A stochastic model for speculative bubbles

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Abstract

This paper aims to provide a simple modelling of speculative bubbles and derive some quantitative properties of its dynamical evolution. Starting from a description of individual speculative behaviours, we build and study a second order Markov process, which after simple transformations can be viewed as a *turning* two-dimensional Gaussian process. Then, our main problem is to obtain some bounds for the *persistence rate* relative to the return time to a given price. In our main results, we prove with both spectral and probabilistic methods that this rate is almost proportional to the turning frequency ω of the model and provide some explicit bounds. In the continuity of this result, we build some estimators of ω and of the pseudo-period of the prices. At last, we end the paper by a proof of the quasi-stationary distribution of the process, as well as the existence of its persistence rate.

Keywords: Speculative bubble; Persistence rate; Gaussian Process; Diffusion Bridge; Statistics of processes.

MSC2010: Primary: 60J70, 35H10, 60G15, 35P15.

1 Introduction

The evolution of prices in markets such as real estate is a popular subject of investigation. The purpose of this paper is to propose a stochastic model which, at the same time, is simple enough to be studied mathematically and accounts for periodicity phenomena induced by speculation.

One commonly talks of financial bubble when, due to speculation of traders or owners, an asset price exceeds an asset fundamental value. These owners then expect to resell the asset at an even higher price in the future. There exist a lot of famous historical examples such as the Dutch Tulip Mania (1634-1637), the Mississippi bubble (1718-1720) or the "Roaring '20s that preceded the 1929" crash. We refer to [13] for a general remainder on historical bubbles. Through some more recent events, one can observe that this phenomena is certainly actual. Think for instance to the Internet bubble which bursted in March 2000 after having led astronomical heights and lost more than 75% of its value, and to the housing bubble encountered in the United States (2000-2010) or in European countries (Spain, Ireland, France ...) (see *e.g.* [16] or [12]).

A huge litterature exists on speculative bubbles and it seems nearly impossible to quote all the related numerous previous works. We point out that our goal here is not to detail a general model flexible enough to take into account several complex economic realities. However, our approach is to propose a very simple tractable model from a mathematical point of view: the natural equilibrium

price is assumed to be 0 all along our temporal evolution, we do not consider any inflation nor credit crunch [10] and there is no regulating effect of any federal bank [3]. At last, the processes introduced in the paper will be supposed time-homogeneous.

According to Shiller [16] (see also [14]), the mechanism of creation of speculative bubbles is the following: *"If asset prices start to rise strongly, the success of some investors attracts public attention that fuels the spread of the enthusiasm for the market: (often, less sophisticated) investors enter the market and bid up prices. This "irrational exuberance" heightens expectations of further price increases, as investors extrapolate recent price action far into the future. The markets meteoric rise is typically justified in the popular culture by some superficially plausible "new era" theory that validates the abandonment of traditional valuation metrics. But the bubble carries the seeds of its own destruction; if prices begin to sag, pessimism can take hold, causing some investors to exit the market. Downward price motion begets expectations of further downward motion, and so on, until the bottom is eventually reached."*

In the previous citation, two phenomenas are exhibited: on the one hand, the investors have a tendency to follow the forecasting rule which consists in deciding that the price will increase if it has (strongly) increased in the past. On the other hand, the actions of the investors have certainly a self-reinforcing effect. In this paper, we assume more or less that the dynamics of the market is dictated by these two phenomenas. However, we (necessarily) assume that there is also a general mean-reverting force and there exists randomness in the decisions of the investors. Then, our model is obtained as the limit of the mean dynamics of all the investors when the number of these investors tends to infinity (see next paragraph for more details).

Let us also precise that our setting corresponds to the so-called rational bubbles under symmetric information paradigm described in [4, 18] for instance. In our framework, we are interested in the periodic pattern commonly encountered in such speculative markets, which is of primarily interest as pointed by [9]. We establish that these periodic phenomena are related to a persistence problem which is also an important field of interest from an economic point of view [4, 5]. Note that our model is also simple enough to imagine statistical inference procedures for the estimation of several key parameters. Hence, even if our work comes from a probabilistic motivation, it also opens the way of statistical procedures to test bubble formation. This last statistical point is shortly discussed in the end of our paper and seems challenging for future works (some numerical results show that the standard likelihood estimation does not seem well suited to approach the unknown parameters in such a model).

