N-dimensional Colonel Blotto Game with Asymmetric Battlefield Values

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17.12.2009

Abstract

This paper describes a geometrical method for constructing equilibrium distribution in the Colonel Blotto game with asymmetric battlefield values. It generalises to the n-dimensional case a construction method first described by Gross and Wagner. The proposed method does particularly well in instances of the Colonel Blotto game in which the battlefield weights satisfy some clearly defined regularity conditions. Though these conditions constrain the set of games in which this method reliably generates equilibrium strategies, they are less restrictive than the condition of symmetry across all battlefields, hitherto common in the literature. The paper also explores the parallel between these conditions and the integer partitioning problem in combinatorial optimisation.

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1 Introduction

Budget-constrained multidimensional allocation problems were amongst the very first ones considered in game theory. The first version can be found in Borel and Ville [1]. This problem and similar ones later came to be known as "Colonel Blotto" games, after Gross and Wagner's approach [4] to the allocation problem.

In the simplest version of the Colonel Blotto game, two generals want to capture three equally valued battlefields. Each general disposes of one divisible unit of military resources. The generals have to simultaneously allocate these resources among the three battlefields. A battlefield is captured by a general if he allocates more resources there than his opponent. The goal of each general is to maximise the number of captured battlefields.

In that game, a pure strategy for player i is a 3-dimensional allocation vector $\mathbf{x}_i = (x_1^i, x_2^i, x_3^i)$ where x_k^i is the amount of resources allocated to the kth city. The set of all pure strategies is the 2-dimensional simplex Δ^2. A mixed strategy is a trivariate distributions function $F : \Delta^2 \to [0, 1]$.

This version of the game was considered in Borel's course on probability [1] at the university of Paris in 1936-37. The solutions given by Borel reappear in Gross's and Wagner's unpublished research memorandum (1950) [4].

They state that a mixed strategy F constitutes a symmetric equilibrium of the game if all one-dimensional margins of F are uniform over $[0, \frac{2}{3}]$. One geometrical approach to building such a distribution F consist of projecting a sphere, together with a uniform generic point belonging to its surface, onto the disc inscribed in an equilateral triangle.

Gross and Wagner conjecture that this geometrical method of generating the equilibrium distribution extends to Colonel Blotto games with more than three equally valued battlefields. This extension is formalised in Laslier and Picard (2002) [7]. It is worth noting that Weinstein (2005) [9] presents a different geometric approach for case of $n \geq 3$ equally valued battlefields.

Roberson (2006) [8] addresses the question of whether the univariate marginal distributions of the equilibrium strategies (n-variate distributions) are necessarily uniform for symmetric battlefield weights but possibly asymmetric budgets, and finds that they indeed have to be. That paper does not, however, solve the Blotto Game with asymmetric battlefield values. Another related paper is Kvasov (2007) [6]. It looks at a variation of the Blotto Game in which the allocation of resources is costly, and there too, battlefields are symmetric.

The present paper generalises Gross and Wagner's geometric approach to construct equilibrium distributions of the n-dimensional Colonel Blotto game with asymmetric battlefield weights. The difficulty lies in inscribing a circle within an irregular n-gon. The necessary and sufficient conditions for this relate to the integer partitioning problem, a well-known problem of combinatorial optimisation.

The next section describes the model, then generalises the proofs of the existing literature to describe knows equilibria of this game. Section 3 presents geometrical methods of constructing equilibrium distributions. It describes Borel's solutions as formulated in Gross and Wagner (1950), then Laslier and Picard's geometric construction method. Section 4 constitutes the main contribution of this paper. It shows how, and under which conditions, this method can be extended to asymmetric n-dimensional cases. The conditions are related to a constrained version of the NP-complete "integer partitioning problem". Section 5 illustrates the construction method using the example of US presidential elections. The final section concludes.

2 Model and Equilibrium

Two players with identical budget normalised to one decide how to allocate their resources across n battlefields indexed by $k \in \{1, ..., n\}$. The absolute value of battlefield k is the positive integer E_k. For all k, denote $e_k = E_k / \sum_{k=1}^{n} E_k$ the relative value of battlefield k and note that $\sum_{k=1}^{n} e_k = 1$. To make the game non-trivial, assume that $0 < e_k < 1/2$, or equivalently that $0 < E_k < \sum_{j \neq k} E_j$, for all $k = 1, ..., n$.

Player $i \in \{1, 2\}$ chooses a nonnegative vector of allocations $\mathbf{x}_i = (x_1^i, ..., x_n^i)$ where x_k^i is the amount of resources allocated to battlefield k. Player i wins in battlefield k if his resources in that battlefield, x_k^i, exceeds the resources x_k^j of the other player. Ties are resolved by flipping a coin. Both players are budget-constrained so the sum of a player's resources allocated across all battlefields cannot exceed that player's budget of 1.

A pure strategy of player i is an n-dimensional vector $\mathbf{x}^i$ satisfying the budget constraint. Denote $\mathcal{S}^i$ the set of pure strategies of player i:

$$\mathcal{S}^i = \left\{ \mathbf{x} \in [0, 1]^n : \sum_{k=1}^{n} x_k \leq 1 \right\}$$

Both players seek to maximise the aggregate value of captured battlefields. The function

$g : \mathcal{S}^i \times \mathcal{S}^j \to \mathbb{R}$ measures the excess aggregate value of battlefields captured by player i if he plays the pure strategy $\mathbf{x}^i$ while player j plays $\mathbf{x}^j$:

$$g(\mathbf{x}^i, \mathbf{x}^j) = \sum_{k=1}^{n} e_k \; sgn(x_k^i - x_k^j),$$

with $sgn(u) = 1$ if $u > 0$, 0 if $u = 0$ and -1 if $u < 0$.

A mixed strategy of player i is an n-variate joint distribution function $F^i : S^i \to [0, 1]$. Denote F_k^i the kth one-dimensional margin of F^i, i.e. the unconditional distribution of x_k^i. For each $k = 1, ..., n$, F_k^i maps $[0, 1]$ into itself. Define the payoff to a mixed strategy as the mathematical expectation of $g(\mathbf{x}^i, \mathbf{x}^j)$ with respect to the strategy F^i.

