayes Correlated Equilibrium)

A probability distribution $\nu \in \Delta(\Delta(A \times T) \times \Theta)$ is a Bayes Correlated Equilibrium (BCE) of (G, S) if

$$\int_{\xi \in \Delta(A \times T), \theta \in \Theta} u(a, \text{marg}_A \xi, \theta) \xi(a, t) d\nu \geq \int_{\xi \in \Delta(A \times T), \theta \in \Theta} u(a', \text{marg}_A \xi, \theta) \xi(a, t) d\nu; \quad (22)$$

for each $t \in T$, $a \in A$ and $a' \in A$; and

$$\text{marg}_{\Delta(T) \times \Theta} \nu = \pi \circ \psi. \quad (23)$$

In the special case of a null information system (so there are no signals), then the obedience condition (22) for $\mu \in \Delta(\Delta(A) \times \Theta)$ will be

$$\int_{g \in \Delta(T), \theta \in \Theta} u(a, g, \theta) g(a) d\mu \geq \int_{g \in \Delta(T), \theta \in \Theta} u(a', g, \theta) g(a) d\mu;$$

for each $a \in A$ and $a' \in A$ while the consistency condition (23) will be

$$\text{marg}_{\Theta} \mu = \psi.$$

8 Discussion

8.1 Payoff Type Spaces

In Bergemann and Morris (2005) and later work, we studied a robust mechanism environments in a setting where agents knew their own "payoff types", there was common knowledge of how utilities depended on the profile of payoff types, but agents were allowed to have any beliefs and higher order beliefs about others' payoff types. In Bergemann and Morris (2007), we discussed a game theoretic framework underlying this work. Here we briefly how this environment maps into the setting of this paper.

Suppose that Θ is a product space with $\Theta = \Theta_1 \times \dots \times \Theta_I$. Consider the special information structure where agent i 's set of possible signals is Θ_i , and each agent i observes the realization $\theta_i \in \Theta_i$, so $S^{**} = \left((\Theta_i)_{i=1}^I, id \right)$, where id is the identity map $id : \Theta \rightarrow \Theta$ with $id(\theta) = \theta$ for all θ . Now the set of Bayes Correlated Equilibria of a game (G, S) describe all the distributions over payoff type profiles and actions consistent with the common prior and common knowledge of rationality. Bergemann and Morris (2007) - in the language of this paper - is an analysis of the structure of Bayes Correlated Equilibria with the special information structure S^{**} .

8.2 Signed Covariance

Chwe (2006) analyzes statistical implications of incentive compatibility in general, and in particular statistical implications of correlated equilibrium play. We can state his main observation in the language of our paper. Fix any basic game G . Fix any Bayes Correlated Equilibrium $\mu \in \Delta(A \times \Theta)$ of the basic game (i.e., the game with the null information structure). Fix a player i and action $a_i^* \in A_i$. Consider the random variable $I_{a_i^*}$ on $A \times \Theta$ that takes value 1 if a_i^* is played and 0 otherwise. Fix any other action $a_i' \in A_i$. Let $\Pi_{a_i^*, a_i'}$ be the random variable on $A \times \Theta$ equal to the payoff gain to player i of choosing action a_i^* rather than a_i' . Then, conditional on a_i^* or a_i' being played, the random variables $I_{a_i^*}$ and $\Pi_{a_i^*, a_i'}$ have positive covariance. This is the content of the main result in Chwe (2006). As he notes, this is not merely a re-writing of the incentive compatibility constraints, since these are linear in probabilities while the covariance is quadratic in probabilities. Thus his signed conditional covariance result is a necessary property of second order statistics of a Bayes Correlated Equilibrium.

We sketch a formal statement and proof. The formal definitions of the random variables $I_{a_i^*}$ and $\Pi_{a_i^*, a_i'}$ are

$$\begin{aligned} I_{a_i^*}(a, \theta) &= \begin{cases} 1, & \text{if } a_i = a_i^* \\ 0, & \text{otherwise} \end{cases} \\ \Pi_{a_i^*, a_i'}(a, \theta) &= u_i((a_i^*, a_{-i}), \theta) - u_i((a_i', a_{-i}), \theta) \end{aligned}$$

