

Subjective Risk, Confidence, and Ambiguity

Christian P. Traeger

Department of Agricultural & Resource Economics, UC Berkeley

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Abstract: The paper extends a dynamic version of the classical von Neumann-Morgenstern setting to incorporate a degree of confidence in or subjectivity of probabilistic beliefs. It provides a simple axiomatic characterization of a new preference representation that addresses ambiguity from a simple perspective, employing only basic tools from risk analysis. Conceptually, the paper renders the concept of smooth ambiguity aversion more precise and extends it to a more general notion of aversion to the subjectivity of belief. The representation maintains the normatively desirable axioms of the standard setting including the von Neumann-Morgenstern axioms and time consistency.

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Correspondence:

Department of Agricultural & Resource Economics

207 Giannini Hall #3310

University of California

Berkeley, CA 94720-3310

E-mail: traeger@berkeley.edu

1 Introduction

The paper extends the standard recursive utility model by introducing a degree of confidence into probabilistic beliefs. The idea of attributing a degree of confidence to lotteries goes back to Ellsberg's (1961) suggestion for solving the paradox today carrying his name. In distinction to most of the ambiguity literature that took up the challenge of the Ellsberg paradox, the present paper keeps the concept of probabilities rather than extending it to capacities or sets of priors. Moreover, instead of abandoning independence, I show that labeling lotteries by their degree of confidence makes it possible to capture Ellsberg type and more general behavior in a setting building on the classical von Neumann & Morgenstern (1944) axioms. A different perspective on the confidence index is as a label for the degree of subjectivity of a lottery.

For the special case of two degrees of subjectivity I obtain a generalized version of the smooth ambiguity model by Klibanoff, Marinacci & Mukerji (2009). It is more general in two respects. First, the current model does not assume a restrictive two stage hierarchical structure of subjective lotteries over objective lotteries, but permits any composition of objective and subjective lotteries in an arbitrary amount of layers. More importantly, the current model relaxes the assumption that objective lotteries are evaluated intertemporally risk neutral, meaning that risk aversion to objective risk is only driven by aversion to intertemporal consumption fluctuations while risk aversion to subjective lotteries incorporates as well intrinsic risk aversion. My generalized framework incorporates both, intrinsic risk aversion to objective as well as to subjective risk. Relating the two gives a better understanding and a more precise definition of the measure of smooth ambiguity aversion promoted in Klibanoff et al. (2009). The two degree of subjectivity version of the model facilitates a three-fold disentanglement of dimensions of preference. One way to span these dimensions is in terms of intertemporal substitutibility, aversion to objective risk, and ambiguity aversion. Alternative coordinates for these dimensions are offered.

I extend the concept of smooth ambiguity aversion to situations with an arbitrary number of subjectivity labels. Here, a generalized form of ambiguity aversion translates into an aversion to the degree of subjectivity of (or the lack of confidence into) a probabilistic belief. A vantage of the current formulation as opposed to other representations is that the present work detaches uncertainty attitude from the lottery

level. Because other papers do not introduce the notion of subjectivity or confidence explicitly, they have to make evaluation depend on the layer in a compound lottery in which it takes place. In my presentation a lottery will be evaluated independently of the level of a decision tree in which it shows up, as long as the assigned degree of subjectivity coincides. Three more aspects of the paper distinguish it from most of its relatives in the decision theoretic literature on ambiguity. All of these aspects aim at a broad reception and applicability of the model. First, the paper develops an as simple as possible representation whose application only uses tools from standard risk theory. Second, the paper presents the axioms in a framework as close as possible to those by von Neumann & Morgenstern (1944), the arguably best known axiomatic framework on decision making under uncertainty among economists. Third, in addition to describing observed behavior under uncertainty, the paper aims at a representation that also serves as a decision support model. For this purpose, I base the representation on normatively attractive axioms including time consistency and the von Neumann-Morgenstern axioms. The paper delivers more than an extended treatment of explaining Ellsberg (1961) type behavior. The model incorporates a dimension into decision processes that has been identified as missing also in the policy arena. For example, the latest report of the International Panel on Climate Change takes a first step to distinguishing between confidence and likelihood (IPCC 2001, Box TS.1, p 22). While both are connected in the end to probabilistic beliefs, the report clearly expresses the need to distinguish between probabilities that are well known, or widely believed in, as opposed to those probabilities that are only based on very recent and scattered explorations or little facts. However, currently these distinctions end in the science part of the report and are not integrated into the economic evaluation. The paper outlines a possible framework for doing so.

The closest relative to my model is the mentioned paper by Klibanoff et al. (2009) together with its predecessors and variants including Segal (1990), Klibanoff, Marinacci & Mukerji (2005), Seo (2009), and Ergin & Gul (2009). I already pointed out the major differences to Klibanoff et al.'s (2009) paper and will discuss them in detail in section 5.1. These differences apply to all of the above papers. Following this introduction, section 2 introduces the technical setting of the paper. Section 3 summarizes the axioms underlying the representation. Section 4 states the representation. In section 5, I discuss the representation, relate it to the literature, and use it to render more precise and extend the notion of smooth ambiguity aversion. Section

6 gives a brief sketch how the model can be used in the context of climate change evaluation. Section 7 concludes. All proofs are gathered in the appendix.

2 The Setting

Time is discrete with a planning horizon $T \in \mathbb{N}$. In the usual abuse of notation T will at the same time denote the set $\{0, \dots, T\}$. Current outcomes in period $t \in T$ are described as elements x of a connected compact metric space X^* . These elements represent consumption levels or more general descriptions of welfare relevant characteristics. To avoid repetition, I introduce several definitions using a generic compact metric space X instead of X^* . The Borel σ -algebra on X is denoted $\mathcal{B}(X)$. Let S be a finite index set. The decision maker employs the index $s \in S$ to distinguish between lotteries (denoting general uncertain situations) that differ in terms of subjectivity of or confidence into the probabilistic belief. For every $s \in S$, I denote by $\Delta_s(X)$ a space of Borel probability measures on X that describe a lottery with degree of subjectivity s . Formally, these different lottery spaces are a family $\{(\Delta(X), s)\}_{s \in S}$. Each space $\Delta_s(X)$ is equipped with the Prohorov metric giving rise to the topology of weak convergence. For notational convenience, I introduce an element $s^0 \notin S$ and define $\bar{S} = S \cup s^0$ and, under slight abuse of notation, $\Delta_{s^0}(X) = X$. I introduce higher order lotteries inductively over the parameter $n \in N = \{0, 1, \dots, N\}$ defining the maximal depth of the decision tree.¹ Let $Z^0(X) = Y^0(X) = X$. In the first induction step I define for $n > 0$ the lottery spaces $Y_s^n(X) = \Delta_s(Z^{n-1}(X))$ for all $s \in \bar{S}$. It describes a decision tree of maximal depth n with a root² lottery of subjectivity s . In the second induction step, I define the general choice space $Z^n(X) = \cup_{s \in \bar{S}} Y_s^n(X)$, which collects decision trees with different degrees of subjectivity in the root. Note that inclusion of s^0 when forming the (disjoint) union allows the decision tree to have branches of differing length. The spaces $Z^n(X)$ are equipped with the (disjoint) union topology and, thus, compact. In a static setting the decision maker's choice objects

¹I refer by the name decision tree also to an “uncertainty tree”, which simply represents uncertainty. Here, the actual choice is that for a particular decision or “uncertainty” tree. Decision nodes could be introduced at any point in the “uncertainty” trees the same way as done in Kreps & Porteus (1978). However, no additional insights would be gained from doing so and the more complicated notation would rather be obstructive.

²The root of a decision tree is its first element. By root lottery I therefore denote the “outermost” lottery or the lottery corresponding to the root of the decision tree that describes the composed lottery.

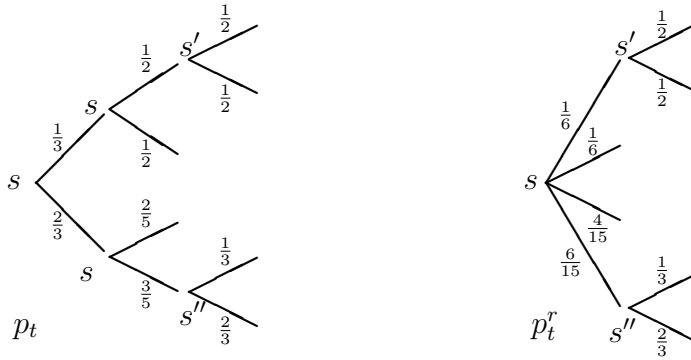


Figure 1: Example of two decision trees, $p_t \in Z^3(X^* \times P_{t+1})$ and $p_t^r \in Z^2(X^* \times P_{t+1})$, depicting uncertainty resolving in period t . Each uncertainty node is labeled with the degree of subjectivity of the corresponding lottery. The leaves of the trees are omitted and would consist of differing elements $(x_t, p_{t+1}) \in X^* \times P_{t+1}$. Lottery p_t^r differs from lottery p_t only and that the root lottery is collapsed with the subsequent layer of uncertainty sharing the same degree of subjectivity. A decision maker satisfying axiom A1 is indifferent between the two depicted decision trees.

would be described as the elements $z \in Z^N(X^*)$. These elements represent arbitrary concatenations of lotteries with differing degrees of subjectivity with a maximal concatenation length (decision tree depth) of N . An example for $N = 3$ with simple probabilities is depicted in Figure 1.