At last, it is generally empirically observed that the bubbles bursts are fasten than bubbles formations. Our model can be generalized to more complex settings where such burst's and formation's timing could be different using a mixture of memory weights $\Gamma_{k,b}$ with $k \geq 2$ (see next paragraph for more details).

1.1 Modeling of speculation

Let us designate by $X := (X_t)_{t \geq 0}$ the temporal evolution of the relative price of a commodity with respect to another one. For instance it can be the difference between the price of the mean square meter of real estate in a particular town and the price of the ounce of gold or the mean salary of a month of work. Let the units be chosen so that, in the mean over a long time period, this relative price is zero. We assume that three mechanisms are at work for the evolution of X:
- Economic reality plays the role of a restoring force, trying to draw X back toward zero. At least as a first approximation, it is natural to assume that this force is linear, whose rate will be denoted $a > 0$.
- Speculation is reinforcing a tendency observed for some times in the past. We make the hypothesis that the weight of past influences is decreasing exponentially fast in time, with rate $b > 0$. The typical length the observation time window will be given by b^{-1}.
- Uncertainty is modeled by a Brownian motion of volatility $c > 0$, which is a traditional assumption

for randomness coming from a lot of small unpredictable and independent perturbations, due to the functional central limit theorem.

Putting together these three leverages, we end up with a law of evolution of X described by the stochastic differential equation

$$\forall \, t \geq 0, \qquad dX_t \;\; = \;\; -aX_t dt + \left(b \int_0^t \exp(b(s-t))\, dX_s \right) dt + c\, dB_t \tag{1}$$

assuming for instance that initially, $X_0 = 0$. Because of the presence of the Brownian motion $(B_t)_{t \geq 0}$ in the r.h.s., the trajectories of X are not differentiable with respect to the time parameter $t \geq 0$. But for the purpose of a heuristic interpretation, let us pretend they are, so we can consider $X'_t := \frac{dX_t}{dt}$. Assume furthermore that the "origin" of time was chosen so that before it, X was zero, namely, in the above economic interpretation, the two commodities had their prices tied up at their relative equilibrium point before time 0. This enables us to define $X'_t = X_t = 0$ for any $t \leq 0$. The middle term of the r.h.s. of (1) can then be rewritten as

$$
\begin{aligned}
b \int_0^t \exp(b(s-t))\, dX_s \;\; &= \;\; \int_0^{+\infty} X'_{t-s}\, b \exp(-bs)\, ds \\
&= \;\; \mathbf{E}[X'_{t-\sigma}]
\end{aligned}
\tag{2}
$$

where σ is distributed as an exponential variable of parameter b and where $\mathbf{E}$ stands for the expectation with respect to σ (i.e. not with respect to the randomness underlying X, the corresponding expectation will be denoted $\mathbb{E}$). Thus X has a drift taking into account its past tendencies X', but very old ones are almost forgotten, due to the exponential weight. Indeed, since $\mathbf{E}[\sigma] = b$, tendencies older than a time of order b don't contribute much.

The equation (1) can be seen as the limit evolution of the means of (relative) prices predicted by a large number $N \in \mathbb{N}$ of speculative agents. Assume that each agent $n \in [\![N]\!] := \{1, ..., N\}$ has his own idea of the evolution of the prices, designated by $X(n) := (X_t(n))_{t \geq 0}$. The mean process $\bar{X} := (\bar{X}_t)_{t \geq 0}$ is defined by

$$\forall \, t \geq 0, \qquad \bar{X}_t \;\; := \;\; \frac{1}{N} \sum_{n \in [\![N]\!]} X_t(n).$$