The following proposition generalises existing results on the form of equilibria in Blotto games to the case of asymmetric battlefield weights. The proof is relegated to Appendix 7.1.

Proposition 1. *Consider the Colonel Blotto Game with asymmetric battlefield weights.*

(i) This game has no pure strategy Nash equilibrium

(ii) Both players meet their resource constraint in equilibrium.

(iii) Let F^* be a probability distribution of $\boldsymbol{x} \in \Delta^{n-1}$ such that each vector coordinate x_k ($k = 1, ..., n$) is uniformly distributed on $[0, 2e_k]$. Then (F^*, F^*) constitutes a symmetric Nash equilibrium.

The first point implies that an equilibrium, if it exists, must be in mixed strategies. The second point guarantees that the support of any equilibrium strategy is the $(n-1)$-dimensional simplex.

Point three states that having univariate margins that are uniform on $[0, 2e_k]$ is a sufficient condition for a mixed strategy with support Δ^{n-1} to constitute a symmetric Nash equilibrium.

Roberson [8] shows that for homogeneous battlefield values ($\forall k \; e_k = 1/n$) uniform univariate margins are also a *necessary* condition for equilibrium.

Is it always possible to build a joint distribution satisfying the properties of F^*? The following section describes the geometric construction method of Gross and Wagner, and later Laslier and Picard, while section 4 generalises it to accommodate asymmetric battlefield values. We obtain conditions under which this construction method always produces a joint distribution satisfying the properties of F^*.

3 Multivariate Distributions - known cases

The aim is to construct a n-variate distribution function F^* from given one-dimensional margins and given the equilibrium restrictions on the support of F^*. Indeed, in equilibrium candidates only use strategies in the $(n-1)$-dimensional simplex, Δ^{n-1}, which does not include the whole of $\times_{k=1}^{n}[0, 2e_k]$. Were it otherwise, it would be possible to construct a joint distribution with any correlation properties.

So the restriction of the support of F^* given its margins limits the number of possible interactions between resource allocations to different battlefields. So far, I have not been able to fully characterise the set of possible correlations satisfying the restrictions on F^*.

This section presents a geometrical method of constructing F^* that this paper will refer to as the *generalised disk solution*, in reference to the *disk* solution presented in Gross and Wagner [4] and later with some modifications in Laslier and Picard [7].

Note that because this is not the only way to construct multivariate distributions satisfying the restrictions above, this method might not describe the entire set of F^*s even in cases where the method is applicable.

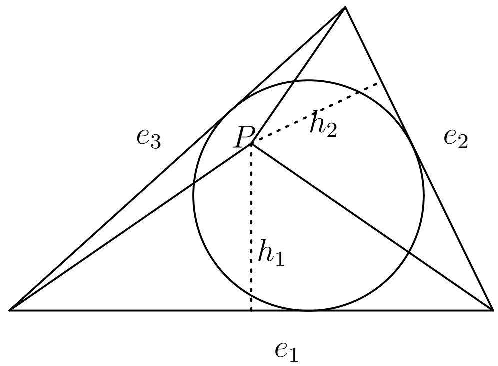
3.1 Triangle Solution - Gross and Wagner 1950

First, consider the case presented in Gross and Wagner [4] for $n = 3$ asymmetric battlefield weights. The following process generates three dimensional vectors $\mathbf{x} = (x_1, x_2, x_3)$ in the two dimensional simplex Δ^2 such that each x_k is distributed uniformly over $[0, 2e_k]$.

Think of the triangle of sides[2] e_1, e_2, e_3, as belonging to the plane with z-coordinate zero in the three-dimensional space (x, y, z). Inscribe a disk of centre O and radius r within that triangle. This disk is the projection (onto the plane $(x, y, 0)$) of the sphere $\mathcal{S}$ of centre

[2] For simplicity we identify a side of the triangle with its length. So we use e_k to refer both to a segment and to its length. Note also that this triangle always exists since $e_k < 1/2\ \forall k$.

O and radius r belonging to the three dimensional space (x, y, z). Finally, let R be a generic point that is uniformly distributed on the surface of the sphere $\mathcal{S}$, and let P be the projection of R onto the plane.



The triangle solution

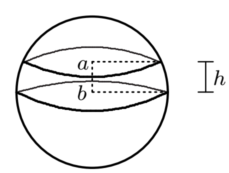
For all k, h_k is the distance of P from the side e_k. In the three-dimensional space, it is also the distance of R from $\mathcal{P}_k$, the vertical plane tangent to the sphere of centre O and which projects onto the side e_k.

If R is uniformly distributed on the surface of the sphere, what is the distribution of h_k? For all $t \in [0, 2r]$, the *spherical cap of height* t is the region of the sphere $\mathcal{S}$ that lies between the vertical plane $\mathcal{P}_k$, and the vertical plane parallel to $\mathcal{P}_k$ and at a distance t away from it. Then, for all $t \in [0, 2r]$, $\Pr(h_k < t) = \Pr(R \in \text{cap of height } t)$, and since R is uniformly distributed on the surface of the sphere, this probability equals the surface area of the cap[3] of height t, $t \in [0, 2r]$, divided by the total surface area of the sphere:

$$\Pr(h_k < t) = \frac{2\pi \int_o^t r \ dx}{2\pi \int_o^{2r} r \ dx} = \frac{t}{2r},$$

and so h_k is distributed uniformly on $[0, 2r]$.

[3] Note that the result of this sub-section is largely driven by the following property of spheres: Consider the spherical segment of height h. Its surface (excluding the bases) is called a zone. Its mathematical expression is $2\pi \int_a^b r \ dx = 2\pi r h$. Note that this area is *independent of the vertical position* of the zone.



Back in the two-dimensional plane, call A_k the area of the triangle of height h_k and side e_k sustended by P. For all k, $A_k = e_k h_k/2$. Since $h_k \sim U[0, 2r]$, it must be that $A_k \sim U[0, 2re_k/2] \equiv U[0, re_k]$.

