Liquidity and Information in Order Driven Markets

Ioanid Roşu*

October 31, 2009

Abstract

This paper proposes a dynamic model of an order driven market with asymmetric information and stochastic fundamental value. In equilibrium, informed traders submit market orders only when they see a fundamental value far from the public price; otherwise, they submit limit orders. Under fairly general assumptions, the price impact of a market order is about four times larger than the price impact of a limit order; this ratio is independent of the parameters of the model. The price impact of a market order does not depend on the fraction of informed traders. Surprisingly, a higher fraction of informed traders generates smaller bid-ask spreads. The ratio of intra-day price volatility to the average spread can be used to estimate the probability of informed trading. (JEL C7, D4, G1)

Keywords: Bid-ask spread, price impact, information acquisition, volatility, trading volume, limit order book, waiting costs.

*University of Chicago, Booth School of Business, `irosu@chicagobooth.edu`. The author thanks Peter DeMarzo, Doug Diamond, Peter Kondor, Juhani Linnainmaa, Christine Parlour, Uday Rajan, Pietro Veronesi for helpful comments and suggestions. He is also grateful to participants at the NBER meeting, October 2008; and to seminar audiences at Chicago, Stanford, Berkeley, Urbana-Champaign, Toronto Economics, Bank of Canada, and the Hong Kong Institute for Monetary Research.

1 Introduction

This article studies the role of information in order-driven markets, where trading is done via limit orders and market orders in a limit order book.[1] Given the importance of order-driven markets,[2] there have been relatively few models to describe price formation in these markets. This is due to the difficulty of the problem. Because in order-driven markets there is no centralized decision maker, prices arise from the interaction of a large number of traders, each of which can be fully strategic. The presence of traders with superior information about the asset value further complicates the problem.

This paper proposes a tractable dynamic model of an order-driven market with asymmetric information. It incorporates four key features. First, in contrast with most models with asymmetric information in which the true asset value is assumed constant, we allow the fundamental value to move over time according to a diffusion process.[3] This captures well the fundamental uncertainty about the value of the asset. Also, it underlines the important role of informed traders in price discovery under changing market conditions.

Second, the model has no order cancellation costs or monitoring costs. Traders arrive randomly to the market, but once they arrive they are fully strategic and can change their order at any time if they wish to do so. By contrast, most existing models assume that these costs are so large that traders never cancel or revise their orders.[4]

Third, both the informed and uninformed traders in this model rationally revise their orders based on all available information. In the case of the uninformed, this is the information contained in the past limit and market orders, and in the case of the informed traders, it is the public information together with their privately observed fundamental value. Given that

[1] A limit order is a price-contingent order to buy (sell) if the price falls below (rises above) a prespecified price. A sell limit order is also called an *offer*, while a buy limit order is also called a *bid*. The limit order book (or simply the book) is the collection of all outstanding limit orders. The lowest offer in the book is called the *ask price*, or simply *ask*, and the highest bid is called the *bid price*, or simply *bid*.

[2] Today more than half of the world's stock exchanges are order-driven, with no designated market makers (e.g., Euronext, Helsinki, Hong Kong, Tokyo, Toronto), while in many hybrid markets designated market makers have to compete with a limit order book (NYSE, Nasdaq, London).

[3] Another paper which allows the fundamental value to be stochastic is Goettler, Parlour, and Rajan (2009). They numerically solve for the equilibrium in a model where the fundamental value changes according to a Poisson process.

[4] This feature is also present in Roşu (2009), but in his symmetric information model there is no fundamental value and prices lie in an exogenously given fundamental band.

agents update rationally, they must have other reasons to trade besides profit; otherwise, according to the "No-Trade" Theorem of Milgrom and Stokey (1982) there would be no trading. Therefore, we assume that at the moment of entry the uninformed traders compare their expected profit with a private cost ranging over some interval containing zero. This implies that part of the trading population derives a private benefit from trading.[5]

Fourth, as in Foucault, Kadan, and Kandel (2005), traders are risk-neutral and have waiting costs: they lose utility proportionally to the expected waiting time. Based on that, traders are patient if they have small waiting costs, or impatient if they have high waiting costs. As explained in Roşu (2009), this assumption generates a limit order book with orders on different levels even in the absence of asymmetric information.[6] In order to focus on the effects of asymmetric information, we assume that the waiting costs of the patient traders are at least an order of magnitude smaller than the costs resulting from asymmetric information. In other words, we assume that those traders are very patient.

Under this system of assumptions, we derive the equilibrium shape of the limit order book and its evolution in time in relation to the fundamental value. The first set of results concerns the nature of the equilibrium and the strategy of the informed traders. We note that, alongside the fundamental value process there is also the public price, or the *efficient price*, which is the uninformed traders' expectation of the fundamental value conditional on all public information (the order flow).

The limit order book always moves up and down along with the efficient price: after each order, all traders modify their limit orders up or down to take into account the update of the efficient price due to the information contained in that order. If we remove the dependence on the efficient price, the equilibrium is stationary Markov and depends only on the number of sellers and the number of buyers in the book. The value functions (expected utilities) of the limit order sellers and buyers in the book satisfy a system of difference equations. The system can only be solved numerically, and we derive several properties of the equilibrium that way. Fortunately, the average properties of the limit order book, and the behavior of

[5]This benefit can be thought as coming from hedging needs or liquidity shocks.

[6]For example, a trader who places a sell limit order close to the ask waits less on average for execution, but gets a lower price. A sell limit trader far from the ask gets a better expected price, but has to wait more. Since agents can fully undercut each other, in equilibrium all these traders have the same expected utility, and this makes the model easier to solve than models that do not allow for order cancellation.

the book around the average values can be expressed in closed form. Because the limit order traders are very patient, this average behavior happens with probability almost equal to one.

The strategy of a patient informed trader depends on how far the fundamental value, v, is from the efficient price, v^e. The optimal order of the informed trader is: a buy market order (BMO) if v is above v^e plus a cutoff value; a buy limit order (BLO) if v is between v^e and v^e plus the cutoff; a sell limit order (BLO) if v is between v^e minus the cutoff and v^e; and a sell market order (SMO) if v is below v^e minus the cutoff. The cutoff value is proportional to the *efficient volatility,* which is the conditional volatility of the fundamental value given all public information.

Put differently, an informed trader who observes an extreme fundamental value submits a market order, while one who observes a moderate fundamental value submits a limit order.[7] To understand why, we note that in the absence of private information a patient trader would use a limit order in order to take advantage of the bid-ask spread. But an informed trader who observes, e.g., a fundamental value well above the efficient price realizes that the order book will drift upwards due to the action of future informed traders. This reduces the expected profit from a buy limit order, thus making the buy market order more attractive. If the fundamental value is above a cutoff, cashing in on the informational advantage with a market order is better than waiting to be compensated for the limit order.

Once the informed trader makes the initial order choice, the behavior mimics that of an uninformed trader. In other words, the equilibrium is pooling: if the informed trader observes an extreme fundamental value, then he submits a market order and exits. But if submitting a limit order is optimal (because the fundamental value is not far from the efficient price), the informed trader does not want to lose the temporary informational advantage by deviating from the behavior of the uninformed; nor does he want to cancel the limit order and submit a market order instead.

The next set of results regards the price impact of a trade and the efficient price process. Since both limit orders and market orders carry information about the fundamental value, the efficient price adjusts after each type of order. A key result is that all types of orders (BMO, BLO, SLO, SMO) are equally likely on average. First, if market orders were more

[7] This makes rigorous an intuition present, e.g., in Harris (1998), Bloomfield, O'Hara, and Saar (2005), Hollifield et al. (2006).

likely than limit orders, the bid-ask spread would increase as limit orders were consumed by market orders. As the bid-ask spread increased, limit orders would become more likely, to the point where market orders and limit orders are equally likely. Second, if buy orders were more likely than sell orders, price impact would increase on average the efficient price to the point at which buy and sell orders are equally likely. Therefore, all order types are equally likely. This argument also shows that the midpoint between the bid and the ask is close to the efficient price, so it provides a good empirical proxy for it.

The average price impact of each type of order has a particularly simple form: Δ for BMO; $u\Delta$ for BLO; $-u\Delta$ for SLO; and $-\Delta$ for SMO; where $u \approx 0.2554$ is a constant, and the price impact parameter Δ is proportional to the fundamental volatility and inversely proportional to the square root of total trading activity (the sum of arrival rates of informed and uninformed traders). This implies that the price impact of a limit order is on average about four times smaller than the price impact of a market order, and that this ratio ($\frac{1}{u} \approx 3.912$) is independent of all the variables in the model. This result can be tested empirically.

The situation is different in states of the limit order book far away from the average, in which the bid-ask spread is large and the midpoint is relatively far from the efficient price. In that case, limit orders have the opposite price impact, e.g., a buy limit order has a negative price impact. This situation only occurs with a very small probability.

A surprising result is that the price impact parameter Δ is independent of the ratio of the arrival rates of informed and uninformed traders (the *information ratio*). First, a lower information ratio decreases the probability of trading with an informed agent, and therefore price impact should be smaller. But a smaller information ratio also makes efficient prices less precise, i.e., the efficient volatility is larger. For this reason, price impact should be larger. On balance, these two effects cancel each other.

The result above compares the average price impact in limit order books corresponding to different values of the information ratio. If one fixes the information ratio and studies the dependence of the price impact on the bid-ask spread, we show numerically that the price impact of a market order is decreasing in the size of the spread. This is because when spreads are larger, informed traders are less likely to submit market orders. This result can be tested empirically.

The presence of informed traders ensures that the efficient price approximates the fundamental value. The speed of convergence is increasing in the information ratio: more informed traders make the efficient price converge more quickly to the fundamental value. This suggests that the unobservable information ratio could be estimated by measuring the speed at which prices come back after a large order that moves the efficient price away from the fundamental value.[8] Moreover, in a stationary equilibrium, the volatility of the efficient price is approximately equal to the fundamental volatility. This result shows that the volatility of the bid-ask spread midpoint is a proxy for the fundamental volatility.

Another consequence of asymmetric information is order flow autocorrelation. Biais, Hillion and Spatt (1995) find that in the Paris Bourse the submission of each type of order makes the subsequent submission of an order of the same type more likely; they call this the "diagonal effect." In this paper, because of the presence of uninformed traders, price impact does not fully correct the pricing error (i.e., the difference between the efficient price and the fundamental value). Thus, the pricing error is autocorrelated, and therefore the order flow is also autocorrelated. Moreover, order flow autocorrelation has an inverted U-shape: it first increases in the information ratio as long as the information ratio stays below one, and it decreases above one.

The third set of results concerns the shape of the limit order book, as measured by the bid-ask spread or price impact, in relation to trading activity, volatility, and information asymmetry. Consistent with previous literature, we show that lower trading activity and higher fundamental volatility generate larger spreads.[9] Smaller trading activity implies that the volatility between two consecutive orders is higher, hence the uncertainty about the fundamental value is higher, which translates into wider bid-ask spreads. A similar argument shows that a higher fundamental volatility causes a larger spread and price impact. These predictions are tested by Linnainmaa and Roşu (2009), who use instrumental variables to generate exogenous variation in trading activity and fundamental volatility.

A surprising result is that the average bid-ask spread is decreasing in the information ratio.

[8]The work of Pastor and Stambaugh (2003) shows that an aggregate measure of liquidity using such price reversals can be used as a fourth factor, besides the three Fama–French factors, in a model of expected stock returns.

[9]See Foucault, Kadan and Kandel (2005), Foucault (1999), and Roşu (2009).

In a static model such as Glosten (1994), in which the fundamental value is constant, a larger percentage of informed traders increases adverse selection and makes spreads larger. But with a moving fundamental value, a larger percentage of informed traders makes efficient prices converge more quickly to the fundamental value. As the bid-ask spread measures the public uncertainty in the fundamental value, more precise prices translate into smaller spreads.

This suggests another way of estimating the information ratio. The ratio of the intra-day volatility of the spread midpoint to the average bid-ask spread depends only on the information ratio, and not on the fundamental volatility or trading activity. This ratio therefore can be used as a measure of the probability of informed trading, in the spirit of Easley, Hvidkjaer, and O'Hara (2002).

Since both the informed and uninformed traders maximize expected utility, we analyze welfare and information acquisition in this model. The expected utility of both the patient uninformed traders and the informed traders (who are assumed patient) is decreasing in the information ratio. This is true for the patient uninformed traders because their compensation comes essentially from the bid-ask spread, which is decreasing in the information ratio. The informed traders are worse off when the information ratio is higher because of the increase in competition among them. By contrast, the impatient uninformed traders submit market orders in equilibrium, and therefore benefit from the smaller bid-ask spread resulting from a higher information ratio.

If informed traders can pay a fixed cost to acquire the right to observe the fundamental value upon their arrival to the market, they choose to become informed only until to the point where the benefit from doing so exactly equals the information cost. This means that the information ratio is one-to-one with the cost of acquiring information.

The paper is organized as follows. Section 2 describes the model. Section 3 solves for the equilibrium limit order book and describes the efficient price process. Section 4 discusses welfare and information acquisition. Section 5 analyzes a numerical solution of the general case. Section 6 discusses empirical implications of the model. Section 7 concludes.

1.1 Background Literature

The choice between market orders and limit orders has been analyzed in various contexts, see, e.g., Chakravarty and Holden (1995), Cohen, Maier, Schwartz, and Whitcomb (1981), Handa and Schwartz (1996), Kumar and Seppi (1993). Dynamic models of order-driven markets include Foucault (1999), Foucault, Kadan, and Kandel (2003), Parlour (1998), Goettler, Parlour, and Rajan (2005, 2009), Back and Baruch (2007), and Roşu (2009). The price behavior in limit order books has been analyzed theoretically by Biais, Martimort, and Rochet (2000), Glosten (1994), OHara and Oldfield (1986), Rock (1990), and Seppi (1997). Models that analyze liquidity traders, the dynamics of prices and trades and the convergence of prices to the fundamental value include Glosten and Milgrom (1985), Kyle (1985), Admati and Pfleiderer (1988), Easley and O'Hara (1987).

While in our model informed traders use market orders and limit orders equally likely, much of the literature on order driven markets assumes that informed traders only use market orders. See, for example, Glosten (1994) and Rock (1996). The argument is usually that, in order to take advantage of their temporary informational advantage, informed traders prefer to submit market orders and get immediate execution. Sandås (2001) rejects the static conditions implied by the Glosten (1994) model, and also finds that liquidity providers earn superior returns. Harris and Hasbrouck (1996) obtain a similar result for the NYSE SuperDOT system.

Harris (1998) considers the dynamic order submission decisions of some stylized traders, and shows that informed traders with a tight deadline submit market orders, while those with a more distant deadline submit limit orders. Similarly, Kaniel and Liu (2006) propose a model where informed traders do submit limit orders when information is long-lived.

Because our model has a moving fundamental value and competing informed traders, the information is naturally short lived. However, we show that in equilibrium informed traders may still prefer to submit limit orders, if the privately observed fundamental value is not too far from the public (efficient) price. This is because there are benefits from supplying liquidity: limit order traders essentially gain from the existence of the bid-ask spread. If informed traders are risk-neutral and the fundamental value is close to the efficient price, there is no reason why they should not want to get the expected profit from supplying liquidity.

Using experimental asset markets, Bloomfield, O'Hara and Saar (2005) show that informed traders tend to use more limit orders than liquidity traders do. Moreover, informed liquidity supply increases over time. The behavior they observe in these markets is similar to the theoretical one predicted by our model: informed traders take liquidity with market orders when the value of their information is high, and the provide liquidity otherwise. Our model provides an equilibrium justification for the extent to which informed traders provide or demand liquidity: ex ante the likelihood of all types of orders, buy or sell, limit or market, must be the same.

Goettler, Parlour, and Rajan (2009) numerically solve a dynamic model of the limit order book with asymmetric information. Informed traders, who decide whether to acquire information at the beginning, observe the fundamental value of the asset, while the uninformed traders observe it with a lag. Investors also have a private valuation on top of their estimate of the fundamental value. They find that on average limit orders are submitted by informed traders.