I construct the general choice space in the intertemporal setting recursively. In the last period, choices are $p_T \in P_T = Z^N(X^*)$. Preceding choice spaces are defined by $P_{t-1} = Z^N(X^* \times P_t)$ for all $t \in \{1, \dots, T\}$. Thus, at the beginning of every period uncertainty is described as a composition of lotteries with differing degrees of subjectivity over current outcomes and over the uncertainty that describes the decision maker's future starting in the next period. I call the choice object $p_t \in P_t$ in period t a generalized temporal lottery. They extend Kreps & Porteus's (1978) concept of a temporal lottery. I define the rank n of a lottery $p_t \in P_t$ by the function $\hat{n} : \cup_{t \in T} P_t \rightarrow N$ with $\hat{n}(p_t) = n$ if $p_t \in Y_s^n(X^* \times P_{t+1})$ for some $s \in S$, $t \in T$, and $n \geq 1$, and $\hat{n}(p_t) = 0$ otherwise. The rank captures the level of compoundedness or concatenation of a lottery, which corresponds to the depth of its representing decision tree (within a given period). I define the function $\hat{s} : \cup_{t \in T} P_t \rightarrow \bar{S}$ by $\hat{s}(p_t) = s$ if $p_t \in Y_s^n(X^* \times P_{t+1})$ for some $s \in S$, $t \in T$, and $n \geq 1$, and by $\hat{s}(p_t) = s^0$ otherwise. It maps every uncertain choice object into the degree of subjectivity of its root lottery and assigns s^0 to a degenerate root lottery. For a degenerate lottery

$p_t = (x_t, p_{t+1}) \in Z^0(X^* \times P_{t+1})$ I introduce the notation

$$p_t(B) = (x_t, p_{t+1})(B) = \delta_{(x_t, p_{t+1})}(B) = \begin{cases} 1 & \text{if } (x_t, p_{t+1}) \in B \\ 0 & \text{if } (x_t, p_{t+1}) \notin B \end{cases}$$

for all $B \in \mathcal{B}(X^* \times P_{t+1})$.³ The space $P_t^s = \{p_t \in P_t \mid \hat{s}(p_t) \in \{s, s^0\}\}$ denotes the space of all compound period t lotteries in which the root lottery has a degree of subjectivity s (as in Figure 1) and includes the certain outcomes. I define the following composition of two lotteries with coinciding degree of subjectivity. For any $s \in S$, $p_t, p'_t \in P_t^s$ and $\alpha \in [0, 1]$ I define a probability α mixture by the operation $\oplus_s^\alpha : P_t^s \times P_t^s \rightarrow P_t^s$ that maps $(p_t, p'_t) \mapsto p_t \oplus_s^\alpha p'_t \in Y_s^{\max\{\hat{n}(p_t), \hat{n}(p'_t), 1\}}$ defined by

$$p_t \oplus_s^\alpha p'_t(B) = \alpha p_t(B \cap Z^{\max\{\hat{n}(p_t)-1, 0\}}(X^* \times P_{t+1})) + (1 - \alpha) p'_t(B \cap Z^{\max\{\hat{n}(p'_t)-1, 0\}}(X^* \times P_{t+1}))$$

for all $B \in \mathcal{B}(Z^{\max\{\hat{n}(p_t), \hat{n}(p'_t), 1\}-1}(X^* \times P_{t+1}))$. Note that the lottery resulting from this mixture lives in the same space as the lottery of p_t and p'_t with the higher rank.

Whenever the root lottery $p_t \in P_t$ shares the same degree of subjectivity with the subsequent layer of uncertainty (as on the left hand side in Figure 1) I define a reduced lottery that collapses the same degree of subjectivity uncertainty into a single layer. Hereto I define for any lottery $p_t \in \Delta_s(Y_s^n(X^* \times P_{t+1}))$ of rank $n + 1$ the reduced lottery $p_t^r \in Y_s^n(X^* \times P_{t+1})$ of rank n by

$$p_t^r(B) = \int_{Y_s^n(X^* \times P_{t+1})} \tilde{p}_t(B) dp_t(\tilde{p}_t) \tag{1}$$

for all $B \in \mathcal{B}(Z^{n-1}(X^* \times P_{t+1}))$. An example is given in Figure 1 where the lottery p_t^r collapses the root lottery and the subsequent layer of uncertainty sharing the same degree of subjectivity in lottery p_t into a single layer of uncertainty.

The space $\mathbf{X} = X^{*T+1} \subset P_0$ characterizes the set of all certain consumption paths faced in the present. A consumption paths $\mathbf{x} \in \mathbf{X}$ is written $\mathbf{x} = (x_0, \dots, x_T)$. Given $\mathbf{x} \in \mathbf{X}$, I define $(\mathbf{x}_{-i}, x) = (x_0, \dots, x_{i-1}, x, x_{i+1}, \dots, x_T) \in \mathbf{X}$ as the consumption path that coincides with \mathbf{x} in all but the i^{th} period, in which it yields outcome x . I denote

³I have not assigned a degree of subjectivity to $(x_t, p_{t+1})(\cdot) = \delta_{(x_t, p_{t+1})}(\cdot)$. It is by itself not a lottery that is part of the choice space P_t , but only a notational object used in defining choice objects – it only describes a possible first entry of the choice objects of type $(\cdot, s) \in \{(\Delta(X), s)\}_{s \in S}$.

the set of certain consumption paths faced in period t by $X^t = X^{T-t+1} \subset P_t$. In every period $t \in T$ the decision maker's preferences \succeq_t are a binary relation on P_t .

Further Remarks: The operator \oplus_s^α mixes same degree of subjectivity lotteries within a given level of compoundedness (which is given by the lottery with the higher rank). Instead, I can as well define a composition where the mixture of two such lotteries elevates compoundedness by one level. Herto, for any $n \in N$, $s \in S$, and $p_t \in Z^{i < n}(X^* \times P_{t+1})$, I define the lottery $\delta_{p_t}^{n,s} \in Y_s^n$ by

$$\delta_{p_t}^{n,s}(B) = \begin{cases} 1 & \text{if } p_t \in B \\ 0 & \text{if } p_t \notin B \end{cases}$$

for all $B \subset Z^{n-1}(X^* \times P_{t+1})$. For any $s \in S$, $\alpha \in [0, 1]$, and $p_t, p'_t \in P_t^s$ with $n^* = \max\{\hat{n}(p_t), \hat{n}(p'_t)\} + 1 \leq N$, I define an elevating probability α mixture by the operation $\odot_s^\alpha : P_t \times P_t \rightarrow P_t^s$ that maps $(p_t, p'_t) \mapsto p_t \odot_s^\alpha p'_t \in Y_s^{n^*}$ defined by

$$p_t \odot_s^\alpha p'_t(B) = \alpha \delta_{p_t}^{n^*,s}(B) + (1 - \alpha) \delta_{p'_t}^{n^*,s}(B) \quad (2)$$

for all $B \in \mathcal{B}(Z^{n^*-1}(X^* \times P_{t+1}))$.

Moreover, if both mixtures share the same degree of subjectivity, it lies at hand to assume that a decision maker does not care whether probabilities are manipulated at the same lottery level or whether the manipulation takes place at an elevated level. Such an assumption corresponds to the statement

$$p_t \odot_s^\alpha p'_t \sim_t p_t \oplus_s^\alpha p'_t \quad \text{for all } p_t, p'_t \in P_t^s \text{ with } \hat{n}(p_t), \hat{n}(p'_t) < N. \quad (3)$$

Indifference in equation (3) is a special case of an axiom requiring indifference to the reduction of same degree of subjectivity lotteries introduced in the next section.

3 Axioms

The first axiom makes the decision maker indifferent to the reduction of same degree of subjectivity lotteries. Using the notation of a reduced lottery introduced in equation (1) such an assumption writes as

A1 (indifference to reduction of lotteries with same degree of subjectivity)

For all $t \in T$, $s \in S$, $n < N$, $p_t \in \Delta_s(Y_s^n(X^* \times P_{t+1}))$: $p_t \sim_t p_t^r$.

A decision maker who satisfies axiom A1 is indifferent between the two lotteries depicted in Figure 1. The literature discussed in the introduction lives of the fact that axiom A1 is not satisfied. These papers need to distinguish evaluation of lotteries on different levels because the level at which the mixture takes place is the only way they can distinguish between e.g. objective and subjective lotteries. I favor tying difference in uncertainty attitude directly to subjectivity and confidence as opposed to the level or order in which uncertainty strikes the agent. That step makes it possible to impose axiom A1 (and satisfy equation 3) without collapsing the representation to the standard von Neumann-Morgenstern representation one losing the additional dimension of decision making.

The following three axioms mostly replicate the standard von Neumann & Morgenstern (1944) axioms for the compact metric space setting (e.g. Grandmont 1972).

A2 (weak order) For all $t \in T$ preferences \succeq_t are transitive and complete, i.e.:

- transitive: For all $p_t, p'_t, p''_t \in P_t$: $p_t \succeq p'_t$ and $p'_t \succeq p''_t \Rightarrow p_t \succeq p''_t$
- complete: For all $p_t, p'_t \in P_t$: $p_t \succeq p'_t$ or $p'_t \succeq p_t$.

A3 (independence) For all $s \in S$, $\alpha \in [0, 1]$, and $t \in T$:

$$\text{For all } p_t, p'_t, p''_t \in P_t^s: \quad p_t \succeq_t p'_t \quad \Rightarrow \quad p_t \oplus_s^\alpha p''_t \succeq_t p'_t \oplus_s^\alpha p''_t \quad .$$

A4 (continuity) For all $t \in T$, for all $p_t \in P_t$:

$$\{p'_t \in P_t : p'_t \succeq p_t\} \text{ and } \{p'_t \in P_t : p_t \succeq p'_t\} \text{ are closed in } P_t \quad .$$

The independence axiom is the only axiom that is slightly modified and I might call it “independence with respect to same degree of subjectivity mixing”. Requiring the same degree of subjectivity for the lotteries $p_t, p'_t, p''_t \in P_t^s$ and the \oplus_s^α operator is a technical assumption to permit a meaningful mixing at the same lottery level. The fact that mixing is required to take place at the same lottery level will be further discussed in a remark at the end of this section. There, I also discuss an alternative independence axiom that mixes lotteries differing degrees of subjectivity at a higher level.