Letting $A = A_1 + A_2 + A_3 = (e_1 + e_2 + e_3)r/2 = r/2$ be the total area of the triangle, we assimilate the fractions x_1, x_2, x_3, which are assumed to belong to the two dimensional simplex, to the fractions A_1/A, A_2/A, A_3/A, which belong to the two dimensional simplex by construction. So for all k, x_k=$A_k/A = 2A_k/r$. Then finally, since $A_k \sim U[0, re_k]$, it must be that $x_k \sim U[0, 2re_k/r]$, i.e. $x_k \sim U[0, 2e_k]$. QED.

Note that this construction is unique as there is only one cyclical permutation of 3 objects, if we account for the orientation of the cycle (i.e. treat $\{x, y, z\}$ and $\{z, y, x\}$ as equivalent).

3.2 Regular n-gon - the *disk* solution - Laslier and Picard 2002

As n increases beyond three, note that different orderings of the e_k's create different supports for the equilibrium strategy. Moreover for $n \geq 4$ it is not possible to inscribe a circle in *any* n-gon. Irregular n-gons are the object of the next section.

Let us first consider the case of regular n-gons. As supported by the *disk* solution, it is possible to construct a multivariate distribution F^* for the case in which all states carry the same value: $e_k = 1/n$ for all k. Then, regardless of n, it is possible to inscribe a circle within the n-gon ; and following the same method as in the triangle case, the process generates n-dimensional vectors $\mathbf{x} = (x_1, ..., x_n)$ belonging to the $(n-1)$-dimensional simplex, such that each x_k is distributed uniformly over $[0, 2/n]$.

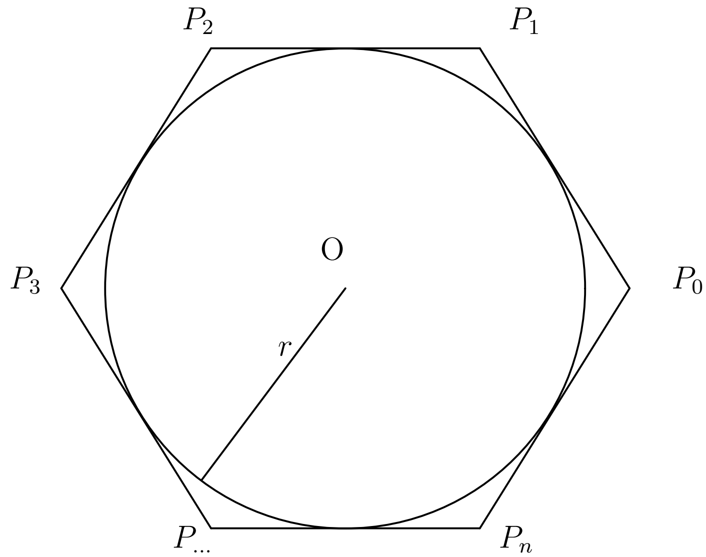
In the two-dimensional, oriented plane, consider the regular n-gon $\{P_0, ..., P_{n-1}\}$ centered at zero such that

$$P_k = \left(\rho \; cos\tfrac{(2k-1)\pi}{n}, \rho \; sin\tfrac{(2k-1)\pi}{n}\right) = \rho \; e^{i\frac{(2k-1)\pi}{n}}.$$

The disk that is inscribed within this n-gon is centered at zero and has radius r such that

$$\left|\frac{P_k + P_{k+1}}{2}\right| = \frac{\rho}{2}\sqrt{2(1 + cos\frac{4\pi}{n})} = r.$$

This disk is the projection onto the plane of the sphere centered at zero of radius r. To generate the n-dimensional vector $\mathbf{x}$, use the method corresponding to the three-dimensional case described above.

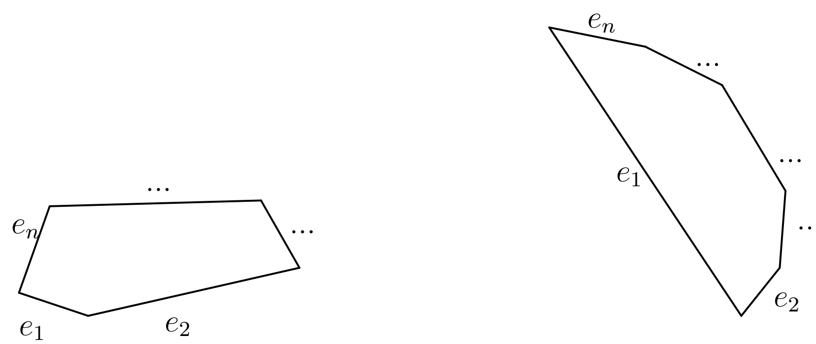


Regular n-gon

Note that there are as many *disk* solutions as there are ways to order n objects in a circle without taking into account the orientation of the circle, i.e. $(n-1)!/2$. Even though all sides have the same length, meaning that the n-gons $\{e_1, e_2, e_3, e_4\}$ and $\{e_3, e_2, e_1, e_4\}$ say, look identical, the correlations of vector coordinates deriving from the resulting joint distributions will be different.

4 Multivariate Distributions - Irregular case

In this section, we present a novel construction method for the case where battlefield values differ. Note that if there exists an n-gon with sides of lengths corresponding to the battlefield values and that admits an inscribed circle, we can use the method for constructing F^* described above. But as noted in the previous section, for $n \geq 4$ it is not possible to inscribe a circle in *any* n-gon. Roughly, the figure needs to be sufficiently regular. Indeed, for some $\{e_k\}_{k=1}^n$, it may never be possible to inscribe a circle in an n-gon of sides e_k regardless of the ordering. This is the case for instance if one e_k is much larger than all the others.



Ill-behaved n-gons

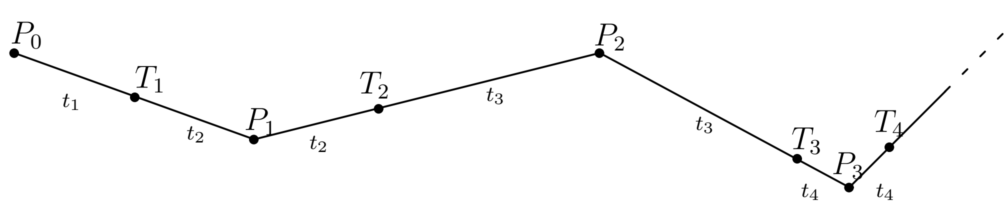
The next sub-section describes how to construct an irregular n-gon admitting an incircle, assuming this is possible. Then, sub-section 4.2 presents the necessary and sufficient conditions on battlefield weights guaranteeing it is possible to construct an irregular n-gon admitting an incircle.