Foucault (1998) proposes a model with a moving fundamental value. Orders have a one-period life and they are also subject to picking-off risks. These risks are also present in Handa and Schwartz (1996). In our model there are no picking-off risks, as we assume that there are no monitoring costs, so traders can immediately and fully adjust their limit orders to incorporate the information contained in upcoming orders.

Kumar and Seppi (1993) propose a one-period model which assumes an exact factor structure for the uninformed order submission strategies. This factor structure therefore also determines the structure of the informed traders' strategies. Similar to Foucault (1999) and the present paper, they find that limit order books are thinner in more volatile markets.

The empirical evidence on the information content of limit orders has been mixed. Biais, Hillion, and Spatt (1995) bring evidence in favor of an information story, showing that, e.g., a buy limit order tends to increase the quotes. In contrast, Griffiths, Smith, Turnbull, and White (2000) find that non-marketable limit orders have a significant price impact in the opposite direction. Our paper allows for this possibility, namely in off-center limit order books with sufficiently large spreads.

Hollifield, Miller, and Sandås (2004) test empirically whether optimal order submission is a monotone function of of the trader's private value for the asset. Traders' private values are assumed not get updated while the orders are outstanding. Hollifield, Miller, Sandås, and Slive (2006) use data from the Vancouver exchange to estimate the gains from trade. They find that traders with more extreme private values usually submit orders with low execution risk and low picking-off risk, while traders with moderate private values submit limit orders with higher execution risk and higher picking-off risk.

2 The Model

Consider a market for an asset which pays no dividends. The time horizon is infinite, and trading in the asset takes place in continuous time.

2.1 Trading

The only types of trades allowed are market orders and limit orders, which are executed with no delays. There is no cost of cancellation for limit orders.[10] However, for technical reasons we assume that, compared with order modifications, order cancellations can be done only after an infinitesimal delay.[11] Since new orders arrive at discrete, albeit random times, we call the interval between two orders a *period*.

Trading is based on a publicly observable limit order book, which is the collection of all the limit orders that have not yet been executed. The limit orders are subject to the usual price priority rule; and, when prices are equal, the time priority rule is applied. If several market orders are submitted at the same time, only one of them is executed, at random, while the other orders are canceled.

2.2 Prices

The buy and sell prices for the asset are determined as the bid and ask prices resulting from trading based on the rules given above. Prices can take any value, i.e., the tick size is zero.

[10] In most financial markets cancellation of a limit order is free, although one may argue that there are still *monitoring* costs. The present model ignores such costs, but one can take the opposite view that there are infinite cancellation / monitoring costs. See, e.g., Foucault, Kadan and Kandel (2005).

[11] This is in order to prevent market manipulation of the type described in the discussion after Proposition 8.

The fundamental value of the asset, or *full-information price*, v_t, moves according to a diffusion process with no drift and constant volatility: $\mathrm{d}v_t = \sigma_v\,\mathrm{d}W_t$, where W_t is a standard Brownian motion. We denote by

$$\begin{aligned} v_t^e &= \quad \mathsf{E}\{v_t \mid \mathcal{J}_t\} &&= \textit{efficient price}, \\ \sigma_t^e &= \left(\mathsf{Var}\{v_t \mid \mathcal{J}_t\}\right)^{1/2} &&= \textit{efficient volatility}, \end{aligned}$$

where $\mathcal{J}_t$ is the public information at t, as defined below. Assume that the fundamental value is normally distributed $v_t \sim N\big(v_t^e, (\sigma_t^e)^2\big)$, and that (v_t^e, σ_t^e) is a Markov process given the order flow, i.e., the changes in v^e and σ^e due to the future order flow depend only on their current values and on the current limit order book.[12]

2.3 Agents

The market is composed of two types of agents: informed traders and uninformed traders. Both types of traders can be patient and impatient, in a sense to be precisely described below. They trade at most one unit, after which they exit the model forever. The traders' types are fixed from the beginning and cannot change.

At any time t all agents know the complete history of market and limit orders, and they know the current and past values of the efficient price v^e and the efficient volatility σ^e. These form the public information, $\mathcal{J}_t$.

All agents in this model are risk-neutral, so their instantaneous utility function is linear in price. Traders discount the future in a way proportional to the expected waiting time. If τ is the random execution time and P_τ is the price obtained at τ, the expected utility of a seller is $f(t) = \mathsf{E}_t\{P_\tau - v_\tau - r(\tau - t)\}$, where v_τ is the fundamental value at τ. (The expectation operator takes as given the strategies of all the players.) Similarly, the expected utility of a buyer is $g(t) = \mathsf{E}_t\{v_\tau - P_\tau - r(\tau - t)\}$. The expected utility is also called the *value function*.

The discount coefficient r is constant.[13] It can take only two values: if it is low, the corresponding traders are called *patient*, otherwise they are *impatient*. As in Roşu (2009), we

[12] With a normal prior $v_t \sim N\big(v_t^e, (\sigma_t^e)^2\big)$, Proposition 12 shows that the posterior distribution of v when a new order arrives is not exactly normal, but is well approximated by the normal distribution. A similar result holds for the assumption that the process (v^e, σ^e) is Markov.

[13] The nature of waiting costs is intentionally vague in this paper. One can interpret it as an opportunity cost of trading. Another interpretation is that waiting costs reflect traders' uncertainty aversion: if uncertainty increases with the time horizon, an uncertainty averse trader loses utility by waiting.

assume that the discount coefficient of the impatient trader is high enough, which implies the impatient agents always submit market orders. Then by r we denote only the time discount coefficient of the patient agents.

All traders have a small one-time private benefit from trading, realized at the time of order execution. This assumption is made in order to avoid no-trading regions for the informed traders. The assumption is not strong: Proposition 15 shows that it is sufficient to consider a private benefit as small as $\Delta/2$, where Δ is the average price impact of a market order.

Informed Traders

Besides the public information, $\mathcal{J}_t$, defined above, an informed trader who arrives at time t observes the fundamental value, v_t. Based on this, the informed trader decides whether to enter the market, and if so, whether to use a buy or sell, market or limit order. Informed traders only observe the fundamental value upon their arrival. For simplicity, we assume that there are only *patient* informed traders. The informed traders arrive to the market according to independent Poisson processes with the arrival intensity rate $2\lambda^{\mathcal{I}}$.

Uninformed Traders

Uninformed traders are either patient or impatient, and are either buyers or sellers. Therefore, there are four types of uninformed traders: patient buyers, patient sellers, impatient buyers, and impatient sellers. All four types are assumed to arrive to the market according to independent Poisson processes with the same arrival intensity rate, $\lambda^{\mathcal{U}}$.[14]

The decision to enter the market is based on a private one-time cost of trading, c, uniformly distributed on the interval $[-C, C]$. This means that each trader who arrives at the market makes the entry decision based on their private cost, c. For example, a seller who expects a utility $f - v^e$ from trading would only enter the market if $f - v^e \geq c$.[15] A negative cost c means that the agent has a private benefit of trading. This is done to avoid the "No-Trade"

[14] By definition, a Poisson arrival with intensity λ implies that the number of arrivals in any interval of length T has a Poisson distribution with parameter λT. The inter-arrival times of a Poisson process are exponentially distributed with the same parameter, λ. The average time until the next arrival is then $1/\lambda$.

[15] Alternatively, one can think of $v^e + c$ as the seller's total valuation for the asset, public and private. The seller trades only if the expected payoff, f, is above the total valuation, $v^e + c$.

Theorem of Milgrom and Stokey (1982), which says that there is no trading if agents trade only for profit reasons.

This assumption also generates a downward-sloping demand: the probability that an uninformed impatient buyer enters the market decreases with the offer price. For this reason, the distribution of the cost c also determines the extreme values of the limit order book: Table 1 shows that, e.g., the sell limit order furthest away from the ask is placed at a distance usually equal to about $2C/3$ from the ask.

2.4 Strategies

Since this is a model of continuous trading, it is desirable to set the game in continuous time. There are also technical reasons why that is useful: in continuous time, with Poisson arrivals the probability that two agents arrive at the same time is zero, and this simplifies the analysis of the game. Setting the game in continuous time requires extra care; see Roşu (2009) for details. The concept of equilibrium is Markov perfect equilibrium (see Fudenberg and Tirole (1991)).

3 The Equilibrium Limit Order Book

3.1 Intuition

Suppose the limit order book is empty and a patient seller labelled "1" decides to enter the market and place a sell limit order at a price a_1 which is determined as a monopoly price.[16] When a second patient seller labelled "2" arrives, the two sellers must now compete. If seller 2 undercut seller 1 at a price a_2 immediately below a_1, then seller 1 would respond by undercutting further. So in equilibrium a_2 should be significantly lower than a_1, to the point that seller 1 has no incentive to undercut. This implies that the two sellers have the same expected utility: seller 1 gets a higher price (a_1) but waits longer, while seller 2 gets a lower price (a_2) but her order gets executed first. The limit order submission strategy is always

[16] Seller 1 relies on the probability that some impatient buyer will place a buy market order and clear the limit order at a_1. If the price a_1 is high, the limit order takes on average a longer time to execute: as mentioned in the previous section, there is a downward-sloping demand from impatient buyers. So the price a_1 depends on the trade-off between getting a higher execution price and waiting longer.

spread-improving: seller 2 submits the limit order at a lower level than seller 1, seller 3 at a lower level than seller 2, and so on. Moreover, no seller has an incentive to deviate and change the order in the queue.

So far we have assumed that neither seller has superior information. But even in the presence of asymmetric information it can be shown that the equilibrium is pooling, i.e., that the informed traders do not reveal their type. Indeed, suppose that seller 1 has more information: at the time of entry he saw a fundamental value v significantly lower than the efficient price v^e. Because he knows that future informed traders are likely to also observe low values of v_t, seller 1 expects the limit order book to drift down on average. So he would prefer to undercut seller 2 now and get a higher expected utility. In the absence of asymmetric information, seller 2 would respond to this out-of-equilibrium behavior by moving her order to a_1: she knows that by placing a limit order below a_2 seller 1 would get a lower expected utility than her, so she has no incentive to further undercut. But in the presence of asymmetric information, seller 2 realizes that seller 1 must be informed, so the best response is to undercut seller 1 by an infinitesimal amount. This way, whatever information seller 1 has, seller 2 now gets essentially the same expected utility by undercutting. So seller 1 has no incentive to reveal his type.

Asymmetric information does make a difference in the decision of seller 1: since he does not reveal his type but still expects the limit order book to drift down, he has an extra incentive to submit a sell market order rather than waiting with a sell limit order. This leads to the following optimal strategy of the informed trader 1: if the observed v is very low, submit a sell market order; if the observed v is moderately low, submit a sell limit order. (The strategy is symmetric on the buy side.) This implies that both market orders and limit orders carry information: e.g., a sell market order potentially comes from an informed trader who observed a very low v. This means that a sell market order decreases the efficient price, v^e. Similarly, a sell limit order decreases the efficient price, although usually by a smaller amount.

Given that the equilibrium is pooling, up to changes in the efficient price due to order flow, the only state variables that matter are m, the number of sell limit orders; and n, the number of buy limit orders in the book. The limit sellers in the book have the same expected utility (value function) $v^e + f_{m,n}$, and the limit buyers have the same value function $v^e + g_{m,n}$.

Because the arrival of each type of order changes the states (m, n), the value functions f and g satisfy some recursive difference equations. The search for the equilibrium then reduces to finding the solution for these difference equations and understanding its properties.

3.2 Equilibrium

We start with the arrival rate of uninformed sellers, $\lambda^{\mathcal{U}}$; the arrival rate of (patient) informed traders, $\lambda^{\mathcal{I}}$; the time discount coefficient of patient traders, r; and the fundamental volatility, σ_v, i.e., the volatility of the fundamental value, v_t. We define a set of numbers and parameters that are used extensively throughout the text.

Definition 1. *We denote by $\Phi(\cdot)$ the cumulative density for the standard normal distribution $N(0,1)$. We define the following numbers:*

$$\begin{aligned}
\alpha &= -\Phi^{-1}(1/4) &&\approx 0.6745,\\
\beta &= \frac{\sqrt{2\pi}\, \mathrm{e}^{\alpha^2/2}}{4} &&\approx 0.7867,\\
u &= \mathrm{e}^{\alpha^2/2} - 1 &&\approx 0.2554.
\end{aligned}$$

We define also the following parameters:[17]

$$\begin{aligned}
\lambda &= 2\lambda^{\mathcal{U}} + 2\lambda^{\mathcal{I}} &&= \textit{trading activity},\\
i &= \lambda^{\mathcal{I}}/\lambda^{\mathcal{U}} &&= \textit{information ratio},\\
\varepsilon &= r/\lambda &&= \textit{granularity},\\
\sigma &= \sigma_v/\sqrt{\lambda} &&= \textit{inter-trade volatility},\\
\Delta &= \sqrt{\tfrac{2}{1+u^2}}\,\sigma \approx 1.3702\,\sigma &&= \textit{price impact parameter},\\
\sigma_e &= \beta\,\tfrac{i+1}{i}\,\Delta &&= \textit{efficient volatility parameter}\\
k &= \tfrac{a_k i}{1+a_k i}, \quad a_k \approx 1.2729 &&= \textit{information decay parameter}.
\end{aligned}$$

We assume that the trading activity λ, the information ratio i, the patience coefficient r, and the fundamental volatility σ_v are exogenous, and we study how the equilibrium shape of the limit order book and the behavior of the efficient price v^e depend on them.

[17] We could define total trading activity as the sum of the arrival rates of all types of agents, $\lambda = 4\lambda^{\mathcal{U}} + 2\lambda^{\mathcal{I}}$. But because on average only half of the uninformed traders who arrive to the market decide to place an order, it is more appropriate to define total trading activity as the actual arrival rate of orders: $\lambda = 2\lambda^{\mathcal{U}} + 2\lambda^{\mathcal{I}}$.

The next theorem describes the equilibrium limit order book. For simplicity of exposition, the proof of the theorem is broken into Propositions 1–9. The results are interdependent, in that most of them depend on the other results being true. Propositions 1, 2, 3, and 5 refer to the equilibrium in general. The other results focus on a set of states which occurs with probability almost equal to one.[18] We also require that the efficient volatility σ^e be equal to its steady-state value.[19] We call this the *average limit order book*.

The results in this section assume, first, that the asset is relatively liquid: trading activity λ is high compared to fundamental volatility σ_v, or equivalently $\Delta = \frac{\sigma_v}{\sqrt{\lambda}}$ is small; and second, that the patient traders are very patient: $\varepsilon = \frac{r}{\lambda}$ is much smaller than Δ (see Proposition 4 for more details); this can be done by choosing a small patience coefficient r. The results of Theorem 1 are approximate statements: the statement about the probability of various order types holds with an error of the order of Δ; the other statements hold with an error smaller than the order of Δ.

Theorem 1. *Under the assumptions above, there exists a pooling stationary Markov perfect equilibrium for which, in the average limit order book:*

- *Upon observing a fundamental value v when the efficient price is v^e, the informed trader submits: a buy market order (BMO) if $v - v^e > \alpha\sigma_e$; a buy limit order (BLO) if $v - v^e \in (0, \alpha\sigma_e)$; a sell limit order (SLO) if $v - v^e \in (-\alpha\sigma_e, 0)$; and a sell market order (SMO) if $v - v^e < -\alpha\sigma_e$. Here $\sigma_e = \beta(\frac{1}{i} + 1)\Delta$ is the efficient volatility parameter.*
- *All order types (BMO, BLO, SLO, SMO) are equally likely ex ante, with probability $\frac{1}{4}$.*
- *After each order the whole limit order book moves, along with the efficient price v^e, by: $\Delta_{BMO} = \Delta$, $\Delta_{BLO} = u\Delta$, $\Delta_{SLO} = -u\Delta$, $\Delta_{SMO} = -\Delta$. In particular, $\frac{\Delta_{BMO}}{\Delta_{BLO}} = \frac{1}{u} \approx 3.9152$.*
- *The average bid-ask spread is $\bar{s} = k\alpha\sigma_e + \Delta(1 + u(1-k))$, where $k = \frac{a_k i}{1+a_k i}$ is the information decay parameter. $\bar{s}$ is increasing in fundamental volatility σ_v, and is decreasing in trading activity λ and information ratio i. The average ask price is $v^e + \bar{a} = v^e + \frac{\bar{s}}{2}$, and the average bid price is $v^e + \bar{b} = v^e - \frac{\bar{s}}{2}$.*

[18]This set of states is defined by Case 4 of Proposition 3, with the additional requirement that the bid-ask spread be less than the order of Δ. For more details, see Proposition 4 and the discussion that precedes it.