In order to match the predominant time-additive framework for certain intertemporal choice, I add additive separability on certain consumption paths. I employ the

axiomatization of Wakker (1988).⁴

A5 (certainty separability)

i) For all $x, x' \in X$, $x, x' \in X^*$ and $t \in T$:

$$(x_{-t}, x) \succeq_1 (x'_{-t}, x) \Leftrightarrow (x_{-t}, x') \succeq_1 (x'_{-t}, x')$$

ii) If $T = 1$ additionally: For all $x_t, x'_t, x''_t \in X^*$, $t \in \{0, 1\}$

$$(x_0, x_1) \sim_1 (x'_0, x'_1) \wedge (x'_0, x'_1) \sim_1 (x''_0, x''_1) \Rightarrow (x_0, x_1) \sim_1 (x''_0, x''_1) .$$

Wakker (1988) calls part *i*) of the axiom coordinate independence. It requires that the choice between two consumption paths does not depend on period t consumption, whenever the latter coincides for both paths. Part *ii*) is known as the Thomsen condition. It is required only if the model is limited to $T = 2$ periods.⁵ Preferences in different periods are related by the following consistency assumption adapted from Kreps & Porteus (1978).

A6 (time consistency) For all $t \in \{0, \dots, T - 1\}$:

$$(x_t, p_{t+1}) \succeq_t (x_t, p'_{t+1}) \Leftrightarrow p_{t+1} \succeq_{t+1} p'_{t+1} \quad \forall x_t \in X^*, p_{t+1}, p'_{t+1} \in P_{t+1} .$$

The axiom is a requirement for choosing between two consumption plans in period t , both of which are degenerate and yield a coinciding outcome in the respective period. For these choice situations, axiom A6 demands that in period t , the decision maker prefers the plan that gives rise to the lottery that is preferred in period $t + 1$.

Further Remarks: I pointed out that the operator \oplus_s^α and, thus, the independence axiom A3, mixes same degree of subjectivity lotteries within at a given lottery level. In the remark of the preceding section I defined an alternative mixture composition \odot_s^α where the mixture of two lotteries elevates the level of compoundedness by one. An alternative to axiom A3 is the following axiom

⁴Other axiomatizations of additive separability include Koopmans (1960), Krantz, Luce, Suppes & Tversky (1971), Jaffray (1974a), Jaffray (1974b), Radner (1982), and Fishburn (1992).

⁵In the case of two periods parts *i*) and *ii*) can also be replaced by the single requirement of triple cancellation (see Wakker 1988, 427).

A3' (elevating independence) For all $s \in S$, $\alpha \in [0, 1]$, $t \in T$, and $p_t, p'_t, p''_t \in P_t$ with $\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t) < N$: $p_t \succeq_t p'_t \Rightarrow p_t \odot_s^\alpha p''_t \succeq_t p'_t \odot_s^\alpha p''_t$

It differs from axiom A3 in two respects. First, it no longer requires the lotteries p_t, p'_t , and p''_t to share a common degree of subjectivity. Second, it creates the lottery mixture on a higher level than either of the individual lotteries. The first change makes it stronger, however, the second change disconnects the levels of the primitive lotteries and the mixed lottery. Under the assumption of indifference to the reduction of same degree of subjectivity lotteries (axiom A1) it is easily verified that indifference between the \oplus_s^α and the \odot_s^α operations holds in the sense of equation (3).⁶ Therefore, under assumption A1, axiom A3' implies axiom A3,⁷ and axiom A3 implies axiom A3' restricted to same degree of subjectivity lotteries.

It might be less obvious that axiom A3 together with axiom A2 is already enough to imply axiom A3' for same degree of subjectivity lotteries. The reason is that axiom A3 already contains an assumption of indifference to the reduction of degenerate lotteries. In axiom A3 choose lotteries $p, p', p'' \in P_t^s$ satisfying $\hat{n}(p) = n < N$ and $\hat{n}(p') = \hat{n}(p'') = n + 1$. Then, a $\alpha = 1$ mixture of the lotteries delivers

$$p_t \succeq_t p'_t \Rightarrow p_t \oplus_s^\alpha p''_t \succeq_t p'_t \oplus_s^\alpha p''_t \Rightarrow \delta_{p_t} \succeq_t p'_t$$

where $\delta_{p_t} \in \Delta_s(\cdot)$. By completeness of preferences (axiom A2) and repeated application I obtain

$$\delta_{p_t}^{n^*,s} \sim_t p_t \tag{4}$$

for $s \in S$ and $n^* > n$. Thus, for arbitrary lotteries $p_t, p'_t, p''_t \in P_t^s$ and $n^* = \max\{\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t)\} + 1 \leq N$, I find

$$\begin{aligned} p_t \succeq_t p'_t &\Rightarrow \delta_{p_t}^{n^*,s} \succeq_t \delta_{p'_t}^{n^*,s} \Rightarrow \delta_{p_t}^{n^*} \oplus_s^\alpha \delta_{p''_t}^{n^*,s} \succeq_t \delta_{p'_t}^{n^*,s} \oplus_s^\alpha \delta_{p''_t}^{n^*,s} \\ &\Rightarrow p_t \odot_s^\alpha p''_t \succeq_t p'_t \odot_s^\alpha p''_t \end{aligned}$$

using first equation (4) and then axiom A3. Note, however, that axioms A3' and A3 together do not imply equation (3). For indifference in equation (3) I need to impose axiom A1.

⁶Use the definition of \odot_s^α along with equation (2) and equation (1).

⁷For lotteries satisfying $\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t) < N$. Otherwise the elevating independence axiom creates a mixture outside of the preference domain.

4 The Representation

The representation recursively constructs a welfare function $\hat{u}_t : X^* \times P_{t+1} \rightarrow U_t \subset \mathbb{R}$ evaluating degenerate outcomes in every period. Within a period, the representation recursively evaluates the different layers of uncertainty (branches of the decision tree in Figure 1). All uncertainty nodes are labeled by their degree of subjectivity. The risk aversion when evaluating a lottery at a particular node is tied to its degree of subjectivity. This risk aversion can be captured by a set of continuous functions $\hat{f}_t = (f_t^s)_{s \in S}$, $f_t^s : \mathbb{R} \rightarrow \mathbb{R}$. I call these functions *uncertainty aggregation weights*. Given a continuous bounded function $\hat{u}_t : X^* \times P_{t+1} \rightarrow U \subset \mathbb{R}$ evaluating degenerate outcomes and a set of uncertainty aggregation weights $\hat{f}_t = (f_t^s)_{s \in S}$, I define the *generalized uncertainty aggregator* $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} : P_t \rightarrow \mathbb{R}$ recursively by setting $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t}(x_t, p_{t+1}) = \hat{u}_t(x_t, p_{t+1})$ for degenerate lotteries $p_t = (x_t, p_{t+1}) \in P_t$ and then inductively increasing its domain to lotteries of rank $\hat{n}(p_t) = 1, 2, \dots, N$ by defining⁸

$$\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t = \left(f_t^{\hat{s}(p_t)} \right)^{-1} \int_{Z^{\hat{n}(p_t)-1}(X^* \times P_{t+1})} f_t^{\hat{s}(p_t)} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p'_t dp_t(p'_t). \quad (5)$$

Graphically, the expression $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p'_t$ captures the evaluation of all the subtrees that lead into the node making up lottery p_t . Each of these possible outcomes is weighted with the uncertainty weighting function $f_t^{\hat{s}(p_t)}$ corresponding to the degree of subjectivity of the lottery p_t . The integral sums over these weighted subtree evaluation and, finally, the function $\left(f_t^{\hat{s}(p_t)} \right)^{-1}$ is applied to renormalize the expression making the generalized uncertainty aggregator a generalized mean.

Theorem 1: The sequence of preference relations $(\succeq_t)_{t \in T}$ satisfies axioms A1-A6 if, and only if, for all $t \in T$ there exist a set of strictly increasing and continuous functions $\hat{f}_t = (f_t^s)_{s \in S}$, $f_t^s : \mathbb{R} \rightarrow \mathbb{R}$, and a continuous and bounded function $u_t : X^* \rightarrow U \subset \mathbb{R}$ such that by defining recursively the functions $\hat{u}_T = u_T$ and $\hat{u}_{t-1} : X^* \times P_t \rightarrow \mathbb{R}$ by

$$\hat{u}_{t-1}(x_{t-1}, p_t) = u_{t-1}(x_{t-1}) + \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t \quad (6)$$

it holds for all $t \in T$ and all $p_t, p'_t \in P_t$

$$p_t \succeq_t p'_t \Leftrightarrow \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t \geq \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p'_t. \quad (7)$$

⁸The sign \circ emphasizes the functional composition of $f_t^{\hat{s}(p_t)} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p'_t = f_t^{\hat{s}(p_t)} [\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p'_t]$.

Preferences $(\succeq_t)_{t \in T}$ over the space of generalized temporal lotteries can be represented by the sequence $(\hat{f}_t, u_t)_{t \in T}$. The functions \hat{f}_t inform the generalized uncertainty evaluation where risk aversion depends on the degree of subjectivity of a lottery (equation 5). The functions u_t represent per period utility and inform the recursive construction of the intertemporal utility or value function \hat{u}_t (equation 6). Note that the representation in Theorem 1 is linear in every time step. In a setting where lotteries would not be distinguished by their degree of subjectivity, the setting of this paper would relate closely to Kreps & Porteus (1978). In their representation, Kreps & Porteus (1978) use a linear uncertainty aggregation at the expense of a non-linear time aggregation. I show in Traeger (2007) how to shift the non-linearity between the time and the risk dimension in such a setting. In the current setting, however, lotteries vary in their degree of subjectivity. Here, giving up linearity in the time step in equation (6) would only facilitate the linearization of f_t^s for one $s \in S$ and would not permit a linear aggregation over uncertainty in general. Thus, I consider the employed linearization over time to be the more reasonable representation. Finally, note that affine transformation of the functions \hat{f}_t^s are allowed in the representation. Affine transformation of the functions u_t are restricted to a common multiplicative constant in different periods and have to be accompanied with a coinciding transformation of the functions $(\hat{f}_t^s)^{-1}$.⁹

Further Remarks: The representation building on axioms A1 to A6 satisfies as well elevating independence mixing lotteries of differing degrees of subjectivity A3', which might be a desirable property for a normative application of the representation. Axiom A1 is responsible for connecting the uncertainty weights on the different layers permitting a unique set \hat{f}_t evaluating lotteries independently of their level in the decision tree.