The expectations of $I_{a_i^*}$, $\Pi_{a_i^*, a_i'}$ and their product, under μ , conditional on the event $\{a_i^*, a_i'\}$ occurring, are:

$$\begin{aligned}
E_\mu(I_{a_i^*} | \{a_i^*, a_i'\}) &= \frac{\sum_{a_{-i}, \theta} \mu((a_i^*, a_{-i}), \theta)}{\sum_{a_{-i}, \theta} \mu((a_i^*, a_{-i}), \theta) + \sum_{a_{-i}, \theta} \mu((a_i', a_{-i}), \theta)} \\
E_\mu(\Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) &= \frac{\left\{ \begin{aligned} &\sum_{a_{-i}, \theta} \mu((a_i^*, a_{-i}), \theta) (u_i((a_i^*, a_{-i}), \theta) - u_i((a_i', a_{-i}), \theta)) \\ &+ \sum_{a_{-i}, \theta} \mu((a_i', a_{-i}), \theta) (u_i((a_i^*, a_{-i}), \theta) - u_i((a_i', a_{-i}), \theta)) \end{aligned} \right\}}{\sum_{a_{-i}, \theta} \mu((a_i^*, a_{-i}), \theta) + \sum_{a_{-i}, \theta} \mu((a_i', a_{-i}), \theta)} \\
E_\mu(I_{a_i^*} \Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) &= \frac{\sum_{a_{-i}, \theta} \mu((a_i^*, a_{-i}), \theta) (u_i((a_i^*, a_{-i}), \theta) - u_i((a_i', a_{-i}), \theta))}{\sum_{a_{-i}, \theta} \mu((a_i^*, a_{-i}), \theta) + \sum_{a_{-i}, \theta} \mu((a_i', a_{-i}), \theta)}
\end{aligned}$$

Now the incentive compatibility condition that that player i prefers a_i^* to a_i' when advised to play a_i^* can be written as

$$E_\mu(I_{a_i^*} \Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) \geq 0 \quad (24)$$

while that incentive compatibility condition that player i prefers a_i' to a_i^* when advised to play a_i' is

$$E_\mu((1 - I_{a_i^*}) \Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) \leq 0$$

which can be re-written as

$$E_\mu(I_{a_i^*} \Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) \geq E_\mu(\Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}). \quad (25)$$

Now the covariance of $I_{a_i^*}$ and $\Pi_{a_i^*, a_i'}$, conditional on $\{a_i^*, a_i'\}$, is

$$E_\mu(I_{a_i^*} \Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) - E_\mu(I_{a_i^*} | \{a_i^*, a_i'\}) E_\mu(\Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\})$$

If $E_\mu(\Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) \geq 0$, (24) and $E_\mu(I_{a_i^*} | \{a_i^*, a_i'\}) \geq 0$ imply that this is non-negative. If $E_\mu(\Pi_{a_i^*, a_i'} | \{a_i^*, a_i'\}) \leq 0$, (25) and $E_\mu(I_{a_i^*} | \{a_i^*, a_i'\}) \leq 1$ imply that this is non-negative.

9 Appendix

9.1 Example

The following is a robust example of non-redundant information systems where S' is not a non-communicating garbling of S in the sense of Lehrer, Rosenberg, and Shmaya (2011) and Lehrer, Rosenberg, and Shmaya (2010), but S' is less informed than S in the sense of Definition 26 and thus - by Theorem 1 - S' is BCE-richer than S .

Suppose that there is uniform prior on $\Theta = \{\theta_1, \theta_2\}$. Information system S has $T_1 = \{t_{11}, t_{12}\}$, $T_2 = \{t_{21}, t_{22}\}$ and $\pi : \Theta \rightarrow \Delta(T)$ given by

$\pi(t \theta_1)$	t_{21}	t_{22}	$\pi(t \theta_2)$	t_{21}	t_{22}
t_{11}	$\frac{4}{9}$	$\frac{2}{9}$	t_{11}	$\frac{1}{9}$	$\frac{1}{9}$
t_{12}	$\frac{2}{9}$	$\frac{1}{9}$	t_{12}	$\frac{2}{9}$	$\frac{2}{9}$

This information system simply has each agent observing a conditionally independent signal with "accuracy" $\frac{2}{3}$.

Information system S' has $T'_1 = \{t'_{11}, t'_{12}\}$, $T'_2 = \{t'_{21}, t'_{22}\}$ and $\pi' : \Theta \rightarrow \Delta(T')$ given by

$\pi'(t' \theta_1)$	t'_{21}	t'_{22}	$\pi'(t' \theta_2)$	t'_{21}	t'_{22}
t'_{11}	$\frac{13}{27}$	$\frac{2}{27}$	t'_{11}	$\frac{1}{27}$	$\frac{11}{27}$
t'_{12}	$\frac{2}{27}$	$\frac{10}{27}$	t'_{12}	$\frac{11}{27}$	$\frac{4}{27}$

Consider the following mapping $\sigma : T \times \Theta \rightarrow \Delta(T')$.