4.1 Irregular n-gon - the *modified disk* solution

Consider the n-vector $\mathbf{e} = (e_1, ..., e_n)$ of battlefield weights, and define the n-vector $\gamma = (\gamma_1, ..., \gamma_n)$ to be a reordering of $\mathbf{e}$ satisfying conditions described in section 4.2. Let k, the index of the coordinates of γ, be congruent modulo n.

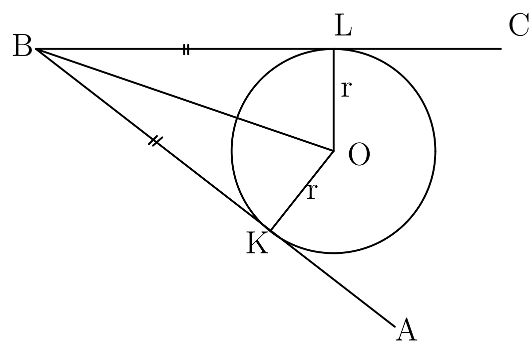
Given γ, consider the following method of constructing an irregular n-gon of ordered sides γ_1, γ_2, etc, such that a circle is inscribed in it.

For $k = 1, ..., n$, let Γ be a string of n connected segments $[P_{k-1}, P_k]$ of length γ_k with the following equidistance property: let T_k be a point of the segment $[P_{k-1}, P_k]$ such that, for each k, the distances $||T_k P_k||$ and $||P_k T_{k+1}||$ are the same, denoted t_k. The points T_k will be the tangency points between the n-gon and the circle inscribed in it.



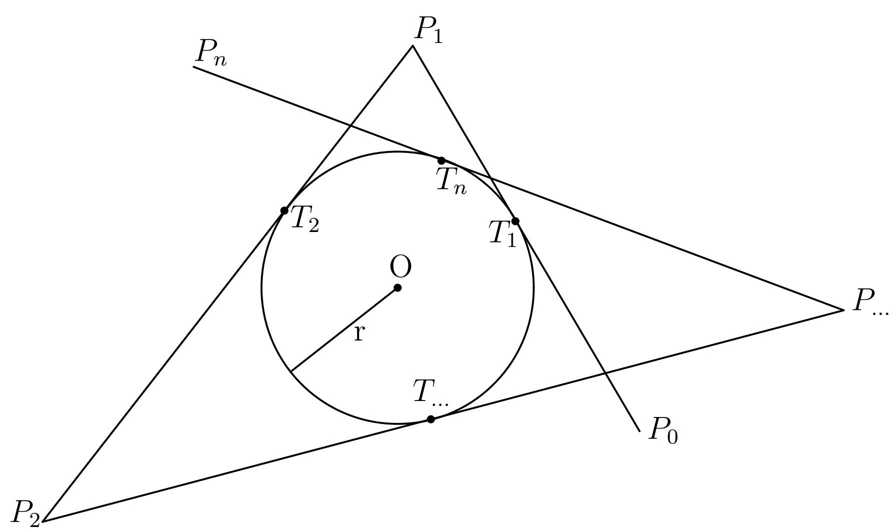
The set Γ

Consider the disk (O, r) and two connected segments $[AB]$ and $[BC]$. Let both segments be tangent to the circle, and let K and L be the points of tangency of $[AB]$ and $[BC]$ respectively. It is a well known result that the distances $||K - B||$ and $||B - L||$ are then necessarily equal.



Equidistance

Accordingly, if a sequence of connected segments can be wrapped around a circle (regardless of the number of times the sequence goes around the circle) in such a way that all segments are tangent to that circle, then the points of tangency of two consecutive segments are equidistant from the point common to both segments.



Wrapping Γ_n *around a circle*

This equidistance property is, by construction, satisfied by the set Γ. So Γ can be wrapped around *any* circle (O, r). The number of times we can wrap this set of connected segments around a circle depends on r. Theorem 2 states that there is only one value of r for which we can wrap a given Γ around a circle, such that $P_n = P_0$, closing the n-gon. Denote θ_k the angle $(P_{k-1}, 0, P_k)$.

Theorem 2. *For a given* Γ, $\sum_{k=1}^n \theta_k = f(r)$ *where* f *is a continuous, strictly monotone function. Therefore,* r^* *satisfying* $f(r^*) = 2\pi$ *is unique.*

Proof. Denote a_k the angle $(T_k, 0, P_k)$. Then $\sum_{k=1}^n \theta_k = 2\sum_{k=1}^n a_k$. The function $\sin^{-1}$ is defined (and monotonically increasing) on $[-1, 1]$, and since for all $x \in \mathbb{R}^{+*}$, $0 < x/\sqrt{x^2 + r^2} \leq x/\sqrt{x^2} = 1$, so

$$\sin a_k = \frac{t_k}{\sqrt{t_k^2 + r^2}} \quad \Leftrightarrow \quad a_k = \sin^{-1}\left[\frac{t_k}{\sqrt{t_k^2 + r^2}}\right],$$

and

$$\sum_{k=1}^{n} \theta_k = 2\sum_{k=1}^{n} \sin^{-1}\left[\frac{t_k}{\sqrt{t_k^2 + r^2}}\right] = f(r),$$

which is strictly decreasing, and hence invertible in r for all n. The proposition follows. ■

Note that r^* depends on the particular choice of t_k so that any vector $\mathbf{e}$ may be associated with several r^*.

We now present the conditions on γ that are necessary and sufficient for the existence of a set Γ, and hence for the existence of an n-gon of sides given by γ and admitting an inscribed circle.

4.2 Necessary and sufficient conditions

When the n-gon is regular, it is always possible to inscribe a circle within it. As we deviate from the regular n-gon, what are sufficient conditions on $\{e_k\}_{k=1}^n$ and on the ordering of the sides of the irregular n-gon that need to be satisfied to ensure that a circle can be inscribed within it?