5 Discussion of the Representation

The discussion of the representation in Theorem 1 proceeds in two steps. First, I analyze a restricted version of the model limiting the space S to only two degrees of

⁹Which implies composing the function \hat{f}_t^s with the inverse of the affine transformation from the right to obtain the new representing sequence \hat{f}_t^s .

subjectivity. This restricted version of the model is a straight forward generalization of Klibanoff et al.’s (2009) smooth ambiguity setting. I show that Klibanoff et al.’s (2009) definition of smooth ambiguity aversion is somewhat “ambiguous” and render the definition more precise. Moreover, I disentangle intertemporal substitutability from risk aversion and ambiguity aversion. Then, I proceed to discuss the general setting with an arbitrary number of degrees of subjectivity in the lottery space. In particular, I generalize the definition of smooth ambiguity aversion in this setting to characterize aversion against the degree of subjectivity of a lottery.

5.1 A binary classification of subjectivity or confidence

I start with interpreting a special case of the representation that is obtained by restricting the degree of subjectivity to $\#S = 2$. I associate the two elements $s \in S = \{subj, obj\}$ with subjective and objective beliefs. Two further restrictions transform it into the smooth ambiguity model of Klibanoff et al. (2009) – translated into the von Neumann-Morgenstern setting. First, the evaluation of objective lotteries in Klibanoff et al.’s (2009) setting is (intertemporally) risk neutral in the sense that $f_t^{obj} = id$ is linear in (or rather absent from) their representation. This latter point will be discussed in detail further below. Second, Klibanoff et al. (2009) restrict the level of compoundedness of the lotteries to $N = 2$ and impose a hierarchy of beliefs implying that decision makers can only face subjective lotteries over objective lotteries, but not vice versa. For example, a situation where a decision maker flips a coin to decide whether he takes a riskless action or enters a subjective lottery cannot be captured in such a setting.¹⁰ In contrast, the representation in Theorem 1 permits an arbitrary sequence of subjective and objective lotteries (within every period).

Maintaining all of these restrictions, the first interesting insight to be gained is that representation Theorem 1 only requires a minimal deviation from the standard von Neumann-Morgenstern setting and preserves even the independence axiom, only labeling lotteries by their degree of subjectivity. Thus, explicitly introducing the dimensions that Ellsberg (1961) already found missing in the Savage framework, i.e. a degree of confidence or subjectivity of belief, leads straight forwardly from von Neu-

¹⁰A similar lottery could be captured, though, under the assumption that the subjective lottery follows the objective lottery with one period of delay. As I will explain below, however, a period of delay will also introduce aversion to intertemporal substitution.

mann & Morgenstern (1944) to a model of smooth ambiguity aversion. The next insight concerns the the interpretation of Klibanoff et al.’s (2009) smooth ambiguity aversion. Hereto, I briefly relate the representation in Theorem 1 to the generalized isoelastic model of Epstein & Zin (1989) and Weil (1990). A priori, a decision maker’s propensity to smooth consumption over time is a different preference characteristic than his risk aversion. However, the intertemporally additive expected utility standard model implicitly assumes that these quite different dimensions of preference coincide. Epstein & Zin (1989) and Weil (1990) observed that in a one commodity version of Kreps & Porteus’s (1978) recursive utility model of temporal lotteries these two dimensions of preference can be disentangled. In Traeger (2007) I show, in a setting corresponding to an $\#S = 1$ version of the current model, that the function f_t measures the difference between Arrow Pratt risk aversion and aversion to intertemporal substitution.¹¹ I name f_t a measure of intertemporal risk aversion. It measures the part of risk aversion that is not simply a cause of a decision maker’s propensity to smooth over time, but due an intrinsic aversion to risk. The concept of intertemporal risk aversion is not limited to the one-commodity setting of the Epstein & Zin (1989) framework, but generalizes to arbitrary dimensions and to settings without a naturally given measure scale of the good under observation. The following axiomatic characterization is put forth in Traeger (2007). For two given consumption paths $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^t$, I define the ‘best of combination’ path $\mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}')$ by $(\mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}'))_\tau = \operatorname{argmax}_{x \in \{\mathbf{x}_\tau, \mathbf{x}'_\tau\}} u_\tau(x)$ and the ‘worst off combination’ path $\mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}')$ by $(\mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}'))_\tau = \operatorname{argmin}_{x \in \{\mathbf{x}_\tau, \mathbf{x}'_\tau\}} u_\tau(x)$ for all $\tau \in \{t, \dots, T\}$.¹² In every period the consumption path $\mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}')$ picks out the better outcome of \mathbf{x} and \mathbf{x}' , while $\mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}')$ collects the inferior outcomes. A decision maker is called (weakly)¹³ *intertemporal risk averse* in period t if and only if for all consumption paths $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^t$

$$\mathbf{x} \sim \mathbf{x}' \quad \Rightarrow \quad \mathbf{x} \succeq_t \frac{1}{2} \mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}') + \frac{1}{2} \mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}'), \quad (8)$$

where $\frac{1}{2} \mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}') + \frac{1}{2} \mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}')$ denotes a lottery with equal chance between the paths $\mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}')$ and $\mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}')$. The premise states that a decision maker is indif-

¹¹As there is only one type of risk, there also is only one function f_t in every period used for uncertainty aggregation.

¹²In Traeger (2007) I show how these paths can be defined purely in terms of preferences.

¹³Analogously, a strict intertemporal risk averse decision maker can be defined by assuming in addition that there exists some period t^* such that $u(\mathbf{x}_{t^*}) \neq u(\mathbf{x}'_{t^*})$ and requiring a strict preference \succ rather than the weak preference \succeq in equation (8).

ferent between the certain consumption paths \mathbf{x} and \mathbf{x}' . Then, an intertemporal risk averse decision maker prefers the consumption path \mathbf{x} (or equivalently \mathbf{x}') with certainty over a lottery that yields with equal probability either a path combining all the best outcomes or a path combining all the worst outcomes. The cited paper shows that the function f_t in the representation is concave if and only if equation (8) holds. In the certainty additive representation employed here, intertemporal risk aversion can as well be understood as risk aversion with respect to utility gains and losses.

The definition of intertemporal risk aversion extends straight forwardly to a setting with differing degrees of risk aversion to objective and subjective lotteries. I characterize intertemporal risk aversion to objective lotteries by requiring for all $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^t$

$$\mathbf{x} \sim \mathbf{x}' \quad \Rightarrow \quad \mathbf{x} \succeq_t \mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}') \oplus_{obj}^{\frac{1}{2}} \mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}') \quad (9)$$

implying concavity of f_t^{obj} , and similarly intertemporal risk aversion to subjective lotteries by requiring for all $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^t$

$$\mathbf{x} \sim \mathbf{x}' \quad \Rightarrow \quad \mathbf{x} \succeq_t \mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}') \oplus_{subj}^{\frac{1}{2}} \mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}') . \quad (10)$$

implying concavity of f_t^{subj} . Klibanoff et al. (2009) implicitly assume that $f^{obj} = \text{id}$ corresponding to indifference in equation (9). This assumption implies that uncertainty evaluation with respect to objective (or first order) lotteries is intertemporal risk neutral. Only when it comes to subjective lotteries, Klibanoff et al. (2009) introduce a non-trivial function f^{subj} and, thus, allow for intertemporal risk aversion. Now Klibanoff et al. (2009) define ambiguity aversion by the concavity of f_t^{subj} , in a setting assuming $f_t^{obj} = \text{id}$. This concept earned the name smooth ambiguity aversion in the decision theoretic literature. Releasing the restriction $f_t^{obj} = \text{id}$ sheds more light onto this definition. In principle, there are two sensible ways of extending Klibanoff et al.'s (2009) representation to incorporate the missing non-linearity f_t^{obj} . The representation I have chosen in Theorem 1 introduces the function f_t^{obj} in such a way that it measures intertemporal risk aversion with respect to objective lotteries without changing the interpretation that f_t^{subj} measures intertemporal risk aversion with respect to subjective lotteries. Given the hierarchical order of subjective over objective lotteries in Klibanoff et al.'s (2009) setting, I could as well introduce a function $f_t^{amb} \equiv f_t^{subj} \circ (f_t^{obj})^{-1}$ to eliminate f_t^{subj} from the representation. Observe the following transformation of the representing equation (7) where p_t and p'_t are different

subjective lotteries over the set of objective lotteries whose representatives are \tilde{p}_t

$$\begin{aligned}
 p_t \succeq_t p'_t &\Leftrightarrow \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t \geq \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p'_t \\
 &\Leftrightarrow (f_t^{subj})^{-1} \int_{Z^1(X^* \times P_{t+1})} dp_t(\tilde{p}_t) f_t^{subj} \circ (f_t^{obj})^{-1} \int_{X^* \times P_{t+1}} d\tilde{p}_t(x_t, p_{t+1}) f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1}) \\
 &\geq (f_t^{subj})^{-1} \int_{Z^1(X^* \times P_{t+1})} dp'_t(\tilde{p}_t) f_t^{subj} \circ (f_t^{obj})^{-1} \int_{X^* \times P_{t+1}} d\tilde{p}_t(x_t, p_{t+1}) f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1}) \\
 &\Leftrightarrow (f_t^{amb})^{-1} \int_{Z^1(X^* \times P_{t+1})} dp_t(\tilde{p}_t) f_t^{amb} \int_{X^* \times P_{t+1}} d\tilde{p}_t(x_t, p_{t+1}) f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1}) \\
 &\geq (f_t^{amb})^{-1} \int_{Z^1(X^* \times P_{t+1})} dp'_t(\tilde{p}_t) f_t^{amb} \int_{X^* \times P_{t+1}} d\tilde{p}_t(x_t, p_{t+1}) f_t^{obj} \circ \hat{u}_t(x_t, p_{t+1})
 \end{aligned}$$

This new function $f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ then measures the additional aversion to subjective risk as opposed to objective risk. For this interpretation, note that $f_t^{subj} \circ (f_t^{obj})^{-1}$ concave is a definition of f_t^{subj} being more concave than f_t^{obj} (Hardy, Littlewood & Polya 1964).¹⁴ Because in Klibanoff et al.'s (2009) setting it is $f_t^{obj} = \text{id}$, their definition of ambiguity aversion does not pin down whether smooth ambiguity aversion should be captured by intertemporal aversion to subjective risk captured in f_t^{subj} and characterized by the lottery choice (10) or whether it should be characterized by the functions f_t^{amb} measuring the additional risk aversion to subjective risk as opposed to objective risk. I suggest to call the latter a measure of smooth ambiguity aversion.