For simplicity, we assume as above that all these processes were also defined for negative times and that

$$\forall \, t \leq 0, \, \forall \, n \in [\![N]\!], \qquad X_t(n) \;\; = \;\; \bar{X}_t \;\; = \;\; 0.$$

At any time $t \geq 0$, each agent $n \in [\![N]\!]$ has access to the whole past history $(\bar{X}_s)_{s \leq t}$ of the mean prices (say, which is published a particular institute or website). But to handle this wealth of information, agent n has chosen, once for all, a time window length $\Upsilon(n) > 0$ and he computes the ratio $(\bar{X}_t - \bar{X}_{t-\Upsilon(n)})/\Upsilon(n)$ in order to decide what is the present tendency of the prices. Then he interferes that this tendency contributes to the infinitesimal evolution of his estimate of prices $dX_t(n)$ via the term $(\bar{X}_t - \bar{X}_{t-\Upsilon(n)})/\Upsilon(n)\, dt$, speculating that what has increased (respectively decreased) will keep on increasing (resp. decreasing). Nevertheless, as everyone, he also undergoes the strength of the economic reality with rate $a > 0$, which adds a term $-aX_t(n)dt$ to his previsions. Furthermore, he cannot escape vagaries of life, good or bad, which disturb his evaluations with the infinitesimal increment $c\sqrt{N}dB_t(n)$, where $B(n) := (B_t(n))_{t \geq 0}$ is a standard Brownian motion. The factor $\sqrt{N}$ may seem strange at first view, but it accounts for the fact that the consequences of random events are amplified by a large population. Alternately, it could be argued that $\sqrt{N}dB_t(n)$ decompose into $\sum_{m \in [\![N]\!]} dB_t(n, m)$, where $(B_t(n, m))_{t \geq 0}$, for $n, m \in [\![N]\!]$, are

independent Brownian motions standing respectively for the random perturbations induced by m on n (including a self-influence $(B_t(n, n))_{t \geq 0}$). It follows that

$$\forall\, t \geq 0, \qquad dX_t(n) \;=\; -aX_t(n)dt + \frac{\bar{X}_t - \bar{X}_{t-\Upsilon(n)}}{\Upsilon(n)} dt + c\sqrt{N} dB_t(n)$$

and we deduce that

$$\forall\, t \geq 0, \qquad d\bar{X}_t \;=\; -a\bar{X}_t dt + \left(\frac{1}{N} \sum_{n \in [\![N]\!]} \frac{\bar{X}_t - \bar{X}_{t-\Upsilon(n)}}{\Upsilon(n)} \right) dt + c\frac{1}{\sqrt{N}} \sum_{n \in [\![N]\!]} dB_t(n)$$

Let us assume that all the $\Upsilon(n)$, for $n \in [\![N]\!]$, and all the $B(m)$, for $m \in [\![N]\!]$ are independent. A first consequence is that the process $\bar{B} = (\bar{B}_t)_{t \geq 0}$ defined by

$$\forall\, t \geq 0, \qquad \bar{B}_t \;:=\; \frac{1}{\sqrt{N}} \sum_{n \in [\![N]\!]} B_t(n)$$

is a standard Brownian motion.

Next, under the hypothesis that all the $\Upsilon(n)$, $n \in [\![N]\!]$ have the same law as a random variable Υ, we get by the law of large numbers, that almost surely,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \in [\![N]\!]} \frac{\bar{X}_t - \bar{X}_{t-\Upsilon(n)}}{\Upsilon(n)} \;=\; \mathbf{E}\left[\frac{\bar{X}_t - \bar{X}_{t-\Upsilon}}{\Upsilon} \right]$$

where $\mathbf{E}$ stands for the expectation with respect to Υ only. Thus letting N go to infinity, $\bar{X}$ ends up satisfying the same evolution equation as X, if the law of Υ is such that

$$\forall\, t \geq 0, \qquad \mathbf{E}\left[\frac{X_t - X_{t-\Upsilon}}{\Upsilon} \right] \;=\; b \int_0^t \exp(b(s-t))\, dX_s \tag{3}$$

almost surely with respect the trajectory $(X_s)_{s \in \mathbb{R}}$.