First note that the restriction $e_k < 1/2\ \forall k$ guarantees that a convex n-gon with sides of lengths given by $\{e_k\}_{k=1}^n$ exists.

This section describes conditions for reordering the coordinates of the n-vector $\mathbf{e} = (e_1, ..., e_n)$ to form the n-vector $\gamma = (\gamma_1, ..., \gamma_n)$. Recall that k, the index of the coordinates of γ, is congruent modulo n. The conditions are necessary and sufficient to be able to inscribe a circle in the irregular convex n-gon with ordered sides given by γ, and from there, to build an equilibrium strategy F^*.

It will be shown that some vectors $\mathbf{e}$ will not admit any reordering γ satisfying these conditions so that it will not be possible to build a distribution with the properties of F^* using the geometric method.

To be able to build such a set Γ, the vector γ needs to satisfy the following restrictions **(P1)** and **(P2)**, that are divided in sub-cases depending on whether n, the number of battlefields, is odd or even.

(P1E) If n is even, then

$$\sum_{i=1}^{n}(-1)^i\gamma_{k+i} \;=\; 0.$$

(P2E) If n is even, then for any k, there exists a constant $c > 0$ such that for $\nu = 1, 2, ..., \frac{n}{2}$,

$$\max_{\nu}\{\sum_{i=1}^{2\nu}(-1)^{i+1}\gamma_{(k+i)}\} \;<\; c \;<\; \min_{\nu}\{\sum_{i=1}^{2\nu-1}(-1)^{i+1}\gamma_{(k+i)}\}.$$

(P1O) If n is odd, then for any k,

$$t_k = \frac{1}{2}\sum_{i=0}^{n-1}(-1)^{i+1}\gamma_{k+i}.$$

(P2O) If n is odd, then for any k,

$$\gamma_k > \left\|\sum_{i=1}^{n-1}(-1)^{i+1}\gamma_{k+i}\right\|.$$

These restrictions are all derived from the fact that by definition, $\gamma_k = t_k + t_{k+1}$, and from the two following requirements:

1)	Congruence	$\forall k,\ t_{k+n} = t_k.$
2)	Fit	$\forall k,\ 0 < t_k < \gamma_k.$

(P1) and **(P2)** hold if and only if congruence and fit are satisfied. The details can be found in appendix 7.2.

Congruence and fit are necessary and sufficient conditions for γ for generate a set Γ as defined in section 4.1. It follows that these properties of γ are necessary and sufficient for the resulting Γ to generate at least one n-gon admitting an incircle. Of course, they are all satisfied when all the coordinates of γ are the same - corresponding to the case of Laslier's and Picard's regular n-gon. The following theorem is the main result of this section:

Theorem 3. *If for a vector* $\mathbf{e}$ *of battlefield weights we can find a reordering* γ *satisfying* ***(P1)*** *and* ***(P2)****, then we can construct an irregular n-gon with an inscribed circle of radius* r^**.*

The radius r^* is defined in theorem 2. In the remainder of this section, we provide some insight into these properties and in particular (section 4.3), ask how easy they are to satisfy.

Conditions **(P2E)** and **(P1O)**, relate to the tangency points between the inscribed circle and the n-gon. They ensure that if t_k belongs to the interval $(0, \gamma_k)$, then t_{k+1}, which is equal to $\gamma_k - t_k$, belongs to the following interval, $(0, \gamma_{k+1})$. We can see that while for n odd, the conditions on the length t_k are very strict (equality), for n even it will be sufficient for t_k to belong to the interval defined in **(P2E)**:

(P2E)' If n is even, then for all k,

$$t_k \in \Big(\max_{\nu}\{\sum_{i=1}^{2\nu}(-1)^{i+1}\gamma_{(k+i)}\} \ , \min_{\nu}\{\sum_{i=1}^{2\nu-1}(-1)^{i+1}\gamma_{(k+i)}\}\Big)$$

So for a given γ, if n is even, it is possible to build and infinity of sets Γ as long as **(P2E)'** is satisfied, while for n odd, there exists a unique Γ with distances t_k satisfying **(P1O)**.

The remaining two conditions, **(P1E)** and **(P2O)**, are discussed in the next sub-section.

4.3 The constrained integer partitioning problem

It is clear that while some vectors $\mathbf{e}$ may admit several corresponding vectors γ, others may admit none. Indeed, the properties are all regularity restrictions on the ordering of the coordinates of γ and impose some balance. Notice that **(P1E)** can be rewritten as:

(P1E)' If n is even, then

$$\sum_{i=1}^{\frac{n}{2}}\gamma_{(k+2i)} = \sum_{i=1}^{\frac{n}{2}}\gamma_{(k+2i-1)} = \frac{1}{2},$$

and that **(P2O)** can be rewritten as:

(P2O)' If n is odd, then for any k,

$$\gamma_k > \left\| \sum_{i=1}^{\frac{n-1}{2}} \gamma_{(k+2i)} - \sum_{i=1}^{\frac{n-1}{2}} \gamma_{(k+2i-1)} \right\|.$$

So the two conditions are similar in requiring that the n-gon generated by γ is balanced in the sense that the summed length of odd sides and the summed length of even sides are equal (for n even) or close in a precise sense (for n odd).

As a brief digression, note that they also can be interpreted as the requirement that there exists a coalition of states such that each state in that coalition and each state in the complement coalition is pivotal. Pivotality is not a very apt concept here, as players are maximising their plurality. It would be more fitting in a context where players maximise their probability of winning.

More can be said about when **(P1E)** and **(P2O)** may be satisfied by noting that these conditions are related to the *constrained integer partitioning problem*, a classic problem of combinatorial optimisation. The exercise consists in partitioning n integers into two subsets of given cardinalities such that the *discrepancy*, the absolute value of the difference of their sums, is minimized.

(P1E) corresponds to the constrained partitioning problem in which the cardinality of the two resulting subsets is $n/2$ and the discrepancy is equal to zero. A partition with a discrepancy of zero is called a *perfect partition*. **(P2O)** corresponds to n instances of a more relaxed version of the constrained partitioning problem just described: for each $k = 1, ..., n$, the aim is to partition $n-1$ integers into two subsets of equal cardinality, such that the discrepancy is less than γ_k.