Definition 1: A decision maker exhibits (strict) smooth ambiguity aversion in period t if the function

$$f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$$

in the preference representation of Theorem 1 is (strictly) concave.

I follow Klibanoff et al. (2009) in defining the term by means of characteristics of the representation. However, (strict) concavity of the function f_t^{amb} is purely a characteristic of preferences and does not depend on a particular version of the representation (even though choices of \hat{u}_t and \hat{f}_t are generally not unique). An axiomatic characterizations of smooth ambiguity aversion in terms of preferences and choices

¹⁴Hereto observe that f_t^{amb} concave and $f_t^{subj} = f_t^{amb} \circ (f_t^{obj})$ implies that f_t^{subj} is a concave transformation of f_t^{obj} .

follows. Employing equations (9) and (10) the condition $f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ concave translates smooth ambiguity aversion in period t into the requirement that for all $\mathbf{x}, \mathbf{x}', \mathbf{x}'' \in \mathbf{X}^t$

$$\mathbf{x} \sim \mathbf{x}' \succeq_t \mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}') \oplus_{obj}^{\frac{1}{2}} \mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}') \quad \Rightarrow \quad \mathbf{x} \succeq_t \mathbf{x}^{\text{high}}(\mathbf{x}, \mathbf{x}') \oplus_{subj}^{\frac{1}{2}} \mathbf{x}^{\text{low}}(\mathbf{x}, \mathbf{x}') .$$

However, ambiguity aversion can be characterized simpler by recognizing that the intertemporal aspect of the risk comparison can as well be dropped.

Proposition 1: A decision maker exhibits (strict) smooth ambiguity aversion in the sense of Definition 1 if, and only if, for all $\mathbf{x}, \mathbf{x}' \in \mathbf{X}^t$

$$\mathbf{x} \oplus_{obj}^{\frac{1}{2}} \mathbf{x}' \succeq_t (\succ_t) \mathbf{x} \oplus_{subj}^{\frac{1}{2}} \mathbf{x}'$$

For the one-commodity setting¹⁵ the model gives rise to a three-fold disentanglement that can be expressed in terms of 6 diffing related concepts (sharing three degrees of freedom):

- the functions u_t characterize aversion to intertemporal substitution,
- the functions f_t^{subj} characterize intertemporal risk aversion to objective risk,
- the functions f_t^{obj} characterize intertemporal risk aversion to subjective risk,
- the functions $f_t^{amb} = f_t^{subj} \circ (f_t^{obj})^{-1}$ characterize smooth ambiguity aversion,
- the functions $g_t^{obj} \equiv f_t^{obj} \circ u_t^{-1}$ measure Arrow Pratt risk aversion with respect to objective lotteries, and
- the functions $g_t^{subj} \equiv f_t^{subj} \circ u_t^{-1}$ measure Arrow Pratt risk aversion with respect to subjective risk.

It follows immediately that in the one-commodity setting smooth ambiguity aversion can as well be expressed as the difference in Arrow Pratt risk aversion with respect to subjective risk and Arrow Pratt risk aversion with respect to objective risk:

$$f_t^{amb} = g_t^{subj} \circ (g_t^{obj})^{-1} .$$

¹⁵Only in the one-commodity setting the inverse of u and the Arrow Pratt measure of risk aversion as well as the measure of intertemporal substitution are unidimensional and well defined.

5.2 The general representation and aversion to the subjectivity of belief

A unique measure of ambiguity aversion is tied to the setting with $\#S = 2$. In general, a decision maker will not always be able to make a binary classification of subjective versus objective lotteries or of only two classes of confidence in beliefs. While objective probabilities are generally classified as those derived from symmetry reasonings or long-run, high frequency observations, subjective risk is basically any probabilistic belief not obtained in that way, leaving a wide field of beliefs for a single category. For example the odds based on a somewhat shorter time series or a slightly irregular coin, a horse race lottery, the odds of a 2°C global warming by 2050 due to climate change, or weather characteristics in Tomboctou on November 22nd 2010. In general, different decision makers are likely to classify different lotteries in different categories. A useful characterization of a decision maker's preferences for the general setting with $\#S > 2$ is as follows. Assume that the decision maker has a complete order over the elements in S in terms of subjectivity. Let $s \triangleright s'$ denote that a lottery labeled s is more subjective than a lottery labeled s' .

Definition 2: A decision maker is (strictly) averse to subjectivity of belief if

$$s \triangleright s' \Leftrightarrow f_t^s \circ (f_t^{s'})^{-1} \text{ (strictly) concave } \quad \forall s, s' \in S .$$

Alternatively, the situation $s \triangleright s'$ can be interpreted as a decision maker being less confident in lotteries of category s than in lotteries of categories s' . Then, aversion to subjectivity of belief is equivalent to aversion to a lack of confidence in beliefs. Definition 1 of smooth ambiguity aversion is the special case of aversion to the subjectivity of belief (or to the lack of confidence) in the case $\#S = 2$. The characterization in terms of preferences carries over straight forwardly to the generalization.

Proposition 2: A decision maker exhibits (strict) aversion to the subjectivity of belief in the sense of Definition 2 if, and only if, for all $\mathbf{x}, \mathbf{x}' \in X^t$ and $s, s' \in S$ with $s \triangleright s'$

$$\mathbf{x} \oplus_{s'}^{\frac{1}{2}} \mathbf{x}' \succeq_t (\succ_t) \mathbf{x} \oplus_s^{\frac{1}{2}} \mathbf{x}' .$$

With respect to the broader literature on ambiguity it is interesting to analyze how the description of the Ellsberg (1961) paradox would differ for a decision maker employing

the representation of Theorem 1 as opposed to the multiple prior approach or the Choquet expected utility approach. In the setting of the Ellsberg paradox a decision maker has to bet on the color of a ball that is drawn from an urn. The crucial feature of the various variants of the setting can be reflected by the following simplified choice situation. In one urn, the decision maker knows that half of the balls are red. In another urn, the decision maker only knows that it contains nothing but red and blue balls. In the first case, the urn draw gives him what I would consider an objective probability of $\frac{1}{2}$ that a red ball is randomly drawn. For the second urn, the principle of insufficient reason would render him a probability of $\frac{1}{2}$ as well. However, a good fraction of the individuals in comparable settings tend to prefer betting on the first urn where they know the number of red balls.¹⁶ The Choquet approach to explaining the paradoxical preference for the urn with the known amount of red balls abandons the concept of a probability and replaces it with a non-additivity set function. The latter captures the decision maker's ambiguity about the red balls in the second urn. Choquet integrating over the capacities induces aversion to ambiguity. The multiple prior approach instead attaches a range of different probability distributions to drawing a red ball from the second ball and, e.g. in the simplest such approach formulated by Gilboa & Schmeidler (1989), evaluates the bet by the worst expected outcome possible within the range of priors. The Klibanoff et al. (2009) approach assigns a second order probability distribution to the urn with the unknown number of balls. The way to think about this latter approach is that each possible number of balls corresponds to an objective or first order lottery. Not knowing the number of balls then translates into a second order or subjective lottery over the first order lotteries. Obviously, the Ellsberg paradox can be handled the same way by means of the representation in Theorem 1. However, there is an alternative way to describe the behavior by means of representation theorem 1. The decision maker attaches a probability of a half to the event drawing a red ball for both urns. However, he labels the urn where he knows the number of balls to be an objective lottery and he labels the lottery where the probability of a half is only obtained from the principle of insufficient reason to be a subjective lottery. If the decision maker is averse to the subjectivity of probabilistic beliefs, he prefers to bet on the "objective" urn.

¹⁶Note that the real versions of the Ellsberg (1961) paradox are set up slightly more elaborate in order to assure that no possible probability assigned to the distributions in the urn can explain the described choice within the standard risk setting.

Note that in the current setting not only those decision makers who are ambiguity neutral, but also those who consider all of the involved urn lotteries to be objective would not exhibit the “paradoxical behavior” predicted by Ellsberg (1961), a behavior that is generally only observed for some fraction of the participants of an according experiment.

6 A Sketch of Possible Applications

I briefly sketch out two possible applications of the model. One is quasi-static in that it uses the representation only for an evaluation of a given future. The other application is dynamic and relates the model to a particular possibility of tying the model to Bayesian updating. Both examples are drawn from the context of climate change economics. In the first example, a decision tree for a given period in the future starts with the root lottery capturing uncertainty about the stock of greenhouse gases in the atmosphere. For every given pollution stock there is a subtree describing uncertainty about the temperature in the same period. For a given (average) temperature there is uncertainty about precipitation. Given precipitation, there is uncertainty about agricultural yield. Given agricultural yield there is uncertainty about market prices and so on. Now, given this decision tree, the decision maker has to assign his degree of confidence or of subjectivity to each of these lotteries. For example, for the subtrees that correspond to low emission stocks, he might be more confident into the probability distributions over temperatures and precipitation. In the subtrees corresponding to a very high realization of the greenhouse gas stock, the decision maker is likely to consider his probabilistic estimates of the temperature change or the precipitation distribution as less reliable, labeling it more subjective. Assume that the decision maker is averse to subjectivity of belief as formalized in definition 2. Then, he attaches a relatively lower value to the more subjective subtrees stemming from a higher perturbation of the climate system than would a decision maker who does not distinguish lotteries by their confidence or subjectivity. Thus, a first conjecture in such a context would be that a decision maker with aversion to the subjectivity of belief would be willing to invest more into measures keeping him in a climate region that he can predict more confidently.