$\sigma(t' t, \theta)$	(t'_{11}, t'_{21})	(t'_{11}, t'_{22})	(t'_{12}, t'_{21})	(t'_{12}, t'_{22})
$(\theta_1, t_{11}, t_{21})$	$\frac{2}{3}$	0	0	$\frac{1}{3}$
$(\theta_1, t_{11}, t_{22})$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$
$(\theta_1, t_{12}, t_{21})$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
$(\theta_1, t_{12}, t_{22})$	$\frac{1}{3}$	0	0	$\frac{2}{3}$
$(\theta_2, t_{11}, t_{21})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0
$(\theta_2, t_{11}, t_{22})$	0	$\frac{2}{3}$	$\frac{1}{3}$	0
$(\theta_2, t_{12}, t_{21})$	0	$\frac{1}{3}$	$\frac{2}{3}$	0
$(\theta_2, t_{12}, t_{22})$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

Observe first that if θ is drawn according to its uniform prior, t is drawn according to S and t' is drawn

according to σ , we get the following distribution $\nu \in \Delta(T' \times T \times \Theta)$:

$\nu(t', t, \theta)$	(t'_{11}, t'_{21})	(t'_{11}, t'_{22})	(t'_{12}, t'_{21})	(t'_{12}, t'_{22})
$(\theta_1, t_{11}, t_{21})$	$\frac{8}{54}$	0	0	$\frac{4}{54}$
$(\theta_1, t_{11}, t_{22})$	$\frac{2}{54}$	$\frac{2}{54}$	0	$\frac{2}{54}$
$(\theta_1, t_{12}, t_{21})$	$\frac{2}{54}$	0	$\frac{2}{54}$	$\frac{2}{54}$
$(\theta_1, t_{12}, t_{22})$	$\frac{1}{54}$	0	0	$\frac{2}{54}$
$(\theta_2, t_{11}, t_{21})$	$\frac{1}{54}$	$\frac{1}{54}$	$\frac{1}{54}$	0
$(\theta_2, t_{11}, t_{22})$	0	$\frac{4}{54}$	$\frac{2}{54}$	0
$(\theta_2, t_{12}, t_{21})$	0	$\frac{2}{54}$	$\frac{4}{54}$	0
$(\theta_2, t_{12}, t_{22})$	0	$\frac{4}{54}$	$\frac{4}{54}$	$\frac{4}{54}$

Observe that the marginal of ν on $T' \times \Theta$ is:

$\text{marg}_\nu(t', \theta_1)$	t'_{21}	t'_{22}	$\text{marg}_\nu(t', \theta_2)$	t'_{21}	t'_{22}
t'_{11}	$\frac{13}{54}$	$\frac{2}{54}$	t'_{11}	$\frac{1}{54}$	$\frac{11}{54}$
t'_{12}	$\frac{2}{54}$	$\frac{10}{54}$	t'_{12}	$\frac{11}{54}$	$\frac{4}{54}$

This is distribution resulting from drawing θ according to the uniform prior and drawing t' according to S' .

Also observe that σ satisfies the property that there exist $\phi_1 : T_1 \rightarrow \Delta(T'_1)$ and $\phi_2 : T_2 \rightarrow \Delta(T'_2)$ such that

$$\begin{aligned} \sum_{t'_2} \sigma((t'_1, t'_2) | (t_1, t_2), \theta) &= \phi_1(t'_1 | t_1) \text{ for each } t_2 \text{ and } \theta \\ \sum_{t'_1} \sigma((t'_1, t'_2) | (t_1, t_2), \theta) &= \phi_2(t'_2 | t_2) \text{ for each } t_1 \text{ and } \theta \end{aligned}$$

This is true for the following (ϕ_1, ϕ_2) :

$\phi_1(t'_1 t_1)$	t'_{11}	t'_{12}	$\phi_2(t'_2 t_2)$	t'_{21}	t'_{22}
t_{11}	$\frac{2}{3}$	$\frac{1}{3}$	t_{21}	$\frac{2}{3}$	$\frac{1}{3}$
t_{12}	$\frac{1}{3}$	$\frac{2}{3}$	t_{22}	$\frac{1}{3}$	$\frac{2}{3}$

Thus from each individual's point of view, under S' , he is simply observing a noisy version (with accuracy $\frac{2}{3}$) of the original signal (with accuracy $\frac{2}{3}$). Note in particular that this implies that each player i attaches probability $\frac{5}{9}$ to θ_1 if he observe t'_{i1} and probability $\frac{4}{9}$ to θ_1 if he observe t'_{i2} .