Contrary to the first guess which could be made, Υ should not be distributed according to an exponential law of parameter b:

Lemma 1 *For any continuous semi-martingale $X = (X_t)_{t \in \mathbb{R}}$ with $X_t = 0$ for $t \leq 0$, (3) is satisfied if Υ is distributed as a gamma law $\Gamma_{2,b}$ of shape 2 and scale b, namely if*

$$\forall\, t \geq 0, \qquad \mathbf{P}[\Upsilon \in dt] \;=\; \Gamma_{2,b}(dt) \;:=\; b^2 t \exp(-bt)\, dt$$

Proof

By continuity of X, it is sufficient to check the almost sure equality of (3) for any fixed $t \geq 0$. Then denote $\widetilde{X}_s = X_t - X_{t-s}$, for $s \geq 0$, so that

$$\mathbf{E}\left[\frac{X_t - X_{t-\Upsilon}}{\Upsilon} \right] \;=\; b^2 \int_0^{+\infty} \frac{X_t - X_{t-s}}{s} s \exp(-bs)\, ds$$

$$=\; b^2 \int_0^{+\infty} \widetilde{X}_s \exp(-bs)\, ds$$

The fact that X is a semi-martingale enables to integrate by parts and we find

$$b^2 \int_0^{+\infty} \widetilde{X}_s \exp(-bs)\, ds \;=\; -b\left[\widetilde{X}_s \exp(-bs) \right]_0^{+\infty} + b \int_0^{+\infty} \exp(-bs)\, d\widetilde{X}_s$$

$$=\; b \int_{-\infty}^t \exp(-b(t-s))\, dX_s$$

$$=\; b \int_0^t \exp(-b(t-s))\, dX_s$$

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■

Remark 2 Conversely, for the process X defined by (1) and $X_t = 0$ for $t \leq 0$, the validity of (3) implies (under an integrability assumption) that the law of Υ is the gamma distribution $\Gamma_{2,b}$. Actually, denote by G the distribution of Υ and assume that $\int_0^{+\infty} s^{-1} G(ds) < +\infty$. Then, by the previous result, (3) reads

$$\forall t \geq 0, \quad \int_0^{+\infty} \frac{X_t - X_{t-s}}{s} G(ds) = \int_0^{+\infty} \frac{X_t - X_{t-s}}{s} \Gamma_{2,b}(ds) \quad a.s. \tag{4}$$

Now, let $t > 0$. Since the above equality holds almost surely, it follows from Girsanov Theorem (see *e.g.* [15], Chapter 8), that we can replace $(X_s)_{s \in [0,t]}$ by c times a Brownian motion (and next by linearity take $c = 1$). Then, the main argument is the support Theorem (see *e.g.* [17]), which yields in particular that for every positive t and ε, for every $\mathcal{C}^1$-function $\varphi : (-\infty, t] \to \mathbb{R}$ such that $\varphi(u) = 0$ on $\mathbb{R}_-$,

$$\mathbb{P}(\sup_{s \in [0,t]} |X_s - \varphi(s)| \leq \varepsilon) > 0. \tag{5}$$

Let φ be such a function. By (4) and (5), we obtain that for every positive ε,

$$\left| \int_0^{+\infty} \frac{\varphi(t) - \varphi(t-s)}{s} (G(ds) - \Gamma_{2,b}(ds)) \right| \leq 2\varepsilon \int_0^{+\infty} \frac{1}{s} (G(ds) + \Gamma_{2,b})(ds)$$

and it follows that for every $\mathcal{C}^1$-function with $\varphi(u) = 0$ on $\mathbb{R}_-$,

$$\int_0^{+\infty} \frac{\varphi(t) - \varphi(t-s)}{s} G(ds) = \int_0^{+\infty} \frac{\varphi(t) - \varphi(t-s)}{s} \Gamma_{2,b}(ds), \tag{6}$$