[19]This means that σ^e does not change when traders arrive at time intervals of equal length $\frac{1}{\lambda}$. By focusing on steady-state results, we also ignore the role of time between transactions; this issue is addressed in Section 3.3.

- *The speed of convergence of v^e to v (the inverse autocorrelation of the pricing error $v^e - v$) equals* $\frac{1}{1-\frac{4}{\pi}\frac{1+u^2}{(1+u)^2}\left(\frac{i}{i+1}\right)^2} \approx \frac{1}{1-0.8606\left(\frac{i}{i+1}\right)^2}$*, and is increasing in i.*

Proof. See Propositions 1–9. □

We now analyze the equilibrium in more detail. The next result shows that the value functions of the uninformed traders satisfy a system of difference equations, and that the bid and ask prices depend on the value functions in a simple way. This allows one to find where orders are placed in the limit order book, i.e., to determine the shape of the book. A consequence of the next proposition is that the equilibrium "decouples," i.e., the value functions of the traders in the book do not depend on the efficient price v^e or on the efficient volatility σ^e.

Proposition 1. *The limit order book evolves according to a Markov perfect equilibrium in which the state variables are (m, n, v^e, σ^e), with m the number of sell limit orders n the number of buy limit orders, v^e the efficient price, and σ^e the efficient volatility. The arrival of the next order moves the order book to another state depending on the order type: BMO to $(m-1, n, v^e + \Delta_{BMO}, \sigma^e_{BMO})$, BLO to $(m, n+1, v^e + \Delta_{BLO}, \sigma^e_{BLO})$, SLO to $(m+1, n, v^e + \Delta_{SLO}, \sigma^e_{SLO})$, SMO to $(m, n-1, v^e + \Delta_{SMO}, \sigma^e_{SMO})$, where Δ_j, σ^e_j are described in Proposition 5. The expected utility of the uninformed limit sellers equals $v^e + f_{m,n}$, and the expected utility of the uninformed limit buyers equals $v^e + g_{m,n}$, where $f_{m,n}$ and $g_{m,n}$ do not depend on v^e or σ^e, and satisfy a system of difference equations (3) in the Appendix. The ask and bid prices in state (m, n, v^e, σ^e) are given, respectively, by*

$$\begin{aligned} v^e + a_{m,n} &= v^e + f_{m-1,n} + \Delta_{BMO}, \\ v^e + b_{m,n} &= v^e - g_{m,n-1} - \Delta_{SMO}. \end{aligned}$$

Proof. See the Appendix. □

The difference equations that f and g satisfy are too complicated to be solved directly, so in Section 5 we explain how to find an approximate numerical solution for the value functions. For now, the more important quantity is the *difference* between the value functions from the perspective of the informed and uninformed traders.

Because traders are risk-neutral, in the rest of the section we often omit the efficient volatility component from the state variables (m, n, v^e, σ^e). We denote by $Q^{\mathcal{I}} = Q^{\mathcal{I}}(v, m, n, v^e)$

the expected selling price minus the waiting costs from the point of view of an informed trader, after the submission of a sell limit order, assuming that the informed trader observes a fundamental value v and that the sell limit order has moved the book to the state (m, n, v^e). Similarly, we denote by $Q^{\mathcal{U}} = Q^{\mathcal{U}}(m, n, v^e) = v^e + f_{m,n}$ the expected selling price minus the waiting costs from the point of view of an uninformed trader.[20] Then the next proposition shows that $Q^{\mathcal{I}} \approx Q^{\mathcal{U}} + k(v - v^e)$ for some number $k \in (0, 1)$ which is increasing in the information ratio i. The intuition is: if an informed trader sees, e.g., a low fundamental value, v, then he knows that the future informed traders are also likely to see low fundamental values, and so their orders are likely to push the efficient price—and hence all other prices—down. This knowledge makes the expected selling price for the informed trader $Q^{\mathcal{I}}$ lower than the corresponding value $Q^{\mathcal{U}}$ for the uninformed trader. The parameter k is a measure of information decay: if the information ratio i is high, future informed traders arrive more quickly and diminish the informational advantage of the current informed trader. Moreover, they push the efficient price more quickly towards the fundamental value.

In the course of the proof of the next result, we see that the formula $Q^{\mathcal{I}} \approx Q^{\mathcal{U}} + k(v - v^e)$ in expected execution prices (minus waiting costs) translates to expected utilities into the following formula: $u^{\mathcal{I}}_{SLO} \approx a_{m,n} - \Delta_{BMO} - (v - v^e - \Delta_{SLO})(1 - k)$, where $u^{\mathcal{I}}_{SLO} = u^{\mathcal{I}}_{SLO}(v, m, n, v^e)$ is the expected utility from the point of view of an informed trader *before* submitting a sell limit order in state (m, n, v^e).[21] We now compute the expected utility of the informed traders and the patient uninformed traders.

Proposition 2. *Consider a limit order book with m sellers and n buyers, with ask price $a = a_{m,n}$ and bid price $b = b_{m,n}$. If $i = \lambda^{\mathcal{I}}/\lambda^{\mathcal{U}}$ is the information ratio, we define the information decay parameter $k = \frac{a_k i}{1 + a_k i}$. Suppose an informed trader arrives when the book is in state (m, n, v^e), and observes a fundamental value v. Then, depending on the order submitted, the expected utility of the informed trader is*

[20] Here we assume that the informed and uninformed traders behave the same way until the order executes, i.e., we assume that the equilibrium is pooling, which according to Proposition 8 is true.

[21] Due to price impact there is a difference between the utility before and after submitting an order. In the previous proposition, e.g., $g_{m,n}$ represents the expected utility from waiting in the book *after* the *BLO* is submitted. If instead we want the expected utility of an uninformed trader *before* he submits a *BLO*, this equals $u^{\mathcal{U}}_{BLO} = g_{m+1,n} - \Delta_{BLO}$, which takes into account the adverse impact of the *BLO* on the efficient price.

$$
\begin{aligned}
u^{\mathcal{I}}_{BMO} &= v - (v^e + a), \\
u^{\mathcal{I}}_{BLO} &\approx (v - v^e - \Delta_{BLO})(1-k) - (b - \Delta_{SMO}), \\
u^{\mathcal{I}}_{SLO} &\approx -(v - v^e - \Delta_{SLO})(1-k) + (a - \Delta_{BMO}), \\
u^{\mathcal{I}}_{SMO} &= (v^e + b) - v.
\end{aligned}
$$

The expected utility of a patient uninformed trader satisfies similar formulas, with $v = v^e$ and $k = 0$: $u^{\mathcal{U}}_{BMO} = -a$; $u^{\mathcal{U}}_{BLO} = -\Delta_{BLO} - (b - \Delta_{SMO})$; $u^{\mathcal{U}}_{SLO} = \Delta_{SLO} + (a - \Delta_{BMO})$; $u^{\mathcal{U}}_{SMO} = b$.

Proof. See the Appendix. □

Next we discuss the strategy of an informed trader. Consider a limit order book in state (m, n, v^e) with ask price $v^e + a$ and bid price $v^e + b$. The bid-ask spread and the spread midpoint are, respectively, s and $v^e + p$, where

$$
\begin{aligned}
s &= a - b, \\
p &= \tfrac{a+b}{2}.
\end{aligned}
$$

We denote by Δ_{BMO}, Δ_{BLO}, Δ_{SLO}, Δ_{SMO} the corresponding change in the efficient price v^e after each type of order. Let also $\Delta'_1 = -\Delta_{SMO} + (1-k)\Delta_{BLO}$, $\Delta'_2 = \Delta_{BMO} - (1-k)\Delta_{SLO}$, $\Delta' = (\Delta'_1 + \Delta'_2)/2$, and $\Delta'' = (\Delta'_1 - \Delta'_2)/2$. The equilibrium behavior of the informed traders in all possible states is given by the following proposition.

Proposition 3. *In the setting of the previous theorem, consider the decision of an informed trader who has just arrived at time t in state (m, n, v^e), when the efficient price is $v^e = v^e_t$. Suppose the informed trader sees a fundamental value $v = v_t$. Then in equilibrium the informed trader submits the following type of order:*

> **Case 1:** $-|kp + \Delta''| > s - \Delta'$. *Then the order is: SMO if $v - v^e < p$; or BMO if $v - v^e > p$.*
>
> **Case 2:** $\frac{1}{1-k}|kp + \Delta''| > s - \Delta' > -|kp + \Delta''|$ *and* $kp + \Delta'' > 0$. *Then the order is: SMO if $v - v^e < \frac{-s+\Delta'_2}{k}$; SLO if $v - v^e \in \left(\frac{-s+\Delta'_2}{k}, \frac{a-\Delta'_2/2}{1-k/2}\right)$; or BMO if $v - v^e > \frac{a-\Delta'_2/2}{1-k/2}$.*
>
> **Case 3:** $\frac{1}{1-k}|kp + \Delta''| > s - \Delta' > -|kp + \Delta''|$ *and* $kp + \Delta'' < 0$. *Then the order is: SMO if $v - v^e < \frac{b+\Delta'_1/2}{1-k/2}$; BLO if $v - v^e \in \left(\frac{b+\Delta'_1/2}{1-k/2}, \frac{s-\Delta'_1}{k}\right)$; or BMO if $v - v^e > \frac{s-\Delta'_1}{k}$.*

Case 4: $s - \Delta' > \frac{1}{1-k}|kp + \Delta''|$. *Then the order is: SMO if* $v - v^e < \frac{-s+\Delta'_2}{k}$, *SLO if* $v - v^e \in \left(\frac{-s+\Delta'_2}{k}, \frac{p+\Delta''}{1-k}\right)$, *BLO if* $v - v^e \in \left(\frac{p+\Delta''}{1-k}, \frac{s-\Delta'_1}{k}\right)$ *and BMO if* $v - v^e > \frac{s-\Delta'_1}{k}$.

In all cases BMO has a positive price impact, and SMO has a negative price impact. Assume s, p *sufficiently large compared with* Δ''. *Then, in Case 4, BLO has a positive price impact, while SLO has a negative price impact. In Case 2, SLO has a positive price impact, and in Case 3, BLO has a negative price impact.*

The uninformed patient buyer submits BLO if $s > -\Delta_{SMO} + \Delta_{BLO}$, *and BMO if* $s < -\Delta_{SMO} + \Delta_{BLO}$. *The uninformed patient seller submits SLO if* $s > \Delta_{BMO} - \Delta_{SLO}$, *and SMO if* $s < \Delta_{BMO} - \Delta_{SLO}$.

Proof. See the Appendix. □

The intuition for this result is as follows. We note that in all cases the informed trader uses market orders if the fundamental value is far enough from the efficient price, v^e. Case 1 represents a situation in which the spread, s, is small and the spread midpoint, $v^e + p$, is relatively centered (close to v^e). Then the informed trader has no incentive to submit a limit order to take advantage of the spread, and is therefore better off placing market orders. Case 2 represents a situation where the spread midpoint is relatively off-center and positive, and the spread is sufficiently large. Then the informed trader never submits *BLO*: there is a lot of competition on the buy side (since the bid is closer to v^e than the ask), while there is not as much competition on the bid side. In that case, the spread is wide enough to support *SLO*, but not wide enough to support *BLO*. Case 3 is symmetric to Case 2. Finally, Case 4 represents a situation with relatively centered prices and larger spreads. In Case 4 all four types of orders are possible.

Proposition 3 shows that in Case 4, *BLO* typically has a positive price impact, and *SLO* has a negative price impact. This is because the spread midpoint is relatively centered and the spread is wide. This means that, e.g., *SLO* indicates a fundamental value in the negative range. By contrast, in Case 2, *SLO* typically has a *positive* price impact. Case 2 can be imagined as an off-center situation where the bid price is close to the efficient price, but the ask price is well above the efficient price. This leads to a wide spread, and therefore the informed has a good incentive to submit *SLO*, while *BLO* is never optimal as there is too much competition on the bid side. But this also implies that *SLO* is usually optimal when the fundamental value is positive: the cutoff for *BMO* is large and positive, while the cutoff for *SMO* is close to the efficient price. This shows that the price impact of an *SLO* in this case is positive. This is surprising: a sell limit order is usually a negative signal.

From the previous result one can also see how the general equilibrium operates. Small spreads (as in Case 1) lead to more market orders, hence to larger spreads. Large spreads (as in Case 4) make market orders very unlikely (as the cutoffs $\frac{-s+\Delta_2'}{k}$ and $\frac{s-\Delta_1'}{k}$ are large in absolute value), which leads to more limit orders, hence to smaller spreads. In steady state this tends to make market orders and limit orders equally likely in equilibrium. It also tends to bring the order book to Case 4, since this is the only case in which the informed trader submits all four types of orders.

Proposition 4. *Assume that $\varepsilon = \Delta^{2+\delta}$, with $\delta > 0$. Then Case 4 of Proposition 3 occurs with probability almost equal to one. The same is true if we further restrict Case 4 to include only states in which the bid-ask spread is smaller than the order of Δ, the price impact parameter. Under these conditions, the bid-ask midpoint is approximately equal to the efficient price. All four types of orders (BMO, BLO, SLO, SMO) have approximately the same ex ante probability, $\frac{1}{4}$.*

Proof. See the Appendix. □

If an order book satisfies Case 4 of Proposition 3 with a bid-ask spread of the order of Δ, and if the efficient volatility σ^e is equal to its steady-state value, we call it the *average limit order book.* Starting with Proposition 6, we consider only the average limit order book. Before that, we prove one more general result, about the changes in efficient price and efficient volatility due to the four types of orders.

Proposition 5. *Consider a limit order book in a state where the fundamental value cutoffs for the optimal strategy of the informed trader are w_1, w_2, w_3, as described in Proposition 3, i.e., we assume that BMO is optimal for $v - v^e \in (w_1, \infty)$, BLO for $v - v^e \in (w_2, w_1)$, SLO for $v - v^e \in (w_3, w_2)$, and SMO for $v - v^e \in (-\infty, w_3)$. Then the efficient price v^e changes after each type of order by the following amounts:*

$$\Delta_{BMO} = \sigma^e \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w_1}{\sigma^e}\right)^2}}{\frac{1}{4i}+1-\Phi\left(\frac{w_1}{\sigma^e}\right)} \qquad \Delta_{BLO} = \sigma^e \frac{\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{w_2}{\sigma^e}\right)^2} - e^{-\frac{1}{2}\left(\frac{w_1}{\sigma^e}\right)^2}\right)}{\frac{1}{4i}+\Phi\left(\frac{w_1}{\sigma^e}\right)-\Phi\left(\frac{w_2}{\sigma^e}\right)}$$

$$\Delta_{SLO} = \sigma^e \frac{\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{1}{2}\left(\frac{w_3}{\sigma^e}\right)^2} - e^{-\frac{1}{2}\left(\frac{w_2}{\sigma^e}\right)^2}\right)}{\frac{1}{4i}+\Phi\left(\frac{w_2}{\sigma^e}\right)-\Phi\left(\frac{w_3}{\sigma^e}\right)} \qquad \Delta_{SMO} = -\sigma^e \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{w_3}{\sigma^e}\right)^2}}{\frac{1}{4i}+\Phi\left(\frac{w_3}{\sigma^e}\right)}$$

where σ^e is the efficient volatility, and $\Phi(\cdot)$ is the standard normal cumulative density. After an order of type $j \in \{BMO, BLO, SLO, SMO\}$ the efficient volatility changes from σ^e to σ^e_j given by Equation (5) in the Appendix.

