Reputation for Quality*

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Abstract

We propose a new model of firm reputation that interprets reputation directly as the market belief about product quality. Quality is persistent and is determined endogenously by the firm's past investments. We analyse how investment incentives depend on the firm's reputation and derive implications for reputational dynamics.

Reputational incentives depend on the specification of market learning. When consumers learn about quality through good news, incentives decrease in reputation and there is a unique work-shirk equilibrium with convergent dynamics. When learning is through bad news, incentives increase in reputation and there is a continuum of shirk-work equilibria with divergent dynamics. More generally, for any imperfect learning process with Brownian and Poisson signals, there exists a work-shirk equilibrium with cyclical dynamics if costs of investment are low. This equilibrium is essentially unique if market learning contains a good news or Brownian component.

1 Introduction

In most industries firms can invest into the quality of their products through human capital investment, research and development, organisational change, and other channels. While imperfect monitoring by customers gives rise to a moral hazard problem, the firm can share in the created value by building a reputation for quality, justifying premium prices. This paper analyses the investment incentives in such a market, characterising how they depend on the current reputation of the firm and the market information structure, and analyzing the resulting reputational dynamics.

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Our key innovation is to model reputation directly as the market belief about the firm's endogenous product quality. As quality is determined by past investments, it is persistent and can serve as a Markovian state variable. This is in contrast to repeated games models, which do not have a state variable, and reputation models in which the state variable is exogenous. As a consequence, long-term reputational dynamics in our model are endogenously driven by reputational incentives, rather than fading out or trailing exogenous shocks.

The model captures key features of many important industries. In labor markets such as those for academics, artists and advertising executives, agents spend much of their time investing in skills and perfecting their trade. Their reputation and future compensation, however, depends heavily on their best paper, performance or campaign. In the computer industry, component manufacturers invest heavily into research and development while customers are only able to observe the performance of the entire computer. Customers therefore often learn about the quality of the product through newsworthy incidents, such as Dell's 2006 recall of 4 million Sony lithium-ion batteries.[1] In the car industry, firms devote considerable resources to improving quality standards through organisational change and new production processes. Since these investments are not observable, customers only learn about the true quality slowly, through consumer reports and the media.[2]

In the model, illustrated in Figure 1, one long-lived firm sells a product of high or low quality to a continuum of identical short-lived consumers. Product quality is a stochastic function of the firm's past investments. Quality then determines future prices through imperfect market learning: a high quality product generally leads to a higher consumer utility than a low quality product, but learning is obstructed by noise. At each point in time, consumers' willingness to pay is determined by the market belief that the quality is high, x_t, which we call the *reputation* of the firm. This reputation changes over time as a function of (a) the equilibrium beliefs of the firm's investments, and (b) market learning about the product quality.

Motivated by the Levy Decomposition Theorem, we suppose that market learning has two components: (1) a Brownian motion capturing continuous information such as consumer reports, and (2) a Poisson process capturing discrete events such as product failures. A Poisson event is a *good news* signal if it indicates high quality, and a *bad news* signal if it indicates low quality. Market learning is *imperfect* if no Poisson signal perfectly reveals the firm's quality.[3]

[1] See "Dell to Recall 4m Laptop Batteries", Financial Times, 15th August 2006.

[2] See "Detroit Carmakers on a Journey to Recover Reputation", Financial Times, 24th December 2008.

[3] There are many examples of these learning processes. Continuous updating may occur as drivers learn about the build-quality of a car, as clients learn about the skills of a consultancy, and as callers learn about the customer service of a telephone service provider. Good news signals may occur in academia when a paper becomes famous, in the bio-tech industry when a trial succeeds, and for actors when they win an Oscar. Bad news signals may occur in the computer industry when batteries explode, in the financial sector when a borrower defaults, and for doctors when they are sued for medical malpractice.

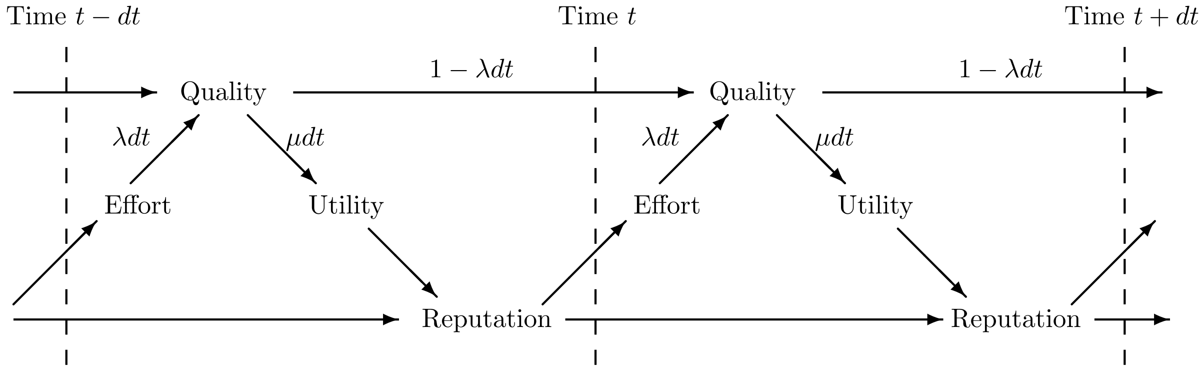

Figure 1: **Gameform.** Effort controls quality through a Poisson process with arrival rate λ. Consumer utility is a stochastic function of quality and serves as noisy signal in updating reputation. The model nests learning through good news, bad news or Brownian news as special cases.

In a Markovian equilibrium the firm's value is a function of its product quality and its reputation. As illustrated in Figure 1, both quality and reputation move slowly and therefore can be interpreted as assets, which the firm builds up at times, and which it depletes at other times. Reputation is valuable because it determines the firm's revenue. Quality in turn is valuable because a high quality product yields higher expected utility to customers, increasing the firm's future reputation. Crucially, as quality is persistent, this reputational payoff does not take the form of an immediate one-off reputational boost but it accrues to the firm as a stream of future *reputational dividends.* Theorem 1 formalizes this idea by writing the asset value of quality, i.e. the difference in value between a high and low quality firm, as the net present value of its future reputational dividends. This formula is important because it is precisely this value of quality which incentivizes the firm to invest.

Our main result, Theorem 2, characterizes equilibria under the assumption that the cost of effort is small and market learning is imperfect. We prove existence of a *work-shirk* equilibrium in which the firm works when its reputation falls below some cutoff x^*. Equilibrium beliefs induce a convergent reputational drift towards x^*, leading to cyclical long-term dynamics.[4] The asymmetry of the work-shirk equilbrium also hinges on this reputational drift. At extreme levels of reputations, $x \approx 0$ or 1, market learning is slow and the reputational dividend is small. At the top, work is not sustainable: If the firm is believed to work, its reputation stays high and reputational dividends stay small, undermining incentives to invest. In contrast, at the bottom, work is sustainable: If the firm is believed to work, its reputation drifts up and reputational dividends increase, generating incentives to invest.

Theorem 2 also shows that the work-shirk equilibrium is essentially unique if market learning satisfied condition (HOPE). This condition requires that reputation will drift up with positive probability, even if the firm is believed to be shirking. It is satisfied when market learning has a

[4]We also call these dynamics convergent, as the distribution $F(x_t)$ of reputation converges to a steady state that is independent of the initial reputation x_0.

non-trivial good news or Brownian component. For an intuition, consider the incentives around a *shirk-work* cutoff, where the firm shirks below the cutoff and works above it. At the cutoff reputational dynamics are divergent; under condition (HOPE) investment incentives are large, implying that the cutoff type strictly prefers to work. To the contrary, without (HOPE), adverse beliefs below a shirk-work cutoff are self-fulfilling and support a continuum of *shirk-work-shirk* equilibria.

In Section 5 we analyse learning processes with Poisson events that perfectly reveal high quality or low quality. These learning specifications are highly tractable and allow for closed-form solutions for any level of costs. Furthermore, they are useful for applications and illustrate the forces at work under more general learning processes. In the perfect good news case there is a unique work-shirk equilibrium. The reputational dividend is the possibility of a product breakthrough that reveals the firm's high quality and boosts its reputation to 1. Since the benefit of such a reputational boost decreases in the firm's reputation, so do investment incentives. The work-shirk beliefs imply cyclical reputational dynamics: A firm with low reputation works, eventually jumps to reputation 1 where it starts shirking; the firm's reputation then drifts down until it hits the cutoff and starts working again.

The perfect bad news case is the polar opposite of perfect good news. There is a continuum of shirk-work equilibria. The reputational dividend is insurance against a product breakdown that reveals the firm's low quality and destroys its reputation. Since the benefit of such insurance increases in the firm's reputation, so do investment incentives. The shirk-work beliefs imply divergent reputational dynamics: A firm with reputation below the cutoff shirks forever, causing its reputation to fall to 0; a firm with reputation above the cutoff works forever, causing its reputation to approach 1. To reconcile this result with the unique work-shirk equilibrium of Theorem **??**, note that the possibility of perfect bad news sustains effort at the top and allows for a "full work" equilibrium, while the failure of (HOPE) allows for shirk-work cutoffs at the bottom and for multiple equilibria.

Our firm controls future quality through investments that affect quality through a Poisson process. In Section 6, we connect our analysis to the standard models in the literature, where quality is chosen in every period (e.g. Klein and Leffler (1981), Mailath and Samuelson (2001)), by taking the quality obsolescence rate λ to infinity. With complete information, an increase in λ front-loads the returns to investment and increases investment incentives. With incomplete information, there is a countervailing effect: For large values of λ, equilibrium beliefs dominate market learning in determining reputational dynamics. In a work-shirk profile, market beliefs rapidly drift towards the cutoff and expected reputational dividends vanish. Thus, there are no work-shirk equilibria for high λ, but full shirking is an equilibrium. In contrast, a shirk-work cutoff that induces divergent reputational dynamics and can incentivize effort when λ is high. We thus find that effort disappears under perfect good news, while any shirk-work cutoff can be sustained under perfect bad news.

1.1 Theoretical Literature

Our paper links classical models of reputation with exogenous types and models of repeated games. In contrast to the repeated games literature, we suppose there is a state variable which links the periods. In contrast to reputation models, we suppose the state variable is the quality of the firm's product rather than some exogenous ability type of the firm (see figure 2).

In their reputation paper, Mailath and Samuelson (2001) consider a firm that sells a good of unknown quality. There are two types of firms: a competent firm who can choose high or low effort, and an inept firm who can only choose low effort. The actual product quality is then a noisy function of the firm's effort. From the consumer's perspective, utility is determined by the probability the firm is competent (the firm's reputation) multiplied by the probability that a competent firm exerts effort.

Mailath and Samuelson derive a striking result: there is a unique Markov perfect equilibrium in pure strategies in which the competent firm always chooses low effort. When the reputation is close to 1, it is impossible to sustain high effort for the same reason as in our paper. Effort then unravels from the top: If the firm is known to be shirking when its reputation passes some cutoff, it has no incentive to exert effort just below this cutoff since an increase in reputation leads to a collapse in the price. In contrast, in our paper, product quality is persistent. Thus, expected quality and price drift down continuously when the firm starts to shirk, and unravelling is prevented.

Holmström (1999) examines a signal-jamming model where an agent of unknown ability can exert effort to confuse the learning of her employer. When the agent's type is constant, the employer gradually learns the agent's ability, and effort declines over time. When the agent's type exogenously changes over time, some effort level is sustained in the stationary equilibrium.

Tadelis (1999) studies reputation as a adverse selection, rather than a moral hazard phenomenon. Quality is exogenous, but firms can buy and sell brand names that can carry reputation because change of ownership is not observed. Consumers learn about quality through good news shocks and Tadelis (1999) accordingly finds that in equilbrium it cannot be that good names are only bought by high quality firms.

There is a wider literature on reputation models with moral hazard and fixed types, surveyed in Bar-Isaac and Tadelis (2008). A number of these papers examine how a firm's incentives to exert effort vary over its lifecycle. First, incentives are low towards the end of the firm's life (Kreps et al. (1982), Diamond (1989)). Second, incentives are low when updating is slow (Benabou and Laroque (1992), Mailath and Samuelson (2001)). Third, when reputation can be lost with one piece of bad news, incentives increase in the level of reputation (Diamond (1989)). Together these papers help explain how demand varies across firms and over time (Foster, Haltiwanger, and

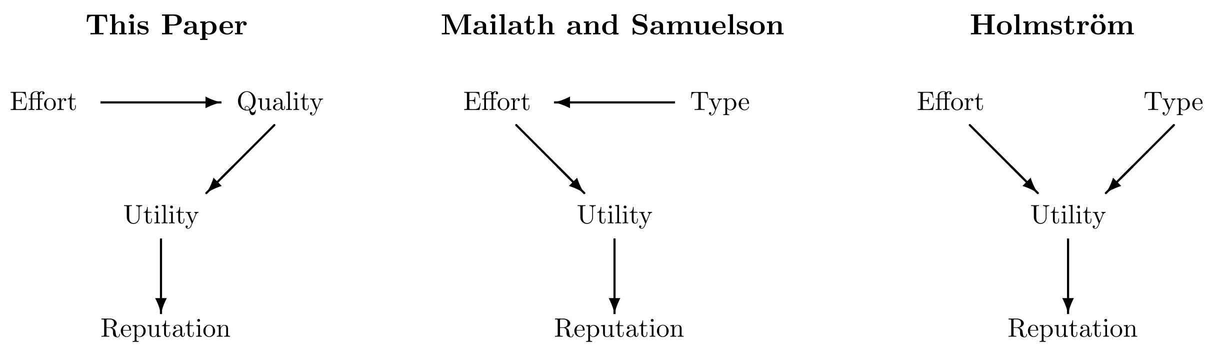

Figure 2: **Relation to Literature.** This figure shows the relationships between this paper, Mailath and Samuelson (2001) and Holmström (1999).

Syverson (2008)).[5] Many of these results can be understood through our trichotomy of good, bad and Brownian learning: With good news learning, firms with low reputations try to build, or buy a reputation (Tadelis (1999)). With bad news learning, firms with high reputations have high incentives to maintain them (Diamond (1989)). With Brownian learning, reputational incentives are hump-shaped as in Mailath and Samuelson (2001).[6]

Cripps, Mailath, and Samuelson (2004) show that reputation is a short-run phenomenon, when interpreted as a public belief about some exogenous uncertain type. The public posterior is a bounded martingale and by the martingale convergence theorem learning and incentives fade out eventually. Cripps, Mailath, and Samuelson (2004) write that "... [A]model of long-run reputations should incorporate some mechanism by which the uncertainty about types is continually replenished". Our stochastic investment into quality is such a mechanism. Unlike in models of purely exogenous shocks, such as Holmström (1999), in which reputation simply trails these shocks, the reputational dynamics of our model are driven by the forward-looking reputational incentives.[7]

In contrast to the repeated games literature (e.g. Fudenberg, Kreps, and Maskin (1990)), our model is distinguished by an evolving state variable. Effort directly feeds through to future reputation and revenue in our model, rather than preventing deliberate punishment by the counterparty in a repeated prisoners' dilemma. However, the models are connected by a common limit: As the frequency of play in a repeated game increases it approaches a continuous-time game, where quality is chosen at any $t \in \mathbb{R}$. This is also the limit of our reputation model as we take the quality obsolescence rate λ to ∞.

In a repeated prisoners' dilemma with frequent actions, Abreu, Milgrom, and Pearce (1991) and

[5]Industry dynamics have been analysed with complete information models with exogenous firm types (Jovanovic (1982), Hopenhayn (1992)) or endogenous capital accumulation (Ericson and Pakes (1995)). The difference between these two approaches is analogous to the distinction between our paper and classical reputation papers.

[6]In this last case, however, our work-shirk equilibrium is in contrast to the "low-high-low" incentives of Mailath and Samuelson (2001). This is because our reputational dividends accrue to the firm over time.

[7]Liu (2009) gives an alternative explanation of long-run reputational dynamics that is driven by imperfect, costly recall and lack of a public posterior.

Sannikov and Skrzypacz (2007) show that only discrete, "bad news" signals that indicate defection can sustain cooperation. Brownian, or good news signals are too noisy to deter defections without destroying all surplus by punishments on the equilibrium path. Thus, sustained cooperation depends on the information structure in the same way as in our model. While the common limit already suggests this analogy, our model highlights an alternative mechanism that distinguishes the role of bad news signals in overcoming moral hazard, namely divergent reputational dynamics.

Finally, our paper is related to contract design with persistent effort. Fernandes and Phelan (2000) suppose that an agent's output depends today on effort both today and yesterday, and derive a recursive formulation to solve for the principal's optimal contract. Jarque (2008) shows that the problem is much simpler when output depends on the geometric sum of past efforts and the cost of effort is linear. Unlike these papers, our consumers simply react to the firm's actions, rather than designing contingent contracts.