We have now established that S' is a "non-coordinated Θ -dependent garbling" of S and thus that, for every game G , the set of Bayes Correlated Equilibria of (G, S') contains the Bayes Correlated Equilibria of (G, S) .

However, S' is not a "non-coordinated garbling" of S . To show this, we would have to show that there exists $\phi : T \rightarrow \Delta(T')$, $\phi_1 : T_1 \rightarrow \Delta(T'_1)$ and $\phi_2 : T_2 \rightarrow \Delta(T'_2)$ such that

$$\pi'(t'|\theta) = \sum_t \phi(t'|t) \pi(t|\theta)$$

and

$$\begin{aligned} \sum_{t'_2} \phi((t'_1, t'_2) | (t_1, t_2)) &= \phi_1(t'_1 | t_1) \text{ for each } t_2 \text{ and } \theta \\ \sum_{t'_1} \phi((t'_1, t'_2) | (t_1, t_2)) &= \phi_2(t'_2 | t_2) \text{ for each } t_1 \text{ and } \theta \end{aligned}$$

In order for this to hold, (ϕ_1, ϕ_2) would have to be as defined above. Let us focus on the probability of a fixed profile of S' signals (t'_{11}, t'_{21}) and write α_{jk} for the probability of (t'_{11}, t'_{21}) conditional on (t_{1j}, t_{2k}) under ϕ , i.e.,

$$\alpha_{jk} = \nu((t'_{11}, t'_{21}) | (t_{1j}, t_{2k}))$$

Now observe that in order to satisfy the above marginal conditions, we must have

$$\begin{aligned} \frac{1}{3} &\leq \alpha_{11} \leq \frac{2}{3} \\ 0 &\leq \alpha_{12} \leq \frac{1}{3} \\ 0 &\leq \alpha_{21} \leq \frac{1}{3} \\ 0 &\leq \alpha_{22} \leq \frac{1}{3} \end{aligned}$$

But

$$\pi'((t'_{11}, t'_{21}) | \theta_1) = \sum_t \phi((t'_{11}, t'_{21}) | t) \pi(t | \theta_1)$$

requires we must have

$$\frac{13}{27} = \frac{4}{9}\alpha_{11} + \frac{2}{9}\alpha_{12} + \frac{2}{9}\alpha_{21} + \frac{1}{9}\alpha_{22}.$$

But this requires $\alpha_{11} = \frac{2}{3}$, $\alpha_{12} = \frac{1}{3}$, $\alpha_{21} = \frac{1}{3}$ and $\alpha_{22} = \frac{1}{3}$. However,

$$\pi'((t'_{11}, t'_{21}) | \theta_2) = \sum_t \phi((t'_{11}, t'_{21}) | t) \pi(t | \theta_2)$$

requires we must have

$$\frac{1}{27} = \frac{1}{9}\alpha_{11} + \frac{2}{9}\alpha_{12} + \frac{2}{9}\alpha_{21} + \frac{4}{9}\alpha_{22}.$$

which is a contradiction.

9.2 Proof of Theorem 1

We will need a number of intermediate results to prove Theorem 1.

9.2.1 Higher Order Beliefs for a Fixed Information System

Fix an information system $S = \left((T_i)_{i=1}^I, \pi \right)$. For a type $t_i \in T_i$, write $\hat{\pi}_i^1[t_i] \in \Delta(\Theta)$ for his posterior under a uniform prior on Θ , so

$$\hat{\pi}_i^1[t_i](\theta) = \frac{\sum_{t_{-i} \in T_{-i}} \pi((t_i, t_{-i}) | \theta)}{\sum_{\theta' \in \Theta, t_{-i} \in T_{-i}} \pi((t_i, t_{-i}) | \theta')}.$$

Write $\Pi_i^1 \subseteq \Delta(\Theta)$ for the range of $\hat{\pi}_i^1$ and π_i^1 for a typical element of Π_i^1 .

Write $\hat{\pi}_i^2(t_i) \in \Delta\left(\Theta \times \left(\times_{j \neq i} \Pi_j^1\right)\right)$ for his belief over Θ and the first order beliefs of other players, so

$$\hat{\pi}_i^2[t_i](\theta, \pi_{-i}^1) = \frac{\sum_{\{t_{-i} \in T_{-i} | \hat{\pi}_j^1(t_j) = \pi_j^1 \text{ for each } j \neq i\}} \pi((t_i, t_{-i}) | \theta)}{\sum_{\theta' \in \Theta, \{t_{-i} \in T_{-i} | \hat{\pi}_j^1(t_j) = \pi_j^1 \text{ for each } j \neq i\}} \pi((t_i, t_{-i}) | \theta')}.$$

Write $\Pi_i^2 \subseteq \Delta\left(\Theta \times \left(\times_{j \neq i} \Pi_j^1\right)\right)$ for the range of $\hat{\pi}_i^2$ and π_i^2 for a typical element of Π_i^2 .