Proof. See the Appendix. □

This implies that the efficient price changes, Δ_{BMO}, Δ_{BLO}, Δ_{SLO}, Δ_{SMO}, depend on the cutoffs for the fundamental value w_1, w_2, w_3. Proposition 3 shows that, in turn, the cutoffs depend on the efficient price changes. The endogenous nature of these variables is the reason why we have to look for numerical solutions to the general problem. (This is done in Section 5.) Fortunately, in the case of the average limit order book one gets closed form solutions.

First, we note that Proposition 4 implies that in the average limit order book the cutoffs w_1, w_2, w_3 are so that all four types of orders are equally likely. Because the fundamental value v is perceived as normal with mean v^e and standard deviation σ^e, the ex ante probabilities that an informed trader submits a certain type of order are: $\mathsf{P}_{BMO} = 1 - \Phi(\frac{w_1}{\sigma^e})$, $\mathsf{P}_{BLO} = \Phi(\frac{w_1}{\sigma^e}) - \Phi(\frac{w_2}{\sigma^e})$, $\mathsf{P}_{SLO} = \Phi(\frac{w_2}{\sigma^e}) - \Phi(\frac{w_3}{\sigma^e})$, $\mathsf{P}_{SMO} = \Phi(\frac{w_3}{\sigma^e})$. Since all orders are equally likely, we can determine the cutoffs: $w_1 = v^e + \alpha\sigma^e, w_2 = v^e, w_3 = v^e - \alpha\sigma^e$, where α is given, as in Definition 1, by $\Phi(-\alpha) = \frac{1}{4}$. Substituting these cutoffs in Proposition 5, we get $\Delta_{BLO} = u\Delta_{BMO}$, $\Delta_{SLO} = -u\Delta_{BMO}$, $\Delta_{SMO} = -\Delta_{BMO}$, where $u = e^{\alpha^2/2} - 1$. This implies that all efficient price changes are constant multiples of Δ_{BMO}.

The next result gives necessary and sufficient conditions for stationarity of the equilibrium in the average limit order book, and determines the value of Δ_{BMO} as the price impact parameter $\Delta = \sqrt{\frac{2}{1+u^2}}\, \sigma$ from Definition 1.

Proposition 6. *The equilibrium is stationary in the average limit order book if and only if $\Delta_{BMO} = \Delta$, or equivalently if the inter-trade volatility $\sigma = \sqrt{\frac{1+u^2}{2}}\, \Delta_{BMO}$, or equivalently if the efficient volatility $\sigma^e = \sigma_e$.*

Proof. See the Appendix. □

There are two ways of looking at this stationarity result. One, given the assumption of normality of v made by the uninformed ($v \sim N(v^e, \sigma^e)$), is to require that the variance of the prior does not change, i.e., $\mathsf{Var}(v_{t+1} - v^e_{t+1}) = \mathsf{Var}(v_t - v^e_t)$. Now, each order contains information, so it reduces the variance of the prior. But the passage of time increases the prior variance because the noise of the fundamental value v from the perspective of the uninformed increases with time. Then stationarity means that these two effects cancel each other out, and the efficient volatility, σ^e, stays constant.

Another way is to observe that the volatility of the efficient price is a good proxy for the fundamental volatility. (This is proved in Proposition 9.) Because of that, one should expect the inter-trade volatility σ to be equal to the volatility of the efficient price change. This change equals $\pm\Delta_{BMO}$ or $\pm u\Delta_{BMO}$, with all four possible changes equally likely (as proved in Proposition 4). But the variance of this change is $\frac{(1+u^2)}{2}\,\Delta^2_{BMO}$, so it should be equal to the inter-trade variance, σ^2.

Corollary 1. *If the equilibrium is stationary, then in the average limit order book the optimal strategy of an informed trader is to submit: BMO if $v - v^e \in (\alpha\sigma_e, \infty)$; BLO if $v - v^e \in (0, \alpha\sigma_e)$; SLO if $v - v^e \in (-\alpha\sigma_e, 0)$; SMO if $v - v^e \in (-\infty, -\alpha\sigma_e)$. The efficient price changes corresponding to all types of orders are:*

$$\Delta_{BMO} = \Delta, \quad \Delta_{BLO} = u\Delta, \quad \Delta_{SLO} = -u\Delta, \quad \Delta_{SMO} = -\Delta, \tag{1}$$

where Δ is the price impact parameter from Definition 1.

Proof. From the discussion following Proposition 5, the optimal cutoffs are: $w_1 = v^e + \alpha\sigma^e$, $w_2 = v^e$, $w_3 = v^e - \alpha\sigma^e$, and the efficient price changes satisfy $\Delta_{BLO} = u\Delta_{BMO}$, $\Delta_{SLO} = -u\Delta_{BMO}$, $\Delta_{SMO} = -\Delta_{BMO}$. Proposition 6 implies that $\Delta_{BMO} = \Delta$ and $\sigma^e = \sigma_e$. □

Corollary 2. *The price impact of any type of order in the average limit order book does not depend on the information ratio, i.*

Proof. From Proposition 6, $\Delta = \sqrt{\frac{2}{1+u^2}}\,\sigma = \sqrt{\frac{2}{1+u^2}}\,\frac{\sigma_v}{\sqrt{\lambda}}$. Since σ_v and λ are exogenous, the price impact parameter Δ does not depend on the information ratio i. But according to Proposition 5 the price impact of any type of order depends only on Δ. □

This result is surprising. One might expect that a higher information ratio increases the probability of trading with an informed agent, and therefore the price impact, Δ_{BMO}, of a buy market order should be larger. But one also needs to take into account that a higher information ratio also makes prices more precise, i.e., the efficient volatility, σ_e, is smaller. For this reason, the price impact should also be smaller. On balance, these two effects cancel and the efficient price adjustment, Δ_{BMO}, does not depend on i.

In the case of the average limit order book, the next proposition computes the average bid-ask spread, and the average bid and ask prices.

Proposition 7. *In the average limit order book*

$$\begin{aligned}
\bar{s} &= k\alpha\sigma_e + \Delta(1 + u(1-k)) &&= \textit{the bid-ask spread,} \\
v^e + \bar{a} &= v^e + \tfrac{\bar{s}}{2} &&= \textit{the ask price,} \\
v^e + \bar{b} &= v^e - \tfrac{\bar{s}}{2} &&= \textit{the bid price,}
\end{aligned}$$

where v^e *is the efficient price, and* $\alpha, \beta, u, \Delta, \sigma_e, i, k$ *are as in Definition 1. The average bid-ask spread can also be written:*

$$\bar{s} = \left(\frac{a_k + a_k i}{1 + a_k i} \alpha\beta + \frac{1 + u + a_k i}{1 + a_k i} \right) \sqrt{\frac{2}{1+u^2}} \frac{\sigma_v}{\sqrt{\lambda}}. \tag{2}$$

It increases in the fundamental volatility, σ_v*, and decreases in the total trading activity,* λ*, and information ratio,* i*.*

Proof. See the Appendix. □

The next result shows that the equilibrium is pooling. This means that an informed trader who submits a limit order does not deviate from the strategy of an uninformed trader. The informed trader can deviate either by undercutting after the initial order submission, or by cancelling the limit order and submitting a market order instead.[22]

Proposition 8. *The equilibrium is pooling.*

Proof. See the Appendix. □

[22] An uninformed trader does not undercut, because that does not change his expected utility. Also, an uninformed trader does not modify the limit order to a market order: once his initial value function is larger than the private one-time cost, c, the subsequent value functions only change by the waiting costs, which are assumed very small. So with probability almost one, his value function stays above c.

The intuition for this result is given in Section 3.1. We note that the pooling result in limit order books is very general. For example, it works also in an informed-only model, as long as the quality of information is different across agents, e.g., if informed traders who arrive later have more precise information that informed traders who arrived earlier (which is the case in our model).

One consequence of the proposition is that informed traders do not switch sides, e.g., from buy limit orders to sell limit orders. This is because doing so reveals that they are informed traders, so the uninformed traders (or less informed traders) would then be better off undercutting the informed trader.

There is one more type of market manipulation that needs to be ruled out. Suppose a newly arrived uninformed impatient seller arrives to the market and finds a bid price, b. Consider the strategy to submit a buy limit order instead. Then, as buy limit orders are usually interpreted as positive signals, the efficient price should increase right away by Δ_{BLO}, and along with it all other limit orders, including the one previously at the bid. Now, the new trader may cancel the buy limit order and submit a sell market order at $b + \Delta_{BLO}$, thus improving the execution price. This type of behavior is ruled out by the assumption made in Section 2: compared with order modifications, order cancellations can be done only after an infinitesimal delay. This means that after the new trader cancels the buy limit order, the traders in the book realize it was just market manipulation, and they have enough time to move their limit orders back to their previous level.

3.3 The Price Process and the Order Flow

In this section we take a closer look at the efficient price process and at the filtration problem that generates it. Recall that in the "average limit order book" the efficient volatility σ^e is equal to its steady-state value, i.e., σ^e remains constant if traders arrive at time intervals of equal length, $\frac{1}{\lambda}$. But in fact orders arrive randomly, according to a Poisson distribution with intensity λ, so the time intervals are also random (exponentially distributed with parameter λ). Later in this section, we explain the role of time in the price process, and investigate how price impact depends on the time elapsed since last transaction.

First, we analyze in more detail the price process for the average limit order book, in which the time between transactions is constant, $\tau = \frac{1}{\lambda}$. We have a fundamental value, v, which moves according to a diffusion process, and an efficient price, v^e, which is brought closer to v by the actions of informed traders. Corollary 1 shows that the efficient price moves only after an order is submitted, by $\pm\Delta$ or $\pm u\Delta$.

In order to describe the joint dynamics of v and v^e, consider the arrival of a trader at time t. This trader is either informed, with probability $\frac{i}{i+1}$, or uninformed, with probability $\frac{1}{i+1}$. If the trader is informed, the order submission depends on how the pricing error $v_t - v_t^e$ compares to three cutoffs $-\alpha\sigma_e$, 0, $\alpha\sigma_e$ (see Corollary 1). If the trader is uninformed, all four order types are equally likely. We denote by b_i the Bernoulli binomial random variable with success probability $\frac{i}{i+1}$, where i is the information ratio; by ψ the 1×4 vector $[1, u, -u, -1]$; by a_4 the discrete random variable which can take the values $1, 2, 3, 4$ with equal probability $\frac{1}{4}$; and by $w_t = v_t - v_t^e$. Then, if $v_{t_+}^e$ is the efficient price immediately after the order is submitted, $v_{t_+}^e = v_t^e + b_i\big[I_{w_t \geq \alpha\sigma_e}\Delta + I_{w_t \in [0,\alpha\sigma_e)} u\Delta + I_{w_t \in [-\alpha\sigma_e, 0)}(-u\Delta) + I_{w_t < -\alpha\sigma_e}(-\Delta)\big] + (1 - b_i)\psi(a_4)\,\Delta$, where I_x the indicator function: $I_x = 1$ if x is true, and $I_x = 0$ if x is false. Using this formula, we derive several properties of the price process.

Proposition 9. *We denote by $\tau = \frac{1}{\lambda}$ the average time interval between orders. Then in the average limit order book,*

1. *The pricing error autocorrelation is $\mathsf{Corr}(v_{t+\tau} - v_{t+\tau}^e, v_t - v_t^e) = 1 - \frac{4}{\pi}\frac{1+u^2}{(1+u)^2}\left(\frac{i}{i+1}\right)^2 \approx 1 - 0.8606\left(\frac{i}{i+1}\right)^2$. The speed of convergence of the efficient price to the fundamental value, $\frac{1}{\mathsf{Corr}(v_{t+\tau} - v_{t+\tau}^e, v_t - v_t^e)}$, is increasing in i.*

2. *Efficient price changes, $v_{t+\tau}^e - v_t^e$, are uncorrelated with each other.*

3. *The volatility of change in the efficient price equals the volatility of change in the fundamental value: $\mathsf{Var}(v_{t+\tau}^e - v_t^e) = \mathsf{Var}(v_{t+\tau} - v_t)$.*[23]

Proof. See the Appendix. □

The pricing error is autocorrelated because of asymmetric information: in the presence of uninformed traders, price impact does not fully correct the pricing error. A small percentage of informed traders (or a small information ratio i) makes prices converge slowly, so the

[23] According to Proposition 6, this equality is equivalent to the equilibrium being stationary.

pricing error is persistent, i.e., its autocorrelation is high. Therefore, the inverse correlation coefficient is a measure of the speed of convergence of the efficient price to the fundamental value.

Despite the fact that the pricing error is autocorrelated, the efficient price changes are uncorrelated. If they were not, uninformed traders would be able to predict the direction of the price change and would further adjust the efficient price. The fact that the volatility of the change in efficient price is the same as the volatility of the change in fundamental value comes from the stationarity of the equilibrium.

We now describe the properties of the price process by taking into account that the time between transactions is random. Recall that the fundamental value follows a diffusion process $\mathrm{d}v_t = \sigma_v \, \mathrm{d}W_t$. We adapt Proposition 5 to deal with different time intervals. Recall that the fundamental value cutoffs for the informed trader are: $w_1 = \alpha\sigma_e$, $w_2 = 0$, $w_3 = -\alpha\sigma_e$, where σ_e is the efficient volatility parameter (a constant depending on the information ratio i). Then at time t the price impact of buy market orders and buy limit orders is given by: $\Delta_{BMO} = \sigma_t^e \, \frac{\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\left(\frac{w_1}{\sigma_t^e}\right)^2}}{\frac{1}{4i}+1-\Phi\left(\frac{w_1}{\sigma_t^e}\right)}$, and $\Delta_{BLO} = \sigma_t^e \, \frac{\frac{1}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{1}{2}\left(\frac{w_2}{\sigma_t^e}\right)^2} - \mathrm{e}^{-\frac{1}{2}\left(\frac{w_1}{\sigma_t^e}\right)^2}\right)}{\frac{1}{4i}+\Phi\left(\frac{w_1}{\sigma_t^e}\right)-\Phi\left(\frac{w_2}{\sigma_t^e}\right)}$.

Proposition 10. *As a function of time since last transaction:*

- *The efficient volatility, σ^e, is increasing;*
- *The price impact of a buy market order, Δ_{BMO}, is increasing;*
- *The price impact of a buy limit order, Δ_{BLO}, is decreasing.*

To get intuition for these results, we note that the efficient volatility σ_t^e is a measure of the public uncertainty about the fundamental value. If the time since last transaction is longer, there is no order flow to reduce the uncertainty, and so the efficient volatility gets larger. This also means that an informed trader arriving at t is more likely to observe an extreme fundamental value. Thus, market orders from the informed traders more likely, and therefore the price impact of a market order is stronger: a market order is more likely to come from an informed trader. Conversely, the effect of a limit order is weaker: a limit order is less likely to come from an informed trader.[24]

[24] In this paper we assume that the average time between transactions, $\frac{1}{\lambda}$, does not change. Suppose instead

We now analyze the order flow and show that in the presence of informed traders the order flow is autocorrelated. This is because informed traders choose their orders based on the privately observed fundamental value v. Since v is autocorrelated (it is a random walk) and since the uninformed traders prevent the efficient price from converging instantly to the fundamental value, the order flow is autocorrelated.

Moreover, we show that the autocorrelation of the order flow has an inverted U-shape with respect to the information ratio i. Proposition 9 shows that the autocorrelation of the pricing error is decreasing in i, but this autocorrelation only affects the orders of the informed traders. When i is small, despite the pricing error autocorrelation being large, the number of informed traders is small. As the latter effect dominates, the order flow autocorrelation is actually increasing in i when i is small. When i is large, the former effect dominates, and the order flow autocorrelation is decreasing in i. This intuition is made rigorous by Proposition 11.