These are computationally difficult problems. The unconstrained partitioning problem is NP-complete, and while some algorithms deliver good approximations of the optimal partition (the partition with the lowest possible discrepancy), the brute force algorithms that compares the discrepancies of all possible partitions is still the best known solution to the problem.

Borgs et al. [2] identify two *phases* of the constrained problem depending on its computational difficulty. They study the typical behaviour of the optimal partition when the n integers are i.i.d. random variables chosen uniformly from the set $\{1, ..., 2^m\}$ for some integer m.

They find that, for m and n tending to infinity in the limiting ratio m/n, with probability tending to one, there exists a perfect partition when $m/n < 1$. They call this the *perfect phase* of the problem. In the *hard phase* of the problem, for $m/n > 1$, the probability of a perfect partition tends to zero and the optimal partition is unique, making computation of the optimal partition more difficult there. Still, the minimum discrepancy, i.e. the discrepancy of the optimal partition, can be bounded from above and below.

While in the limiting case, the phase transition is sharp at 1, in finite cases, the phase transition happens within a specified interval containing 1, and it is not clear whether the transition is sharp. Finally, the number of perfect partitions in the perfect phase is lower than in the limiting case by about twenty percent for a given ratio m/n.

For the purpose of this paper, the results of Borgs et al. allow the conclusion that **(P1E)** and **(P2O)** are likely to be more easily satisfied for $m/n < 1$ than for $m/n > 1$, and that while **(P2O)** may be satisfied for $m/n > 1$, **(P1E)** never is.

	$m/n < 1$	$m/n > 1$
(P1E)	easy	impossible
(P2O)	easy	hard

Finally note the importance of the assumption that battlefield values are integers. Indeed, were battlefield values drawn from $\mathbb{R}$, the condition for n even would hold with probability zero.

5 Application

One compelling illustration of this model is the election of US presidents by electoral college: first, during primaries , two candidates, one Democrat, the other Republican, are chosen to represent their party in the general election, which is then held simultaneously in all 51 US states (50 + D.C). Each state is allocated a number of electoral votes depending on its population[4]. There are 538 electoral votes in total. A candidate gains all electoral votes of a given state if he receives more than half the votes cast in that state. To win the election, a candidate must win at least 270 electoral votes.

[4]For details, see Appendix 7.3

This situation can be modeled as an asymmetric colonel Blotto game under the following three assumptions: (i) presidential candidates face identical budget constraints, (ii) the probability of winning the election in a given state increases with campaigning resource allocated to that state, and (iii) candidates wish to maximise their plurality, rather than the probability of winning the election.

The first two assumptions are the least controversial. In fact assumption (i) is trivially satisfied if we think of the campaigning resource as time spent campaigning in each state.

What if we think of money as the resource? In practice, candidates can choose whether to self-finance their general election campaign, or (since 1976) can accept public funding[5]. To be eligible to receive the public funds, a candidate must limit spending to the donation[6]. So if both candidates are publicly funded, it makes sense to assume that they both face the same resource constraint.

The assumption of equal budgets becomes more trying if at least one of the candidates is self-funded. Indeed, there is considerable evidence that in these cases, budgets differ, as seen in the latest US presidential elections.

The positive relationship between campaign effort and votes is well documented, be it whether campaigning effort is understood to be time spent campaigning in a state (Herr [5]) or financial campaign expenditures in that state (Chapman and Palda, [3]). So assumption (ii) is also pretty unproblematic.

This is not so for the last assumption. In general, one would assume that candidates maximise the probability of their winning the election. Nevertheless one could argue that because presidential elections coincide with Senate and House of representative elections, presidential candidates do campaign so as to maximise the plurality of votes in they favour, not only so as to win the presidential election. This is more believable in cases where one candidate already expects to win with a significant plurality, but surely not when elections are close. Either way, it is fair to say that maximising the plurality in his favour is at least a candidate's secondary objective.

[5]For information on the Public Matching Fund scheme, visit the Federal Election Commission at http://www.fec.gov/.

[6]In essence. More precisely, the candidate may not accept private contributions for the campaign. Private contributions may, however, be accepted for a special account maintained exclusively to pay for legal and accounting expenses associated with complying with the campaign finance law. These legal and accounting expenses are not subject to the expenditure limit. For more detail, see the FEC brochure for Public Funding of Presidential Elections at http://www.fec.gov/pages/brochures/pubfund.shtml.

One strong argument supporting the claim that candidates care at least a little about plurality is that they do indeed campaign in all states, while ignoring small states (states with few electoral votes, that have little chance of being pivotal) would be consistent with the strategy of a candidate solely trying to maximise his probability of winning the election.

So we can think of the US general election game as a Colonel Blotto game. In both cases candidates choose how to allocate a fixed amount of resources across states. Strategic considerations arise because of the positive relationship between campaign effort and votes. By spending more in a state than his opponent, a candidate increases his chances to win that state.

In this section we look for a solution to a Colonel Blotto game in which each state has a value corresponding to its relative number of electoral votes. The distribution of electoral votes across states is shown in Appendix 7.3.

Two candidates with budgets $X_A = X_B = X$ decide how to allocate their campaigning funds across $n = 51$ states indexed by $k \in 1, ..., n$. The value of state k is e_k which corresponds to the number of electoral votes allocated to state k as a fraction of the total number of electoral votes, 538. For instance, the state of Alabama has 9 electoral votes, so for that state, $e = 9/538$. Accordingly $e_k < 1$ for all k and $\sum_{k=1}^{n} e_k = 1$.

Candidate i's plurality, i.e. the number of electoral votes won minus the number of electoral votes lost is measured by the function $g_i : \mathcal{S}_i \times \mathcal{S}_i \to \mathbb{R}$ defined in section 2.

Since this fits exactly into the setup of section 2, the results of all following sections hold, including the existence of one equilibrium distribution. Indeed, consider the vector γ_n presented in Appendix 7.4. It is such that each e_k corresponds to the number of electoral votes allocated to state k as a fraction of the total number of electoral votes, 538. For clarity, we multiply all numbers back by 538. Note that this solution uses the current distribution of electoral votes (i.e. the third column in table 7.3), but that the construction method works equally well for the other two distributions.