the result being available for all positive t. Denoting $r = \varphi(t)$ and $h(s) = \varphi(t) - \varphi(t-s)$ for all $s \in [0,t]$, we get that for all $r \in \mathbb{R}$ and all $\mathcal{C}^1$ function $h : [0,t] \to \mathbb{R}$ with $h(0) = 0$,

$$r \int_t^{+\infty} \frac{1}{s} (G - \Gamma_{2,b})(ds) + \int_0^t h(s) \frac{G - \Gamma_{2,b}}{s} ds = 0$$

namely

$$\int_t^{+\infty} \frac{1}{s} (G - \Gamma_{2,b})(ds) = 0$$

and G and $\Gamma_{2,b}$ coincide on $(0,t]$. Since this is true for all $t > 0$, we get that G and $\Gamma_{2,b}$ coincide on $(0,+\infty)$. Because they are both probability measures, they cannot differ only on $\{0\}$, so $G = \Gamma_{2,b}$. In fact this proof can be extended to any continuous semi-martingale whose martingale part is non-degenerate.

□

The law $\Gamma_{2,b}$ has the same rate b of exponential decrease of the queues at infinity as the exponential distribution $\Gamma_{1,b}$ of σ in (2). The most notable difference between these two distributions is their behavior near zero: it is much less probable to sample a small values under $\Gamma_{2,b}$ than under $\Gamma_{1,b}$. Furthermore, $\Gamma_{2,b}$ is a little more concentrated around its mean $2/b$ than $\Gamma_{1,b}$ around its mean $1/b$, their respective relative standard deviations being $1/2$ and 1. These features are compatible with the previous modeling: the chance is small that an agent looks shortly in the past to get an idea of the present tendency of X and the dispersion of the lengths of the windows used by the agents may not be very important. These behaviors would be amplified, if instead of $\Gamma_{2,b}$, we had chosen a gamma distribution $\Gamma_{k,b}$ of shape k and scale b, with $k \in \mathbb{N} \setminus \{1,2\}$, for the law of Υ. The limit evolution in this situation is dictated by the stochastic differential equation in $X^{[k]}$ given by

$$\forall t \geq 0, \quad dX_t^{[k]} = -aX_t^{[k]} dt + \left(b(k-2)! \int_0^t g_{b,k}(t-s) \, dX_s^{[k]} \right) dt + c \, dB_t$$

(starting again from $X_0^{[k]} = 0$), where $g_{b,k}$ is the function defined by

$$g_{b,k} \; : \; \mathbb{R}_+ \ni s \; \mapsto \; \exp(-bs) \sum_{l \in [\![0,k-2]\!]} \frac{(bs)^l}{l!}$$

(curiously, the r.h.s. coincides with the probability that a Poisson random variable of parameter bs belongs to $[\![0, k - 2]\!]$).