We define order flow autocorrelation as the percentage difference between the conditional probability of a market order, $\mathsf{P}(BMO_{t+\tau} \mid BMO_t)$, and its unconditional probability, which according to Proposition 4 is equal to 1/4. We denote by I_x the indicator function, ϕ the standard normal density, and Φ the standard normal cumulative density.

Proposition 11. *The order flow autocorrelation is*

$$\frac{\mathsf{P}(BMO_{t+\tau} \mid BMO_t) - 1/4}{1/4} = \int \frac{1 + 4i\, I_{x \geq \alpha}}{1+i} \, \frac{1 + 4i\Phi\left(\frac{\sigma_e x - \Delta - \alpha\sigma_e}{\sigma}\right)}{1+i} \, \phi(x)\, \mathrm{d}x \; - \; 1.$$

It is increasing in i for $i < 1$ and decreasing in i for $i > 1$.

Proof. See the Appendix. □

The next table displays the order flow autocorrelation for selected values of i.

Information ratio	0.01	0.1	0.5	1	5	10	100
Order flow autocorrelation	0.02%	0.85%	5.17%	6.15%	3.86%	3.03%	2.14%

trading activity λ varies over time and traders infer λ from the order flow. Then a larger time since last transaction suggests reduced trading activity, which makes the price impact parameter Δ larger (Δ varies inversely with $\sqrt{\lambda}$), thus the price impact is larger for both market orders and limit orders. This strengthens the results in Proposition 10 for market orders, but weakens it for limit orders.

We now discuss the assumption that the fundamental value is normally distributed and that the price process is Markov. In the average limit order book, assume that the prior distribution of v given v^e is normal: $v_t \sim N(v^e_t, \sigma_e^2)$. Then we compute the posterior distribution given the order flow, and compare it to the normal distribution. Consider an order of type $j \in \{BMO, BLO, SLO, SMO\}$ submitted at time t. We denote by w^u and w^d the upper and lower cutoffs for order j, respectively, as in Corollary 1: if $j = BMO$, $w^d = \alpha\sigma_e, w^u = \infty$; if $j = BLO$, $w^d = 0, w^u = \alpha\sigma_e$; if $j = SLO$, $w^d = -\alpha\sigma_e, w^u = 0$; and if $j = SMO$, $w^d = -\infty, w^u = -\alpha\sigma_e$.

Proposition 12. *If $\tau = \frac{1}{\lambda}$, the density of $v_{t+\tau}$ conditional on v^e_t and order j is:*

$$f(x) \;=\; \frac{\mathrm{e}^{-\frac{1}{2}\frac{(x-v^e_t)^2}{\sigma^2+\sigma_e^2}}}{\sqrt{2\pi(\sigma^2+\sigma_e^2)}} \; \frac{1+4i\left[\Phi\left(\frac{\sigma_e(x-v^e_t)}{\sigma\sqrt{\sigma^2+\sigma_e^2}} - \frac{w^d\sqrt{\sigma^2+\sigma_e^2}}{\sigma\sigma_e}\right) - \Phi\left(\frac{\sigma_e(x-v^e_t)}{\sigma\sqrt{\sigma^2+\sigma_e^2}} - \frac{w^u\sqrt{\sigma^2+\sigma_e^2}}{\sigma\sigma_e}\right)\right]}{1+i},$$

where the cutoffs w^u and w^d for order j are defined as above.

Proof. The computation is similar to the proof of Proposition 11. □

If $F(x)$ and $G(x)$ are two cumulative density functions (CDF), we define the CDF error as $\max_x |F(x) - G(x)|$. The next table displays the CDF error when $F(x)$ is the posterior distribution corresponding to order $j = BMO, BLO$ as in Proposition 12, and $G(x)$ is the normal distribution $N(v^e_t, \sigma_e^2)$.

Information ratio	0.01	0.1	0.5	1	5	10	100
CDF error for *BMO*	0.48%	2.89%	6.56%	6.64%	2.87%	1.80%	1.46%
CDF error for *BLO*	0.50%	3.12%	7.31%	7.45%	4.03%	2.90%	1.60%.

The maximum CDF error for *BMO* equals 6.817% and is attained at $i = 0.895$; the maximum CDF error for *BLO* equals 7.617% and is attained at $i = 0.832$.

The same type of analysis shows that the Markov approximation for the efficient price process v^e_t is good. We perform the analysis only for two successive *BMO*. If $\tau = \frac{1}{\lambda}$, we define the Markov error $\frac{\mathsf{E}(v_{t+\tau} \mid v^e_t, BMO_t, BMO_{t+\tau}) - 2\Delta}{2\Delta}$. The maximum Markov error is 3.990%, attained at $i = 0.206$.

4 Information Acquisition

In this section we assume that, in exchange for a fixed cost γ, patient traders can observe the fundamental value v_t upon entering the market. The decision to become informed is made before observing the actual state of the limit order book, based only on its average properties.[25] We first study how the expected utility of the informed an uninformed traders depends on the information ratio. Recall that, according to Proposition 7, the average bid-ask spread is $\bar{s} = \left(\frac{a_k + a_k i}{1 + a_k i}\alpha\beta + \frac{1+u+a_k i}{1+a_k i}\right)\Delta$, where $\Delta = \sqrt{\frac{2}{1+u^2}}\frac{\sigma_v}{\sqrt{\lambda}}$ does not depend on i.

Proposition 13. *We denote by $U^{\mathcal{I}}$ and $U^{\mathcal{U}}$ the unconditional expected utility of the informed traders and the uninformed (patient) traders, respectively. Then*

$$U^{\mathcal{I}} = \frac{1 + u(1-k)}{2i}, \quad U^{\mathcal{U}} = \frac{\bar{s}}{2} - \Delta(1+u).$$

Both $U^{\mathcal{I}}$ and $U^{\mathcal{U}}$ are decreasing in the information ratio i. The informational advantage, $U^{\mathcal{I}} - U^{\mathcal{U}}$, is also decreasing in i.

Proof. See the Appendix. □

The results are intuitive. An informed trader gets a lower utility from a higher information ratio, as this implies more competition and a quicker loss of informational advantage. But the uninformed traders also have a lower utility, since their compensation is determined by the bid-ask spread, which is decreasing in i. The only traders who benefit from more competition among the informed are the impatient uninformed traders, who have an expected utility of $-\frac{\bar{s}}{2}$, which is increasing in i.

Now, consider the case with few informed traders, i.e., i is low. Their expected utility $U^{\mathcal{I}} = \frac{1+u(1-k)}{2i}$ is very large compared to $U^{\mathcal{U}}$, therefore incoming traders have a strong incentive to acquire information.[26] They do so to the point when there is no more informational advantage from acquiring information.

[25] This assumption is not too restrictive since the average properties of the book occur with probability almost equal to one (see Proposition 4).

[26] When i converges to zero, $\bar{s}$ converges to $(a_k\alpha\beta + 1 + u)\Delta$, which is a finite number. Therefore, $U^{\mathcal{U}}$ is finite, while $U^{\mathcal{I}}$ is infinite.

Proposition 14. *In equilibrium, the information acquisition cost equals the informational advantage:* $\gamma = U^{\mathcal{I}} - U^{\mathcal{U}}$.

Proof. In equilibrium, a trader is indifferent between acquiring and not acquiring information. Therefore, if we include the cost γ, the expected utilities for the informed and uninformed are equal: $U^{\mathcal{I}} - \gamma = U^{\mathcal{U}}$. $\square$

This result implies the equivalence between having an exogenous information ratio, i, and an exogenous information acquisition cost, γ.

In the final part of this section, we discuss the utility of the informed trader conditional on observing the fundamental value.[27]

Proposition 15. *In the average limit order book, the expected utility of the informed trader after observing a fundamental value* v *is* ($w = v - v^e$)*:* $w - \frac{\bar{s}}{2}$, *if* $w \geq \alpha\sigma_e$; $w(1-k) + \frac{\bar{s}}{2} - \Delta(1 + u(1-k))$, *if* $w \in [0, \alpha\sigma_e)$; $-w(1-k) + \frac{\bar{s}}{2} - \Delta(1 + u(1-k))$, *if* $w \in [-\alpha\sigma_e, 0)$; *and* $-w - \frac{\bar{s}}{2}$, *if* $w < -\alpha\sigma_e$. *The minimum value is* $-\frac{\Delta}{2}\,\frac{a_k i(1-\alpha\beta)+(1+u)-a_k\alpha\beta}{1+a_k i}$, *attained at* $w = 0$.

Proof. See the Appendix. $\square$

This implies that the minimum expected utility of an informed trader is at least $-\frac{\Delta}{2}$, regardless of the information ratio i. In Section 2, the informed traders are assumed to receive a private benefit from trading, so that it is always optimal for them to trade. The previous result shows that it is sufficient to consider a private benefit of $\frac{\Delta}{2}$.

5 Numerical Results

In this section we discuss an approximate solution to the general case described in Propositions 3 and 5. This allows us to determine several variables of interest: the maximum number of limit orders in the book, the endogenous limits of the book, the minimum bid-ask spread, and the standard deviation of the spread.

The difficulty in solving the difference equations resulting from the model (Equation (3) in the Appendix) comes from the fact that they depend on two variables: the number of sellers,

[27] The value computed in Proposition 13, $U^{\mathcal{I}}$, is the expected utility of the informed computed *before* observing the fundamental value.

m, and the number of buyers, n. Our method is to search for an approximate solution which considers only the variation in m. This is suggested by Proposition 4, which shows that the average limit order book occurs with probability almost equal to one. Thus, we can study only one side of the limit order book, assuming that the other side is in the average case. We normalize $C = 1$. Thus, in Equation (3) we set $f_{m,n+1} = f_{m,n-1} = f_{m,n}$, and omit the dependence on n. We denote by $f'_m = f_m + \Delta_{SLO}$. We get the following equations: $f_m\left(\frac{1-a_m}{2} + \frac{1+f'_{m+1}}{2} + 2i(\mathsf{P}_{BMO} + \mathsf{P}_{SLO})\right) = f_{m+1}\left(\frac{1+f'_{m+1}}{2} + 2i\,\mathsf{P}_{SLO}\right) + f_{m-1}\left(\frac{1-a_m}{2} + 2i\,\mathsf{P}_{BMO}\right) - \varepsilon(2+2i)$, and $f_{m-1} = a_m - \Delta_{BMO}$. The last equation makes us able to replace f_{m-1} by a_m, and get the recursive equation only in terms of a_m. The probability of BMO and SLO from the informed trader is computed using the cutoffs from Proposition 3. The cutoffs depend on the bid-ask spread s and the midpoint p. Here, $a = a_m$ and $b = \bar{b} = -\frac{\bar{s}}{2}$, so we compute $s = a - \bar{b}$, $p = \frac{a+\bar{b}}{2}$.

To solve the new recursive equation for a_m, we start with an arbitrary value $a_3 \in (\bar{b}, 1)$. Then we compute a_2 as a function of a_1 (and the fixed a_3) from the recursive equation. In state $m = 1$ the patient seller is a monopolist, so he chooses a level a_1 that maximizes his expected utility, $f_1 = a_2 - \Delta_{BMO}$. This produces the optimal level a_1, which is the top order in the limit order book. From this we also obtain a_2. We construct recursively the sequence $a_1, a_2, a_3, \ldots, a_M$, with a_M the last value such that the following properties hold: the expected utility $f_M \geq \bar{b}$ (otherwise the seller at a_M is better off submitting a market order at the bid; and the sequence f_m is decreasing. We choose the largest value of a_3 for which the sequence f_m also goes below $\bar{b}$ (i.e., $f_{M+1} < \bar{b}$).

Table 1 describes variables extracted from the numerical solution: the maximum number of sell limit orders in the book (M), the highest level in the book (a_1), the minimum bid-ask spread, and the standard deviation of the bid-ask spread. For comparison, we also include the average bid-ask spread. The numerical solution depends on several parameters: Δ, the price impact parameter; ε, the granularity parameter; and i, the information ratio. We choose $\Delta = 10^{-2}$, and let ε take three different values $\varepsilon = 10^{-6}, 10^{-8}, 10^{-10}$. The information ratio $i = 0.01, 0.1, 0.5, 1, 5, 10, 100$.

<< Table 1 about here >>

We find that the maximum number of traders in the book, M, is in general decreasing in the information ratio, i: according to Proposition 13, the expected utility of all patient traders (informed or uninformed) is decreasing in i, so fewer traders are willing to submit limit orders as i increases. The maximum number of traders is increasing in the granularity parameter $\varepsilon = \frac{r}{\lambda}$: smaller waiting costs makes it possible to accommodate more patient traders in the book. The highest level of a SLO in the book, a_1, is between $\frac{C}{2}$ and $\frac{2C}{3}$ (recall that we normalized $C = 1$). For most values of i, $a_1 \approx \frac{2C}{3}$.

The minimum bid-ask spread is increasing in i. The intuition is the same as for the average bid-ask spread: more informed traders generate more precise prices and smaller spreads. The bid-ask spread standard deviation, $\sigma(s)$, displays a U-shape in the information ratio: as i increases, the average spread is smaller, which decreases spread volatility; but the number of limit orders in the book is also smaller, which increases spread volatility. Also, $\sigma(s)$ is decreasing in the ε: with smaller waiting costs, the book is more dense, and the bid-ask spread volatility is smaller.

Finally, we check numerically that the price impact of a market order is decreasing in the size of the spread. In the context of a one-sided limit order book, this translates into the price impact of a BMO being smaller if m is smaller (hence, the bid-ask spread $a_m - \bar{b}$ is wider). We observe that when the spread is very large (m is small), the price impact of a market order, Δ_{BMO}, is very close to zero. This is because a large spread reduces the incentive of an informed trader to submit market orders. If market orders are unlikely to come from informed traders, their price impact is small.

6 Empirical Discussion

In this section we discuss testable implications of the model. The exogenous parameters are: the arrival rate of informed traders, $2\lambda^{\mathcal{I}}$; the arrival rate of each type of uninformed traders, $\lambda^{\mathcal{U}}$; and the volatility of the fundamental value, σ_v. All these parameters are unobservable, but the sum $\lambda = 2(\lambda^{\mathcal{I}} + \lambda^{\mathcal{U}})$ is observable: it is the arrival rate of all types of orders. Therefore, we choose as exogenous parameters: the total trading activity, λ; the information ratio, $i = \frac{\lambda^{\mathcal{I}}}{\lambda^{\mathcal{U}}}$; and the fundamental volatility, σ_v.

We can estimate the fundamental volatility by using the efficient price. This is unobservable, but Proposition 4 shows that the bid-ask midpoint is a good proxy for it. Moreover, Proposition 9 shows that the volatility of the efficient price between two trades, $\sigma(v^e_{t+\tau} - v^e_t)$, is equal to the inter-trade fundamental volatility, $\sigma = \frac{\sigma_v}{\sqrt{\lambda}}$. Thus, we can approximate the fundamental volatility, σ_v, by the volatility of the bid-ask midpoint, multiplied by the square root of trading activity.

Before we discuss the information ratio, we analyze the dependence of the bid-ask spread on the parameters of the model. Proposition 7 shows that more trading activity causes bid-ask spreads to narrow. In Foucault, Kadan and Kandel (2005) and Roşu (2009) this occurs because of waiting costs: when trading is more frequent, limit order traders wait less and accept to be compensated with smaller spreads. In this paper, waiting costs are assumed small compared to asymmetric information costs, so the channel is different: more trading activity implies that the volatility between two consecutive orders is lower, and less public uncertainty about the fundamental value narrows the spreads. Also, larger fundamental volatility widens the bid-ask spread. The intuition, as in Foucault (1999), is that the bid-ask spread is a measure of the public uncertainty about the fundamental value.[28] Linnainmaa and Roşu (2009) test these predictions in the Helsinki Stock Exchange by using instrumental variables: deseasonalized sunshine for trading activity, and lagged CBOE volatility index (VIX) for fundamental volatility.