This vector satisfies the conditions **(P1)** and **(P2)** for n odd ($n = 51$). Note that within the framework of section 4.3, the 51 partitioning problems corresponding to this exercise are in the perfect phase. Here, the greatest of the $n = 51$ integers is 55, the number of electoral votes for the state of California. So we can treat the electoral votes as n i.i.d integers chosen uniformly from the set $\{1, ..., 2^m\}$ with $m = 6$, in which case $m/n \simeq$

$6/50 \ll 1$ (perfect phase) so that the partitioning problem should be relatively easy to solve. Indeed, a solution can be easily found heuristically, as shown in Appendix 7.4. This illustrates one possible equilibrium of the US general elections game.

6 Conclusion and Open Questions

This paper describes a geometrical method for constructing equilibrium distribution in the Colonel Blotto game with asymmetric battlefield values. The appeal of geometrical methods for constructing n-dimensional distributions subject to restrictions on their support and their margins lies in the relative simplicity with which they describe complicated multi-dimensional objects. The drawback is that they may fail to generate the full set of distributions satisfying given restrictions on support and margins. This downside is limited when that set is well defined, as it is here, so that the exercise becomes to generate instances of these well-defined objects.

The method presented in this paper generalises to the n-dimensional case a construction method first proposed by Gross and Wagner. It does particularly well in instances of the Colonel Blotto game in which the battlefield weights satisfy some clearly defined regularity conditions (Section 4.2). Though these conditions constrain the set of games in which this method reliably generates equilibrium strategies, they are less restrictive than the condition of symmetry across all battlefields (Laslier and Picard). Moreover, their implications suggest directions for further research.

Noticing that the conditions on the reordering γ can be interpreted as the requirement that there exists a coalition such that every battlefield is pivotal suggests a parallel between behaviour of candidates seeking to maximise plurality and candidates seeking to maximise probability of victory, though this paper leaves the exact relationship between these games an open question.

Finally, the restrictions on the support of equilibrium distributions limit the number of possible correlations across x_k's. This captures the idea that even though it is intuitive that more resources are likely to be allocated to battlefields with greater weight, the solution suggests that allocations to different battlefields interact in a particular way. Looking more carefully at possible correlations across x_k's could be interesting from the empirical point of view.

7 Appendix

7.1 Proof of Proposition 1

Proof of (i) and (ii): Straightforward.

Proof of (iii): To prove this point, it is sufficient to show that the payoff to any pure strategy $\mathbf{y} \in \mathcal{S}^i$ against F^* is non-positive. First we show that the expected payoff to player i from playing F^* against F^* is zero. Let $\mathbf{x}^i = (x_1^i, ..., x_n^i)$ and $\mathbf{x}^j = (x_1^j, ..., x_n^j)$ be generated by F^*. Accordingly, for all $k = 1, ..., n$, x_k^i and x_k^j are drawn from the uniform distribution over $[0, 2e_k X]$ and $Pr(x_k^j < x_k^i) = F_k^*(x_k^i) = \frac{x_k^i}{2e_k}$. So given $\mathbf{x}^i$, for all $k = 1, ..., n$,

$$E[sgn(x_k^i - x_k^j)|\mathbf{x}_i] = 2F^*(x_k^i) - 1 = \frac{x_k^i}{e_k} - 1.$$

And hence, for all $k = 1, ..., n$,

$$\begin{aligned} E[sgn(x_k^i - x_k^j)] &= \int_0^{2e_k} \left(\frac{t}{e_k} - 1\right) \, dF_k^*(t) \\ &= \frac{1}{e_k} \int_0^{2e_k} \left(\frac{t}{e_k} - 1\right) \, dt \end{aligned}$$

which is zero for all $k = 1, ..., n$ so that:

$$E[g(F^*, F^*)] = \sum_{k=1}^{n} e_k \cdot E[sgn(x_k^i - x_k^j)] = 0.$$

Now consider the payoff to player i of playing an arbitrary pure strategy $\mathbf{y} \in \mathcal{S}^i = \Delta^{n-1}$ against F^*. Since for all $k = 1, ..., n$, $e_k < \frac{1}{2}$ and F_k^* is the uniform distribution on $[0, 2e_k]$, $F_k^*(y_k) = y_k/2e_k$ if $y_k \in [0, 2e_k]$ and $F_k^*(y_k) = 1$ if $y_k > 2e_k$. So

$$\begin{aligned} E[sgn(y_k - x_k^j)|\mathbf{y}] &= 2F_k^*(y_k) - 1 \\ &= 2 \min \left\{1, \frac{y_k}{2e_k}\right\} - 1. \end{aligned}$$

Hence:

$$\begin{aligned} E[g(\mathbf{y}, F^*)] &= \sum_{k=1}^{n} e_k \; min \left\{1, \frac{y_k}{e_k} - 1\right\} \\ &\leq \sum_{k=1}^{n} e_k \Big(\frac{y_k}{e_k} - 1\Big) \end{aligned}$$

The last term equals $\sum_{k=1}^{n} y_k - \sum_{k=1}^{n} e_k$ which is zero since $\mathbf{y} \in \Delta^{n-1}$ and $\sum_{k=1}^{n} e_k = 1$ by construction. So $g(\mathbf{y}, F^*) \leq 0 = g(F^*, F^*)$ for all $\mathbf{y} \in \mathcal{S}_i$. ■

7.2 Restrictions on γ, the reordering of e

In this appendix, I illustrate how to derive the conditions **(P1)** and **(P2)** from the property $t_k + t_{k+1} = \gamma_k$, and the requirements:

$$\begin{array}{lll} \textbf{1)} & \text{Congruence} & \forall k,\ t_{k+n} = t_k \\ \textbf{2)} & \text{Fit} & \forall k,\ 0 < t_k < \gamma_k \end{array}$$