1.2 Empirical Literature

A number of empirical papers examins the importance of reputation in internet auctions (eBay). Resnick, Zeckhauser, Swanson, and Lockwood (2006) find that a new seller obtains significantly lower prices than a seller with a good feedback score. Cabral and Hortaçsu (2009) similarly find that a seller with negative feedback obtains significantly lower prices. More interestingly, Cabral and Hortaçsu (2009) study sellers' reactions to negative feedback. They find that a seller who receives negative feedback becomes more likely to receive additional negative feedback, and is more likely to exit. This suggests that either underlying quality is correlated over time, or a seller who receives negative feedback exerts less effort, as in our bad news case.

Studies have also examined the role of reputation in other markets. In the airline industry, a crash reduces the stock market value of the airline and manufacturer in question, reduces demand for all aviation, but increases the value of firms who compete directly with the crashed airline (Chalk (1987), Borenstein and Zimmerman (1988), Bosch, Eckard, and Singal (1998)). In the restaurant market, the introduction of grade cards increased investments in hygiene, and had the biggest effect on non-chain restaurants (Jin and Leslie (2003, 2009)). In the vehicle emission testing market, garages with higher pass rates can demand higher prices (Hubbard (1998, 2002)). In all of these cases, firms make investments that affect the quality of the product, and hence their reputation. While these studies demonstrate the importance of maintaining a reputation, there is little evidence on the effect of reputation on the firm's investment incentives, as examined in this paper.

2 Model

Timing: Time $t \in [0, \infty)$ is continuous and infinite. The common interest rate is $r \in (0, \infty)$.

Firm and Consumers: There is one firm and a continuum of consumers. At any point in time t the firm's product can have high or low quality, $\theta_t \in \{L = 0, H = 1\}$. The expected instantaneous value of the product to the consumer equals $\theta_t dt$. Consumers learn about the firm's quality though a stochastic process $dZ_t = dZ(\theta_t, \varepsilon_t)$. Motivated by the Levy decomposition theorem, we suppose Z_t is generated by a Brownian motion and a finite number of Poisson processes. The Brownian motion $Z_{B,t}$ is generated by

$$dZ_{B,t} = \mu_B \theta_t dt + dW_t$$

where W_t is the Wiener process. The Poisson process are indexed by $y \in \{1, \ldots, Y\}$ with arrival rates $\mu_{\theta,y}$. Signal y is good news if $\mu_y := \mu_{H,y} - \mu_{L,y} > 0$ and bad news if $\mu_y < 0$. We say that dZ is *perfect good (bad) news learning* if there is no Brownian component, $\mu_B = 0$, and a single Poisson signal that reveals high (low) quality perfectly, i.e. $\mu_{H,y} = \mu$ and $\mu_{L,y} = 0$ (resp. $\mu_{H,y} = 0$ and $\mu_{L,y} = \mu$). Market learning is imperfect if $\mu_{\theta,y} \in (0, \infty)$ for each signal.[8]

Strategies: At time t the firm chooses effort $\eta_t \in [0, 1]$ at cost $c\eta_t dt$. Product quality θ_t is a function of past effort $(\eta_s)_{0 \le s \le t}$ via a Poisson process with arrival rate λ that models quality obsolescence. Absent a shock, quality is constant: $\theta_{t+dt} = \theta_t$, while at a shock, previous quality becomes obsolescent and quality is determined by the level of investment: $Pr(\theta_{t+dt} = H) = \eta_t$. This implies $\Pr(\theta_t = H) = \int_0^t \lambda e^{\lambda(s-t)} \eta_s ds + e^{-\lambda t} \Pr(\theta_0 = H)$.[9] We assume that this Poisson process is independent of market learning ε_t.

Information: The signal dZ_t is public information, while actual product quality θ_t is observed only by the firm. The market belief about product quality $x_t = \Pr(\theta_t = H)$ at time t is called the firm's reputation.

Reputation Updating: The reputation increment $dx_t = x_{t+dt} - x_t$ is governed by the signal dZ_t

[8]Experienced utility is generated by these signals and carries no additional information. For example, with only a Brownian motion, we can let dZ_t/μ_B measure the instantaneous utility of the agent.

[9]This formulation provides a tractable way to allow the firm's type to depend on its past investments. One can view effort as the choice of absorptive capacity, determining the ability of a firm to recognise new external information and apply it to commercial ends (Cohen and Levinthal (1990)). Equivalently, one could assume the firm first sees the new technology arrive and chooses whether to adopt it at cost $k = c/\lambda$. Yet another equivalent interpretation is that a low-quality firm chooses the arrival rate of high quality from $[0; \lambda]$ at marginal cost c/λ, and a high-quality firm can abate the intensity λ of the arrival of low quality at marginal cost c/λ.

and beliefs about effort $\widetilde{\eta}_t$. By independence, dx_t can be decomposed additively:

$$dx_t = \lambda(\widetilde{\eta}_t - x_t)dt + x_t(1-x_t)\frac{\Pr(dZ_t|H) - \Pr(dZ_t|L)}{x_t \Pr(dZ_t|H) + (1-x_t)\Pr(dZ_t|L)}. \tag{2.1}$$

Let $d_\theta x_t$ be the increments conditional on quality θ. We evaluate (2.1) explicitly in Section 4.

Profit and Consumer Surplus: Firm and consumers are risk-neutral. At time t the firm sets price equal to the expected value x_t. While consumers get utility 0 in expectation, the firm's instantaneous profit is $(x_t - c\eta_t)dt$ and its discounted present value is thus given by:

$$V_\theta\left(x;\eta,\widetilde{\eta}\right) := \int_{t=0}^{\infty} e^{-rt}\mathbb{E}_{\theta_0=\theta,x_0=x,\eta,\widetilde{\eta}}\left[x_t - c\eta_t\right]dt. \tag{2.2}$$

Markov-Perfect-Equilibrium: We assume Markovian beliefs $\widetilde{\eta} = \widetilde{\eta}\,(x)$ and show below that optimal effort $\eta = \eta\,(x)$ is independent of history and current product quality θ. A Markov-Perfect-Equilibrium $\langle \eta, \widetilde{\eta}\rangle$ consists of a Markovian effort function $\eta : [0,1] \to [0,1]$ for the firm and Markovian market beliefs $\widetilde{\eta} : [0,1] \to [0,1]$ such that 1) $\eta \in \eta^*\left(\widetilde{\eta}\right) := \arg\max_\eta \left\{V_\theta\left(x;\eta,\widetilde{\eta}\right)\right\}$ maximizes firm value $V_\theta\left(x;\eta,\widetilde{\eta}\right)$, and 2) market beliefs are correct: $\widetilde{\eta} = \eta$. In a Markovian equilibrium η, we will write the firm's value as a function of its quality and its reputation: $V_\theta(x)$.

2.1 Optimal Investment Choice

In principle, the firm's effort choice η as well as market beliefs $\widetilde{\eta}$ could depend on the entire public history $Z^t = (Z_s)_{0\leq s<t}$, as well as the private history $\theta^t = (\theta_s)_{0\leq s<t}$ and time t. We assume that market beliefs $\widetilde{\eta}$ are Markovian because we think of the continuum of consumers as sharing their experience in a sufficient, yet incomplete manner, e.g. through consumer reports. For Markovian beliefs $\widetilde{\eta}$, all payoff relevant parameters at time t depend on the history only via the current product quality θ_t and the firm's reputation x_t. Thus, the optimal effort choice of the firm only depends on these two parameters.

The benefit of effort in $[t; t+dt]$ is the probability of a technology shock hitting, λdt, times the difference in value functions $\Delta\,(x) := V_H\,(x) - V_L\,(x)$, which we call the *value of quality.* The marginal cost of investment is c, and thus optimal effort $\eta\,(x)$ is given by

$$\eta\,(x) = \begin{cases} 1 & \text{if } c < \lambda\Delta\,(x)\,, \\ 0 & \text{if } c > \lambda\Delta\,(x)\,. \end{cases} \tag{2.3}$$

As quality after the shock is independent of previous quality, so is optimal effort.

Lemma 1 summarizes this discussion:

Lemma 1 *For Markovian beliefs $\widetilde{\eta}(x)$ there is an optimal Markovian effort function $\eta(x)$ that depends solely on the firm's reputation but not on its product quality. Additionally, $\eta(x)$ satisfies the "bang-bang" equation (2.3).*

Equation (2.3) makes the model tractable and is the reason that we assume the cost of effort to be independent of product quality and past effort. An implication of equation (2.3) is that our results are not driven by the asymmetric information about product quality θ, but solely by the unobserved investment η into future quality.

2.2 Cutoff Equilibria and Reputational Dynamics

We call an equilibrium *work-shirk*, if there exists a cutoff x^* such that a firm with low reputation $x < x^*$ exerts effort, $\eta(x) = 1$, whereas a firm with a high reputation $x > x^*$ does not, $\eta(x) = 0$. The opposite case, where low reputations shirk and high reputations work, is called a *shirk-work* equilibrium.

Work-shirk equlibria and shirk-work equilibria have opposite reputational dynamics. Net of market learning, dynamics $dx = \lambda(\widetilde{\eta}_t - x_t)dt$ are convergent in a work-shirk equilibrium, i.e. $dx > 0$ for $x < x^*$ and $dx < 0$ for $x > x^*$, but divergent in a shirk-work equilibrium. We will see in section 6 that for high values of λ incentives disappear in work-shirk profiles, but not in shirk-work profiles.

We consider two effort profiles η, η' as close if their value functions are close. Formally, let $dist(\eta, \eta') := \sup_{x\in(0,1),\theta} \left\{\left|V_{\theta,\eta}(x) - V_{\theta,\eta'}(x)\right|\right\}$. This (pseudo) metric captures a fundamental asymmetry between work-shirk and shirk-work cutoffs. A work-shirk profile with cutoff $x^* = 1-\varepsilon$ close to 1 is close to the *full-work* profile, i.e. $\eta(x) = 1$ for all x: For small x_0, the trajectories are identical for a long time; thus the values are close. For large x_0, learning is slow and both values are close to 1. To the contrary, a shirk-work profile with cutoff $x^* = \varepsilon$ close to 0, is not similar to the full-work profile, i.e. $\eta(x) = 0$ for all x: For $x_0 < x^*$ sufficiently small, the value under shirk-work is close to 0 because the firm is stuck in the shirk-region. The value under full work is greater because reputation will drift up due to favorable beliefs.

2.3 Welfare

Suppose product quality is publicly observed. Then the benefit of exerting effort equals the obsolescence rate λ, times the price differential 1, divided by the effective discount rate $r + \lambda$. Thus first-best effort is given by:

$$\eta = \begin{cases} 1 & \text{if } c < \frac{\lambda}{r+\lambda} \\ 0 & \text{if } c > \frac{\lambda}{r+\lambda} \end{cases} . \tag{2.4}$$

There is no equilibrium with positive effort if $c > \frac{\lambda}{r+\lambda}$: Effort decreases welfare and the firm makes negative profits as consumers receive zero utility in equilibrium. The firm therefore prefers to shirk at all levels of reputation, thereby guaranteeing itself a non-negative payoff.

We thus restrict attention in the paper to the case $c < \frac{\lambda}{r+\lambda}$.

3 Value of Quality

A firm's value $V_\theta(x)$ is a function of its reputation x and its quality θ. While reputation directly determines revenue, quality derives its value indirectly, through its effect on reputation. More precisely, Theorem 1 shows that the value of quality can be written as a present value of future reputational dividends.

Lemma 2 shows that, when the firm is choosing its effort optimally, the value function is increasing in reputation. To prove the lemma, we need to rule out the possibility that a firm with a higher initial reputation may shirk, lose its product quality, and fall behind a firm with a lower initial reputation. We do so by supposing the firm with the higher reputation mimics the firm with the lower reputation, thereby staying ahead in all states of the world.

Lemma 2 *Given an optimal response to market beliefs $\eta^*(\widetilde{\eta})$, the value function of the firm $V_\theta(x;\eta^*(\widetilde{\eta}),\widetilde{\eta})$ is strictly increasing in its reputation x and increasing in market beliefs $\widetilde{\eta}$.*

Proof. See Appendix A.2. □

Lemma 2 implies that across equilibria η, η', with $\eta'(x) \geq \eta(x)$ for all x, the firm's value is increasing in effort $V_\theta(x;\eta',\eta') \geq V_\theta(x;\eta,\eta)$.

To analyze the asset value of quality $\Delta(x)$, we decompose it into (a) the immediate benefit of having high quality, called the *reputational dividend*, and (b) the continuation benefit of a high quality product.

$$\begin{aligned}\Delta(x) &= (1-rdt)(1-\lambda dt)\mathbb{E}_x\left[V_H(x+d_Hx) - V_L(x+d_Lx)\right] \\ &= (1-rdt-\lambda dt)\mathbb{E}_x\left[(V_H(x+d_Hx) - V_H(x+d_Lx)) + \Delta(x+d_Lx)\right].\end{aligned} \tag{3.1}$$

The first line uses the principle of dynamic programming, while the second adds and subtracts $V_H(x+d_Lx)$. Intuitively, for a fixed initial reputation x, current profits are completely determined by the firm's reputation. Rather than increasing current profits, high quality increases tomorrow's reputation.

Integrating (3.1) yields equation (3.2) in Theorem 1, which expresses the asset value of quality as the discounted sum of future reputational dividends. This expression serves as a work-horse throughout the paper.

Theorem 1 *Fix any Markovian beliefs $\widetilde{\eta}$ and a Markovian best response $\eta^*(\widetilde{\eta})$. Then two closed-*

form expressions for the value of quality $\Delta(x)$ are given by:

$$\Delta(x) = \int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}[D_H(x_t)]dt, \tag{3.2}$$
$$= \int_0^\infty e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=H}[D_L(x)]dt, \tag{3.3}$$

where $\theta^t = L$ is short for $\theta_s = L$ for all $s \in [0;t]$, and the reputational dividend $D_\theta(x)$ is defined by

$$D_\theta(x) := \mathbb{E}[V_\theta(x + d_H x) - V_\theta(x + d_L x)]/dt.$$

Proof. To integrate up (3.1), fix x and set $\psi(t) := \mathbb{E}_{x_0=x,\theta^t=L}[\Delta(x_t)]$. Up to terms of order $o(dt)$ we have

$$\begin{aligned}
-d\left(\psi(t)e^{-(r+\lambda)t}\right) &= -e^{-(r+\lambda)t}\left(\psi(t+dt) - \psi(t) - (r+\lambda)dt\psi(t)\right) \\
&= e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}\left[-\mathbb{E}_{x_t}\left[\Delta(x_t + d_L x_t)\right] + (1 + (r+\lambda)dt)\Delta(x_t)\right] \\
&= e^{-(r+\lambda)t}\mathbb{E}_{x_0=x,\theta^t=L}\left[D_H(x_t)\right]dt
\end{aligned}$$

and (3.2) follows.

Equation (3.3) follows from the alternative decomposition of (3.1) when we add and subtract $V_L(x + d_H x)$ instead of $V_H(x + d_L x)$. $\square$

While standard reputation models incentivise effort by an immediate effect on the firm's reputation, effort in our model pays off through quality with a delay. Once quality is established, it is persistent and generates a stream of reputational dividends until it becomes obsolete. We must thus evaluate the reputational incentives at future levels of reputation x_t, rather than the current level x.

Corollary 1 *Fix any Markovian beliefs $\widetilde{\eta}$ and a Markovian best response $\eta^*(\widetilde{\eta})$. For a given reputation x, a high-quality firm has a higher value than a low-quality firm, i.e. $V_H(x) \geq V_L(x)$.*

Proof. By the updating equation (2.1) we have $d_H x \geq d_L x$, by Lemma 2 we get $D_\theta(x) = V_\theta(x_t + d_H x_t) - V_\theta(x_t + d_L x_t) \geq 0$. Finally by Theorem 1 we get $\Delta(x) = V_H(x) - V_L(x) \geq 0$. $\square$

4 Main Result

Our main result, Theorem 2 shows that for imperfect market learning processes and sufficiently small costs, there exists a work-shirk equilibrium and, under condition (HOPE), this equilibrium is essentially unique. We prove Theorem 2 by evaluating the reputational dividends that constitute the value of quality and incentivise effort. The reputational dividend is the sum of a Brownian component (4.2) and a Poisson component (4.4).

For the Brownian component of learning, updating evolves according to

$$d_\theta x = \mu_B x(1-x)\left(\mu_B\left(\theta - x\right)dt + dW\right). \tag{4.1}$$

To calculate the value of quality we apply Itô's formula to get:

$$\mathbb{E}_x[V_H(x + d_\theta x)] = V_H(x) + \mu_B^2 x(1-x)\left(\theta - x\right)V_H'(x)dt + \frac{(\mu_B x(1-x))^2}{2}V_H''(x)dt.$$

The Brownian component of the reputational dividend is thus:

$$D_H(x) = \mathbb{E}_x[V_H(x + d_H x) - V_H(x + d_L x)]/dt = \mu_B^2 x(1-x)V_H'(x). \tag{4.2}$$

The dividend declines to zero in either tail as the factor $x\,(1-x)$ slows down reputational updating.