According to Proposition 7, a higher percentage of informed traders (or, equivalently, a higher information ratio) leads to smaller spreads. This is because in the presence of more informed traders the efficient price converges faster to the fundamental value. Thus, the uncertainty about the fundamental value of the asset is lower, and consequently the spreads are also smaller. This result is in contrast with other models of order driven markets. For example, in Glosten (1994) a higher information ratio leads to higher spreads. This is because in that model the fundamental value does not change over time, and informed traders always submit market orders.[29]

The dependence of the bid-ask spread on the information ratio suggests a method to

[28] The same intuition holds for different types of markets: see for example Kyle (1985) or Glosten and Milgrom (1985).

[29] Under Glosten's assumptions, a higher information ratio increases the probability that a limit order is cleared by an informed market order. Thus, to protect themselves, limit order traders set larger spreads.

estimate the information ratio. One benefit of estimating this parameter is its relation to the probability of informed trading (also called "PIN") of Easley, Hvidkjaer, and O'Hara (2002). Indeed, in our model, the probability of dealing with an informed trader is $\frac{i}{i+1}$, where i is the information ratio. Proposition 7 suggests using the average bid-ask spread to estimate the information ratio. But the bid-ask spread also depends on the price impact parameter, Δ, which is proportional to the inter-trade volatility, $\sigma = \frac{\sigma_v}{\sqrt{\lambda}}$. Thus, the ratio of the bid-ask spread to inter-trade volatility $\frac{\bar{s}}{\sigma} = \left(\frac{1+u^2}{2}\right)^{1/2} \frac{1+a_k i}{(a_k+a_k i)\alpha\beta+(1+u+a_k i)}$ depends only on i, and is increasing in i.

Another way of estimating the information ratio is to use market *resilience*, i.e., the speed of convergence of the efficient price to the fundamental value. According to Proposition 9, a higher information ratio makes prices revert more quickly to fundamentals. This also suggests that our market resilience measure is related to the liquidity measure of Pastor and Stambaugh (2003), which estimates the speed of reversals of prices after trades.

This discussion raises the question why spreads are larger around earnings announcements. According to our results, the answer is not asymmetric information, since by itself it would generate smaller spreads. The explanation must be either lower trading activity, or larger volatility, or a combination of both. The main reason seems to be the high fundamental volatility surrounding these events, since without this type of uncertainty it is not clear why there would be lower trading activity.

We next discuss empirical implications about price impact. In Corollary 1, we find a *price impact ratio* of $\frac{1}{u} \approx 3.912$, i.e., the price impact of a market order is about four times larger than the price impact of a limit order. This fact can be tested empirically, and appears robust to alternative specifications of the model.[30] Moreover, the magnitude of the price impact of a market order, $\Delta = \left(\frac{2}{1+u^2}\right)^{1/2} \frac{\sigma_v}{\sqrt{\lambda}}$, is proportional to the fundamental volatility and inversely proportional to the square root of total trading activity. Another empirical implication is

[30] For example, suppose the fundamental value is not normally distributed around the efficient price, and instead has a "fat tails" distribution, i.e., the distribution has positive excess kurtosis. A standard example is the Pearson type VII family given by the density $f\left(x;\ m = \frac{5}{2} + \frac{3}{b}\right) = \frac{\Gamma(m)}{((2m-3)\pi)^{1/2}\Gamma(m-1/2)} \left(1 + \frac{x^2}{2m-3}\right)^{-m}$, where $\Gamma(\cdot)$ is the Gamma function. For $b > 0$ this yields a one-parameter family with zero mean, unit variance, zero skewness, and arbitrary positive excess kurtosis. If $b = 1$, the excess kurtosis equals 0.990 and the price impact ratio equals 4.166. If $b = 5$, the excess kurtosis equals 3.190 and the price impact ratio equals 4.464. Thus, even with large excess kurtosis the price impact ratio is not much larger than 4.

that the price impact of a market order is negatively correlated with the bid-ask spread: a wider spread provides a bigger incentive to provide liquidity with a limit order, thus reduces the probability of a market order from an informed trader.

According to Proposition 10, as a function of time since last transaction, the price impact of a market order is increasing, and the price impact of a limit order is decreasing. To our knowledge, the result about limit orders is new. In the case of market orders, Hausman, Lo, and MacKinlay (1992) do a probit analysis of the price process and find that, opposite to our theoretical prediction, a longer time since last transaction is correlated with a smaller price impact. This could be due to the fact that both price impact and time since last transaction are determined endogenously.

We note that Proposition 10 refers to the actual time since last transaction, not to the *average* time between transactions (i.e., duration). Most theoretical papers focus on duration when studying in price formation.[31] In Easley and O'Hara (1992), agents only trade when they have a signal ("news"), therefore long durations are likely to be associated with no news, and with small price impacts. In our paper, duration is constant and equal to $\frac{1}{\lambda}$, the inverse of trading activity. Empirically, Dufour and Engle (2000) find that longer durations are correlated with smaller price impact. Since longer durations means lower trading activity, this seems to contradict the fact that lower trading frequency causes higher price impact. This again can be attributed to the endogeneity of variables. Linnainmaa and Roşu (2009) use instrumental variables to generate exogenous variation in trading activity. They find that the opposite result is true.

In our model the order flow is autocorrelated. Proposition 11 suggests that the order flow autocorrelation can be used a proxy for the unobserved information ratio, i, at least when $i < 1$. But the diagonal effect can be explained in at least two other ways. One is by taking the view of Evans and Lyons (2002), who replace private information about fundamentals with private information about order flow. Agents with signals about the future order flow rush to trade in the direction of their signal (front running) and prices adjust in the direction of the orders. To distinguish such a model from ours, one can analyze price reversals: in our model traders are informed about the true value of the asset, so there are no price reversals

[31] See, for example, Diamond and Verrecchia (1987) and Easley and O'Hara (1992).

(i.e., efficient price changes are uncorrelated). Another explanation of the diagonal effect is that uninformed traders optimally "work" their orders, i.e., divide them into smaller orders and execute them over time. In our model, agents trade only one unit so we do not have this effect. But in general multi-unit trading also creates order flow autocorrelation. Such a model can be distinguished empirically from ours if the identity of traders is known.

7 Conclusions

This paper presents a tractable models of an order driven market with both liquidity traders and informed traders. The informed patient traders submit market orders if they observe a fundamental value of the asset far from the efficient (public) price, and a limit order if the fundamental value is close to the efficient price. As a result, both market orders and limit orders carry information. On average, the price impact of a limit order is about four times smaller than the price impact of a market order. In rare cases, in off-center limit order books with wide bid-ask spreads, a buy limit order can have a negative price impact. As a function of time since last transaction, the price impact of a market order is increasing, and the price impact of a limit order is decreasing. The price impact of a market order is decreasing in the size of the spread.

The market displays two types of resilience: a "micro" resilience, related to the shape of the limit order book: the bid and ask prices tend to stay close to the efficient price, which also leads to small values of the bid-ask spread; and a "macro" resilience: the efficient price converges to the fundamental value. The order flow is autocorrelated, and the autocorrelation has an inverted U-shape in the information ratio.

A higher fundamental volatility and a lower trading activity cause the bid-ask spread to widen. In contrast to static models of order driven markets, a higher percentage of informed trader (a higher information ratio) causes *lower* spreads. A higher information ratio also makes prices converge more quickly to the fundamental value. We use these facts to provide an empirical proxy for the probability of informed trading.

Some of the limitations of this model point towards future directions for research. In particular, the model assumes that at all times traders monitor the market and make the right inferences. It would be useful to see how the results of this paper change in the presence of monitoring costs. This would create the risk of limit orders being picked off, which does not occur in our model. Also, in this model the liquidity traders incur a one-time private cost upon their arrival to the market. This is essentially equivalent to a private valuation that does not change over time. While relaxing this assumption arguably does not change the qualitative predictions of our model, it probably leads to a richer model in which some traders cancel their orders.

A Proofs of Results

Proof of Proposition 1: We denote by $f'_{m+1,n} = f_{m+1,n} + \Delta_{SLO}$; and by P_j the probability that an informed trader submits an order $j \in \{BMO, BLO, SMO, SLO, NO\}$. Then the recursive equations for f and g, and the equations for a and b are:

$$\begin{aligned}
f_{m,n} &= \tfrac{\lambda^{\mathcal{U}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}\Big(\frac{1-\frac{a_{m,n}}{C}}{2} f_{m-1,n} + \frac{1+\frac{a_{m,n}}{C}}{2} f_{m,n}\Big) + \tfrac{\lambda^{\mathcal{U}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}\Big(\frac{1-\frac{g'_{m,n+1}}{C}}{2} f_{m,n+1} + \frac{1+\frac{g'_{m,n+1}}{C}}{2} f_{m,n}\Big) \\
&\quad + \tfrac{\lambda^{\mathcal{U}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}\Big(\frac{1+\frac{f'_{m+1,n}}{C}}{2} f_{m+1,n} + \frac{1-\frac{f'_{m+1,n}}{C}}{2} f_{m,n}\Big) + \tfrac{\lambda^{\mathcal{U}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}\Big(\frac{1+\frac{b_{m,n}}{C}}{2} f_{m,n-1} + \frac{1-\frac{b_{m,n}}{C}}{2} f_{m,n}\Big) \\
&\quad + \tfrac{2\lambda^{\mathcal{I}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}\Big(\mathsf{P}_{BMO}\, f_{m-1,n} + \mathsf{P}_{BLO}\, f_{m,n+1} + \mathsf{P}_{SLO}\, f_{m+1,n} + \mathsf{P}_{SMO}\, f_{m,n-1} + \mathsf{P}_{NO}\, f_{m,n}\Big) \\
&\quad - \tfrac{r}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}, \qquad (3) \\
g_{m,n} &= \tfrac{\lambda^{\mathcal{U}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}\Big(\frac{1-\frac{a_{m,n}}{C}}{2} g_{m-1,n} + \frac{1+\frac{a_{m,n}}{C}}{2} g_{m,n}\Big) + \cdots \\
a_{m,n} &= f_{m-1,n} + \Delta_{BMO}, \\
b_{m,n} &= -g_{m,n-1} + \Delta_{SMO}.
\end{aligned}$$

The last two equations describe the ask price and the bid price in terms of the value functions. Suppose the book is in state (m, n, v^e), and a buy market order is submitted. Then the seller at the ask gets the ask price, $v^e + a_{m,n}$, while all the sellers above the ask get $v^e + \Delta_{BMO} + f_{m-1,n}$, which is the expected utility of the sellers in state $(m-1, n, v^e + \Delta_{BMO})$ (the new efficient price is $v^e + \Delta_{BMO}$). Since all sellers have the same expected utility, $a_{m,n} = f_{m-1,n} + \Delta_{BMO}$. Similarly, $b_{m,n} = g_{m,n-1} + \Delta_{SMO}$.

The proof for the recursive equation for f is similar to that of Theorem 1 in Roşu (2009). Suppose the system is in state (m,n) (for brevity, we omit v^e, σ^e). Consider an uninformed patient seller with a sell limit order. His utility depends on what the next trader does. Since traders arrive according to a Poisson process, the next trader arrives after an expected time of $\frac{1}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}$. This explains the last term $r \cdot \frac{1}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}$ from the Equation (3) for f: the agent loses utility proportionally to the expected waiting time. We now explain the second term, involving $f_{m,n+1}$. Suppose the next trader is an uninformed patient buyer, an event which occurs with probability $\frac{\lambda^{\mathcal{U}}}{4\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}$. If she submits a *BLO*, she pays in expectation $v^e+\Delta_{BLO}+g_{m,n+1}$ (taking into account the impact of *BLO* on v^e) for an asset worth v^e in expectation; this implies an expected profit of $-g'_{m,n+1}$, where $g'_{m,n+1} = g_{m,n+1} + \Delta_{BLO}$. Moreover, she incurs a one-time private cost c uniformly distributed on $[-C, C]$. Thus, she enters the market only if $c < -g'_{m,n+1}$, which has an ex ante probability of $\frac{1-\frac{g'_{m,n+1}}{C}}{2}$. In that case, she submits a *BLO*, which sends the system to state $(m, n+1)$. If $c \geq -g'_{m,n+1}$, with ex ante probability of $\frac{1+\frac{g'_{m,n}}{C}}{2}$, she exits the market, and the system remains in state (m,n). The discussion for the other types of uninformed traders is the same.[32]

Suppose now a, b, f, g are of the order of Δ, which is assumed small. Then $\frac{1-\frac{a_{m,n}}{C}}{2} \approx \frac{1}{2}$, etc. The recursive equation for f becomes

$$f_{m,n} = \frac{1+4i\,\mathsf{P}_{BMO}}{4+4i}\, f_{m-1,n} + \frac{1+4i\,\mathsf{P}_{BLO}}{4+4i}\, f_{m,n+1} + \frac{1+4i\,\mathsf{P}_{SLO}}{4+4i}\, f_{m+1,n} + \frac{1+4i\,\mathsf{P}_{SMO}}{4+4i}\, f_{m,n-1} - \varepsilon \qquad (4)$$

up to the order of Δ^2, where $i = \frac{\lambda^{\mathcal{I}}}{\lambda^{\mathcal{U}}}$ and $\varepsilon = \frac{r}{2\lambda^{\mathcal{U}}+2\lambda^{\mathcal{I}}}$. □

Proof of Proposition 2: If an informed trader submits a *SMO*, he sells for $v^e + b_{m,n}$ an asset that is worth v. Thus, the expected utility is $(v^e + b_{m,n}) - v$.

We analyze the submission of a *SLO* by an informed trader. If the book is in state (m, n, v^e), the *SLO* moves it to $(m+1, n, v^e + \Delta_{SLO})$. Suppose we already know that in this state, $Q^{\mathcal{I}} \approx Q^{\mathcal{U}} + k(v - v^e - \Delta_{SLO})$, where $Q^{\mathcal{I}}$ and $Q^{\mathcal{U}}$ are the expected execution prices (minus waiting costs) from the point of view of the informed and uninformed seller, respectively. The expected utility of the informed seller is $u^{\mathcal{I}}_{SLO} = Q^{\mathcal{I}} - v$, and the expected utility of the uninformed seller is $f_{m+1,n} = Q^{\mathcal{U}} - (v^e + \Delta_{SLO})$. The difference of the two expected utilities equals $u^{\mathcal{I}}_{SLO} - f_{m+1,n} = (Q^{\mathcal{I}} - Q^{\mathcal{U}}) - (v - v^e - \Delta_{SLO}) = -(1-k)(v - v^e - \Delta_{SLO})$. We get $u^{\mathcal{I}} = f_{m+1,n} - (1-k)(v - v^e - \Delta_{SLO})$. But $f_{m+1,n} = f_{m-1,n}$, up to the order of Δ^2 (the

[32] Note there are also changes in the efficient price: e.g., the term containing $f_{m-1,n}$ should have instead $f_{m-1,n} + \Delta_{BMO}$. But because the changes in the efficient price cancel out in the aggregate, we omit them.

difference is given by the negligible waiting costs). From the second to last equation in (3), $f_{m-1,n} = a_{m,n} - \Delta_{BMO}$. Putting all these equations together, we get $u^{\mathcal{I}} = -(1-k)(v - v^e - \Delta_{SLO}) + (a_{m,n} - \Delta_{BMO})$.