In the second example, I connect the representation in the simplest possible way

with a Bayesian setting where a decision maker learns about fundamental parameters of his environment. The setting of this paper permits the decision maker to attach a different degree of subjectivity to his priors capturing different uncertainties. For example, let p characterize the average temperature distribution in the central valley in California in 2050, the heart of Californian agricultural production. I model a decision maker who is aware that he can adapt to new information as uncertainty resolves over time. The decision maker employs a regional climate model for California that is coupled to a global climate model. Given his model is correct he obtains a probability distribution for the temperature T . However, there are unknowns θ_1 in the characterization or quality of the regional climate model. Also there are unknowns θ_2 with respect to the quality of the global model to which the regional model is coupled. Given both, θ_1 and θ_2 , the probability distribution for the temperature is given by the likelihood function $l(T|\theta_1, \theta_2)$. Given θ_1 and θ_2 he trusts his model enough to label the lottery $l(T|\theta_1, \theta_2)$ objective. However, he is aware that there are severe issues with regional modeling so he labels the prior $\mu_1(\theta_1|\theta_2)$ over the unknowns θ_1 of the regional model, which might depend on unknowns in the global climate model, to be subjective of degree s . The unknowns of the global model are captured by a prior $\mu_2(\theta_2)$ with a degree of subjectivity $s' \triangleleft s$ somewhere in between the other two distributions. The decision maker can obviously calculate the expected probability distribution over the temperature in 2050 by integrating out the priors to $p(x) = \int \int l(x|\theta_1, \theta_2) d\mu_1(\theta_1|\theta_2) d\mu_2(\theta_2)$. However, from an evaluative perspective there is no use in doing so, because the different layers of uncertainty correspond to different degrees of confidence or subjectivity. Therefore, they have to be evaluated recursively, each with the corresponding degree of aversion. Both priors $\mu_1(\theta_1|\theta_2)$ and $\mu_2(\theta_2)$ can be updated as in any standard Bayesian model of learning. With sufficient information, in the long run, the priors would shrink to a singleton and the decision maker would be left with the objective uncertainty or volatility of the temperature predicted by the model. The described procedure is not the only way to connect the representation of this paper with Bayesian learning. Of course, another interesting reasoning about learning in the present context will be to explicitly model changes in the degree of confidence instead of just shrinking priors. This question opens up a wide alley of future research.

7 Conclusions

The paper presents a model for evaluating scenarios that involve probabilistic beliefs that differ in their degree of subjectivity (or confidence). The evaluation of scenarios only employs simple tools from risk analysis. The representation facilitates a unified framework for representing aversion to intertemporal substitution, aversion to objective risk, aversion to subjective risk, and smooth ambiguity aversion. It respects the normatively desirable axioms of von Neumann & Morgenstern (1944) and of time consistency. Moreover, the representation facilitates a better understanding and a more precise definition of smooth ambiguity aversion as the additional intertemporal risk aversion to subjective as opposed to objective lotteries. The concept of smooth ambiguity aversion is put forth in the literature in a hierarchical and binary context of purely subjective second order beliefs over purely objective first order beliefs. The representation of this paper frees the degree of subjectivity from this straitjacket by incorporating the degree of subjectivity straight into the notion of a lottery. I briefly sketched out two possible applications of the model. The more elaborate application to learning opens up a wide alley for future research.

Appendix

Proof of Theorem 1:

Part I develops the representation for a single layer of uncertainty in a given period. Part II builds the recursive evaluation of a general decision tree within a given period. Part III constructs the intertemporal aggregation. Part IV shows that the axioms are satisfied by the representation.

Part I 1) I denote the underlying choice space in a given period t by $X_t = X^* \times P_{t+1}$ (for the last period is $X_T = X^*$). By axioms A2 and A4 there exists an ordinal representation $\tilde{u}_t : X_t \rightarrow \mathbb{R}$ of preferences $\succeq_t |_{X_t}$, i.e. preferences over degenerate period t choices only. I denote the evaluation function for these degenerate lotteries $p_t \in P_t$ with $\hat{n}(p_t) = 0$ also by

$$V^0(p_t) = \tilde{u}_t(p_t)$$

2) For a given parameter s , axioms A2-A4 on $\Delta_s(X_t)$ are the standard von Neumann-Morgenstern axioms for a compact metric setting that permit an expected utility

presentation on $\Delta_s(X_t)$. The only distinction to the standard presentation is that I formally distinguish an element $p_t \in X_t$ from the degenerate lottery $\delta_{p_t}^{1,s} \in \Delta_s(X_t)$. However, as I pointed out in the context of equation (4) axioms A2 and A3 imply $p_t \sim \delta_{p_t}^{1,s}$ for $p_t \in X_t$, so for employing the standard mixture space arguments the two can be identified. The standard von Neumann-Morgenstern reasoning shows that there exists a particular version of \tilde{u}_t that makes it possible to represent preferences over lotteries in the expected utility form. Instead of using the standard representation, I follow Traeger (2007) and build the representation on an arbitrary function $\tilde{u}_t : X_t \rightarrow \mathbb{R}$ representing degenerate choices $\succeq_t |_{X_t}$. At the current point \tilde{u}_t could be the function singled out by von Neumann-Morgenstern as well as any strictly increasing and continuous transformation of it. For a given parameter s , Theorem 1 in Traeger (2007) translates into the following preference representation:

Given is $\tilde{u}_t : X_t \rightarrow \mathbb{R}$ with $\text{range}(\tilde{u}_t) = U$ representing preferences $\succeq_t |_{X_t}$. Then $\succeq_t |_{\Delta_s X_t}$ satisfies axioms A2-A4 if, and only if, there exists a strictly increasing and continuous function $f_t^s : U \rightarrow \mathbb{R}$ such that

$$V_s^1(p) = (f_t^s)^{-1} \int_{X_t} f_t^s \circ \tilde{u}_t dp$$

represents $\succeq_t |_{\Delta_s(X_t)}$ for all $p \in \Delta_s(X_t)$. Moreover, f and f' both represent \succeq in the above sense if, and only if, there exist $a, b \in \mathbb{R}, a > 0$ such that $f' = af + b$.

3) Undertaking step 2) for all $s \in S$ results in a sequence of increasing and continuous functions $\hat{f}_t = (f_t^s)_{s \in S}, f_t^s : \mathbb{R} \rightarrow \mathbb{R}$, as stated in the theorem, and a representation of $\succeq_t |_{Z_t^1(X_t)}$ by

$$\bar{V}^1(p_t) = \begin{cases} V^0(p_t) = \tilde{u}_t(p_t) & \text{if } \hat{n}(p_t) = 0 \\ V^1(p_t) = (f_t^{\hat{s}(p_t)})^{-1} \int_{X_t} f_t^{\hat{s}(p_t)} \circ \tilde{u}_t dp_t & \text{if } \hat{n}(p_t) = 1 \end{cases}$$

Part II constructs inductively a representation of $\succeq_t |_{Z_t^n(X_t)}$ for $n \in N$.

4) Let $\bar{V}^n : Z_t^n(X_t) \rightarrow \mathbb{R}$ represent $\succeq_t |_{Z_t^n(X_t)}$. By equation (4) I can evaluate degenerate lotteries in $\Delta_s(Z_t^n(X_t))$ just as the corresponding elements in $Z_t^n(X_t)$. That identification makes \bar{V}^n a representation for degenerate lotteries in $Z_t^{n+1}(X_t)$. Thus, for given s , by axioms A2-A4 and Theorem 1 in Traeger (2007), cited in step

2, the lotteries in $\Delta_s(Z_t^n(X_t))$ can be represented by

$$V_s^{n+1}(p_t) = (\tilde{f}_t^s)^{-1} \int_{Z^n(X_t)} \tilde{f}_t^s \circ V^{\hat{n}(\tilde{p}_t)}(\tilde{p}_t) dp_t(\tilde{p}_t)$$

for some strictly increasing and continuous function $\tilde{f}_t^s : \text{range}(V^n) \rightarrow \mathbb{R}$. Employing the representation theorem for each $s \in S$ delivers a representation over the union $Z^{n+1}(X_t) = \cup_{s \in \bar{S}} Y_s^{n+1}(X_t)$ including $Y_{s_0}^{n+1}(X_t) = Z^n(X_t)$ that evaluates lotteries $p \in Z^{n+1}(X_t)$ by

$$\bar{V}^{n+1}(p_t) = \begin{cases} V^0(p_t) = \tilde{u}_t(p_t) & \text{if } \hat{n}(p_t) = 0 \\ V^1(p_t) = (\tilde{f}_t^{\hat{s}(p_t)})^{-1} \int_{X_t} \tilde{f}_t^{\hat{s}(p_t)} \circ \tilde{u}_t dp_t & \text{if } \hat{n}(p_t) = 1 \\ \vdots & \vdots \\ V^{n+1}(p) = (\tilde{f}_t^{\hat{s}(p)})^{-1} \int_{Z^n(X_t)} \tilde{f}_t^{\hat{s}(p)} \circ V^{\hat{n}(\tilde{p}_t)}(\tilde{p}_t) dp_t(\tilde{p}_t) & \text{if } \hat{n}(p_t) = n + 1. \end{cases}$$

5) I show that the \tilde{f}_t^s in V^{n+1} can be chosen to coincide with the f_t^s in V^n (and, thus, in all the $V^{i \leq n}$). Let $p, p', p'' \in P_s^n \subset Z_t^n(X_t)$. Reduction of the following lottery gives

$$\begin{aligned} [\delta_{p_t} \oplus_s^\alpha \delta_{p'_t}]^r(B) &= \int_{Y_s^n(X_t)} \tilde{p}_t(B) d(\delta_{p_t} \oplus_s^\alpha \delta_{p'_t})(\tilde{p}_t) \\ &= \alpha \int_{Y_s^n(X_t)} \tilde{p}_t(B) d(\delta_{p_t})(\tilde{p}_t) + (1 - \alpha) \int_{Y_s^n(X_t)} \tilde{p}_t(B) d(\delta_{p'_t})(\tilde{p}_t) \\ &= \alpha p_t(B) + (1 - \alpha) p'_t(B) \end{aligned}$$

for all $B \in \mathcal{B}(Z^{n-1}(X_t))$ and, thus, $[\delta_{p_t} \oplus_s^\alpha \delta_{p'_t}]^r = p_t \oplus_s^\alpha p'_t$. Then, by axiom A1

$$\delta_{p_t} \oplus_s^\alpha \delta_{p'_t} \sim_t [\delta_{p_t} \oplus_s^\alpha \delta_{p'_t}]^r = p_t \oplus_s^\alpha p'_t$$