Proceeding inductively for $k \geq 2$, write $\hat{\pi}_i^k(t_i) \in \Delta\left(\Theta \times \left(\times_{j \neq i} \Pi_j^{k-1}\right)\right)$ for his belief over Θ and the $(k-1)$ th order beliefs of other players, so

$$\hat{\pi}_i^k[t_i](\theta, \pi_{-i}^{k-1}) = \frac{\sum_{\{t_{-i} \in T_{-i} | \hat{\pi}_j^{k-1}(t_j) = \pi_j^{k-1} \text{ for each } j \neq i\}} \pi((t_i, t_{-i}) | \theta)}{\sum_{\theta' \in \Theta, \{t_{-i} \in T_{-i} | \hat{\pi}_j^{k-1}(t_j) = \pi_j^{k-1} \text{ for each } j \neq i\}} \pi((t_i, t_{-i}) | \theta')}.$$

Write $\Pi_i^k \subseteq \Delta\left(\Theta \times \left(\times_{j \neq i} \Pi_j^{k-1}\right)\right)$ for the range of $\hat{\pi}_i^k$ and π_i^k for a typical element of Π_i^k .

Each $\hat{\pi}_i^k$ generates a partition T_i which becomes more refined as k increases. Since each T_i is finite, the information system has a *depth* K , so that the depth of the information system S is smallest integer K such that

$$\hat{\pi}_i^k(t_i) = \hat{\pi}_i^k(t'_i) \Leftrightarrow \hat{\pi}_i^K(t_i) = \hat{\pi}_i^K(t'_i)$$

for all $k \geq K$. Let $\pi_i^*[t_i]$ be a list of the first K th level beliefs of player i , so

$$\pi_i^*[t_i] = \left(\hat{\pi}_i^k[t_i] \right)_{k=1}^K.$$

Let $\Pi_i^* \subseteq \prod_{k=1, \dots, K} \Pi_i^k$ be the range of π_i^* .

A type space is non-redundant if each type has distinct higher order beliefs, i.e., $t_i \neq t'_i \implies \hat{\pi}_i^K(t_i) \neq \hat{\pi}_i^K(t'_i)$.

For a type $t_i \in T_i$, write $\hat{\pi}_i^1[t_i] \in \Delta(\Theta \times T_{-i})$ for his posterior under a uniform prior on $\Theta \times T_{-i}$, so

$$\hat{\pi}_i^1[t_i](\theta) = \frac{\pi((t_i, t_{-i})|\theta)}{\sum_{\theta' \in \Theta, t'_{-i} \in T_{-i}} \pi((t_i, t'_{-i})|\theta')}.$$

Observe that if information system is non-redundant, then by construction we have that for each i and $t_i, t'_i \in T_i$, $\hat{\pi}_i(t_i) \neq \hat{\pi}_i(t'_i)$.

For any redundant information system, we can construct a canonical "reduced information system" which is non-redundant. Let T_i^* be the (finite) range of $\hat{\pi}_i^\infty$. Define $\pi^{**} : \Theta \rightarrow T^*$ by

$$\pi^{**}(t^*|\theta) = \sum_{\{t \in T | \hat{\pi}_i^K(t_i) = t_i^* \text{ for each } i\}} \pi(t|\theta).$$

By construction, each type in $t_i \in T_i$, will have the same higher order beliefs and type $\hat{\pi}_i^K(t_i) \in T_i^*$ and each type in T_i^* will have distinct higher order beliefs.

9.2.2 Higher Order Belief Equivalence

Write $BCE(G, S) \subseteq \Delta(A \times \Theta)$ for the set of BCE action state distributions of (G, S) (see Definition 6), i.e., the set of distributions on $A \times \Theta$ that can be induced by a BCE of (G, S) .