For the Poisson component of learning, recall that $\mu_y = \mu_{H,y} - \mu_{L,y}$ is the net arrival rate of good news. The reputational increment is given by

$$dx = \sum_y \mu_y x(1-x)\begin{cases} \left(\mu_{H,y}x + \mu_{L,y}\left(1-x\right)\right)^{-1} & \text{at arrival } y, \\ -dt & \text{otherwise.}\end{cases} \tag{4.3}$$

Absent an arrival the second drift term is negative since, when $\mu_y > 0$, no news is bad news.

The Poisson component of the reputational dividend is thus

$$D_H(x) = \mathbb{E}_x[V_H(x + d_H x) - V_H(x + d_L x)]/dt = \sum_y \mu_y\left[V_H\left(x + d_y x\right) - V_H(x)\right], \tag{4.4}$$

where $d_y x := \mu_y x(1-x)/\left(\mu_{H,y}x + \mu_{L,y}\left(1-x\right)\right)$ is the reputation increment at arrival y. If market learning is imperfect $\mu_{\theta,y} \in (0,\infty)$, then the reputational increments $d_y x$ converge to zero in the tails and the dividend declines to zero, just like the Brownian component.

The most robust feature of equilibrium under imperfect learning is thus that there must be a shirk-region at the top. If the firm is believed to be working in an interval around $x = 1$, it is all but certain to have a high reputation in the future, undermining incentives to actually invest. To go beyond this local result, and prove the main result of the paper, Theorem 2, we will focus on the case of sufficiently small costs c.

To concisely state Theorem 2 we say that the learning process dZ satisfies (HOPE), when a firm with any non-zero reputation has a chance of its reputation increasing even under adverse equilibrium beliefs:

$$\Pr\left[dx > 0 | \tilde{\eta} = 0\right] > 0 \qquad \text{for all } x > 0 \tag{HOPE}$$

This holds if (a) there is a non-trivial Brownian component, (b) there are good news signals, or (c) there are only bad news signals but $-\sum_y \mu_y > \lambda$, so that the absence of bad news can outweigh adverse equilibrium beliefs. Theorem 2 shows that, under (HOPE), equilibrium is essentially unique. That is, for any $\epsilon > 0$ and sufficiently low cost c, any two equilibria η, η' are close in the metric defined in Section 2.2, i.e. $dist\left(\eta, \eta'\right) < \epsilon$.

Theorem 2 *For any imperfect learning process, there exists $c > 0$ such that for all $c^* \in (0, c)$:*

(a) There exists a work-shirk equilibrium with cutoff $x^* \in (0, 1)$.

(b) Reputational dynamics converge to a non-trivial cycle in this equilibrium.

(c) With (HOPE), equilibrium is essentially unique.

Proof. See Appendix B. □

Theorem 2 shows that, when costs are low, effort is sustainable at the bottom but not at the top. This fundamental asymmetry is illustrated in the left panel of Figure 3 for the case of Brownian motion. Intuitively, when the firm is believed to be working, the value of quality is zero at $x = 1$ since current dividends are zero and, as the firm's reputation stays at $x = 1$, future dividends are zero. In contrast, the value of quality is positive at $x = 0$ since favorable equilibrium beliefs push the firm's reputation into the interior of $(0, 1)$, where dividends are high. The firm thus invests at $x = 0$ not because of the immediate reputational dividends, which are close to 0, but because of future dividends, when the firm's reputation is sensitive to actual quality.

Imperfect learning about low quality is a necessary condition for Theorem 2. Poisson signals that reveal low quality perfectly, i.e. y with $\mu_{y,H} = 0$, invalidate the argument for "shirking at the top" because $x_{t+dt} = 0$ at arrival of signal y. We study perfect bad news learning in Section 5.2. Imperfect learning about high quality, on the other hand, is merely assumed for symmetry and tractability in the proofs. Proposition 1 replicates the results of Theorem 2 for perfect good news learning.

Figure 3 is almost a proof of Theorem 2(a). Let $\Delta_{x^*}(x)$ be the asset value of quality for a firm with reputation x in a work-shirk profile with cutoff x^*. Arguments in the appendix show that:

$$\Delta_1(x) \begin{cases} > 0 & \text{for } x < 1 \\ = 0 & \text{for } x = 1 \end{cases} \quad \text{and } \Delta_1'(1) < 0.$$

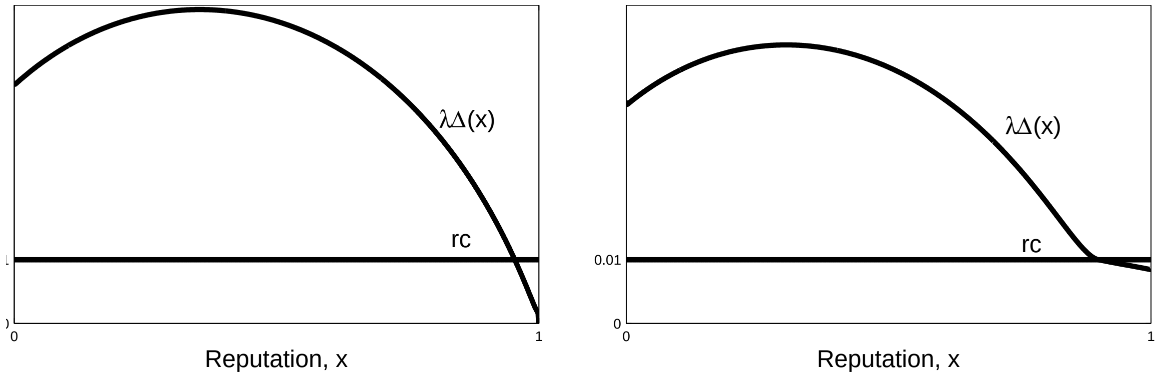

Figure 3: **Asset Value of Quality under Full Effort (left) and in Work-Shirk Equilibrium (right).** This figure assumes that $\mu = 1$, $\lambda = 1$, $r = 1$ and $c = 0.01$. In the work-shirk equilibrium, the resulting cutoff is is $x^* = 0.900$.

and thus for small c there exists x^* such that

$$\lambda \Delta_1(x) \begin{cases} > c & \text{for } x < x^* \qquad \text{(Low reputations work)} \\ = c & \text{for } x = x^* \qquad (x^* \text{ is indifferent)} \\ < c & \text{for } x > x^* \qquad \text{(High reputations shirk).} \end{cases} \tag{4.5}$$

To prove part (a) we just need to replace Δ_1 on the LHS with Δ_{x^*}. The problem with this simple argument is that it implicitly assumes continuity of Δ'_{x^*} as $x^* \to 1$. However, it is not difficult to show that $\lim_{x^* \to 1} \Delta'_{x^*}(1) = 0$ while $\Delta'_1(1) > 0$. As a result, it could be that $\Delta_{x^*}(x)$ is increasing in x for $x > x^*$, contradicting the last condition in (4.5).

To appreciate this complication, consider the marginal value of reputation $V'_\theta(x)$ to a firm with reputation $x \in [x^*, 1]$ where $x^* \approx 1$. A reputational increment dx is valuable to the firm only as long as $x_t|_{x_0=x+dx} > x^*$: As soon as $x_t|_{x_0=x+dx} = x^*$ the increment $x_t|_{x_0=x+dx} - x_t|_{x_0=x}$ vanishes because of the difference of drift to the left of x^* and to the right of x^*. As a consequence, $V'_\theta(x)$ and $D_\theta(x)$ may be minimized at the cutoff x^*. Numerical simulations show that this is actually the case in relevant parameter ranges for a pure Brownian learning process. Thus, we need to take seriously the possibility that $\Delta_{x^*}(x)$ could be minimised at $x = x^*$.

To rule this out, i.e. show that $\lambda\Delta_{x^*}(x^*) > \lambda\Delta_{x^*}(x)$ for $x \in (x^*, 1]$, we need a better understanding of the reputational dynamics dx and the marginal values $V'_{\theta,x^*}(x)$ for $x, x^* \approx 1$. Assume for notational simplicity that learning is pure Brownian without Poisson signals. Then, the dynamics of $(1-x)$ approximate a geometric Brownian motion which is reflected at $(1-x^*)$ by the

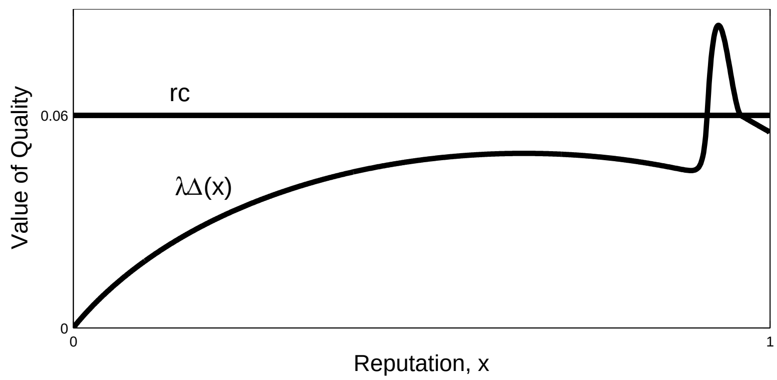

Figure 4: **Shirk-Work-Shirk Equilibrium.** This figure illustrates the asset value of quality in a work-shirk equilibrium, $\Delta_{x^*}(x)$. The straight line equals rc/λ. This figure assumes that $\mu = 1$, $\lambda = 1$, $r = 1$ and $c = 0.06$. The resulting cutoffs are $\underline{x} = 0.910$ and $\overline{x} = 0.958$.

large relative difference in the drift terms. For the high quality firm,

$$\begin{aligned} d_H(1-x) &= -\lambda\,(\eta - x)\,dt - \mu_B^2 x\,(1-x)^2\,dt + \mu_B x\,(1-x)\,dW \\ &\approx \begin{cases} -\lambda\,(1-x)\,dt - \mu_B\,(1-x)\,dW & \text{for } x < x^* \\ \lambda x dt & \text{for } x > x^* \end{cases} \end{aligned}$$

and likewise for $d_L(1-x)$.

This has two implications. First, while the dividend may be minimised at x^*, the value of quality at the cutoff $\Delta_{x^*}(x^*)$ is largely determined by the dividends at $x < x^*$. Second, the marginal value of reputation and the dividend at $x > x^*$ are small in relation to those at $x < x^*$. This is because a reputational increment essentially disappears when $x_t = x^*$ and this happens much sooner for initial reputations $x_0 > x^*$ than for $x_0 < x^*$. Hence for $x > x^*$, $\Delta_{x^*}(x)$ is an average of low dividends while $x_t > x^*$, and a continuation value $\Delta_{x^*}(x^*)$ when x_t hits x^*. This average comes to less than $\Delta_{x^*}(x^*)$, as required.

Slow reputational updating at $x \approx 0$ and $x \approx 1$ suggests another, *shirk-work-shirk* type of equilibrium, where a firm works when its reputation is between two cutoffs, $x \in [\underline{x}, \overline{x}]$ and shirks elsewhere. A firm with a low reputation is trapped in a lower "shirk-hole" in which market learning is too slow to incentivise effort, while a firm above $\underline{x}$ experiences convergent dynamics around $\overline{x}$.

Theorem 2(c) shows that such shirk-work-shirk equilibria disappear for small costs. Intuitively, reputational dividends and the value of quality are bounded below on any interval $[\varepsilon; 1-\varepsilon]$ so, when costs are small, all intermediate reputations prefer to work. At a shirk-work cutoff, $\underline{x}$, working is then profitable for low costs if the firm will escape from the lower shirk region with positive probability. That is, if (HOPE) is satisfied.

If (HOPE) is violated, i.e. if learning is only through bad news and $-\sum_y \mu_y < \lambda$, the work-shirk is not unique but coexists with a continuum of shirk-work-shirk equilibria. The failure of (HOPE) implies that a firm whose reputation drops into the shirk hole will remain there forever. This creates a discontinuity in the value function that incentivizes effort above the cutoff but not in the shirk-hole just below.[10][11]

Investment incentives in a shirk-work-shirk equilibrium are greatest just above the shirk-work cutoff where the divergent reputational drift makes value functions discontinuous. This captures the intuition that one product breakdown can put a reputable firm in the "hot-seat" where one more breakdown would finish the firm off. In such an equilibrium a firm that fails once will try hard but a firm that fails repeatedly gives up.

The above arguments imply that with (HOPE) and small costs, the firm works at all intermediate and low levels of reputation and shirks at some high levels of reputation, i.e. the work-shirk equilibrium is "essentially" unique. Beyond this we can show that there is a only one work-shirk equilibrium, but there may be other "work-shirk-work-shirk" equilibria with additional work-regions at the very top. Nevertheless, any such equilibrium is similar to work-shirk equilibria in that it is characterised by a reputation levels $0 < x^* < 1$ such that $\eta(x) = 1$ for $x < x^*$ and $\eta(x) = 0$ for some $x > x^*$.[12] These equilibria all involve work on $[0, 1 - \epsilon]$, so they converge to the work-shirk equilibrium in the Hausdorff metric as $c \to 0$.

The above analysis relies on the assumption of low costs c to ensure work for intermediate reputations $x \in [\varepsilon; 1 - \varepsilon]$. For higher costs, numerical simulations indicate that shirk-work-shirk equilibria exist also in the pure Brownian case where (HOPE) holds (see Figure 4). As discussed abore, investment incentives in this equilibrium (with $c = 0.06$) are much higher than in the above work-shirk equilibrium (with $c = 0.01$). A firm at the work-shirk cutoff has more to lose when a sequence of bad utility draws can push its reputation into a shirk-region, where it may be stuck forever. This argument makes it unlikely that these two types of equilibrium co-exist for given parameters.

[10] To construct shirk-work-shirk equilibria, we can first choose the lower, shirk-work cutoff low enough so as to discourage work in the shirk-hole, and then reapply the arguments in Appendix B to prove existence of the upper, work-shirk cutoff with the required properties.

[11] This argument implies that the equilibrium correspondence is not lower hemi-continuous. Learning processes with (HOPE) approximate learning processes without (HOPE), e.g. by taking the Brownian component of learning μ_B to zero. While equilibria along the convergent sequence are work-shirk, additional shirk-work-shirk equilibria discontinuously appear in the limit. These shirk-work-shirk equilibria are not close to the work-shirk equilibrium in the metric $dist(\cdot, \cdot)$.

[12] For example, given the parameters in figure 3, there is another equilibrium with working on $[0, 0.900]$, shirking on $[0.900, 0.944]$, working on $[0.944, 0.9605]$ and shirking on $[0.9605, 1]$.

5 Perfect Poisson Learning

The analysis in Section 4 subsumes Poisson processes with imperfect learning. We now turn to Poisson processes that can perfectly reveal the firm's quality. These cases are highly tractable and allow for a more explicit equilibrium characterization: see Appendix D.1 and Appendix D.2. The perfect good news case illustrates and extends Theorem 2, in that equilibrium is work-shirk and unique. The perfect bad news case highlights the limitations of Theorem 2, in that equilibria are shirk-work and not unique. Moreover, these learning models are natural for many applications, making the results interesting in their own right.

5.1 Perfect Good News

Assume that consumers learn about quality θ_t from infrequent product *breakthroughs* that reveal high quality $\theta = H$ with arrival rate μ. Absent a breakthrough, updating evolves deterministically according to:

$$\frac{dx}{dt} = \lambda(\eta(x) - x) - \mu x(1 - x). \tag{5.1}$$

Let x_t be the deterministic solution of the ODE (5.1) with initial value x_0.

The reputational dividend is the value of having a high quality in the next instant. This equals the value of increasing the reputation from x to 1 times the probability of a breakthrough:

$$D_H(x) = \mathbb{E}\left[V_H(x + d_H x) - V_H(x + d_L x)\right]/dt = \mu(V_H(1) - V_H(x))$$

Using equation (3.2), the asset value of quality is:

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu[V_H(1) - V_H(x_t)]dt \tag{5.2}$$

The reputational dividend $V_H(1) - V_H(x_t)$ is decreasing in x_t, so that $\Delta(x_0)$ is decreasing in x_0. It follows that any equilibrium is work-shirk. Intuitively, a breakthrough that increases the firm's reputation to 1 is most valuable for a firm with a low reputation. Thus, investment incentives decrease in reputation and the equilibrium is work-shirk.

The work-shirk beliefs imply that reputational dynamics converge to a cycle. Absent a breakthrough, the firm's reputation converges to a stationary point $\hat{x} = \min\{\lambda/\mu, x^*\}$ where the firm works with positive probability. When a breakthrough occurs, the firm's reputation jumps to 1. The firm is then believed to be shirking, so its reputation drifts down to $\hat{x}$, absent another breakthrough. In the long-run, the firm's reputation therefore cycles over the range $[\hat{x}, 1]$.

Proposition 1 *Under perfect good news learning:*

(a) Every equilibrium is work-shirk.