Translated into the representation I find that

$$\begin{aligned} V_s^{n+1}(\delta_{p_t} \oplus_s^\alpha \delta_{p'_t}) &= (\tilde{f}_t^s)^{-1} \left[\alpha \int_{Z^n(X_t)} \tilde{f}_t^s \circ V^n(\tilde{p}_t) d\delta_{p_t}(\tilde{p}_t) \right. \\ &\quad \left. + (1 - \alpha) \int_{Z^n(X_t)} \tilde{f}_t^s \circ V^n(\tilde{p}_t) d\delta_{p'_t}(\tilde{p}_t) \right] \\ &= (\tilde{f}_t^s)^{-1} \left[\alpha \tilde{f}_t^s \circ (f_t^s)^{-1} \int_{Z^{n-1}(X_t)} f_t^s \circ V^{n-1}(\tilde{p}_t) dp_t(\tilde{p}_t) \right. \\ &\quad \left. + (1 - \alpha) \tilde{f}_t^s \circ (f_t^s)^{-1} \int_{Z^{n-1}(X_t)} f_t^s \circ V^{n-1}(\tilde{p}_t) dp'_t(\tilde{p}_t) \right] \end{aligned}$$

has to equal

$$\begin{aligned}
 V_s^n(p_t \oplus_s^\alpha p'_t) &= (f_t^s)^{-1} \left[\alpha \int_{Z^{n-1}(X_t)} f_t^s \circ V^{n-1}(\tilde{p}_t) dp_t(\tilde{p}_t) \right. \\
 &\quad \left. + (1 - \alpha) \int_{Z^{n-1}(X_t)} f_t^s \circ V^{n-1}(\tilde{p}_t) dp'_t(\tilde{p}_t) \right].
 \end{aligned}$$

Abbreviating $K(p) = \int_{Z^{n-1}(X_t)} f_t^s \circ V^{n-1} dp$, equivalence of the two expressions results in

$$\begin{aligned}
 V_s^{n+1}(\delta_{p_t} \oplus_s^\alpha \delta_{p'_t}) &= V_s^n(p_t \oplus_s^\alpha p'_t) \\
 \Leftrightarrow (\tilde{f}_t^s)^{-1} &\left[\alpha \tilde{f}_t^s \circ (f_t^s)^{-1} K(p_t) + (1 - \alpha) \tilde{f}_t^s \circ (f_t^s)^{-1} K(p'_t) \right] \\
 &= (f_t^s)^{-1} [\alpha K(p_t) + (1 - \alpha) K(p'_t)] \\
 \Leftrightarrow \alpha \tilde{f}_t^s \circ (f_t^s)^{-1} &K(p_t) + (1 - \alpha) \tilde{f}_t^s \circ (f_t^s)^{-1} K(p'_t) \\
 &= \tilde{f}_t^s \circ (f_t^s)^{-1} [\alpha K(p_t) + (1 - \alpha) K(p'_t)].
 \end{aligned}$$

Because preferences are non-degenerate, $K(p)$ can be varied on a continuum and by Hardy et al. (1964,) the continuous function $\tilde{f}_t^s \circ (f_t^s)^{-1}$ has to be linear implying $\tilde{f}_t^s = a f_t^s + b$ for some $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ (on the domain relevant to the representation). As affine transformations of the uncertainty aggregation weights do not change the representation (see step 2), I can choose $\tilde{f}_t^s = f_t^s$.

6) Steps 4) and 5) can be applied inductively for $n \in \{1, \dots, N - 1\}$, yielding a representation for $\succeq_t |_{Z^N(X_t)} = \succeq_t$. Once the uncertainty aggregation weights f_t^s have been shown to coincide at the different levels, the functions \bar{V}^n can as well be constructed inductively by defining $\bar{V}^0 = \tilde{u}_t$ and

$$\bar{V}^n(p_t) = (f_t^{\hat{s}(p_t)})^{-1} \int_{Z^{\hat{n}(p_t)-1}(X_t)} f_t^{\hat{s}(p_t)} \circ V^{\hat{n}(p_t)}(\tilde{p}_t) dp_t(\tilde{p}_t)$$

for $n \in N$ (noting that $\hat{n}(\tilde{p}_t) < n$). Then, for a given sequence of uncertainty weights \hat{f}_t and a given function \tilde{u}_t it is $\mathcal{M}_{\tilde{u}_t}^{\hat{f}_t} p_t = \bar{V}^N(p_t)$. I have established the existence of the sequences \hat{f}_t^s as in the theorem and the existence of some \tilde{u}_t such that the representation equation (7) in the theorem holds.

Part III shows that the sequence $\hat{u}_t, t \in T$ constructed as stated in equation (6) indeed gives rise to a feasible set of Bernoulli utility functions \tilde{u}_t .

7) Recall that the only requirement on the functions \tilde{u}_t is that they have to be an ordinal representation of preferences on the space of degenerate outcomes in period t , i.e. for $\succeq_t |_{X_t}$. Axioms A2, A4, and A5 imply a certainty additive representation for preferences restricted to the subspace of certain consumption paths (Wakker 1988, theorem III.4.1).¹⁷ I denote the corresponding continuous per period utility functions by $u_t : X^* \rightarrow \mathbb{R}$. They are unique up to affine transformations with a coinciding multiplicative constant (and heterogeneous additive constants).

8) For the last period I can choose $\tilde{u}_t = \hat{u}_T = u_T$. I show recursively that $\hat{u}_{t-1}(x_{t-1}, p_t) = u_{t-1}(x_{t-1}) + \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t$ is an (ordinal) representation of $\succeq_{t-1} |_{X_{t-1}}$ given that \hat{u}_t is an (ordinal) representation of $\succeq_t |_{X_t}$. Note that by construction of the uncertainty aggregator $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t}$, a certain consumption path $\mathbf{x}_t = (x_t, x_{t+1}, \dots, x_T)$ is indeed evaluated to $\hat{u}_t(\mathbf{x}^t) = \sum_{\tau=t}^T u_\tau(x_\tau)$. I define a certainty equivalent of a lottery $p_t \in P_t$ to be a lottery $(x_t^{p_t}, p_{t+1}^{p_t}) \in P_t$ that satisfies $(x_t^{p_t}, p_{t+1}^{p_t}) \sim_t p_t$. For any lottery there exists such a certainty equivalent and it does not matter which one is chosen.¹⁸ By the representation already constructed, I know that $\mathcal{M}_{\hat{u}_t}^{\hat{f}_t} p_t = \hat{u}_t(x_t^{p_t}, p_{t+1}^{p_t})$. Moreover, by inductively replacing $p_{t+1}^{p_t}$ with a certainty equivalent, I obtain a certainty equivalent to the lottery p_t that is a certain consumption path, which I denote by $\mathbf{x}_t^{p_t}$.

9) Observe that by time consistency

$$\begin{aligned}
 p_t & \sim_t \mathbf{x}_t^{p_t} \\
 \Leftrightarrow (x_{t-1}, p_t) & \sim_{t-1} (x_{t-1}, \mathbf{x}_t^{p_t})
 \end{aligned}$$

¹⁷A note on the details of the theorem's applicability. If the sets $\{p'_0 \in P_0 : p'_0 \succeq_0 \mathbf{x}\}$ and $\{p'_0 \in P_0 : \mathbf{x} \succeq_0 p'_0\}$ are closed in P_0 for all $\mathbf{x} \in X^{T+1} \subset P_0$, then the sets $\{p'_0 \in P_0 : p'_0 \succeq_0 \mathbf{x}\} \cap X^{T+1} = \{\mathbf{x}' \in X^{T+1} : \mathbf{x}' \succeq_0 \mathbf{x}\}$ and $\{p'_0 \in P_0 : \mathbf{x} \succeq_0 p'_0\} \cap X^{T+1} = \{\mathbf{x}' \in X^{T+1} : \mathbf{x} \succeq_0 \mathbf{x}'\}$ are closed in X^{T+1} endowed with the relative topology for all $\mathbf{x} \in X^{T+1}$. Moreover the relative topology on X^{T+1} is the product topology on X^{T+1} .

¹⁸The existence is most easily observed from the representation already constructed. The uncertainty aggregator is a generalized mean and, thus, the value of any lottery lies between the value of the worst and the best outcome. For more details see induction hypothesis H2 in proof of theorem 2 in Traeger (2007).

and therefore

$$\begin{aligned}
 (x_{t-1}, p_t) & \succeq_{t-1} (x'_{t-1}, p'_t) \\
 \Leftrightarrow (x_{t-1}, \mathbf{x}^{p_t}) & \succeq_{t-1} (x'_{t-1}, \mathbf{x}^{p'_t}) \\
 \Leftrightarrow u_{t-1}(p_t) + \sum_{\tau=t}^T u_\tau(x_\tau^{p_t}) & \geq u_{t-1}(p'_t) + \sum_{\tau=t}^T u_\tau(x_\tau^{p'_t}) \\
 \Leftrightarrow u_{t-1}(p_t) + \mathcal{M}_{p_t}^{\hat{f}_t} \hat{u}_t & \geq u_{t-1}(p'_t) + \mathcal{M}_{p'_t}^{\hat{f}_t} \hat{u}_t
 \end{aligned}$$

Hence $\hat{u}_{t-1} : X^* \times P_t \rightarrow \mathbb{R}$ with $\hat{u}_{t-1}(x_{t-1}, p_t) = u_{t-1}(x_{t-1}) + \mathcal{M}_{p_t}^{\hat{f}_t} \hat{u}_t$ is an (ordinal) representation of $\succeq_{t-1} |_{X^* \times P_t}$. Thus, indeed there exist continuous functions u_t as stated in the theorem so that feasible Bernoulli utility functions \tilde{u}_t used in the representation in part II are given by the functions \hat{u}_t constructed in equation (6).