Lemma 3 *If $S^* = ((T_i^*)_{i=1}^I, \pi^*)$ is the reduced information system of $S = ((T_i)_{i=1}^I, \pi)$, then $BCE(G, S^*) = BCE(G, S)$ for all G .*

Proof. Suppose $\nu^* \in \Delta(A \times T^* \times \Theta)$ is a BCE of (G, S^*) . Define $\sigma^* : T^* \times \Theta \rightarrow \Delta(A)$ by

$$\sigma(a|t^*, \theta) = \frac{\nu^*(a, t^*, \theta)}{\sum_{a' \in A} \nu^*(a', t^*, \theta)}$$

and $\nu \in \Delta(A \times T \times \Theta)$ by

$$\nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \sigma(a|\hat{\pi}^\infty(t), \theta)$$

By construction, ν is a BCE of (G, S) which induces the same action state distribution as ν^* .

Suppose $\nu \in \Delta(A \times T \times \Theta)$ is a BCE of (G, S) . Define $\sigma^* : T^* \times \Theta \rightarrow \Delta(A)$ by

$$\sigma(a|t^*, \theta) = \frac{\sum_{\{t \in T | \hat{\pi}_i^\infty(t_i) = t_i^* \text{ for each } i\}} \nu(a, t, \theta)}{\sum_{\{t \in T | \hat{\pi}_i^\infty(t_i) = t_i^* \text{ for each } i\}} \sum_{a' \in A} \nu(a', t, \theta)}$$

and $\nu^* \in \Delta(A \times T^* \times \Theta)$ by

$$\nu^*(a, t, \theta) = \psi(\theta) \pi^*(t^*|\theta) \sigma^*(a|t^*, \theta)$$

By construction, ν is a BCE of (G, S) which induces the same action state distribution as ν^* . ■

Lehrer, Rosenberg, and Shmaya (2011) report a related result as Theorem 6.1.

9.2.3 Higher Order Beliefs Game

For a fixed non-redundant information system $S = \left((T_i)_{i=1}^I, \pi\right)$ and integer ε , we will construct a finite "higher order beliefs game" $G_{S,\varepsilon}$. This is a variation and simplification of such a game used in Dekel, Fudenberg, and Morris (2006). For any finite set X , the Euclidean distance between two points $\zeta, \zeta' \in \Delta(X)$ is defined as

$$\|\zeta - \zeta'\| = \sqrt{\sum_{x \in X} (\zeta(x) - \zeta'(x))^2}$$

A set of probability distributions $\Xi \subseteq \Delta(X)$ is said to be an ε -grid of $\Delta(X)$ if every point in $\Delta(X)$ is within ε of a point in Ξ . Now let A_i^1 be any ε -grid of $\Delta(\Theta)$ including Π_i^1 . Let A_i^2 be any ε -grid of $\Delta\left(\Theta \times \left(\times_{j \neq i} A_j^1\right)\right)$ including Π_i^2 . Inductively, for each $k = 2, \dots, K$, let A_i^k be any ε -grid of $\Delta\left(\Theta \times \left(\times_{j \neq i} A_j^{k-1}\right)\right)$ including Π_i^{k-1} . Let

$$A_i = \times_{k=1, \dots, K} A_i^k.$$

We assume the prior over states is given by the uniform prior ψ_0 , so

$$\psi_0(\theta) = \frac{1}{\#\Theta}$$

for all $\theta \in \Theta$. We want to give players an incentive to truthfully announces their higher order beliefs. We write $a_i = (a_i^1, \dots, a_i^K)$ for a typical element of A_i . Let

$$u_i((a_i, a_{-i}), \theta) = u_i^1(a_i^1, \theta) + \sum_{k=2}^K u_i^k(a_i^k, a_{-i}^{k-1}).$$

Now let

$$u_i^1(a_i^1, \theta) = 2a_i^1(\theta) - \sum_{\theta' \in \Theta} (a_i^1(\theta'))^2$$

and, for $k \geq 2$,

$$u_i^k(a_i^k, a_{-i}^{k-1}) = 2a_i^k(\theta, a_{-i}^{k-1}) - \sum_{a_{-i}^{k-1} \in A_{-i}^{k-1}} \left(a_i^k(a_{-i}^{k-1})\right)^2$$

This completes the description of the game $G_{S,\varepsilon} = \left((\Pi_i^*, u_i)_{i=1}^I, \psi_0\right)$.

Now observe that each player has an incentive to set each a_i^k as close as possible to his true belief over $\Theta \times \left(\times_{j \neq i} A_j^{k-1} \right)$. By the construction of the action space, each a_i^k will always be within ε of the player's true belief.