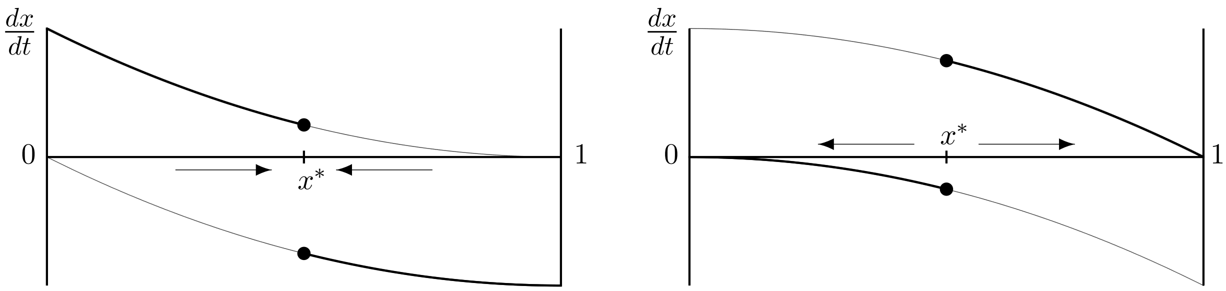

Figure 5: **Reputational dynamics in Good News (left) and Bad News (right).** This figure illustrates how the reputational drift dx/dt, absent a breakthrough, changes with the reputation of the firm, x. These pictures assume $\lambda = \mu$ and $x^* = 1/2$. The dark line shows the drift when beliefs are correct.

(b) Reputational dynamics converge to a non-trivial cycle.

(c) If $\lambda \geq \mu$, the equilibrium is unique.

Proof. Part (a). Reputation x_t follows (5.1), so an increase in x_0 raises x_t at each point in time. Lemma 2 says that $V_H(x)$ is strictly increasing in x, so equation (5.2) implies that $\Delta(x_0)$ is decreasing in x_0. Part (b) follows from (a).

Part (c). Given $\lambda \geq \mu$, the process x_t starting at $x_0 = 1$, falls until it becomes stock at x^*. Equation (**??**) in Appendix D.1 derives a closed-form expression for $\Delta_{x^*}(x^*)$. This implies that $\Delta_{x^*}(x^*)$ is decreasing in the cutoff, implying the equilibrium is unique. □

Suppose that $\lambda \geq \mu$, so the firm's reputation drifts up whenever it is believed to be working (see Figure 5). In this case, the dynamics are stationary at $\hat{x} = x^*$, at which point the firm chooses to work with intensity $\eta(x^*) = x^* \left(1 + \frac{\mu}{\lambda}(1 - x^*)\right)$.

To understand the uniqueness result of Proposition 1(c), suppose the market believes the cutoff is $\tilde{x}$, and denote the firm's best response by $x^*(\tilde{x})$. An increase in $\tilde{x}$ means the firm's reputation will not drift down as far, absent a breakthrough. This change benefits low-quality firms more than high-quality firms, reducing $\Delta(x)$. As a result, $x^*(\tilde{x})$ is decreasing in $\tilde{x}$ and there is a unique fixed point where $x^*(\tilde{x}) = \tilde{x}$.

This result illustrates Theorem 2. When the market learns through perfect good news, the strength of the signal is decreasing in the firm's reputation. This means that not only are investment incentives higher at $x = 0$ than $x = 1$, but that incentives are monotonically decreasing in x.

5.2 Perfect Bad News

Assume that x_t is generated by *breakdowns* that reveal low quality $\theta = L$ with arrival rate μ. Absent a breakdown, updating evolves deterministically according to:

$$\frac{dx}{dt} = \lambda(\eta(x) - x) + \mu x(1-x). \tag{5.3}$$

Let x_t be the deterministic solution of ODE (5.3) with initial value x_0.

The reputational dividend is the value of having a high quality in the next instant. This equals the value of not losing one's reputation times the probability of a breakdown:

$$D_L(x) = \mathbb{E}\left[V_L(x_t + d_H x_t) - V_L(x_t + d_L x_t)\right]/dt = \mu(V_L(x_t) - V_L(0)).$$

Using equation (3.3), the asset value of quality is:

$$\Delta(x_0) = \int_0^\infty e^{-(r+\lambda)t}\mu[V_L(x_t) - V_L(0)]dt. \tag{5.4}$$

The jump size $V_L(x_t) - V_L(0)$ is increasing in x_t, so that $\Delta(x)$ is increasing in x. It follows that any equilibrium is shirk-work. Intuitively, a breakdown that destroys the firm's reputation entirely is most damaging for a firm with a high reputation. Thus, incentives to invest increase in reputation and equilibrium must be shirk-work.

The shirk-work beliefs imply that the reputational dynamics diverge. Consider an equilibrium with $x^* \in (0,1)$, i.e. the firm works if its reputation is below x^* and works if it is above x^*. A firm that starts with reputation above x^* converges to reputation $x = 1$, absent a breakdown. If the firm is hit by such a breakdown while its product quality is still low, it gets stuck in a shirk-hole with reputation $x = 0$. A firm with reputation below x^* initially shirks and may have either rising or falling reputation, depending on parameters. In either case, its reputation will either end up at $x = 0$ or $x = 1$.

To allow for positive effort in some equilibrium we impose the following assumption:

$$\frac{\lambda}{r+\lambda+\mu}\mu(1-c)/r > c \tag{5.5}$$

Proposition 2 *Under perfect bad news learning:*

(a) Every equilibrium is shirk-work.

(b) If $x^* \in (0,1)$ then reputational dynamics diverge to 0 or 1.

(c) Assume (5.5) holds and $\lambda \geq \mu$. There is a non-empty interval $[a,b]$ such that every cutoff $x^* \in [a,b]$ defines an equilibrium.

Proof. Part (a). Reputation x_t follows (5.3), so an increase in x_0 raises x_t at each point in time. Lemma 2 says that $V_L(x)$ is strictly increasing in x, so equation (5.4) implies that $\Delta(x_0)$ is increasing in x_0. Part (b) follows from (a).

Part (c). If $\lambda \geq \mu$, the dynamics are divergent at x^*: if $x_0 = x_t - \epsilon$, then $\lim x_t = 0$; if $x_0 = x_t + \epsilon$, then $\lim x_t = 1$. Thus, to define value functions and Δ at the cutoff x^* we need to specify whether or not x^* works. Denote by $\Delta^-_{x^*}(x)$ (resp. $\Delta^+_{x^*}(x)$) the value of quality at x when x^* is believed to be shirking (resp. working). At $x^* \in (0,1)$ we have $\Delta^-_{x^*}(x^*) = \lim_{x \nearrow x^*} \lambda\Delta_{x^*}(x)$ and $\Delta^+_{x^*}(x^*) = \lim_{x \searrow x^*} \lambda\Delta_{x^*}(x)$. Lemma 2 says that $V_L(x_t)$ is strictly increasing in x_t, so (5.4) implies that

$$\Delta^-_{x^*}(x^*) < \Delta^+_{x^*}(x^*). \tag{5.6}$$

A cutoff $x^* \in (0,1]$ then defines a shirk-work equilibrium iff[13]

$$\lambda\Delta^-_{x^*}(x^*) \leq c \leq \lambda\Delta^+_{x^*}(x^*). \tag{5.7}$$

Equation (5.4) implies that $\Delta^+_{x^*}(x^*)$ and $\Delta^-_{x^*}(x^*)$ are increasing and continuous in x^*. For the lower bound, observe that $\lambda\Delta^-_0(0) = 0$, because a firm with no reputation that is believed to be shirking is stuck at 0 forever. For the upper bound, $\lambda\Delta^+_1(1) = \frac{\lambda}{r+\lambda+\mu}\mu(1-c)/r$ because $V_{L,1}(1) = \frac{r+\lambda}{r+\lambda+\mu}(1-c)/r$, $V_{L,1}(0) = 0$, and $\lambda\Delta^+_1(1) = \frac{\lambda\mu}{r+\lambda}\left(V_{L,1}(1) - V_{L,1}(0)\right)$. Under assumption (5.5), equation (5.7) therefore defines a non-empty interval of cutoffs, $[a,b]$. □

Suppose $\lambda \geq \mu$, so that whenever the firm is known to be shirking its reputation drifts down (see Figure 5). In this case, the region below x^* is a shirk-hole: when a firm's reputation is below the cutoff, it is certain to see its reputation decrease because of the adverse beliefs. Such a firm always shirks, eventually giving rise to a low quality product and a product breakdown destroying whatever is left of its reputation. When a firm's reputation is above the cutoff, favourable market beliefs contribute to an increasing reputation and the firm invests to insure itself against a product breakdown. At the cutoff, the firm works when it is believed to be working and shirks whenever it is believed to be shirking.[14]

Theorem 2(c) shows that there is an interval of equilibrium cutoffs satisfying (5.7). The multiplicity is driven by a discontinuity in the value function at x^*, caused by the divergent reputational dynamics. Intuitively, the market's beliefs become self-fulfilling. If the market believes the firm

[13]The case $x^* = 0$ is more subtle because there are two qualitatively different effort-profiles with cutoff $x^* = 0$.

If $x^* = 0$ is believed to be working, there is no shirk-hole, and the necessary and sufficient condition for equilibrium is that the lowest reputation works, i.e. $c \leq \lambda\Delta^+_0(0)$.

However, if $x^* = 0$ is believed to be shirking, then $x^* = 0$ is a shirk-hole and the equilibrium condition is $c \leq \lambda\Delta^-_0(x)$ for all $x > 0$ (and $c \geq \lambda\Delta^-_0(0)$ which is automatically satisfied because $\lambda\Delta^-_0(0) = 0$). This condition is weaker than (5.7) as $\Delta^+_0(0) < \lim_{x\to 0}\Delta^-_0(x)$: The existence of a shirk-hole increases the value of quality. Formally, this is reflected in equation (5.4) in that $V_L(0) = 0$ if reputation 0 shirks, but $V_L(0) > 0$ if reputation 0 works.

[14]The divergent dynamics imply that there will be path dependence in reputations. This is consistent with the existence of credit traps in financial markets, and may help explain why political scandals have such dramatic effects on politicians careers (Diermeier, Keane, and Merlo (2005)).

is shirking, its reputation falls, undermining its incentive to invest. Conversely, if the market believes the firm is working, its reputation rises, causing the firm to invest in order to protect its appreciating reputation.[15]

The shirk-work equilibrium under perfect bad news learning seems to be at odds with the unique work-shirk equilibrium under "almost perfect" bad news covered in Theorem 2. However, this limit is continuous. Full work, $\eta(x) = 1$ for all x, is an equilibrium under perfect bad news learning when costs are low.[16] This full work profile is approximated by work-shirk equilibria when learning is via bad news signals y with high but finite ratios $\mu_{L,y}/\mu_{H,y}$. With such signals the reputational increment $d_y x$ and the reputational dividend remain high close to $x = 1$. Thus, the work-shirk equilibria with $x^* \approx 1$ for close to perfect learning approximate the full work equilibrium, with $x^* = 1$, under perfect learning.

When $\lambda < \mu$ the dynamics have additional interesting features: Define $\widehat{x} = 1 - \frac{\lambda}{\mu} \in (0,1)$ to be the stationary point in the dynamics when the firm is believed to be shirking. There are two types of equilibria:

1. *Trapped equilibria.* When $\widehat{x} < x^*$, a firm with reputation $x \in (0, x^*)$ finds its reputation converging to $\widehat{x}$, and remains stuck in a shirk-hole. At some point is suffers a breakdown and remains at $x = 0$ thereafter. Since the dynamics are divergent at x^* the value function is discontinuous, and there is an interval of such equilibria.

2. *Permeable equilibria.* When $\widehat{x} > x^*$, a firm with reputation $x \in (0, x^*)$ finds its reputation increasing. If x_t passes x^* before a breakdown hits, the firm starts to work and its reputation may converge to one. Since the value functions are continuous at a permeable cutoff x^*, there is at most one permeable equilibrium.

5.3 Perfect Good and Bad News

If the product can both enjoy breakthroughs revealing high quality with intensity μ_g, and suffer breakdowns revealing low quality with intensity $-\mu_b$, the reputational dividend is given by:

$$\begin{aligned} D_\theta(x) &= \mu_g (V_\theta(1) - V_\theta(x)) - \mu_b (V_\theta(x) - V_\theta(0)) \\ &= -(\mu_b + \mu_g) V_\theta(x) + \mu_g V_\theta(1) + \mu_b V_\theta(0). \end{aligned} \tag{5.8}$$

When good news is more frequent than bad news, $\mu_g > -\mu_b$, the analysis is similar to the perfect good news case of Section 5.1: Reputational dividends and value of quality are decreasing

[15] While outside the model, this multiplicity creates an incentive for firms to invest in marketing in order to shape consumers expectations.

[16] Investment incentives under full work are bounded below by $\Delta_0(x) > \Delta_0^+(0) > 0$. This bound is derived explicitly in equation (**??**) in Appendix D.2.

in reputation and any equilibrium must be work-shirk. However, full work can be an equilibrium now, because the perfect bad news signal incentivizes work even for high reputations.

When bad news is more frequent than good news, $-\mu_b > \mu_g$, the analysis is similar to the perfect bad news case of Section 5.2: Reputational dividends and value of quality are increasing in reputation, and any equilibrium must be shirk-work. However, when costs are low, equilibrium must be full work now, because the perfect good news signal guarantees (HOPE) and rules out shirking at the bottom.

6 Quality Choice

We now connect our analysis to the standard models in the literature, where quality is chosen in every period, by taking the quality obsolescence rate λ to infinity. As discussed in the introduction we find that an increase in λ can be detrimental to incentives despite the benefit of front-loading the returns to effort. In particular we find that the work-shirk equilibria of Theorem 2 disappear as $\lambda \to \infty$.

Theorem 3 *For any learning process, there exists λ^* such that for all $\lambda > \lambda^*$:*

(a) Pure shirking is an equilibrium.

(b) There is no work-shirk equilibrium with cutoff $x^* \in (0;1]$.

Proof. See Appendix C. □

Intuitively, when we fix a learning process dZ_t and choose λ sufficiently high, the reputational dynamics dx_t in (2.1) are dominated by equilibrium beliefs $\lambda(\widetilde{\eta} - x)$, and any effect of learning from actual quality is quickly lost. If beliefs $\widetilde{\eta}$ are work-shirk the firm's future reputation and revenue will be close to the cutoff x^*, irrespectively of its investment, quality, or consumer utility.

More precisely, we can apply Theorem 1 to write the benefits of working as the product of an average reputational dividend and an average time at which the dividend accrues:

$$\lambda\Delta(x) = \lambda \int_0^\infty e^{-(r+\lambda)t}\mathbb{E}[D(x_t)]dt \approx \underbrace{\frac{\lambda}{r+\lambda}}_{\leq 1} \underbrace{D(x_{future})}_{\to 0}$$

The first term $\frac{\lambda}{r+\lambda}$ captures the front-loading effect discussed in the introduction: As λ increases so does the probability that current effort affects quality. At the same time it increases the rate at which this quality becomes obsolete, and the effective discount rate $r + \lambda$. Aggregating these two effects, an increase in λ front-loads returns and increases $\lambda/(r+\lambda)$. By this logic, investment incentives are increasing in λ under complete information, where the dividend from quality is equal to 1.

Under incomplete information to the contrary, investment incentives vanish: While the front-loading effect $\lambda/(r+\lambda)$ is bounded by 1, the reputational dividend of quality vanishes as a function of λ. To see this, we decompose the value of a firm, say that is shirking and has low quality, into current profits and continuation value $V_L(x) = x_t dt + (1 - rdt) V_L(x - dx)$. Rearranging terms, we find that marginal value $V_L'(x) = \frac{1}{dx/dt}(x_t - rV_L(x))$ is decreasing in $dx/dt \approx \lambda$. As the marginal value of reputation vanishes, so does the reputational dividend of quality, which is composed of "jump-terms" $\mu_y(V_\theta(x_y) - V_\theta(x))$ derived from the Poisson component of learning, and a continuous term $\mu_B^2 x(1-x)V_\theta'(x)$ derived from the Brownian component of learning.[17]

For the perfect Poisson learning processes of Section 5, we find:

Proposition 3 *There is λ^* such that for all $\lambda > \lambda^*$:*

(a) Under perfect good news learning, full shirking is the only equilibrium.

(b) Under perfect bad news learning, any cutoff $x^* \in (0;1]$ defines a shirk-work equilibrium, as long as condition (5.5) is satisfied.

Proof. Part (a). By Proposition 1 any equilibrium under good news is work-shirk, but the only such equilibrium for high λ entails full shirking by Theorem 3. More explicitly, we can bound the cutoff type's investment incentives directly,

$$\lambda\Delta_{x*}(x^*) \leq \lambda\Delta_0(0) \leq \frac{\mu}{\lambda+\mu} \; \lambda \int_0^\infty e^{-rt} x_t dt \leq \frac{\mu}{\lambda+\mu}\frac{\lambda}{r+\lambda}.$$

The first inequality comes from equation (**??**), the second from substituting $x^* = 0$ into (**??**), and the third from observing that $x_t \leq e^{-\lambda t}$ when $x^* = 0$.