First, let's develop the first requirement. For n even:

$$\begin{aligned} & t_{k+n} = t_k \\ \Leftrightarrow\ & t_k = \gamma_{k+n-1} - \gamma_{k+n-2} + \gamma_{k+n-3} - ... - \gamma_k + t_k \\ \Leftrightarrow\ & \textstyle\sum_{i=1}^{n} (-1)^i \gamma_{k+i} = 0 \\ \Leftrightarrow\ & \textbf{(P1E)} \end{aligned}$$

For n odd:

$$\begin{aligned} & t_{k+n} = t_k \\ \Leftrightarrow\ & t_k = \gamma_{k+n-1} - \gamma_{k+n-2} + \gamma_{k+n-3} - ... + \gamma_k - t_k \\ \Leftrightarrow\ & 2t_k = \gamma_{k+n-1} - \gamma_{k+n-2} + \gamma_{k+n-3} - ... + \gamma_k \\ \Leftrightarrow\ & 2t_k = \textstyle\sum_{i=1}^{n} (-1)^{i+1} \gamma_{k+i} \\ \Leftrightarrow\ & \textbf{(P1O)} \end{aligned}$$

Now, let's develop the second requirement.

For n odd, from (P1O) we know that $t_k = \frac{1}{2}(\gamma_k - \gamma_{k+1} + \gamma_{k+2} - ... + \gamma_{k+n-1})$. So

$$0 < t_k < \gamma_k$$

$$\Leftrightarrow \quad -\gamma_k < -\gamma_{k+1} + \gamma_{k+2} - ... + \gamma_{k+n-1} < \gamma_k$$

$$\Leftrightarrow \quad \gamma_k > \left\| \sum_{n-1}^{i=1} (-1)^{i+1} \gamma_{k+i} \right\|$$

$$\Leftrightarrow \textbf{(P2E)}$$

For n even, the fit requirement, $\forall k$, $0 < t_k < \gamma_k$ gives us n restrictions:

$$\begin{array}{lccccc} (1) & 0 & < & t_k & < & \gamma_k \\ (2) & 0 & < & t_{k+1} & < & \gamma_{k+1} \\ (3) & 0 & < & t_{k+2} & < & \gamma_{k+2} \\ & & & \vdots & & \\ (n) & 0 & < & t_{k+n-1} & < & \gamma_{k+n-1} \end{array}$$

They can all be simplified to n restrictions on t_k:

$$\begin{array}{llcccl}
(1) & 0 & < & t_k & < & \gamma_k \\
(2) & \gamma_k - \gamma_{k+1} & < & t_k & < & \gamma_k \\
(3) & \gamma_k - \gamma_{k+1} & < & t_k & < & \gamma_k - \gamma_{k+1} + \gamma_{k+2} \\
 & & & \vdots & & \\
(n) & \gamma_k - \gamma_{k+1} + ... + \gamma_{k+n-2} - \gamma_{k+n-1} & < & t_{k+n-1} & < & \gamma_k - \gamma_{k+1} + ... + \gamma_{k+n-2}
\end{array}$$

Notice that t_k faces $n/2$ upper bounds and $n/2$ lower bounds. All n conditions are satisfied if:

$$\max_{\nu}\{\sum_{i=1}^{2\nu}(-1)^{i+1}\gamma_{(k+i)}\} \; < \; t_k \; < \; \min_{\nu}\{\sum_{i=1}^{2\nu-1}(-1)^{i+1}\gamma_{(k+i)}\}$$

and for this to be possible, γ needs to satisfy **(P2E)**.

7.3 Distribution of Electoral Votes (Source: FEC www.fec.gov)

State	1981-1990	1991-2000	2001-2010	State	1981-1990	1991-2000	2001-2010
Alabama	9	9	9	Missouri	11	11	11
Alaska	3	3	3	Montana	4	3	3
Arizona	7	8	10	Nebraska	5	5	5
Arkansas	6	6	6	Nevada	4	4	5
California	47	54	55	New Hampshire	4	4	4
Colorado	8	8	9	New Jersey	16	15	15
Connecticut	8	8	7	New Mexico	5	5	5
Delaware	3	3	3	New York	36	33	31
D.C	3	3	3	North Carolina	13	14	15
Florida	21	25	27	North Dakota	3	3	3
Georgia	12	13	15	Ohio	23	21	20
Hawaii	4	4	4	Oklahoma	8	8	7
Idaho	4	4	4	Oregon	7	7	7
Illinois	24	22	21	Pennsylvania	25	23	21
Indiana	12	12	11	Rhode Island	4	4	4
Iowa	8	7	7	South Carolina	8	8	8
Kansas	7	6	6	South Dakota	3	3	3
Kentucky	9	8	8	Tennessee	11	11	11
Louisiana	10	9	9	Texas	29	32	34
Maine	4	4	4	Utah	5	5	5
Maryland	10	10	10	Vermont	3	3	3
Massachusetts	13	12	12	Virginia	12	13	13
Michigan	20	18	17	Washington	10	11	11
Minnesota	10	10	10	West Virginia	6	5	5
Mississippi	7	7	6	Wisconsin	11	11	10
				Wyoming	3	3	3

7.4 One possible support of the *modified disk* solution applied to US data.

For clarity, all numbers are multiplied by 538.

k	e_k	t_k	t_{k+1}	k	e_k	t_k	t_{k+1}
1	31	2	29	26	3	1	2
2	8	6	2	27	3	2	1
3	9	3	6	28	3	1	2
4	10	7	3	29	4	3	1
5	11	4	7	30	4	1	3
6	17	13	4	31	4	3	1
7	20	7	13	32	4	1	3
8	21	14	7	33	5	4	1
9	27	13	14	34	5	1	4
10	21	8	13	35	5	4	1
11	15	7	8	36	6	2	4
12	15	8	7	37	6	4	2
13	15	7	8	38	7	3	4
14	10	3	7	39	7	4	3
15	7	4	3	40	8	4	4
16	7	3	4	41	9	5	4
17	6	3	3	42	9	4	5
18	5	2	3	43	10	6	4
19	5	3	2	44	10	4	6
20	4	1	3	45	11	7	4
21	3	2	1	46	11	4	7
22	3	1	2	47	11	7	4
23	3	2	1	48	12	5	7
24	3	1	2	49	13	8	5
25	3	2	1	50	34	26	8
-	-	-	-	51	55	29	26

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