It remains to show that in state (m, n, v^e), $Q^{\mathcal{I}} \approx Q^{\mathcal{U}} + k(v - v^e)$. The average time between transactions is $\tau = \frac{1}{\lambda}$, where $\lambda = 2(\lambda^{\mathcal{I}} + \lambda^{\mathcal{U}})$ is the total trading activity. This is called a "period." We denote by $v_l = v_l(v)$ the expected fundamental value next period, conditional on all the public information and on observing l periods before a fundamental value, v; and by σ_l the corresponding standard deviation. (We have $v_0 = v$, $\sigma_0 = 0$.) Let $w_l = v_l - v^e$, where v^e is the current efficient price. We denote by $Q^{\mathcal{I}}_{a,l} = Q^{\mathcal{I}}_{a,l}(v)$ the expected execution price of the a'th sell limit order in the queue ($a = 1$ being at the ask) from the point of view of an informed trader who observed the fundamental value l periods before. If $Q^{\mathcal{U}}$ is the same concept from the point of view of an uninformed seller, this does not depend on a, as the uniformed sellers have the same expected execution price once we include waiting costs. Let $q_{a,l}(w_l, \sigma_l) = Q^{\mathcal{I}}_{a,l} - Q^{\mathcal{U}}$. We note that $q_{0,l} = 0$ for any l, since $Q^{\mathcal{I}}_{0,l} = Q^{\mathcal{U}}$ is the ask price, up to the negligible waiting costs.

The current expected execution price equals the average expected execution price next period (minus waiting costs). This implies

$$\begin{aligned} q_{a,l}(w_l, \sigma_l) \;=\;& \mathsf{P}_{BMO}(w_l, \sigma_l)\big(\Delta + q_{a-1,l+1}(w_{l+1}, \sigma_{l+1})\big) + \mathsf{P}_{BLO}(w_l, \sigma_l)\big(u\Delta + q_{a,l+1}(w_{l+1}, \sigma_{l+1})\big) \\ &+ \mathsf{P}_{SLO}(w_l, \sigma_l)\big(-u\Delta + q_{a+1,l+1}(w_{l+1}, \sigma_{l+1})\big) + \mathsf{P}_{SMO}(w_l, \sigma_l)\big(-\Delta + q_{a,l+1}(w_{l+1}, \sigma_{l+1})\big), \end{aligned}$$

where P_j is the probability of having an order of type $j \in \{BMO, BLO, SLO, SMO\}$ next period, conditional on observing the fundamental value l periods before. We simplify the search for a solution by assuming that we are in the average limit order book. To describe the evolution of (w_l, σ_l), we can use the same arguments as in the proof of Proposition 5, except that we have to account for the fact that the fundamental value changes during until next period by a normal variable, $v_{t+\tau} - v_t \sim N(0, \sigma)$. We define the cutoffs w^u and w^d by: if $j = BMO$, $w^u = \infty$ and $w^d = -w_l + \alpha\sigma_e$; if $j = BLO$, $w^u = -w_l + \alpha\sigma_e$ and $w^d = -w_l$; if $j = SLO$, $w^u = -w_l$ and $w^d = -w_l - \alpha\sigma_e$; if $j = SMO$, $w^u = -w_l - \alpha\sigma_e$ and $w^d = -\infty$. If $j \in \{BMO, BLO, SLO, SMO\}$, we denote by $\Delta_j = \Delta, u\Delta, -u\Delta, -\Delta$, respectively. Then,

(w_l, σ_l) evolve according to: $w_{l+1} = w_l + \sigma_l \mu_j - \Delta_j$, with $\mu_j = \frac{1}{\sqrt{2\pi}}\Big(\mathrm{e}^{-\frac{1}{2}\left(\frac{w^d}{\sigma_l}\right)^2} - \mathrm{e}^{-\frac{1}{2}\left(\frac{w^u}{\sigma_l}\right)^2}\Big)/\Big(\frac{1}{4i} + \Phi\big(\frac{w^u}{\sigma_l}\big) - \Phi\big(\frac{w^d}{\sigma_l}\big)\Big)$; and $(\sigma_{l+1})^2 = \sigma^2 + (\sigma_l)^2\Big[1 + \frac{1}{\sqrt{2\pi}}\Big(\frac{w^d}{\sigma_l}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{w^d}{\sigma_l}\right)^2} - \frac{w^u}{\sigma_l}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{w^u}{\sigma_l}\right)^2}\Big)/\Big(\frac{1}{4i} + \Phi\big(\frac{w^u}{\sigma_l}\big) - \Phi\big(\frac{w^d}{\sigma_l}\big)\Big) - \mu_j^2\Big]$. Also, we compute $\mathsf{P}_j = \frac{1+4i\left(\Phi\left(\frac{w^u}{\sigma_l}\right) - \Phi\left(\frac{w^d}{\sigma_l}\right)\right)}{4+4i}$.

We look for stationary solutions of the recursive equation for q, i.e., solutions which do not depend on l. Moreover, we reduce the number of variables for q by looking at homogenous solutions in $x_l = w_l/\sigma_l$. We simplify the recursive equation for x_l by making $\sigma_{l+1} = \sigma_l = \sigma_e$ (which is the asymptotic limit of σ_l): $x_{l+1} = x_l + \mu_j - \Delta_j/\sigma_e$. We also redefine $q_a(x)$ to equal $q_a(x)/\sigma_e$. We make an initial guess for $q_a^{(0)}(x)$ for all a: $q_a^{(0)}(x) = \frac{a}{a+\ln(1+1/i)+1}x$, where $i = \lambda^{\mathcal{I}}/\lambda^{\mathcal{U}}$ is the information ratio. This satisfies $q_0(x) = 0$ and $q_l(x) \approx x$ for l very large (if the *SLO* is very far from the ask, there is only a very small probability that the *SLO* is executed before the pricing error $v - v^e$ is corrected). Then we define $q_a^{(n+1)}(\cdot)$ to satisfy the recursive equation with $q_a^{(n)}(\cdot)$ on the right hand side. The algorithm converges. We are interested in $q_1(x)$, the expected execution price for a limit order at the ask. Numerically, this is approximately linear: $q_1(x) = kx$, where $k = \frac{a_k i}{1+a_k i}$, and $a_k \approx 1.2729$. We know that at $l = 0$, $q_1(x) = (Q^{\mathcal{I}} - Q^{\mathcal{U}})/\sigma_e$, and $x = (v - v^e)/\sigma_e$. This proves that $Q^{\mathcal{I}} \approx Q^{\mathcal{U}} + k(v - v^e)$. □

Proof of Proposition 3: We denote by $w = v - v^e$. Then, from Proposition 2, the expected utility of an informed trader is: $u^{\mathcal{I}}_{BMO} = w - a$, $u^{\mathcal{I}}_{BLO} = w(1-k) - b - \Delta'_1$, $u^{\mathcal{I}}_{SLO} = -w(1-k) + a - \Delta'_2$, $u^{\mathcal{I}}_{SMO} = -v + b$, where $\Delta'_1 = -\Delta_{SMO} + \Delta_{BLO}(1-k)$ and $\Delta'_2 = \Delta_{BMO} - \Delta_{SLO}(1-k)$. We define $\Delta' = (\Delta'_1 + \Delta'_2)/2$, $\Delta'' = (\Delta'_1 - \Delta'_2)/2$. We write $BMO > BLO$ to indicate that BMO is preferred to BLO. Then $BMO > BLO \iff w > \frac{s-\Delta'_1}{k}$, $BLO > SLO \iff w > \frac{p+\Delta''}{1-k}$, $SLO > SMO \iff w > \frac{-s+\Delta'_2}{k}$, $BMO > SLO \iff w > \frac{a-\Delta'_2/2}{1-k/2}$, $BLO > SMO \iff w > \frac{b+\Delta'_1/2}{1-k/2}$, $BMO > SMO \iff w > p$. An analysis of all possible cases proves the main result.

To analyze the price impact of an order of type j, suppose this is the optimal order when $w = v - v^e \in (w^d, w^u)$. Because w has a normal distribution centered at zero, the price impact $\Delta_j > 0 \iff w^u + w^d > 0$. If $j = BMO$, $w^u = \infty$, so $\Delta_{BMO} > 0$; if $j = SMO$, $w^d = -\infty$, so $\Delta_{SMO} < 0$. We now show that $\Delta_{BLO} > 0$ in Case 4, and $\Delta_{BLO} < 0$ in Case 3 if Δ'' sufficiently small. (A similar analysis works for Δ_{SLO}.) In Case 4, we know that $s - \Delta' > -\frac{1}{1-k}(kp + \Delta'')$. If $j = BLO$, $w^u = \frac{s-\Delta'_1}{k}$, $w^d = \frac{p+\Delta''}{1-k}$. The condition $w^u + w^d > 0$

(positive price impact) is equivalent to $s - \Delta' > -\frac{1}{1-k}(kp + \Delta'') - 2\Delta''$, which is true if Δ'' is sufficiently small compared to s. In Case 3, we know that $-\frac{1}{1-k}(kp + \Delta'') > s - \Delta''$. If $j = BLO$, $w^u = \frac{s-\Delta_1'}{k}$, $w^d = \frac{b+\Delta_1'/2}{1-k/2}$. The condition $w^u + w^d < 0$ (negative price impact) is equivalent to $s - \Delta' < -\frac{1}{1-k}(kp + \Delta'') - 2\Delta''\frac{1-k/2}{1-k}$, which is true if Δ'' is sufficiently small compared to $|p|$.

The statement about the patient uninformed buyers and sellers follows directly from Proposition 2. (For buyers we only need to compare *BLO* with *BMO*, and for sellers we only need to compare *SLO* with *SMO*.) □

Proof of Proposition 4: It is enough to show that Case 4 of Proposition 3 occurs with probability approximately one when the ask and the bid are close to the efficient price, to the order of Δ. This means that $f_{m,n}$, $g_{m,n}$, $a_{m,n}$, $b_{m,n}$ are of the order of Δ. Then the value function of the uninformed traders, f, satisfies the recursive equation (4), up to the order of Δ^2. (See the proof of Proposition 1.) Consider the ask side of the limit order book. The ratio of the arrival rates of *SLO* and *BMO* is $c = \frac{1+4i\,\mathsf{P}_{SLO}}{1+4i\,\mathsf{P}_{BMO}}$. Proposition 2 of Roşu (2009) shows that if $c < 1$, i.e., if market orders arrive faster than limit orders, the average bid-ask spread $\bar{s}$ is wide (of the order of one); and if $c > 1$, $\bar{s}$ is proportional to $\varepsilon \ln(1/\varepsilon)\frac{c(c+1)}{(c-1)\ln(c)}$, where $\varepsilon = \frac{r}{\lambda}$, and the spread standard deviation $\sigma(s)$ is proportional to $\left(\frac{\varepsilon}{(c-1)^3}\right)^{1/2}$. Since $\varepsilon = \Delta^{2+\delta}$, with $\delta > 0$, $\varepsilon \ln(1/\varepsilon)\frac{c(c+1)}{(c-1)\ln(c)}$ is smaller than the order of Δ. It follows that near the average spread $c \approx 1$. This implies $\mathsf{P}_{SLO} = \mathsf{P}_{BMO}$, and similarly, for the bid side, $\mathsf{P}_{BLO} = \mathsf{P}_{SMO}$. These are two equations in two unknowns: the spread s and the midpoint p.

We solve for s and p: $s = \bar{s} = k\alpha\sigma_e + \Delta(1 + u(1-k))$ (as in Proposition 7), and $p = \bar{p} = 0$. For this, we assume $\Delta_1' = \Delta_2' = \Delta' = \Delta(1 + u(1-k))$, $\Delta'' = 0$, and $\sigma^e = \sigma_e$ (these conditions must be checked at the end). We are in Case 4 of Proposition 3, in which all order probabilities are positive, so the cutoffs are $w_1 = \frac{s-\Delta'}{k} = \alpha\sigma_e$, $w_2 = 0$, $w_3 = -\frac{s-\Delta'}{k} = \alpha\sigma_e$. Now it is straightforward to verify that $s = \bar{s}$ and $p = \bar{p}$ satisfy $\mathsf{P}_{SLO} = \mathsf{P}_{BMO}$ and $\mathsf{P}_{BLO} = \mathsf{P}_{SMO}$. Moreover, using the cutoffs, we find that all probabilities are equal to $\frac{1}{4}$. We use Proposition 5 to compute $\Delta_{BMO} = \sigma_e \frac{\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-\frac{1}{2}\alpha^2}}{\frac{1}{4i}+1-\Phi(\alpha)} = \Delta$, etc. This implies the desired values for Δ', Δ_1', Δ_2', Δ''. The condition $\sigma^e = \sigma_e$ comes from Proposition 6: the equilibrium is assumed stationary.

Suppose the spread s is in some vicinity of $\bar{s}$ of length of the order of $\Delta^{1+\delta/4}$. Assume that $c = \frac{1+4i\,\mathsf{P}_{SLO}}{1+4i\,\mathsf{P}_{BMO}} > 1$ on that vicinity. We compute $\mathsf{P}_{BMO} = \Phi\left(-\frac{s-\Delta_1'}{k\sigma^e}\right)$. Since σ^e is of the order of

Δ, $\mathsf{P}_{BMO} - \frac{1}{4} = \Phi\big(-\frac{s-\Delta_1'}{k\sigma^e}\big) - \Phi\big(-\frac{\bar{s}-\Delta_1'}{k\sigma_e}\big)$ is of the order of $\Delta^{\delta/4}$. The same argument holds for P_{SLO}, therefore, $c-1$ is of the order of $\Delta^{\delta/4}$. We know that $\sigma(s)$ is proportional to $\big(\frac{\varepsilon}{(c-1)^3}\big)^{1/2}$ and that $\varepsilon = \Delta^{2+\delta}$, therefore, $\sigma(s)$ is of the order of $\Delta^{1+\delta/2-3\delta/8} = \Delta^{1+\delta/8}$. Because the standard deviation of s is of a smaller order ($\Delta^{1+\delta/8}$) than the length of the vicinity ($\Delta^{1+\delta/4}$), it follows that the probability of s being in this vicinity is almost one.