Part (b). Pick any $x^* > 0$. First, suppose $x_0 > x^*$ and observe that $x_t \geq 1 - e^{-\lambda t}$. Equation (**??**) in Appendix D.2 gives an explicit formula for the value of quality. Taking limits,

$$\lim_{\lambda\to\infty} \lambda\Delta_{x^*}(x) \geq \lim_{\lambda\to\infty} \frac{\lambda\mu}{\lambda+\mu} \int_{t=0}^\infty e^{-rt}(1 - e^{-\lambda t} - c)(1 - e^{-(\lambda+\mu)t})\, dt = \mu(1-c)/r$$

where the final equality uses the fact that the integral converges to $(1-c)/r$. Assumption (5.5) implies that $\mu(1-c)/r > c$. Hence for sufficiently large λ, working is optimal for all $x > x^*$ and any x^*.

[17]Theorem 3 does not show that full shirking is the unique equilibrium in the Brownian case. In this sense, this result is weaker than the one in Sannikov and Skrzypacz (2007). This seems to be a matter of analytic tractability rather than economic substance: Consider the incentives in a shirk-work-shirk equilibrium with cutoffs $\underline{x} < \overline{x}$ for high values of λ. A firm with reputation $\overline{x}$ faces the risk of slipping through the work-region into the shirk hole. Let γ_θ be the intensity with which this happens to a firm with quality θ. The dividend of quality is now a function of the difference in these intensities, $\gamma_L - \gamma_H$. For large values of λ, the size of the work-region $\overline{x} - \underline{x}$ must be small to bound this difference from below. The analysis in Sannikov and Skrzypacz (2007) suggests that this increases γ_H. Even a firm with high quality slips into the shirk hole arbitrarily fast, and $V(\overline{x})$ and $\Delta(\overline{x})$ converge to 0.

Next suppose $x_0 < x^*$ and observe that $x_t \leq e^{-(\lambda-\mu)t}$. Equation (**??**) in Appendix D.2 derives an explicit formula for the asset value of quality. Taking limits,

$$\lim_{\lambda\to\infty} \lambda\Delta_{x^*}(x) \leq \lim_{\lambda\to\infty} \frac{\lambda\mu}{\lambda-\mu} \int_{t=0}^{\infty} e^{-rt} e^{-(\lambda-\mu)t} dt = 0$$

Hence for sufficiently large λ, shirking is optimal for all $x < x^*$ and any x^*. $\square$

The key difference that sustains effort in part (b) is that reputational dynamics are divergent at a shirk-work cutoff x^*, and value functions are discontinuous as discussed in Section 5.2. Thus, while the marginal value of reputation $V'_\theta(x)$ vanishes for all $x \neq x^*$ and the value function approaches $V_\theta(0)$ for $x < x^*$, and $V_\theta(1)$ for $x > x^*$, the reputational dividend approaches $\mu(V_\theta(1) - V_\theta(0))$. The proof shows that under condition (5.5) this exceeds c. High values of λ amplify the multiplicity of equilibria found in the bad news case. Intuitively, a firm that starts below the cutoff finds its reputation falling to zero instantly and gives up, while a firm above the cutoff finds its reputation rising to one instantly and works to stay there.

Theorem 3 has a surprising consequence: Providing more information about the firm's quality may be detrimental to equilibrium effort. More specifically, consider a shirk-work equilibrium under perfect bad news learning. Then improve the learning process by introducing additional perfect good news signals. If the arrival rate of the good news exceeds the arrival rate of the bad news, Section 5.3 shows that equilibria must be work-shirk. If additionally λ is high enough, full shirking is the only equilibrium by Lemma 3.

Ordering signal structures by sufficiency, equilibria by effort, and equilibrium sets by the set order we can summarize:

Corollary 2 *More information can lead to less effort.*

The economics of the counter-example resembles the problem of renegotiation-proofness in a repeated prisoners' dilemma: Under perfect bad news learning a firm with a high reputation works because a breakdown permanently destroys its reputation. Additional good news signals grant the firm a second chance after a breakdown, and undermine incentives to work hard in the first place.

7 Conclusion

This paper develops a new model of reputation, where the firm invests in the quality of its products, which is imperfectly observed by the market. As customers experience the product, the firm's reputation evolves. This evolution, in turn, affects the firm's incentives to invest in quality.

The model forms a bridge between repeated games and classical models of reputation. In contrast to repeated games, different firms may have different capabilities. In contrast to classical models of reputation, firm's capabilities are a function of past decisions and are therefore endogenous. This model seems realistic: The current state of General Motors is a function of its past hiring policies, investment decisions and reorganisations, all of which are endogenous.

Our results highlight the role of the learning process in determining reputational incentives: When the market learns high quality perfectly through product breakthroughs, there is a unique work-shirk equilibrium and convergent dynamics. When the market learns low quality perfectly through product breakdowns, there is a continuum of shirk-work equilibria and divergent dynamics. For imperfect learning processes, there exists a work-shirk equilibrium just like for perfect good news learning, but no shirk-work equilibrium.

There are many ways to extend this model to capture additional important aspects of firm reputation. One can allow the high-quality firms to have lower investment costs than low-quality firms. Under perfect good news learning, equilibria are work-shirk, with a low quality firm's cutoff x_L^* lying below a high quality firm's cutoff, x_H^*, implying that the firm's reputation ultimately cycles over $[x_L^*, 1]$ if $\lambda \geq \mu$. In another variant, one might suppose that people stop buying from the firm when its reputation falls below an exogenous threshold, $\underline{x}$. The value functions now satisfy $V_L(\underline{x}) = V_H(\underline{x}) = 0$; so with Brownian learning the firm shirks when its reputation gets close to $\underline{x}$ as its life expectancy gets short. Beyond firm reputation, we hope that our model may prove useful in other applications, such as finance, trade, political careers, or labor markets.

A General Results

A.1 Log-likelihood Ratio Transformation

For most of the technical proofs in the appendix, we represent reputation by the log-likelihood ratio $\ell(x) = \log(x/(1-x)) \in \mathbb{R} \cup \{-\infty, \infty\}$ of state H rather than the posterior x. The relevant transformation functions are:

$$\ell(x) = \log \frac{x}{1-x} \qquad\qquad x(\ell) = \frac{e^\ell}{1+e^\ell} \qquad\qquad \frac{dx}{d\ell} = \frac{e^\ell}{\left(1+e^\ell\right)^2} = x\,(1-x)$$

When we work in ℓ-space we write $\widehat{V}_\theta(\ell) := V_\theta(x(\ell))$ for value functions, $\widehat{D}_\theta(\ell) := D_\theta(x(\ell))$ for the reputational dividend, $\widehat{\Delta}(\ell) := \Delta(x(\ell))$ for the value of quality, and $\widehat{\eta}(\ell) := \eta(x(\ell))$ for effort. The advantage of this transformation is that reputational updating is more tractable in ℓ-space.

Market Learning: Bayesian updating from signals dZ_t is linear in ℓ-space:

$$\ell_{t+dt} = \ell_t + \log \frac{\Pr(dZ_t|H)}{\Pr(dZ_t|L)}. \tag{A.1}$$

More specifically, the Brownian component of learning $\theta\mu_B dt + dW$ imposes a Brownian motion on reputational dynamics:

$$d_\theta \ell = \mu_B^2 \left(\theta - 1/2\right) + \mu_B dW$$

The Poisson component of learning, with event y arriving at intensity $\mu_{y,\theta}$, affects reputational dynamics via

$$d\ell = \begin{cases} \delta_y & \text{in case of arrival } y, \\ -\sum_y \mu_y dt & \text{absent of an arrival,} \end{cases}$$

where the jump equals $\delta_y = \log(\mu_{y,H}/\mu_{y,L})$.

Equilibrium Beliefs: This part of reputational updating is more complex in ℓ-space:

$$\frac{d\ell}{dt} = \frac{d\ell}{dx}\lambda\left(\widetilde{\eta} - x\right) = \lambda \frac{\left(1+e^\ell\right)^2}{e^\ell}\left(\widetilde{\eta} - \frac{e^\ell}{1+e^\ell}\right) = \begin{cases} \lambda\left(1+e^{-\ell}\right) & \text{for } \widetilde{\eta} = 1, \\ -\lambda\left(1+e^{\ell}\right) & \text{for } \widetilde{\eta} = 0. \end{cases} \tag{A.2}$$

Asymptotic Dynamics: In a work-shirk profile with cutoff $\ell^* \gg 0$, dynamics of high reputations

$0 \ll \ell \leq \ell^*$ are approximately independent of ℓ with

$$d_\theta \ell = \begin{cases} \left(\lambda\left(1+e^{-\ell}\right)+\mu_B^2\left(\theta-1/2\right)-\sum_y \mu_y\right) dt & \text{almost constant drift,} \\ +\mu_B dW & \text{constant wiggle,} \\ +\sum_{y\in Y}\delta_y & \text{constant jumps,} \end{cases} \tag{A.3}$$

while the boundary ℓ^* is approximately reflecting with $d\ell \approx -\lambda e^\ell \approx -\infty$.

Reputational Dividend: $\widehat{D}_\theta(\ell)$ is additive across the Brownian and Poisson component of market learning:

$$\widehat{D}_\theta(\ell) = \frac{\mu_B^2}{2}\widehat{V}_\theta'(\ell) + \sum_y \mu_y\left(\widehat{V}_\theta(\ell+\delta_y) - \widehat{V}_\theta(\ell)\right). \tag{A.4}$$

A.2 Value of Reputation: Reprise

We first give the proof of Lemma 2:

Proof of Lemma 2. Fix θ, $(x', \widetilde{\eta}')$ and $(x'', \widetilde{\eta}'') \geq (x', \widetilde{\eta}')$, i.e. $x'' \geq x'$ and $\widetilde{\eta}''(x) \geq \widetilde{\eta}'(x)$ for all x. Write the best response $\eta^*(\widetilde{\eta}')$ to the Markovian beliefs $\widetilde{\eta}'$ in a non-Markovian way as a function of the public history $\overline{\eta}(dZ^t) = \eta\left(x\left(dZ^t, \widetilde{\eta}', x'\right)\right)$. For any realization of the random processes, denote by $(x_t'', \theta_t'', \eta_t'', dZ_t'')$ the trajectory of reputation, quality, effort and utility given effort $\overline{\eta}(dZ^t)$, initial reputation x'' and market beliefs $\widetilde{\eta}''$, and by $(x_t', \theta_t', \eta_t', dZ_t')$ the corresponding trajectory given $\overline{\eta}, x', \widetilde{\eta}'$.

By construction, effort $\eta_t' = \eta_t''$, quality $\theta_t' = \theta_t''$, and utility $dZ_t' = dZ_t''$ will coincide for all times t. Reputation on the other hand may start at different levels $x'' \geq x'$ and because the updating equation (2.1) implies that $x_{t+dt}(x_t, \widetilde{\eta}_t, dZ_t)$ is increasing in x_t and $\widetilde{\eta}_t$, we get $x_t'' \geq x_t'$.

Thus, by mimicking the effort of the firm with lower initial reputation x', the firm with a strictly higher initial reputation x'' can secure itself a strictly higher value. By Lemma 1 there must be a Markov strategy that is at least as good as this mimicking strategy. $\square$

We now extend Lemma 2 (positive marginal value of reputation in equilibrium) in two directions. Lemma 4 extends the result to non-equilibrium work-shirk profiles and provides an explicit formula for the marginal value of reputation (equation A.5). Lemma 5 provides a lower bound to incremental value that is uniform across equilibrium effort profiles.

We first prove an auxiliary lemma that uses flexible accounting in representing value functions as NPV of future profits. Future profits depend on current quality and effort via the reputational evolution. Lemma 3 shows one way of controlling this evolution, by replacing the actual effort function η by an arbitrary other, not necessarily Markovian, effort function $\overline{\eta}\left(Z^t\right)$.

Lemma 3 *For any effort and belief function $\langle \widehat{\eta}, \widetilde{\eta} \rangle$, and any (non-Markovian) alternative effort function $\overline{\eta}\left(Z^t\right)$ the firm's value equals:*

$$\widehat{V}_\theta(\ell_0) = \mathbb{E}_{\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\left(x\left(\ell_t\right) - c\overline{\eta}\left(Z^t\right)\right) + \left(\widehat{\eta}(\ell_t) - \overline{\eta}\left(Z^t\right)\right)\left(\lambda\widehat{\Delta}(\ell_t) - c\right)\right)dt\right]$$

Proof. Fix θ_0, ℓ_0 and $\widetilde{\eta}$. Consider first a "one shot deviation" from $\widehat{\eta}$, i.e. an alternative effort function $\overline{\eta}$ that differs from $\widehat{\eta}$ only for $t \in [0, dt]$, say $\overline{\eta} = 1$ while $\widehat{\eta} = 0$. A firm that exerts effort according to $\widehat{\eta}$ but whose quality θ_{dt} is governed by $\overline{\eta}$ gains $\lambda\widehat{\Delta}(\ell_0)dt$. Thus, the firm's actual value is the value under the more favorable process $\overline{\eta}$, minus the fair value of the quality subsidy:

$$\widehat{V}_\theta(\ell_0) = \mathbb{E}_{\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(x\left(\ell_t\right) - c\widehat{\eta}(\ell_t)\right)dt\right] - \lambda\widehat{\Delta}(\ell_0)dt.$$

For "multi-period" deviations, we accumulate a term $\left(\widehat{\eta}(\ell_t) - \overline{\eta}\left(Z^t\right)\right)\lambda\widehat{\Delta}(\ell_t)dt$ whenever $\widehat{\eta}(\ell_t) \neq \overline{\eta}\left(Z^t\right)$. Thus, in general we have

$$\begin{aligned}
\widehat{V}_\theta(\ell_0) &= \mathbb{E}_{\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\left(x\left(\ell_t\right) - c\widehat{\eta}(\ell_t)\right) + \left(\widehat{\eta}(\ell_t) - \overline{\eta}\left(Z^t\right)\right)\lambda\widehat{\Delta}(\ell_t)\right)dt\right] \\
&= \mathbb{E}_{\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\left(x\left(\ell_t\right) - c\overline{\eta}\left(Z^t\right)\right) + \left(\widehat{\eta}(\ell_t) - \overline{\eta}\left(Z^t\right)\right)\left(\lambda\widehat{\Delta}(\ell_t) - c\right)\right)dt\right]
\end{aligned}$$

as required. $\square$

To state and prove the next lemma, we write the firm's reputation at time t as a function of its initial reputation ℓ_0, realized utilities Z^t and the Markovian beliefs $\widetilde{\eta}$: $\ell_t = \ell_t\left(\ell_0, Z^t, \widetilde{\eta}\right)$.

Lemma 4 *Fix a work-shirk profile with cutoff ℓ^*:*

(a) Reputational increments are decreasing:

$$\frac{\partial \ell_t}{\partial \ell}(\ell, Z^t, \widetilde{\eta}) < 1.$$

(b) Value $\widehat{V}_\theta(\ell)$ is continuous in reputation ℓ.

(c) If the cutoff ℓ^* weakly prefers to shirk, i.e. $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) \leq c$, the marginal value of reputation is strictly positive:

$$\widehat{V}_\theta'(\ell) > 0 \text{ for all } \ell \in \mathbb{R}.$$

(d) If the cutoff ℓ^* is indifferent, i.e. $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = c$, the marginal value of reputation equals:

$$\widehat{V}_\theta'(\ell) = \int_{t=0}^\infty e^{-rt}\mathbb{E}_{\theta_0=\theta}\left[\frac{e^{\ell_t}}{\left(1+e^{\ell_t}\right)^2}\frac{\partial \ell_t}{\partial \ell}(\ell, Z^t, \widetilde{\eta})\right]dt > 0 \text{ for all } \ell \in \mathbb{R}. \tag{A.5}$$

(e) The value of quality at ℓ^* is strictly positive:

$$\widehat{\Delta}_{\ell^*}(\ell^*) > 0.$$

Proof. For part (a), consider the reputational trajectories ℓ_t, ℓ_t' originating at $\ell < \ell'$. Equations (A.1) and (A.2) imply that $\ell_t' - \ell_t$ is decreasing in t for all work-shirk profiles: Market learning (A.1) shifts ℓ_t and ℓ_t' by the same amount $\log(\Pr(dZ_t|H)/\Pr(dZ_t|L))$, while equilibrium beliefs shrink $\ell_t' - \ell_t$ at rate $\lambda\left(1+e^{-\ell_t'}\right) - \lambda\left(1+e^{-\ell_t}\right) \approx -\lambda e^{-\ell_t}\left(1-e^{-(\ell_t'-\ell_t)}\right) < 0$ in the work-region and similarly in the shirk-region.