9.2.4 Playing the Game

Lemma 4 *For every $\varepsilon > 0$, the game $(G_{S,\varepsilon}, S)$ has a BCE where each player always announces his true higher order beliefs, i.e., if $\nu \in \Delta(A \times T \times \Theta)$ satisfies*

$$\nu(a, t, \theta) = \begin{cases} \psi_0(\theta) \pi(t|\theta), & \text{if } a = \pi^*[t] \\ 0, & \text{otherwise} \end{cases}$$

for all a, t and θ , then ν is a BCE of $(G_{S,\varepsilon}, S)$. In this BCE, each players' actions are restricted to the set $\Pi_i^ \subset A_i$. This BCE induces an action state distribution $\mu \in \Delta(A \times \Theta)$ satisfying*

$$\mu(a, \theta) = \begin{cases} \psi_0(\theta) \pi(t|\theta), & \text{if there exists } t \text{ with } a = \pi^*[t] \\ 0, & \text{otherwise} \end{cases}$$

Now suppose that the game is played with a different information system S' . Does there exists a BCE of the game $(G_{S,\varepsilon}, S')$ where players only choose action profiles in Π^* ? For a distribution $\nu' \in \Delta(\Pi^* \times T' \times \Theta)$, write $\nu'(\cdot | \pi_i^*(t_i), t'_i)$ for the induced distribution over $T_{-i} \times \Theta$ of a type t'_i of player i advised to take action $\pi_i^*(t_i)$, so that

$$\nu'(t_{-i}, \theta | \pi_i^*(t_i), t'_i) = \frac{\sum_{t'_{-i}} \nu'((\pi_i^*(t_i), \pi_{-i}^*(t_{-i})), (t'_i, t'_{-i}), \theta)}{\sum_{t'_{-i}, \tilde{t}_{-i}, \tilde{\theta}} \nu'((\pi_i^*(t_i), \pi_{-i}^*(\tilde{t}_{-i})), (t'_i, t'_{-i}), \tilde{\theta})}$$

Lemma 5 *For each $\varepsilon > 0$, if $\nu' \in \Delta(\Pi^* \times T' \times \Theta)$ is a BCE of $(G_{S,\varepsilon}, S')$, then each player choosing action $\pi_i^*(t_i)$ under ν' has beliefs over $\Pi_{-i}^* \times \Theta$ within ε of $\tilde{\pi}_i(t_i)$, i.e., $\sum_{\theta, a_{-i}, t'_{-i}} \nu'(a, t', \theta) > 0$ implies that*

$$\|\nu'(\cdot | \pi_i^*(t_i), t'_i) - \tilde{\pi}_i(t_i)\| \leq \varepsilon.$$

Proof. A first necessary condition is that player i with type t'_i and recommendation $\pi_i^*(t_i)$ has an incentive to set $a_i^1 = \hat{\pi}_i^2[t_i]$. A necessary condition for this is that his beliefs on Θ are within ε of $\hat{\pi}_i^2[t_i]$. Now for each $k = 2, \dots, K$, a necessary condition is that player i with type t'_i and recommendation $\pi_i^*(t_i)$ has an incentive to set $a_i^k = \hat{\pi}_i^k[t_i]$. A necessary condition for this is that his beliefs on $A_{-i}^{k-1} \times \Theta$ are within ε of $\hat{\pi}_i^k[t_i]$. But this last condition reduces to the condition of the lemma. ■

9.2.5 Completing the Proof

Without loss of generality, we will assume that S and S' are non-redundant: by Lemma 3, redundancies do not change the set of BCE.

If: We first establish that if S' is less informed than S , then S' is BCE richer than S .

First note that if S' is less informed than the reduced information system of S , it is also less informed than S . Let $\nu \in \Delta(A \times T \times \Theta)$ be a BCE of (G, S) . Write $V_i(a_i, a'_i, t_i)$ for the expected utility for agent i under distribution ν if he is type t_i , receives recommendation a_i but chooses action a'_i , so that

$$V_i(a_i, a'_i, t_i) \triangleq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta).$$

Now - by Definition 3 - for each $i = 1, \dots, I$, $t_i \in T_i$ and $a_i \in A_i$, we have

$$V_i(a_i, a_i, t_i) \geq V_i(a_i, a'_i, t_i); \quad (26)$$

and $a'_i \in A_i$; and, by Definition 2, for all $t \in T$ and $\theta \in \Theta$, we have

$$\sum_{a \in A} \nu(a, t, \theta) = \psi(\theta) \pi(t|\theta) \quad (27)$$

Now let σ be a Θ -dependent non-communicating garbling that transforms S to S' and define $\nu' \in \Delta(A \times T' \times \Theta)$ by

$$\nu'(a, t', \theta) = \sum_{t \in T} \nu(a, t, \theta) \sigma(t'|t, \theta). \quad (28)$$