We have just proved that Case 4 of Proposition 3 occurs with probability approximately one when the spread is of the order of $\Delta^{1+\delta/4}$. Moreover, the midpoint $v^e + p$ is close the efficient price, to the order of $\Delta^{1+\delta/4}$. The order probabilities are all equal to $\frac{1}{4}$, to the order of $\Delta^{\delta/4}$. □

Proof of Proposition 5: We know $v \sim N(v^e, (\sigma^e)^2)$. We note that an informed trader submits an order of type $j \in \{BMO, BLO, SLO, SMO\}$ if $v - v^e \in (w^d, w^u)$, for some cutoffs w^d and w^u, which depend on j. As in Equation (4), conditional on the fundamental value v, the probability of an order of type j is $\frac{1+4iI_{v-v^e\in(w^d,w^u)}}{4+4i}$, where I_x is the indicator function. By Bayes' rule, this implies that conditional on an order of type j being submitted, $v = v^e + \sigma^e \frac{1+4iI_{v-v^e\in(w^d,w^u)}}{4+4i\left(\Phi(\frac{w^u}{\sigma^e})-\Phi(\frac{w^d}{\sigma^e})\right)}$, where $\Phi(\cdot)$ is the standard normal cumulative density. This has density $\left(v^e + \sigma^e \frac{1+4iI_{x\in(w^d,w^u)}}{4+4i\left(\Phi(\frac{w^u}{\sigma^e})-\Phi(\frac{w^d}{\sigma^e})\right)}\right)\phi(x)$, where $\phi(\cdot)$ is the standard normal density. We compute the conditional expectation and variance given order j (and the public information up to that point): $\mathsf{E}(v|j) = v^e + \sigma^e \mu_j$, where $\mu_j = \frac{\frac{1}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{1}{2}\left(\frac{w^d}{\sigma^e}\right)^2} - \mathrm{e}^{-\frac{1}{2}\left(\frac{w^u}{\sigma^e}\right)^2}\right)}{\frac{1}{4i}+\Phi\left(\frac{w^u}{\sigma^e}\right)-\Phi\left(\frac{w^d}{\sigma^e}\right)}$, and

$$\mathsf{Var}(v|j) \;=\; (\sigma_j^e)^2 \;=\; (\sigma^e)^2 \left[1 + \frac{\frac{1}{\sqrt{2\pi}}\left(\frac{w^d}{\sigma^e}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{w^d}{\sigma^e}\right)^2} - \frac{w^u}{\sigma^e}\,\mathrm{e}^{-\frac{1}{2}\left(\frac{w^u}{\sigma^e}\right)^2}\right)}{\frac{1}{4i} + \Phi\left(\frac{w^u}{\sigma^e}\right) - \Phi\left(\frac{w^d}{\sigma^e}\right)} - \mu_j^2\right]. \tag{5}$$

□

Proof of Proposition 6: The equilibrium is stationary in the average limit order book if $\mathsf{Var}_t(v_{t+\tau}) = \mathsf{Var}_{t-\tau}(v_t)$ for $\tau = \frac{1}{\lambda}$. We have $\mathsf{Var}_{t-\tau}(v_t) = (\sigma^e)^2$, so we need to show that the variance next period is also equal to $(\sigma^e)^2$. Equation (5) states that $\mathsf{Var}_t(v_t|j) = (\sigma^e)^2\big[\cdots\big]$, which implies that $\mathsf{Var}_t(v_{t+\tau}|j) = \sigma^2 + (\sigma^e)^2\big[\cdots\big]$. Then, $\mathsf{Var}_t(v_{t+\tau})$ is the average corresponding to the four types of orders. When we sum the four terms, the middle terms inside the square brackets cancel each other out, because of the sum of the telescopic series has end terms corresponding to $w^u = \infty$ and $w^d = -\infty$, so the terms are zero. We get $\mathsf{Var}_t(v_{t+\tau}) = \sigma^2 + (\sigma^e)^2\big[1 - \frac{1}{4}(\sum_j \mu_j^2)\big] = \sigma^2 + (\sigma^e)^2 - \frac{1}{4}\sum_j(\sigma^e\mu_j)^2 = \sigma^2 + (\sigma^e)^2 - \frac{1}{4}(2\Delta^2_{_{BMO}} + 2u^2\Delta^2_{_{BMO}})$. (According to the discussion before Proposition 6, all efficient price changes are proportional to $\Delta_{_{BMO}}$.) Thus, $\mathsf{Var}_t(v_{t+\tau}) = (\sigma^e)^2 \iff \sigma^2 = \frac{1+u^2}{2}\Delta^2_{_{BMO}} \iff \sigma = \sqrt{\frac{1+u^2}{2}}\Delta_{_{BMO}}$.

Since by definition $\Delta = \sqrt{\frac{2}{1+u^2}}\ \sigma$, the stationarity of the equilibrium is also equivalent to $\Delta_{_{BMO}} = \Delta$. The only thing left to prove is that this is also equivalent to $\sigma^e = \sigma_e$. Proposition 5 implies that $\Delta_{_{BMO}} = \sigma^e \frac{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\alpha^2/2}}{\frac{1}{4i}+\frac{1}{4}} = \sigma^e \frac{1}{\beta\ \frac{i+1}{i}}$. This implies $\sigma^e = \beta\ \frac{i+1}{i}\ \Delta_{_{BMO}}$. Since by definition, $\sigma_e = \beta\ \frac{i+1}{i}\ \Delta$, $\sigma^e = \sigma_e$ is equivalent to $\Delta_{_{BMO}} = \Delta$. □

Proof of Proposition 7: Proposition 4 shows that all probabilities are equal to $\frac{1}{4}$. We denote by $\Delta' = \Delta(1 + u(1-k))$ and $\Delta'' = 0$. The cutoffs for the four types of orders in the average limit order book (Case 4 of Proposition 3) are $w_1 = \frac{\bar{s}-\Delta'}{k}$, $w_2 = \frac{\bar{p}+\Delta''}{1-k}$, $w_3 = -\frac{\bar{s}-\Delta'}{k}$. Since $\mathsf{P}_{_{SMO}} = \Phi(-\frac{\bar{s}-\Delta'}{k\sigma_e}) = \frac{1}{4}$, we get $\frac{\bar{s}-\Delta'}{k\sigma_e} = \alpha$, which implies $\bar{s} = k\alpha\sigma_e + \Delta'$. From $\mathsf{P}_{_{SLO}} = \Phi(\frac{\bar{p}+\Delta''}{(1-k)\sigma_e}) - \Phi(-\frac{\bar{s}-\Delta'}{k\sigma_e})\frac{1}{4}$ we also get $\frac{\bar{p}+\Delta''}{(1-k)\sigma_e} = 0$, which implies $\bar{p} = 0$. Since $\bar{s} = \bar{a} - \bar{b}$ and $\bar{p} = \frac{\bar{a}+\bar{b}}{2}$, we get $\bar{a} = \frac{\bar{s}}{2}$ and $\bar{b} = -\frac{\bar{s}}{2}$. □

Proof of Proposition 8: We show that informed traders who submit limit orders do not deviate from the strategy of uninformed traders. Thus, they preserve the limit orders in the book and move them up or down along with the efficient price. Deviation means either that an informed trader undercuts after the initial order submission, or that the informed trader cancels the limit order and places a market order instead. As discussed in Section 3.1, the first kind of deviation cannot exist in equilibrium: by undercutting, the informed trader reveals his type; thus, the other traders in the book have an incentive to undercut to the point at which the informed trader no longer has an advantage. The same argument shows that the previously informed traders do not deviate either: they are at a disadvantage compared with the newly arrived informed trader. In conclusion, the only kind of deviation we need to worry about is the modification from a limit order to a market order.

Without loss of generality, assume that the informed trader submits a BLO. This means that the informed observes a fundamental value v_t, so that the pricing advantage $w_t = v_t - v_t^e$ is in $(0, \alpha\sigma_e)$. The efficient price moves from v_t^e to $v_t^e + u\Delta$, thus, the new pricing advantage, w, is in $(-u\Delta, \alpha\sigma_e - u\Delta)$. Suppose the trader who arrives next period (at $t + \tau$, with

$\tau = \frac{1}{\lambda}$) submits a BMO. This order moves the efficient price by Δ, but also moves the expected fundamental value from the perspective of the informed trader (the new trader might be informed). As in the proof of Proposition 2, we compute the new pricing advantage $w' = \mathsf{E}_t(v_{t+\tau}|v_t, BMO_{t+\tau}) - (v_t^e + \Delta)$ by $w' = w + \sigma \frac{\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{1}{2}\rho^2}}{\frac{1}{4i} + \Phi(-\rho)} - \Delta$, where $\rho = \frac{-w+\alpha\sigma_e}{\sigma}$. We use Proposition 2 to compute the expected utility under maintaining BLO, or from submitting BMO: $u^{\mathcal{I}}_{BLO}(w') = (1-k)w' - (\bar{b} + \Delta)$, $u^{\mathcal{I}}_{BMO}(w') = w' - \bar{a}$. (We note that the formula for $u^{\mathcal{I}}$ from maintaining BLO does not have the Δ_{BLO} term.) Then $BLO > BMO \iff w' < \frac{\bar{s}-\Delta}{k}$. One can now verify this inequality numerically, for all information ratios i and $w \in (-u\Delta, \alpha\sigma_e - u\Delta)$.

A similar method works if the next trader submits SLO. If the next trader submits BLO, then the informed trader is in competition with the new trader on the bid side. The new pricing error is $w' = w + \sigma \frac{\frac{1}{\sqrt{2\pi}}\left(\mathrm{e}^{-\frac{1}{2}\rho_1^2} - \mathrm{e}^{-\frac{1}{2}\rho_2^2}\right)}{\frac{1}{4i} + \Phi(-\rho_1) - \Phi(-\rho_2)} - u\Delta$, where $\rho_1 = -\frac{w}{\sigma}$ and $\rho_2 = \frac{-w+\alpha\sigma_e}{\sigma}$. As before, $BLO > BMO \iff w' < \frac{\bar{s}-\Delta}{k_2}$, where k_2 is the information decay parameter for an informed trader who has a limit order on the second level on the bid side. This is computed using the method in Proposition 2, and the verification is again done numerically. If the next trader submits SMO, the buy order of the informed trader is executed, so no analysis needs to be made. One can further check that the informed trader does not deviate when he is not the first at the bid, but the second, or third, etc. $\square$

PROOF OF PROPOSITION 9: Given the expression for the price process:

$$v^e_{t+\tau} = v^e_t + b_i\left[\chi_{w_t \geq \alpha\sigma_e}\Delta + \chi_{w_t \in [0,\alpha\sigma_e)} u\Delta + \chi_{w_t \in [-\alpha\sigma_e, 0)}(-u\Delta) + \chi_{w_t < -\alpha\sigma_e}(-\Delta)\right] + (1-b_i)\psi(a_4)\,\Delta,$$

the results follow by direct computation using the definition of covariance. We note that the result $\mathsf{Var}(v^e_{t+\tau} - v^e_t) = \mathsf{Var}(v_{t+\tau} - v_t)$ translates into $\sigma^2 = \frac{1+u^2}{2}\Delta^2$, which according to Proposition 6 is equivalent to the equilibrium being stationary. $\square$

PROOF OF PROPOSITION 10: We denote by t the time when the last transaction took place, and by T the time since the last transaction. Because the fundamental value follows a diffusion process, $\mathrm{d}v_t = \sigma_v \mathrm{d}W_t$, the increment $v_{t+T} - v_t$ is normally distributed $N(0, \sigma_v^2 T)$. Equation (5) implies that $(\sigma^e_{t+T})^2 = \mathsf{Var}_t(v_{t+T}|j) = \sigma_v^2 T + (\sigma^e)^2\left[\cdots\right]$, is increasing in T. At time $t+T$ the price impact of BMO and BLO is, respectively, $\Delta_{BMO} = \sigma^e_{t+T}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha\sigma_e}{\sigma^e_{t+T}}\right)^2} / \left(\frac{1}{4i} + 1 -\right.$

$\Phi\big(\frac{\alpha\sigma_e}{\sigma^e_{t+T}}\big)\big)$, and $\Delta_{BLO} = \sigma^e_{t+T}\frac{1}{\sqrt{2\pi}}\big(1-\mathrm{e}^{-\frac{1}{2}\left(\frac{\alpha\sigma_e}{\sigma^e_{t+T}}\right)^2}\big)/\big(\frac{1}{4i}+\Phi\big(\frac{\alpha\sigma_e}{\sigma^e_{t+T}}\big)-\frac{1}{2}\big)$. As functions of σ^e_{t+T} we can check that Δ_{BMO} is increasing, and Δ_{BLO} is decreasing. Since σ^e_{t+T} is increasing in T, the same results hold for Δ_{BMO} and Δ_{BLO} as functions of T. □

PROOF OF PROPOSITION 11: We compute the probability of BMO_t conditional on observing v_t, and all public information up to that point. $\mathsf{P}(BMO_t|v_t) = \frac{1+4iI_{v_t>v^e_t+\alpha\sigma_e}}{4+4i}$, where I_x is the indicator function. If $\tau = \frac{1}{\lambda}$ is the average time between orders, $\mathsf{P}(BMO_{t+\tau}|BMO_t, v_t) = \frac{1+4iI_{v_{t+\tau}>v^e_{t+\tau}+\alpha\sigma_e}}{4+4i}$. Then, $\mathsf{P}(BMO_{t+\tau}, BMO_t) = \int \frac{1+4iI_{\frac{v_{t+\tau}-v_t}{\sigma_e}+\frac{v_t-v^e_t}{\sigma_e}>\frac{\Delta}{\sigma_e}+\alpha}}{4+4i}\ \frac{1+4iI_{\frac{v_t-v^e_t}{\sigma_e}>\alpha}}{4+4i}\,\mathrm{d}v_t = \int \frac{1+4iI_{x_2\frac{\sigma}{\sigma_e}+x_1>\frac{\Delta}{\sigma_e}+\alpha}}{4+4i}\ \frac{1+4iI_{x_1>\alpha}}{4+4i}\phi(x_1)\phi(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2$, where $\phi(\cdot)$ is the standard normal density. Then, $\mathsf{P}(BMO_{t+\tau}, BMO_t) = \int \frac{1+4i\Phi\left(\frac{x_1\sigma_e-\Delta-\alpha\sigma_e}{\sigma}\right)}{4+4i}\ \frac{1+4iI_{x_1>\alpha}}{4+4i}\phi(x_1)\,\mathrm{d}x_1$. Since $\mathsf{P}(BMO) = \frac{1}{4}$, it follows that $\frac{\mathsf{P}(BMO_{t+\tau}|BMO_t)-1/4}{1/4} = 16\,\mathsf{P}(BMO_{t+\tau}, BMO_t) - 1$. This proves the desired formula.

□

PROOF OF PROPOSITION 13: We denote by $w_t = v_t - v^e_t$ the pricing advantage, and by $\Delta' = \Delta(1+u(1-k))$. The unconditional expected utility is obtained by integrating out $w \sim N(0,\sigma_e)$ in the conditional expected utility from Proposition 15. Then, $U^{\mathcal{I}} = \frac{1}{4}\big(\frac{2\sigma_e}{\beta} + \frac{2\sigma_e}{\beta}u(1-k)\big) - \Delta' = \frac{\Delta'}{2i}$. The uninformed patient seller submits an SLO and gets expected utility $U^{\mathcal{U}} = -u\Delta + (\bar{a}-\Delta) = \frac{\bar{s}}{2} - \Delta(1+u)$. This is the same expected utility as for the uninformed patient buyer, who submits a BLO. It is straightforward to show that $U^{\mathcal{I}}$ and $U^{\mathcal{U}}$, as well as $U^{\mathcal{I}} - U^{\mathcal{U}}$, are decreasing in i. □

PROOF OF PROPOSITION 15: We denote by $\Delta' = \Delta(1+u(1-k))$. According to Proposition 2, conditional on v_t, the expected utility of the informed trader is: $w - \frac{\bar{s}}{2}$ if $w > \alpha\sigma_e$ (BMO); $w(1-k) + \frac{\bar{s}}{2} - \Delta'$ if $w \in [0, \alpha\sigma_e)$ (BLO); $-w(1-k) + \frac{\bar{s}}{2} - \Delta'$ if $w \in [-\alpha\sigma_e, 0)$ (SLO); and $-w - \frac{\bar{s}}{2}$ if $w < -\alpha\sigma_e$ (SMO). It is straightforward to check that the minimum value is attained at $w = 0$. □

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Table 1: Numerical results for the ask side of the limit order book

This table reports: the maximum number of sell limit orders (*SLO*) in the book; the highest level of a *SLO* in the book; the average bid-ask spread; the minimum bid-ask spread; and the standard deviation of the bid-ask spread. The numerical solution depends on three parameters: the price impact parameter, $\Delta = 10^{-2}$; the granularity parameter, $\varepsilon = 10^{-6}, 10^{-8}, 10^{-10}$; and the information ratio $i = 0.01, 0.1, 0.5, 1, 5, 10, 100$.

		Information Ratio						
		0.01	0.1	0.5	1	5	10	100
Maximum number of *SLO*	$\varepsilon = 10^{-6}$	154	259	164	124	62	43	19
	$\varepsilon = 10^{-8}$	2080	1263	706	555	263	198	99
	$\varepsilon = 10^{-10}$	6269	4386	2952	2285	1106	813	430
Highest *SLO*	All ε	0.6644	0.6233	0.5513	0.5287	0.6424	0.6421	0.6419
Average spread/Δ	All ε	1.9399	1.8975	1.7819	1.7109	1.5859	1.5602	1.5338
Minimum spread/Δ	All ε	1.2431	1.1431	0.8571	0.6768	0.3922	0.3441	0.3172
Spread standard deviation/Δ	$\varepsilon = 10^{-6}$	0.8500	0.4761	0.3441	0.3765	0.4204	0.4362	0.6513
	$\varepsilon = 10^{-8}$	0.1294	0.0546	0.0359	0.0380	0.0418	0.0434	0.0654
	$\varepsilon = 10^{-10}$	0.0174	0.0067	0.0040	0.0039	0.0041	0.0043	0.0065