Turning to points (b) to (e), let $\overline{\eta}\left(Z^t\right) = \widehat{\eta}(\ell_t(\ell', Z^t, \widetilde{\eta}))$ be the effort strategy of a firm with initial reputation ℓ, expressed in a non-Markovian way directly as a function of market signals Z^t. We decompose the incremental value of reputation as follows:

$$\widehat{V}_{\theta,\widehat{\eta}}(\ell') - \widehat{V}_{\theta,\widehat{\eta}}(\ell) = [\widehat{V}_{\theta,\widehat{\eta}}(\ell+d\ell) - \widehat{V}_{\theta,\overline{\eta}}(\ell)] + [\widehat{V}_{\theta,\overline{\eta}}(\ell) - \widehat{V}_{\theta,\widehat{\eta}}(\ell)] \tag{A.6}$$

The first term in (A.6) is the reputational advantage of starting with reputation $\ell' > \ell$, when the firm starting at reputation ℓ mimics the effort of the firm starting at ℓ'. It is determined by the derivative of future reputation with respect to current reputation:

$$\widehat{V}_{\theta,\widehat{\eta}}(\ell') - \widehat{V}_{\theta,\overline{\eta}}(\ell) = \int e^{-rt}\mathbb{E}_{\theta_0=\theta,\overline{\eta}}\left[\left(x\left(\ell_t(\ell', Z^t, \widetilde{\eta})\right) - x\left(\ell_t\left(\ell, Z^t, \widetilde{\eta}\right)\right)\right)\right] dt$$

This term is always positive. Taking the limit as $\ell' \to \ell$ and applying the chain rule gives rise to equation A.5.

The second term in (*) is the net value of shirking whenever $\overline{\eta}\left(Z^t\right) = 0$ and $\widehat{\eta}(\ell_t) = 1$. We will now show that it is of order $o\left(d\ell\right)$.

$$\begin{aligned}
&\widehat{V}_{\theta,\overline{\eta}}(\ell) - \widehat{V}_{\theta,\widehat{\eta}}(\ell) \\
&= \mathbb{E}_{\ell_0=\ell,\theta_0=\theta,\overline{\eta},\widetilde{\eta}}\left[\int_0^\infty e^{-rt}\left(\widehat{\eta}(\ell_t) - \overline{\eta}\left(Z^t\right)\right)\left(c - \lambda\widehat{\Delta}(\ell_t)\right) dt\right] \\
&= \mathbb{E}_{\theta_0=\theta,\overline{\eta}}\left[\int_0^\infty e^{-rt}\left(\widehat{\eta}(\ell_t(\ell, Z^t, \widetilde{\eta})) - \widehat{\eta}(\ell_t(\ell', Z^t, \widetilde{\eta}))\right)\left(c - \lambda\widehat{\Delta}(\ell_t(\ell, Z^t, \widetilde{\eta}))\right) dt\right] \\
&= \mathbb{E}_{\theta_0=\theta,\overline{\eta}}\left[\int_{\ell_t(\ell,Z^t,\widetilde{\eta})<\ell^*<\ell_t(\ell',Z^t,\widetilde{\eta})} e^{-rt}\left(c - \lambda\widehat{\Delta}(\ell_t(\ell, Z^t, \widetilde{\eta}))\right) dt\right] \qquad \text{(A.7)} \\
&\leq \max_{\widetilde{\ell}\in[\ell^*;\ell^*+(\ell'-\ell)]}\left\{\left|c - \lambda\widehat{\Delta}\left(\widetilde{\ell}\right)\right|\right\}\left(\ell' - \ell\right)/2\lambda
\end{aligned}$$

The first equality applies Lemma 3. The second applies the definition of $\overline{\eta}\left(Z^t\right)$. The third uses that the effort functions disagree if and only if the trajectories are on opposite sides of the cutoff, i.e. $\ell_t(\ell, Z^t, \widetilde{\eta}) < \ell^* < \ell_t(\ell + d\ell, Z^t, \widetilde{\eta})$. The final inequality uses that by (a), $\ell_t(\ell + d\ell, Z^t, \widetilde{\eta}) -$

$\ell_t(\ell, Z^t, \widetilde{\eta})$ is decreasing in t and by (A.2) it is decreasing at rate $-\lambda\left(2+e^{-\ell_t(\ell)}+e^{\ell_t(\ell')}\right)$ whenever $\ell_t(\ell, Z^t, \widetilde{\eta}) < \ell^* < \ell_t(\ell', Z^t, \widetilde{\eta})$. This proves (b).

For (c) the integrand is positive when $\lambda\widehat{\Delta}(\ell^*) \leq c$: Reducing effort on the margin is profitable if cost exceeds benefit.

For (d), when $\lambda\widehat{\Delta}(\ell^*) = c$ and $\widehat{\Delta}$ is continuous, the upper bound of (A.5) is of order $o(\ell'-\ell)$ because now $\max_{\ell\in[\ell^*,\ell^*+(\ell'-\ell)]}\left\{\left|c-\lambda\widehat{\Delta}(\ell)\right|\right\}$ is of order $o(1)$.

To prove (e) assume to the contrary that the cutoff is non-positive $\widehat{\Delta}(\ell^*) \leq 0$. A fortiori $\widehat{\Delta}(\ell^*) \leq c$ and by part (c) the marginal value of reputation is positive $\widehat{V}'_H(\ell) > 0$ and so is the reputational dividend $\widehat{D}_H(\ell)$. But then, also the value of quality would be positive $\widehat{\Delta}(\ell^*) > 0$. □

Lemma 4(d) has the flavor of the envelope theorem: when the firm's first-order condition holds at the cutoff, then a change in initial reputation only affects its payoff through the reputational evolution. Intuitively, a firm with a lower initial reputation works more, leading to a gain of $\widehat{\Delta}(\ell)$ when a technology shock hits. This gain is exactly offset by the extra cost born by the firm. The marginal value of reputation $\widehat{V}'_\theta(\ell)$ is thus determined solely by the "durability" of the reputational increment $\ell'_t - \ell_t$.

In important cases we can truncate the integral (A.5) at time T when the reputational evolution hits ℓ^*. When $\ell_t < \ell^* < \ell'_t$ the increment $\ell'_t - \ell_t$ decreases at rate $-\lambda\left(2+e^{-\ell_t}+e^{\ell'_t}\right)$. If the cutoff ℓ^* is "reflecting" because either $\ell^* \approx \pm\infty$ or $\lambda \gg 0$, the reputational increment $\ell'_t - \ell_t$ approximately disappears at T and we can restrict the integral in (A.5) to $t \leq T$.

We now come to the second extension, which we will use to prove essential uniqueness of equilibrium (Lemma 10).

Lemma 5 *There exists a function $\alpha(\ell,\ell') > 0$, that is a lower bound of the incremental value of reputation $\alpha(\ell,\ell') < \widehat{V}_\theta(\ell') - \widehat{V}_\theta(\ell)$ for $\ell < \ell'$, uniformly across cost c and equilibrium effort $\widehat{\eta}$.*

Proof. Let $\ell' > \ell$. In equilibrium the firm with the higher reputation ℓ' cannot benefit by imitating the effort of the firm with the lower reputation via $\overline{\eta}\left(Z^t\right) = \widehat{\eta}(\ell_t(\ell, Z^t, \widetilde{\eta}))$. Therefore,

$$
\begin{aligned}
\widehat{V}_\theta(\ell') - \widehat{V}_\theta(\ell) &\geq \int e^{-rt}\mathbb{E}_{\overline{\eta}}\left[x\left(\ell_t\left(\ell', Z^t, \widetilde{\eta}\right)\right) - x\left(\ell_t\left(\ell, Z^t, \widetilde{\eta}\right)\right)\right] dt \\
&\geq k_1 \frac{e^\ell}{\left(1+e^\ell\right)^2} \int e^{-rt}\mathbb{E}_{\overline{\eta}}\left[\ell_t\left(\ell', Z^t, \widetilde{\eta}\right) - \ell_t\left(\ell, Z^t, \widetilde{\eta}\right)\right] dt \\
&\geq k_1 \frac{e^\ell}{\left(1+e^\ell\right)^2} \int e^{-rt}\left[\ell' - \ell - \lambda\left(2+e^{\ell'_t}+e^{-\ell_t}\right)t\right]_+ dt \\
&\geq k_2 \frac{e^\ell}{\left(1+e^\ell\right)^2} \frac{\left(\ell'-\ell\right)^2}{2\lambda\left(2+e^\ell+e^{-\ell}\right)}
\end{aligned}
$$

The first line bounds $\widehat{V}_\theta(\ell')$ below by the value when exerting effort $\overline{\eta}$. The second line essentially applies the chain rule to factor out $dx/d\ell = e^\ell/\left(1+e^\ell\right)^2$, and the constant k_1 accounts for the possibility that $dx/d\ell$ needs to be evaluated at ℓ_t instead of ℓ, and possibly $dx(\ell_t)/d\ell < dx(\ell)/d\ell$. The second line takes the most pessimistic stance on the evolution of $\ell'_t - \ell_t$ by assuming that $\eta(\ell_t) = 1$ and $\eta(\ell'_t) = 0$ in which case $d(\ell'_t - \ell_t)/dt = -\lambda(2+e^{\ell'_t}+e^{-\ell_t})t$. The third line evaluates the integral over an essentially linear function, and k_2 accounts for $e^{-rt} < 1$ and the possibility that $\lambda(2+e^{\ell'_t}+e^{-\ell_t}) < \lambda(2+e^{\ell}+e^{-\ell})$. $\square$

While the bounds in the proof are far from being tight the argument actually shows, conversely, that $\widehat{V}'_\theta(\ell^*) = 0$ at a work-shirk cutoff ℓ^*, when market learning has no Brownian component, i.e. $\mu_B = 0$.

B Proof of Theorem 2

We prove Theorem 2 in ℓ-space, introduced in Appendix A.1. We show that for sufficiently small c there exists a cutoff ℓ^* such that:

(a) Cutoff is indifferent: $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = c$ (Section B.1, Lemma 6),

(b) Low reputations work: $\lambda\widehat{\Delta}_{\ell^*}(\ell) > c$ for $\ell < \ell^*$ (Section B.2, Lemma 8),

(c) High reputations shirk: $\lambda\widehat{\Delta}_{\ell^*}(\ell) < c$ for $\ell > \ell^*$ (Section B.3, Lemma 9).

Section B.4 shows the essential uniqueness of the work-shirk equilibrium.

B.1 Indifference of Cutoff

We now show that for small costs there exists a high cutoff ℓ^* that satisfies the indifference condition. Since $\widehat{\Delta}$ and $\widehat{V}$ depend on c, we subscript them with c where useful.

Lemma 6 *For every $\ell \in \mathbb{R}$ there exists $c(\ell) > 0$ such that for all $c^* < c$ there exists $\ell^* > \ell$ such that $c^* = \lambda\widehat{\Delta}_{\ell^*,c^*}(\ell^*)$.*

Proof. Fix $\ell \in \mathbb{R}$ and consider $\widehat{\Delta}_{\ell,c}(\ell)$ as a function of $c \in [0, \lambda/(r+\lambda)]$. By Lemma 4(e) we have $\widehat{\Delta}_{\ell,c}(\ell) > 0$ for all c. Since $\widehat{\Delta}_{\ell,c}(\ell)$ is continuous in c, it takes on its minimum $\widehat{\Delta}_{\ell,c'}(\ell) > 0$ at some c'.

Let $c(\ell) = \lambda\widehat{\Delta}_{\ell,c'}(\ell)$ and fix $c^* \in (0, c(\ell))$. Using the definitions of c' and c^*,

$$\lambda\widehat{\Delta}_{\ell,c^*}(\ell) \geq \lambda\widehat{\Delta}_{\ell,c'}(\ell) > c^*,$$

so the firm prefers to work. On the other hand, at $\ell = \infty$:

$$c^* > \lambda \widehat{\Delta}_{\infty, c^*}(\infty) = 0,$$

so the firm prefers to shirk. By continuity of $\widehat{\Delta}_{\ell', c^*}(\ell')$ as a function of $\ell' \in [\ell; \infty]$, there exists $\ell^* \in (\ell, \infty)$ with $c^* = \lambda \widehat{\Delta}_{\ell^*, c^*}(\ell^*)$. $\square$

The daunting array of quantifiers in the statement of this lemma guarantees that we can assume ℓ^* with $c^* = \lambda \widehat{\Delta}_{\ell^*, c^*}(\ell^*)$ as large as necessary in the upcoming arguments.

B.2 Low Reputations Work

We first use equation (A.5) to prove an auxiliary result about the marginal value of reputation:

Lemma 7 *Fix any $\alpha > 0$, $M > 0$ and ℓ_V sufficiently large. Suppose $\ell^* > \ell_V$ is sufficiently high and $c = \lambda \widehat{\Delta}_{\ell^*, c}(\ell^*)$.*

(a) $\widehat{V}'_\theta(\ell)$ is decreasing on $[\ell_V; \ell^*]$.

(b) $\widehat{V}'_\theta(\ell)$ "diminishes" to the right of ℓ^* :

$$\widehat{V}'_\theta(\ell^* - \gamma) > M \widehat{V}'_\theta(\ell^* + \delta) \text{ for all } \gamma \in [\alpha; \ell^* - \ell_V] \text{ and all } \beta > 0.$$

Intuitively, incremental reputation above ℓ^* is less "durable" because it disappears when ℓ_t hits ℓ^* and reputational updating $\frac{d\ell}{dt}$ decelerates from $-\lambda\left(1 + e^{\ell}\right) \approx -\infty$ to $\lambda\left(1 + e^{-\ell}\right) \approx \lambda$. Formally, for $\ell < \ell^*$ let $T(\ell) = \min\{t | \ell_t \geq \ell^*, \ell_0 = \ell\}$ be the *cutoff time*: This is the first time that the reputational dynamics starting at ℓ reach, or exceed, the cutoff ℓ^*. For $\ell > \ell^*$ let $T(\ell) = \min\{t | \ell_t \leq \ell^*, \ell_0 = \ell\}$.

Proof. (a): For high enough ℓ^* and $\ell_0 \in (\ell_V; \ell^*)$, the reputational dynamics ℓ_t are approximately governed by a Brownian motion with constant drift and jumps (A.3). As long as $\ell_t < \ell^*$ we have $\frac{\partial \ell_t}{\partial \ell_0} \approx 1$. When the trajectory hits ℓ^* at time $T(\ell_0)$, then $\frac{\partial \ell_t}{\partial \ell_0} \approx 0$ for $t > T(\ell_0)$ since ℓ^* is reflecting. Using equation (A.5)

$$\widehat{V}'_\theta(\ell_0) = \int_0^\infty e^{-rt} \mathbb{E}_{\ell_0}\left[\frac{e^{\ell_t}}{(1 + e^{\ell_t})^2} \frac{\partial \ell_t}{\partial \ell_0}\right] dt \approx \mathbb{E}_{\ell_0}\left[\int_0^{T(\ell_0)} e^{-rt} \frac{e^{\ell_t}}{(1 + e^{\ell_t})^2} \frac{\partial \ell_t}{\partial \ell_0} dt\right]. \tag{B.1}$$

Since $e^{\ell_t}/(1 + e^{\ell_t})^2$ is strictly decreasing for $\ell_t > 0$, and $T(\ell_0)$ is decreasing in ℓ_0, equation (B.1) is strictly decreasing in ℓ on $[\ell_V; \ell^*]$.

(b): Let $\gamma \in [\alpha; \ell^* - \ell_V]$. When ℓ^* is sufficiently high, the reputational dynamics are given by (A.3). The drift is finite, and the expected cutoff time $\mathbb{E}\left[T(\ell^* - \gamma)\right]$ is bounded below independently of ℓ^*. Thus, $\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}(\ell')\right]$ for $\ell' \in [\ell_V; \ell^* - \alpha]$ is bounded away from 0 as $\ell^* \to \infty$.

Next, consider $\ell^* + \beta$. The expected time until cutoff $\mathbb{E}\left[T\left(\ell^* + \beta\right)\right]$ converges to 0 as $\ell^* \to \infty$, uniformly in δ. This is easier to see for the posterior x_t than for the log-likelihood-ratio ℓ, as $\mathbb{E}\left[\frac{d_\theta x}{dt}\right] = -\lambda x$ is bounded away from 0 while $1 - x^*$ converges to 0. Thus, $\mathbb{E}\left[\frac{d\ell_t}{d\ell_0}\left(\ell^* + \beta\right)\right]$ for $\beta > 0$ converges to 0 as $\ell^* \to \infty$.