By construction, for all $t \in T$ and $\theta \in \Theta$,

$$\begin{aligned} \sum_{a \in A} \nu'(a, t', \theta) &= \sum_{a \in A, t \in T} \nu(a, t, \theta) \sigma(t'|t, \theta), \text{ by (28)} \\ &= \sum_{t \in T} \psi(\theta) \pi(t|\theta) \sigma(t'|t, \theta), \text{ by (27)} \\ &= \psi(\theta) \pi'(t'|\theta), \text{ because } \sigma \text{ is an } \Theta\text{-dependent non-communicating garbling that transforms } S \text{ to } S'. \end{aligned}$$

Thus ν' satisfies the consistency condition (Definition 2) to be a BCE of (G, S') . Write $V'_i(a_i, a'_i, t'_i)$ for the expected utility for agent i under distribution ν' if he is type t'_i , receives recommendation a_i but chooses action a'_i , so that

$$V'_i(a_i, a'_i, t'_i) \triangleq \sum_{a_{-i} \in A_{-i}, t'_{-i} \in T'_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu'((a_i, a_{-i}), (t'_i, t'_{-i}), \theta).$$

and symmetrically write

$$V_i(a_i, a'_i, t_i) \triangleq \sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta).$$

Now ν' satisfies the obedience condition (Definition 3) to be a correlated equilibrium of (u, ψ, S') if for each $i = 1, \dots, I$, $t'_i \in T'_i$ and $a_i \in A_i$,

$$V'_i(a_i, a_i, t'_i) \geq V'_i(a_i, a'_i, t'_i)$$

for all $a'_i \in A_i$. But

$$\begin{aligned} V'_i(a_i, a'_i, t'_i) &= \sum_{a_{-i} \in A_{-i}, t'_{-i} \in T'_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu'((a_i, a_{-i}), (t'_i, t'_{-i}), \theta) \\ &= \sum_{a_{-i} \in A_{-i}, t'_{-i} \in T'_{-i}, \theta \in \Theta, t \in T} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \sigma(t'|t, \theta), \text{ by the definition of } \nu' \\ &= \sum_{a_{-i} \in A_{-i}, t \in T, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \sum_{t'_{-i} \in T'_{-i}} \sigma((t'_i, t'_{-i}) | (t_i, t_{-i}), \theta) \\ &= \sum_{a_{-i} \in A_{-i}, t \in T, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \phi_i(t'_i | t_i), \\ &= \sum_{t_i \in T_i} \phi_i(t'_i | t_i) \left[\sum_{a_{-i} \in A_{-i}, t_{-i} \in T_{-i}, \theta \in \Theta} u_i((a'_i, a_{-i}), \theta) \nu((a_i, a_{-i}), (t_i, t_{-i}), \theta) \right] \\ &= \sum_{t_i \in T_i} \phi_i(t'_i | t_i) V_i(a_i, a'_i, t_i) \end{aligned}$$

Now for each $i = 1, \dots, I$, $t'_i \in T'_i$ and $a_i \in A_i$,

$$\begin{aligned} V'_i(a_i, a_i, t'_i) &= \sum_{t_i \in T_i} \phi_i(t'_i | t_i) V_i(a_i, a_i, t_i), \text{ by (29)} \\ &\geq \sum_{t_i \in T_i} \phi_i(t'_i | t_i) V_i(a_i, a'_i, t_i), \text{ by (26) for each } t_i \in T_i \\ &= V'_i(a_i, a'_i, t'_i), \text{ by (29)} \end{aligned}$$

for each $a'_i \in A_i$. Thus ν' is a BCE of (G, S') . By construction ν' and ν induce the same distribution in $\Delta(A \times \Theta)$. Since this argument started with an arbitrary BCE ν of (G, S) and an arbitrary G , we have $BCE(G, S') \subseteq BCE(G, S)$ for all games G .

Only if: Now we show that if S' is not less informed than S , then there exists a game G and an action state distribution $\mu \in \Delta(A \times \Theta)$ such that μ is a BCE equilibrium action distribution of (G, S) but is not a BCE equilibrium action distribution of (G, S') .

Consider the higher order beliefs game $G(S, \varepsilon)$ described above. There exists a truthtelling equilibrium for (G, S) . Because S' is not less informed than S^* , there does not exist a action type state distribution

$\nu' \in \Delta(T \times T' \times \Theta)$ that induces μ and assigns every type t'_i assigned to play t_i the same belief about others' actions and the state as the type t_i under μ . Thus by compactness, the necessary condition of Lemma 5 for an equilibrium of (G, S') inducing the same action state distribution fails.

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