Part IV proofs necessity of the axioms. Axiom A2 is obviously satisfied. With respect to axiom A3 observe that for all $t \in T$, $p_t, p'_t, p''_t \in P_t$, and $\alpha \in [0, 1]$:

$$\begin{aligned}
 p_t \succeq_t p'_t & \Rightarrow \mathcal{M}_{p_t}^{\hat{f}_t} p_t \geq \mathcal{M}_{p'_t}^{\hat{f}_t} p'_t \\
 & \Rightarrow \left(f_t^{\hat{s}(p_t)} \right)^{-1} \int_{Z^{\hat{n}(p_t)-1}(X^* \times P_{t+1})} f_t^{\hat{s}(p_t)} \circ \mathcal{M}_{p_t}^{\hat{f}_t} \tilde{p}_t dp_t(\tilde{p}_t) \\
 & \geq \left(f_t^{\hat{s}(p'_t)} \right)^{-1} \int_{Z^{\hat{n}(p'_t)-1}(X^* \times P_{t+1})} f_t^{\hat{s}(p'_t)} \circ \mathcal{M}_{p'_t}^{\hat{f}_t} \tilde{p}_t dp'_t(\tilde{p}_t) \\
 & \Rightarrow \left(f_t^{\hat{s}(p_t)} \right)^{-1} \left[\int_{Z^{\hat{n}(p_t)-1}(X^* \times P_{t+1})} \alpha f_t^{\hat{s}(p_t)} \circ \mathcal{M}_{p_t}^{\hat{f}_t} \tilde{p}_t dp_t(\tilde{p}_t) + K \right] \\
 & \geq \left(f_t^{\hat{s}(p'_t)} \right)^{-1} \left[\int_{Z^{\hat{n}(p'_t)-1}(X^* \times P_{t+1})} \alpha f_t^{\hat{s}(p'_t)} \circ \mathcal{M}_{p'_t}^{\hat{f}_t} \tilde{p}_t dp'_t(\tilde{p}_t) + K \right]
 \end{aligned}$$

where

$$K = \int_{Z^{\hat{n}(p''_t)-1}(X^* \times P_{t+1})} (1 - \alpha) f_t^{\hat{s}(p''_t)} \circ \mathcal{M}_{p''_t}^{\hat{f}_t} \tilde{p}_t dp''_t(\tilde{p}_t) .$$

If follows

$$\begin{aligned}
 & \left(f_t^{\hat{s}(p_t)} \right)^{-1} \int_{Z^{n^*-1}(X^* \times P_{t+1})} f_t^{\hat{s}(p_t)} \circ \mathcal{M}_{p_t}^{\hat{f}_t} \tilde{p}_t d(p_t \oplus_s^\alpha p''_t)(\tilde{p}_t) \\
 & \geq \left(f_t^{\hat{s}(p'_t)} \right)^{-1} \int_{Z^{n^*-1}(X^* \times P_{t+1})} f_t^{\hat{s}(p'_t)} \circ \mathcal{M}_{p'_t}^{\hat{f}_t} \tilde{p}_t d(p'_t \oplus_s^\alpha p''_t)(\tilde{p}_t)
 \end{aligned}$$

with $n^* = \max\{\hat{n}(p_t), \hat{n}(p'_t), \hat{n}(p''_t)\}$ and, thus,

$$p_t \oplus_s^\alpha p''_t \succeq_t p'_t \oplus_s^\alpha p''_t .$$

To see that axiom A4 is satisfied note that in the union topology a set is closed if each preimage of the set under the injection maps¹⁹ is closed. Thus, given that the composition $f_t^s \circ \hat{u}$ are continuous (and the topology of weak convergence) the sets in axiom A4 are closed. Axiom A5 is easily observed to be satisfied by recognizing that the evaluation on certain consumption paths reduces to the formula $\hat{u}_t(\mathbf{x}^t) = \sum_{\tau=t}^T u_\tau(x_\tau)$. Finally, axiom A6 is seen to be satisfied by inspecting equation (6). \square

Proof of Proposition 1:

For all $\mathbf{x}, \mathbf{x}' \in X^t$ I have

$$\begin{aligned} \mathbf{x} \oplus_{obj}^{\frac{1}{2}} \mathbf{x}' &\succeq_t \mathbf{x} \oplus_{subj}^{\frac{1}{2}} \mathbf{x}' \\ &\Rightarrow \left(f_t^{obj}\right)^{-1} \left[\frac{1}{2} f_t^{obj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \mathbf{x} + \frac{1}{2} f_t^{obj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \mathbf{x}' \right] \\ &\geq \left(f_t^{subj}\right)^{-1} \left[\frac{1}{2} f_t^{subj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \mathbf{x} + \frac{1}{2} f_t^{subj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \mathbf{x}' \right] . \end{aligned}$$

Defining $K(\mathbf{x}) = f_t^{obj} \circ \mathcal{M}_{\hat{u}_t}^{\hat{f}_t} \mathbf{x} = f_t^{obj} \circ \sum_{\tau=t}^T u_\tau(\mathbf{x}_\tau)$ I find

$$\begin{aligned} &\Rightarrow f_t^{subj} \circ \left(f_t^{obj}\right)^{-1} \left[\frac{1}{2} [K(\mathbf{x})] + \frac{1}{2} [K(\mathbf{x}')] \right] \\ &\geq \frac{1}{2} f_t^{subj} \circ \left(f_t^{obj}\right)^{-1} [K(\mathbf{x})] + \frac{1}{2} f_t^{subj} \circ \left(f_t^{obj}\right)^{-1} [K(\mathbf{x}')] \end{aligned}$$

and, thus, $f_t^{amb} = f_t^{subj} \circ \left(f_t^{obj}\right)^{-1}$ concave by Hardy et al. (1964, 75) on the range relevant for the representation. Analogously I find strict concavity to hold by replacing \succeq_t by \succ_t and \geq by $>$. \square

Proof of Proposition 2:

For every pair $s, s' \in S$ with $s \triangleright s'$ the proof is a copy of the proof of proposition 1. \square

¹⁹The s -th injection map inj_s assigns an element of $\Delta(\cdot)$ to the corresponding element in $\Delta(\cdot, s) = \Delta_s(\cdot)$ (e.g. Cech 1966, 85).

References

- Cech, E. (1966), *Topological Spaces*, John Wiley & Sons, London.
- Ellsberg, D. (1961), 'Risk, ambiguity and the savage axioms', *Quarterly Journal of Economics* **75**, 643–69.
- Epstein, L. G. & Zin, S. E. (1989), 'Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework', *Econometrica* **57**(4), 937–69.
- Ergin, H. & Gul, F. (2009), 'A theory of subjective compound lotteries', *Journal of Economic Theory* **144**, 899–929.
- Fishburn, P. C. (1992), 'A general axiomatization of additive measurement with applications', *Naval Research Logistics* **39**(6), 741–755.
- Gilboa, I. & Schmeidler, D. (1989), 'Maxmin expected utility with non-unique prior', *Journal of Mathematical Economics* **18**(2), 141–53.
- Grandmont, J.-M. (1972), 'Continuity properties of a von Neumann-Morgenstern utility', *Journal of Economic Theory* **4**, 45–57.
- Hardy, G., Littlewood, J. & Polya, G. (1964), *Inequalities*, 2 edn, Cambridge University Press. first published 1934.
- IPCC (2001), *Climate Change 2001: The Scientific Basis, Technical Summary*, Secretariat of the International Panel of Climate Change, Geneva. Edited by J.T. Houghton, Y. Ding, D.J. Griggs, M. Noguer and P.J. van der Linden and D. Xiaosu.
- Jaffray, J.-Y. (1974a), Existence. Propriétés de Continuité, Additivité de Fonctions d'Utilité sur un Espace Partiellement ou Totalelement Ordonné, PhD thesis, Université der Paris, VI.
- Jaffray, J.-Y. (1974b), 'On the existence of additive utilities on infinite sets', *Journal of Mathematical Psychology* **11**(4), 431–452.
- Klibanoff, P., Marinacci, M. & Mukerji, S. (2005), 'A smooth model of decision making under ambiguity', *Econometrica* **73**(6), 1849–1892.

- Klibanoff, P., Marinacci, M. & Mukerji, S. (2009), 'Recursive smooth ambiguity preferences', *Journal of Economic Theory* **144**, 930–976.
- Koopmans, T. C. (1960), 'Stationary ordinal utility and impatience', *Econometrica* **28**(2), 287–309.
- Krantz, D., Luce, R., Suppes, P. & Tversky, A. (1971), *Foundations of measurement. Vol. I. Additive and polynomial representations*, Academic Press, New York.
- Kreps, D. M. & Porteus, E. L. (1978), 'Temporal resolution of uncertainty and dynamic choice theory', *Econometrica* **46**(1), 185–200.
- Radner, T. (1982), *Microeconomic Theory*, Academic Press, New York.
- Segal, U. (1990), 'Two-stage lotteries without the reduction axiom', *Econometrica* **58**, 349–377.
- Seo, K. (2009), 'Ambiguity and second-order belief', *Econometrica* **77**(5), 1575–1605.
- Traeger, C. (2007), 'Intertemporal risk aversion - or - wouldn't it be nice to know whether robinson is risk averse?'. CUDARE Working Paper No 1103.
- von Neumann, J. & Morgenstern, O. (1944), *Theory of Games and Economic Behaviour*, Princeton University Press, Princeton.
- Wakker, P. (1988), 'The algebraic versus the topological approach to additive representations', *Journal of Mathematical Psychology* **32**, 421–435.
- Weil, P. (1990), 'Unexpected utility in macroeconomics', *The Quarterly Journal of Economics* **105**(1), 29–42.