For large values of ℓ^* we can ignore in (A.5) all terms with $t > T\left(\ell_0\right)$. Since $e^{\ell}/(1+e^{\ell})^2$ is decreasing in $\ell > 0$ we get bounds $e^{\ell_t(\ell^*-\gamma)}/(1+e^{\ell_t(\ell^*-\gamma)})^2 \geq e^{\ell^*}/(1+e^{\ell^*})^2 \geq e^{\ell_t(\ell^*+\beta)}/\left(1+e^{\ell_t(\ell^*+\beta)}\right)^2$ and equation (A.5) implies

$$\frac{\widehat{V}_\theta'(\ell^* - \gamma)}{\widehat{V}_\theta'(\ell^* + \beta)} \geq \frac{\frac{e^{\ell^*}}{(1+e^{\ell^*})^2}\mathbb{E}\left[\int_0^{T(\ell^*-\gamma)} e^{-rt}\frac{d\ell_t}{d\ell_0}(\ell^* - \gamma)dt\right]}{\frac{e^{\ell^*}}{(1+e^{\ell^*})^2}\mathbb{E}\left[\int_0^{T(\ell^*+\beta)} e^{-rt}\frac{d\ell_t}{d\ell_0}(\ell^* + \beta)dt\right]} \geq const.\frac{\mathbb{E}\left[T(\ell^* - \gamma)\right]}{\mathbb{E}\left[T\left(\ell^* + \beta\right)\right]}$$

Therefore, $\widehat{V}_\theta'(\ell^* - \gamma)/\widehat{V}_\theta'(\ell^* + \beta)$ diverges as $\ell^* \to \infty$, uniformly over all $\gamma \in [\alpha; \ell^* - \ell_V]$ and $\beta > 0$. $\square$

Lemma 8 shows that firms with low reputations work. For reputations $\ell \in [\ell_\Delta, \ell^*]$ for some ℓ_Δ defined below, the optimality of working follows directly by showing that $\widehat{\Delta}(\ell)$ is decreasing on $[\ell_\Delta; \ell^*]$. For reputations $\ell < \ell_\Delta$ the result follows from the closeness of $\widehat{\Delta}_{\ell^*}(\cdot)$ and $\widehat{\Delta}_\infty(\cdot)$.

Lemma 8 *Assume ℓ^* is large, costs c are small and $\lambda\widehat{\Delta}_{\ell^*}(\ell^*) = c$. Then $\lambda\widehat{\Delta}_{\ell^*}(\ell) > c$ for all $\ell < \ell^*$.*

Proof. *Claim 1.* For any $\alpha > 0$, there exists ℓ_D sufficiently large such that $\widehat{D}_H(\cdot)$ is strictly decreasing on $[\ell_D; \ell^* - \alpha]$ for any $\ell^* > \ell_D$.

Proof. By (A.4) $\widehat{D}_H(\cdot)$ is composed of $\widehat{V}_H'$-terms and jump terms. The former are taken care off by Lemma 7(a). For the jump terms, pick α,ℓ_V and $M = 1$ as in Lemma 7, and choose $\ell_D = \ell_V + \max_y\left\{-\delta_y\right\}$. We need to show $\widehat{D}_H\left(\ell\right) > \widehat{D}_H\left(\ell'\right)$ for all $\ell < \ell'$ in $[\ell_D; \ell^* - \alpha]$ and all $y \in Y$.

First consider good news events $y \in Y^+$ with $\delta_y, \mu_y > 0$ and assume wlog that $\ell + \delta_y > \ell'$. Then

$$\begin{aligned}
\widehat{D}_H\left(\ell\right) - \widehat{D}_H\left(\ell'\right) &= \mu_y\left(\int_\ell^{\ell+\delta_y}\widehat{V}_H'\left(\widetilde{\ell}\right)d\widetilde{\ell} - \int_{\ell'}^{\ell'+\delta_y}\widehat{V}_H'\left(\widetilde{\ell}\right)d\widetilde{\ell}\right) \\
&= \mu_y\int_\ell^{\ell'}\left(\widehat{V}_H'\left(\widetilde{\ell}\right) - \widehat{V}_H'\left(\widetilde{\ell} + \delta_y\right)\right)d\widetilde{\ell} \\
&> 0
\end{aligned}$$

where the inequality follows pointwise for every $\widetilde{\ell}$ from Lemma 7(a) if $\widetilde{\ell}+\delta_y < \ell^*$ and from Lemma 7(b) if $\widetilde{\ell}+\delta_y > \ell^*$.

Now consider bad news events $y \in Y^-$ with $\delta_y, \mu_y < 0$. Then

$$\begin{aligned}\widehat{D}_H(\ell) - \widehat{D}_H(\ell') &= -\mu_y \left(\int_{\ell+\delta_y}^{\ell} \widehat{V}_H'\left(\widetilde{\ell}\right) d\widetilde{\ell} - \int_{\ell'+\delta_y}^{\ell'} \widehat{V}_H'\left(\widetilde{\ell}\right) d\widetilde{\ell} \right) \\ &= -\mu_y \int_{\ell+\delta_y}^{\ell} \left(\widehat{V}_H'\left(\widetilde{\ell}\right) - \widehat{V}_H'\left(\widetilde{\ell} + (\ell' - \ell)\right)\right) d\widetilde{\ell} \\ &> 0\end{aligned}$$

where the inequality follows pointwise for every $\widetilde{\ell}$ by Lemma 7(a) because $\ell + \delta_y \geq \ell_V$.

Claim 2. For any $\varepsilon > 0$, there exists $\alpha > 0$, ℓ_D arbitrarily high, and $\ell_\Delta > \ell_D$ sufficiently high such that for any $\ell^* > \ell_\Delta$ and $\ell' \in (\ell_\Delta; \ell^*)$, we have

$$(r+\lambda) \int e^{-(r+\lambda)t} \Pr(\ell_t' \in [\ell_D; \ell^* - \alpha]) dt \geq 1 - \varepsilon.$$

Proof. This is because the reputational dynamics $d\ell_t$ in $[\underline{\ell}, \ell^*]$ are approximately governed by a Brownian motion with constant drift and jumps (A.3) reflected at ℓ^*.

Claim 3. There exists ℓ_Δ sufficiently large such that $\widehat{\Delta}(\cdot)$ is strictly decreasing on $[\ell_\Delta; \ell^*]$ for any $\ell^* > \ell_\Delta$.

Proof. Pick ℓ_D as in Claim 1 and $\ell_\Delta > \ell_D$ as in Claim 2. Claim 1 states that $\widehat{D}_H(\cdot)$ is strictly decreasing on $[\ell_D; \ell^* - \alpha]$ and claim 2 states that $\ell_t \in [\ell_D; \ell^* - \alpha]$ with probability close to 1. The claim follows because the reputational evolution ℓ_t is monotone, and the value of quality $\widehat{\Delta}(\ell)$ is the integral over the dividends $\widehat{D}_H(\ell_t)$.

Claim 4. Assume that $\lambda \widehat{\Delta}_{\ell^*}(\ell^*) = c$ and fix any $\overline{\ell}$. Then $\widehat{\Delta}_{\ell^*}(\cdot)$ converges to $\widehat{\Delta}_\infty(\cdot)$ uniformly on $[-\infty, \overline{\ell}]$ as $\ell^* \to \infty$.

Proof. As $\ell^* \to \infty$, $\widehat{\Delta}_{\ell^*}(\ell)$ converges pointwise to $\widehat{\Delta}_\infty(\ell)$ for all ℓ. Let $\ell^* \gg \overline{\ell}$. For any $\ell < \overline{\ell}$, equations (A.5) and (A.4) imply that, $\widehat{V}_{\theta,\ell^*}'(\ell)$ and $\widehat{D}_{\theta,\ell^*}(\ell)$, and thus $\widehat{\Delta}_{\ell^*}(\ell)$, depend on ℓ^* only on trajectories ℓ_t that reach ℓ^*. The future discounted probability of these trajectories converges to 0 as $\ell^* \to \infty$, so the convergence is uniform for $\ell < \overline{\ell}$.

Proof of Lemma. Choose $0 \ll \overline{\ell} \ll \ell^*$. Claim 3 implies that $\widehat{\Delta}_{\ell^*}(\ell)$ is strictly decreasing in ℓ for $\ell \in [\overline{\ell}, \ell^*)$. Since $\lambda \widehat{\Delta}_{\ell^*}(\ell^*) = c$, we have

$$\lambda \widehat{\Delta}_{\ell^*}(\ell) > c \quad \text{for } \ell \in [\overline{\ell}, \ell^*).$$

The function $\widehat{\Delta}_\infty(\cdot)$ is bounded away from 0 on $[-\infty, \bar{\ell}]$. Hence Claim 4 implies that $\widehat{\Delta}_{\ell^*}(\ell)$ is bounded away from zero. For small costs c, we get

$$\lambda \widehat{\Delta}_{\ell^*}(\ell) > c \quad \text{for } \ell \in [-\infty, \bar{\ell}],$$

as required. $\square$

B.3 High Reputations Shirk

Lemma 9 shows that firms with high reputations shirk.

Lemma 9 *Suppose ℓ^* is large and $\lambda\widehat{\Delta}(\ell^*) = c$. Then $\lambda\widehat{\Delta}(\ell') < c$ for all $\ell' > \ell^*$.*

Proof. The idea of the proof is to develop $\widehat{\Delta}(\ell')$ into dividends $\widehat{D}(\ell'_t)$ and a continuation value $e^{-(r+\lambda)T}\widehat{\Delta}(\ell'_T)$ for a "small" stopping time T, and show that the dividends $\widehat{D}(\ell'_t)$ are small compared to dividends of $\widehat{\Delta}(\ell^*)$, while the continuation value $e^{-(r+\lambda)T}\widehat{\Delta}(\ell'_T)$ is no bigger than the respective term of $\widehat{\Delta}(\ell^*)$. There are two possible comparisons:

(a) Develop $\widehat{\Delta}(\ell^*)$ until T and compare dividends and appreciation of continuation values separately, i.e. show $\widehat{D}(\ell'_t) < \widehat{D}(\ell^*_t)$ and $\widehat{\Delta}(\ell'_T) - \widehat{\Delta}(\ell') \leq \widehat{\Delta}(\ell^*_T) - \widehat{\Delta}(\ell^*)$.

(b) Let $T = T(\ell')$ be the time when ℓ'_t first hits ℓ^* (so that $\widehat{\Delta}(\ell'_T) = \widehat{\Delta}(\ell^*)$) and compare the dividend $\widehat{D}(\ell'_t)$ to the annuity value of $\widehat{\Delta}(\ell^*)$.

Below we show $\widehat{D}(\ell'_t) < \widehat{D}(\ell^*_t)$ for $t \approx 0$ and use comparison (a), if there is some Poisson event $y \in Y^-$ signalling bad news, i.e. $\mu_y = \mu_{H,y} - \mu_{L,y} < 0$. On the other hand we can show $\widehat{D}(\ell'_t) < (r+\lambda)\widehat{\Delta}(\ell^*)$ and use comparison (b), if there is no such bad news Poisson event y.

Case (a): Assume that there is a bad news Poisson event $y \in Y$, with $\mu_y < 0$. Assume by contradiction, that $\ell' > \ell^*$ maximizes $\widehat{\Delta}(\ell)$ on $[\ell^*, \infty]$. We develop $\widehat{\Delta}(\ell')$ and $\widehat{\Delta}(\ell^*)$ until T and subtract $e^{-(r+\lambda)T}\widehat{\Delta}(\ell')$ to express the rental value of $\widehat{\Delta}(\ell')$ as sum of dividends and appreciation:

$$\begin{aligned}
(r+\lambda)\, T\widehat{\Delta}(\ell') &= \int_0^T e^{-(r+\lambda)t}\mathbb{E}_{\theta^t=L}\left[\widehat{D}_H(\ell'_t)\right]dt + \left(\widehat{\Delta}(\ell'_T) - \widehat{\Delta}(\ell')\right), \\
(r+\lambda)\, T\widehat{\Delta}(\ell^*) &= \int_0^T e^{-(r+\lambda)t}\mathbb{E}_{\theta^t=L}\left[\widehat{D}_H(\ell^*_t)\right]dt + \left(\widehat{\Delta}(\ell^*_T) - \widehat{\Delta}(\ell^*)\right),
\end{aligned}$$

Let $T > 0$ be sufficiently small[18] with $\ell'_t > \ell^*$ for $t \leq T$ and $\ell^*_T < \ell^*$. Then the appreciation $\widehat{\Delta}(\ell^*_T) - \widehat{\Delta}(\ell^*)$ is positive by claim 2 in the proof of Lemma 8, while $\widehat{\Delta}(\ell'_T) - \widehat{\Delta}(\ell')$ is negative by choice of ℓ'.

[18]Technically, T is a stopping time, but for notational simplicity we treat it as a real number here.

We now argue that $\widehat{D}_H(\ell_t^*) > \widehat{D}_H(\ell_t')$. We first compare the jump terms for bad news events $y \in Y^-$ with $\mu_y, \delta_y < 0$. Because of the reflecting dynamics we can assume that $\ell_t^* + \delta_y < \ell_t' + \delta_y < \ell_t^* < \ell^* < \ell_t'$. Just like in the proof of claim 1 in Lemma 8 we get

$$\mu_y \left(\widehat{V}_H(\ell_t^* + \delta_y) - \widehat{V}_H(\ell_t^*) \right) - \mu_y \left(\widehat{V}_H(\ell_t' + \delta_y) - \widehat{V}_H(\ell_t') \right) = -\mu_y \int_{\ell_t^*}^{\ell_t'} \left(\widehat{V}_H'(\ell + \delta_y) - \widehat{V}_H'(\ell) \right) d\ell. \tag{*}$$

The RHS is strictly positive because $\widehat{V}_H'(\ell + \delta_y)$ exceeds $\widehat{V}_H'(\ell)$ by Lemma 7(a) when $\ell < \ell^*$ and by Lemma 7(b) when $\ell > \ell^*$.

The (*) term dominates the good news and Brownian component of $\widehat{D}_H(\ell_t')$, i.e. $\mu_y \int_{\ell_t'}^{\ell_t' + \delta_y} \widehat{V}_H'\left(\widetilde{\ell}\right) d\widetilde{\ell}$ for $y \in Y^+$ and $\widehat{V}_H'(\ell_t')\mu_B^2/2$. This is because $\widehat{V}_H'(\ell_t^* + \delta_y)/\widehat{V}_H'(\ell_t')$ diverges as $\ell^* \to \infty$ (Lemma 7(b)).

Case (b): Assume that there is no bad news Poisson event, i.e. $\mu_y \geq 0$ for all $y \in Y$. We develop $\widehat{\Delta}(\ell')$ until cutoff time $T(\ell')$. As there are no downward jumps in ℓ_t', we have $\ell_t' > \ell^*$ for $t < T$.

$$\begin{aligned} \widehat{\Delta}(\ell') - \widehat{\Delta}(\ell^*) &= \mathbb{E}\left[\int_0^{T(\ell')} e^{-(r+\lambda)t} \widehat{D}_H(\ell_t')\, dt + e^{-(r+\lambda)T(\ell')} \widehat{\Delta}(\ell^*) \right] - \widehat{\Delta}(\ell^*) \\ &= \mathbb{E}\left[\int_0^{T(\ell')} \left(e^{-(r+\lambda)t} \widehat{D}_H(\ell_t') - (r+\lambda)\widehat{\Delta}(\ell^*) \right) dt \right] \end{aligned} \tag{B.2}$$

To show that the integrand is negative, we develop $(r+\lambda)\widehat{\Delta}(\ell^*)$ into future reputational dividends $\widehat{D}_H(\ell_t^*)$, that will on average exceed $\widehat{D}_H(\ell_t')$:

$$\begin{aligned} (r+\lambda)\widehat{\Delta}(\ell^*) &= (r+\lambda)\int_0^\infty e^{-(r+\lambda)t}\mathbb{E}\left[\widehat{D}_H(\ell_t^*)\right] dt \\ &\geq (r+\lambda)\int_0^\infty e^{-(r+\lambda)t} \begin{pmatrix} \Pr(\ell_t^* \in [\ell_D; \ell^* - \alpha]) \\ * \inf_{\ell \in [\ell_D; \ell^* - \alpha]} \left\{ \widehat{D}_H(\ell) \right\} \end{pmatrix} dt \\ &\geq (r+\lambda)\int_0^\infty e^{-(r+\lambda)t} \begin{pmatrix} 1 - \varepsilon \\ * M \sup_{\ell > \ell^*} \left\{ \widehat{D}_H(\ell) \right\} \end{pmatrix} dt \\ &\geq \sup_{t \leq T(\ell')} \left\{ \widehat{D}_H(\ell_t') \right\} \end{aligned}$$

The third line uses that for $\alpha > 0$ sufficiently small, and ℓ^* sufficiently large we get $\Pr(\ell_t^* \in [\ell_D; \ell^* - \alpha]) = 1 - \varepsilon$ by claim 3 in the proof of Lemma 11, while by choosing ℓ^* large enough, we get $\inf_{\ell \in [\ell_D; \ell^* - \alpha]} \left\{ \widehat{D}_H(\ell) \right\} > M \sup_{\ell > \ell^*} \left\{ \widehat{D}_H(\ell) \right\}$, for any M. $\square$

B.4 Essential Uniqueness

We prove Theorem 2(c) in two steps. Lemma 10 shows that for low costs, intermediate reputations prefer to work in any tentative equilibrium effort profile. Lemma 11 shows that if market learning ensures hope and intermediate reputations work, so do low reputations: a firm with reputation just below a tentative shirk-work cutoff hopes to achieve an intermediate reputation in the future and thus prefers to work.

Lemma 10 *There exists a function $\beta(\ell) > 0$, such that the value of quality is bounded below $\beta(\ell) < \widehat{\Delta}(\ell)$, uniformly across costs c and equilibrium effort profiles $\widehat{\eta}$.*

Proof. If the Poisson component of market learning is non-trivial, i.e. there is $y \in Y$ with $\delta_y \neq 0$, then the reputational dividend is uniformly bounded below by Lemma 5, and so is $\widehat{\Delta}(\ell)$.

If market learning is pure Brownian and the value of quality equals $\widehat{\Delta}(\ell) = \int e^{-(r+\lambda)t}\mathbb{E}\left[\mu_B^2 \widehat{V}_\theta'(\ell_t)/2\right] dt$, this argument still holds. While Lemma 5 does not prove $\widehat{V}_\theta'(\ell_t) > 0$ for every ℓ_t, the volatility of the Brownian motion distributes ℓ_t over some interval $[\ell; \ell']$ and the expectation $\mathbb{E}\left[\frac{\mu_B^2}{2}\widehat{V}_\theta'(\ell_t)\right]$ is then bounded below by $k\left(\widehat{V}_\theta(\ell') - \widehat{V}_\theta(\ell)\right)$. □

Lemma 11 *Fix $\overline{\ell} > 0$. If market learning ensures hope and costs are sufficiently low, then a firm with reputation below $\overline{\ell}$ works in equilibrium.*

Proof. From Lemma 10 we know that the firm prefers to work for $\ell \in \left[-\overline{\ell}; \overline{\ell}\right]$. By contradiction, assume that there is equilibrium shirking in $\left[-\infty; -\overline{\ell}\right]$ and let the highest shirk-work cutoff be $\ell_* < -\overline{\ell}$. We develop $\widehat{\Delta}(\ell_* - \varepsilon)$ until the first time T when $\ell_t = -\overline{\ell}$:

$$\widehat{\Delta}(\ell_* - \varepsilon) \geq \mathbb{E}[e^{-(r+\lambda)T}]\widehat{\Delta}(\ell_T). \tag{B.3}$$

As $\widehat{\Delta}(\ell_T) = \widehat{\Delta}(-\overline{\ell})$ is bounded below by $\beta\left(-\overline{\ell}\right)$ we just need to show that $\mathbb{E}[e^{-(r+\lambda)T}] > 0$, independently of effort $\widehat{\eta}$ and costs c: By the assumption that market learning ensures hope, and by choosing $-\overline{\ell}$, and thus ℓ_*, low enough, the firm's initial reputation $\ell_* - \varepsilon$ will rise above ℓ_* with positive probability. Once $\ell_t > \ell_*$, equilibrium beliefs $\widetilde{\eta} = 1$ will push reputation $\ell_{t'}$ to $-\overline{\ell}$ in finite time with positive probability.

Thus the RHS of (B.3) is strictly positive, and by choosing c small enough we get $\lambda\widehat{\Delta}(\ell_* - \varepsilon) > c$, and the desired contradiction. □

C Quality Choice

Proof of Lemma 3. We will show that the dividend $\widehat{V}_\theta'(\ell)$ approaches 0 uniformly over all reputations and all work-shirk effort profiles. That is, $\lim_{\lambda\to\infty}\sup_{\ell^*\in[-\infty,\infty],\ell\in\mathbb{R}}\widehat{V}_{\theta,\ell^*}'(\ell)=0$, where $\ell^*=\pm\infty$ captures the full shirk (resp. work) profile.

To do so, fix ϵ and let $\ell^{**}>0$ solve $e^{\ell^{**}}/\left(1+e^{\ell^{**}}\right)^2=\epsilon$. First, consider a cutoff ℓ^* in the tail, i.e. $|\ell^*|>\ell^{**}$.

$$\begin{aligned}
\lim_{\lambda\to\infty}\sup_{|\ell^*|>\ell^{**},\ell_0\in\mathbb{R}}\widehat{V}_{\theta,\ell^*}'(\ell_0) &= \lim_{\lambda\to\infty}\sup_{|\ell^*|>\ell^{**},\ell_0}\mathbb{E}\left[\int e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial\ell_t}{\partial\ell_0}dt\right]\\
&\le \lim_{\lambda\to\infty}\sup_{|\ell^*|>\ell^{**},\ell_0}\mathbb{E}\left[\int e^{-rt}\left(\frac{\Pr(|\ell_t|<\ell^{**})}{4}+\frac{\Pr(|\ell_t|\ge\ell^{**})e^{\ell^{**}}}{(1+e^{\ell^{**}})^2}\right)dt\right]\\
&\le \varepsilon.
\end{aligned}$$

The first line applies Lemma 4 (d). The second line uses $e^\ell/(1+e^\ell)^2\le 1/4$ and Lemma 4 (a). The first term under the integral vanishes because $\lim_{\lambda\to\infty}\sup_{\ell_0}\Pr\left(|\ell_t|<\ell^{**}\right)=0$ for all $t>0$. The second term is bounded by ε.

Next, suppose that $|\ell^*|\le\ell^{**}$. Suppose the process ℓ_t hits ℓ^* at time T. Using equation (A.5) we get:

$$\begin{aligned}
\lim_{\lambda\to\infty}\sup_{|\ell^*|<\ell^{**},\ell_0\in\mathbb{R}}\widehat{V}_\theta'(\ell_0) &= \lim_{\lambda\to\infty}\sup_{|\ell^*|<\ell^{**},\ell_0}\mathbb{E}\left[\int e^{-rt}\frac{e^{\ell_t}}{(1+e^{\ell_t})^2}\frac{\partial\ell_t}{\partial\ell_0}dt\right]\\
&\le \lim_{\lambda\to\infty}\sup_{|\ell^*|<\ell^{**},\ell_0}\frac{1}{4}\mathbb{E}\left[\int_{t=0}^{T}e^{-rt}dt+\int_{t=T}^{\infty}e^{-rt}\frac{\partial\ell_t}{\partial\ell_0}dt\right]\\
&= 0.
\end{aligned}$$

The first two lines are as above. The first integral vanishes because $\sup_{\ell_0,|\ell^*|<\ell^{**}}\mathbb{E}\left[T\right]\to 0$ as $\lambda\to\infty$. The second integral vanishes because reputational increments disappear at an absorbing boundary, i.e. $E[\partial\ell_t/\partial\ell_0(\ell^*)|t\ge T]\to 0$ as $\lambda\to\infty$.

Thus the marginal value of reputation $\widehat{V}'(\ell)$ uniformly approaches 0 as $\lambda\to\infty$. The reputational dividend in the general Poisson & Brownian case is given by

$$\widehat{D}_\theta(\ell)=\sum_y(\mu_{H,y}-\mu_{L,y})\left(\widehat{V}_\theta\left(\ell+\log\frac{\mu_{H,y}}{\mu_{L,y}}\right)-\widehat{V}_\theta(\ell)\right)+\mu_B^2\widehat{V}_\theta'(\ell).$$

□

D Perfect Poisson Learning

In this appendix we solve the perfect learning specifications of Section 5 explicitly by calculating value functions in closed form. This approach highlights the analytic tractability of these learning specificications and delivers a more explicit understanding of value functions and the value of quality. Some of the derived expressions are also used in the proofs of Section 5.

We assume throughout that $\lambda \geq \mu$, so that the drift of the firm's reputation is determined by market beliefs.

D.1 Perfect Good News

Shirk-region, above the cutoff $x \geq x^*$: Suppose $x_0 = 1$ and let x_t solve the dynamics (5.1), absent a breakthrough. x_t is strictly decreasing until it stops at $x_t = x^*$. The firm weakly prefers to shirk and we assume it always does so. With a low quality product, reputation is deterministic and firm value is given by:

$$V_L(x_s) = \int_{t=0}^{\infty} e^{-rt} x_{t+s} dt$$

With a high quality product dynamics are more complicated, because the reputation jumps to 1 at a μ-shock and quality disappears at a λ-shock:

$$\begin{aligned} V_H(x_s) &= \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} [x_{t+s} + \lambda V_L(x_{t+s}) + \mu V_H(1)] dt \\ &= \int_{t=0}^{\infty} x_{t+s} e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\mu+\lambda)t} \right] dt + \frac{\mu}{r+\lambda+\mu} V_H(1), \end{aligned} \tag{D.1}$$

where we rewrote the $\lambda V_L(x_{t+s})$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda V_L(x_{t+s}) dt = \frac{\lambda}{\lambda+\mu} \int_{t=0}^{\infty} x_{t+s} e^{-rt} [1 - e^{-(\mu+\lambda)t}] dt.$$

We evaluate (D.1) at $x_s = 1$, and rearrange

$$V_H(1) = \frac{r+\lambda+\mu}{r+\lambda} \int_{t=0}^{\infty} x_t e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\mu+\lambda)t} \right] dt.$$

The value of quality is the difference of the value functions (D.1) and (**??**)[19]

$$\Delta(x_s) = \frac{\mu}{r+\lambda} \int_{t=0}^{\infty} x_t e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\mu+\lambda)t} \right] dt - \frac{\mu}{\lambda+\mu} \int_{t=0}^{\infty} x_{t+s} e^{-rt} \left[1 - e^{-(\mu+\lambda)t} \right] dt.$$

[19]Alternatively, we could compute the reputational dividend from the value functions and plug it into the dividend formula for the value of quality (5.2). Explicit calculations show that both approaches lead to the same result.

When $x_s = x^*$, we get

$$\Delta(x^*) = \frac{\mu}{r+\lambda} \int_{t=0}^{\infty} (x_t - x^*) e^{-rt} \left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\mu+\lambda)t} \right] dt.$$

Quality at x^* is valuable because of the possibility that reputation jumps from x^* to $x_0 = 1$. The terms in brackets capture the possibilities of λ and μ shocks while x_t descends from 1 to x^*.

Work-region, below the cutoff $x \le x^*$: Next, suppose $\tilde{x}_0 = 0$ and let $\tilde{x}_t$ solve the dynamics (5.1), absent a breakthrough. $\tilde{x}_t$ is strictly increasing until it stops at $\tilde{x}_t = x^*$. The firm weakly prefers to work and we assume it always does so. With a high quality product, the firm's reputation drifts up until $\tilde{x}_t = x^*$, or a μ-shock hits:

$$V_H(\tilde{x}_s) = \int_{t=0}^{\infty} e^{-(r+\mu)t} [(\tilde{x}_{t+s} - c) + \mu V_H(1)] dt.$$

With a low quality product, the firm's reputation drifts up until $\tilde{x}_t = x^*$, or a λ-shock hits:

$$\begin{aligned} V_L(\tilde{x}_s) &= \int_{t=0}^{\infty} e^{-(r+\lambda)t} [(\tilde{x}_{t+s} - c) + \lambda V_H(\tilde{x}_{t+s})] dt \\ &= \int_{t=0}^{\infty} (\tilde{x}_{t+s} - c) \left[\frac{\lambda}{\lambda-\mu} e^{-(r+\mu)t} - \frac{\mu}{\lambda-\mu} e^{-(r+\lambda)t} \right] dt + \frac{\lambda}{r+\lambda} \frac{\mu}{r+\mu} V_H(1), \end{aligned} \tag{D.2}$$

where we rewrote the $\lambda V_H(\tilde{x}_{t+s})$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda)t} \lambda V_H(\tilde{x}_{t+s}) dt = \frac{\lambda}{\lambda-\mu} \int_{t=0}^{\infty} (\tilde{x}_{t+s} - c) e^{-rt} (e^{-\mu t} - e^{-\lambda t}) dt + \frac{\lambda}{r+\lambda} \frac{\mu}{r+\mu} V_H(1).$$

The value of quality is the difference of the value functions (**??**) and (D.2):

$$\Delta(\tilde{x}_s) = \frac{r}{r+\lambda} \frac{\mu}{r+\mu} V_H(1) - \frac{\mu}{\lambda-\mu} \int_{t=0}^{\infty} (\tilde{x}_{t+s} - c) e^{-rt} (e^{-\mu t} - e^{-\lambda t}) dt$$

The first term captures the value of the high quality firm's breakthroughs while the second term captures the opportunity cost.

D.2 Perfect Bad News

Work-region, above the cutoff $x \ge x^*$: First assume that $x^* > 0$, so that $V_L(0) = 0$. Consider starting at x_0, just above x^*, and let x_t solve the dynamics (5.3), absent a breakdown. x_t is strictly increasing and converges to 1. With a high quality product, reputation is deterministic and firm value equals:

$$V_H(x_s) = \int_{t=0}^{\infty} e^{-rt} (x_{t+s} - c)] \, dt.$$

With a low quality product dynamics are more complicated, because the reputation jumps to 0 at a μ-shock and quality improves at a λ-shock:

$$V_L(x_s) = \int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t}[(x_{t+s} - c) + \lambda V_H(x_{t+s}) + \mu \cdot V_L(0)]\, dt. \tag{D.3}$$

$$= \int_{t=0}^{\infty} e^{-rt}(x_{t+s} - c)\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}\right] dt, \tag{D.4}$$

where we rewrote the $\lambda V_H(x_{t+s})$-term by changing the order of integration:

$$\int_{t=0}^{\infty} e^{-(r+\lambda+\mu)t} \lambda V_H(x_{t+s})\, dt = \frac{\lambda}{\mu+\lambda} \int_{t=0}^{\infty} e^{-rt}(x_{t+s} - c)[1 - e^{-(\lambda+\mu)t}]\, dt.$$

The value of quality is the difference of the value functions (**??**) and (D.4),

$$\Delta(x_s) = \frac{\mu}{\lambda+\mu} \int_{t=0}^{\infty} e^{-rt}(x_{t+s} - c)(1 - e^{-(\lambda+\mu)t})\, dt.$$

The cost of low quality is the loss of reputation when the μ-shock hits before the λ-shock.

Shirk-region, below the cutoff $x \geq x^*$: Next, consider starting at $\tilde{x}_0$ just below x^*, and and let $\tilde{x}_t$ solve the dynamics (5.3), absent a breakdown. $\tilde{x}_t$ is strictly decreasing and converges to 0. With a low quality product, reputation is deterministic and firm value equals:

$$V_L(\tilde{x}_s) = \int_{t=0}^{\infty} e^{-(r+\mu)t} \tilde{x}_{t+s}\, dt.$$

With a high quality product, quality disappears at a λ-shock and the firm's value function is

$$V_H(\tilde{x}_s) = \int_{t=0}^{\infty} e^{-(r+\lambda)t}[\tilde{x}_{t+s} + \lambda V_L(\tilde{x}_{t+s})]\, dt.$$

$$= \int_{t=0}^{\infty} \tilde{x}_{t+s}\left[\frac{\lambda}{\lambda-\mu} e^{-(r+\mu)t} - \frac{\mu}{\lambda-\mu} e^{-(r+\lambda)t}\right] dt, \tag{D.5}$$

where we rewrote the $\lambda V_L(\tilde{x}_{t+s})$-term by changing the order of integration:

$$\int_{t=0}^{\infty} \lambda V_L(\tilde{x}_{t+s}) = \frac{\lambda}{\lambda-\mu} \int_{t=0}^{\infty} e^{-rt} \tilde{x}_{t+s}(e^{-\mu t} - e^{-\lambda t})dt.$$

The value of quality is the difference of the value functions (D.5) and (**??**):

$$\Delta(x_s) = \frac{\mu}{\lambda-\mu} \int_{t=0}^{\infty} e^{-rt} x_{t+s}(e^{-\mu t} - e^{-\lambda t})\, ds.$$

Again, the cost of low quality is the loss of reputation when the μ-shock hits before the λ-shock.

Full work, i.e. $\eta(x) = 1$ for all x : Finally, suppose $x^* = 0$ so the firm always works. Consider $x_0 = 0$ and let x_t denote the dynamics (5.3), absent a breakdown. The value function for the high quality firm is given by (**??**). The value function of the low quality firm (D.4) becomes

$$V_L(x_s) = \int_{t=0}^{\infty} e^{-rt}(x_{t+s} - c)\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}\right] dt + \frac{\mu}{r+\lambda+\mu}V_L(0).$$

Setting $s = 0$, we obtain

$$V_L(0) = \frac{r+\lambda+\mu}{r+\lambda}\int_{t=0}^{\infty} e^{-rt}(x_{t+s} - c)\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\lambda+\mu)t}\right] dt.$$

The value of quality is the difference of (**??**) and (**??**):

$$\Delta(x_s) = \frac{\mu}{\lambda+\mu}\int_{t=0}^{\infty} x_{t+s}e^{-rt}\left[1 - e^{-(\lambda+\mu)t}\right] dt - \frac{\mu}{r+\lambda}\int_{t=0}^{\infty} x_t e^{-rt}\left[\frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu}e^{-(\mu+\lambda)t}\right] dt.$$

Cost terms cancel since both high- and low quality firms always work. This equation parallels equation (**??**) in the good-news case. When $s = 0$, this becomes

$$\Delta(0) = \frac{\mu}{\lambda+\mu}\int_{t=0}^{\infty} e^{-rt}x_t\left[\frac{r}{r+\lambda} - \frac{r+\mu+\lambda}{r+\lambda}e^{-(\lambda+\mu)t}\right] dt.$$

The value of quality realizes when a μ-shock hits before the first λ-shock.

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