For $k = 2$, we recover (1) and $X^{[2]} = X$. The stochastic process X is clearly not Markovian, but we will see in the sequel that it is a Markov process of order 2: it is sufficient to add another real component to X to get a Markov process. It can be shown more generally that $X^{[k]}$ is a Markov process of order k: $k - 1$ real components must be added to make it a Markov process. While this observation provides opportunities of better modelings, the investigation of $X^{[k]}$ for $k > 2$ (as well as the extension to non-integer values of k) is deferred to a future paper. Here we will concentrate on the properties of X, but before presenting the results obtained, let us give some simulations of X in Figure 1.
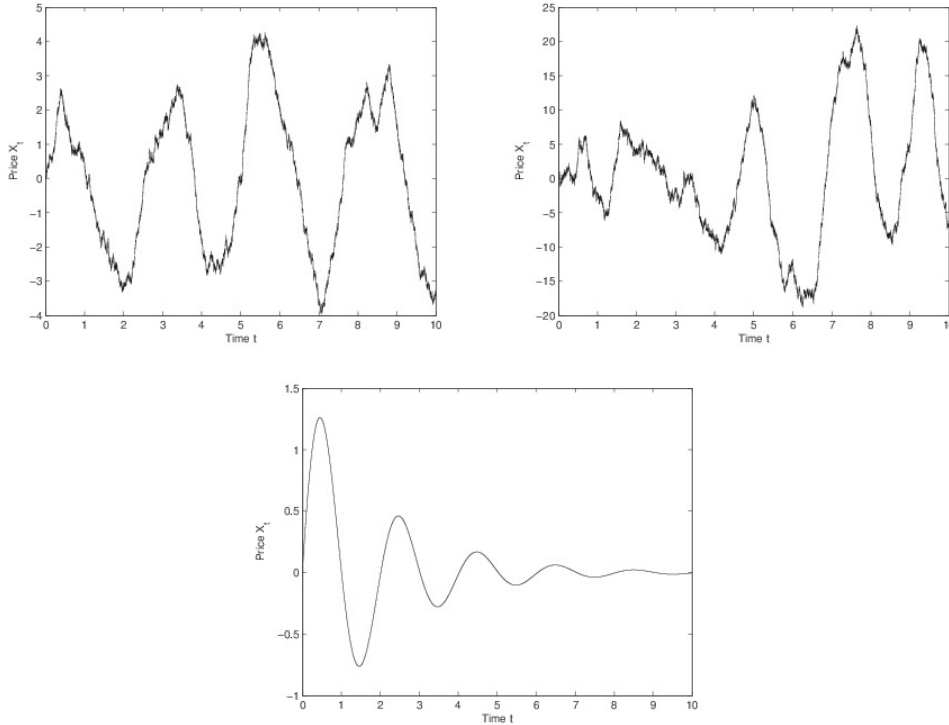

Figure 1: Several trajectories for various parameters (top left: $a = 1, b = 5, c = 1$, top right: $a = 1, b = 10, c = 5$, bottom: $a = 1, b = 10, c = 0$).

A periodic structure appears, as that observed in practice in the forming of speculative bubbles. The process X shows some regularity in returning to its equilibrium position, trend which seems to be only slightly perturbed by the noise. The variety of the trajectories is apparently less rich than that experienced by traditional Ornstein-Ulhenbeck processes, suggesting a concentration of the trajectory laws around some periodic patterns. Figure 2 shows the density of the return time of the process $(X)_{t \geq 0}$ to its equilibrium price 0. These results have been obtained using a large number of Monte-Carlo simulations. One may remark in Figure 2 that the tail of the return time to equilibrium state is much smaller for our bubble process than the one of the O-U. process with the same invariant measure on the X coordinate and with the same amount of injected randomness (namely through a standard Brownian motion). The purpose of this paper is to quantify these behaviors.
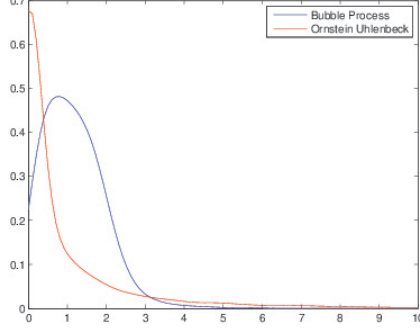
Figure 2: Density function of the return time of the Ornstein-Uhlenbeck process, as well as the one of the bubble process. The O-U process is set to have the same invariant measure as the bubble process and the same amount of injected randomness.

1.2 Results

As already mentioned, the process X whose evolution is driven by (1) is not Markovian. Nevertheless, it is not so far away from being Markovian: consider the process $Y := (Y_t)_{t \geq 0}$ defined by

$$\forall \, t \geq 0, \qquad Y_t \ := \ b \int_0^t \exp(b(s-t)) \, dX_s - bX_t$$

The process $Z := (Z_t)_{t \geq 0} := ((X_t, Y_t)^*)_{t \geq 0}$ (where * stands for the transpose operation) is then Markovian and its evolution is dictated by the simple 2-dimensional stochastic differential equation

$$\forall \, t \geq 0, \qquad dZ_t \ = \ AZ_t \, dt + C \, dB_t \tag{7}$$

starting from $Z_0 = 0$ and where

$$A \ := \ \begin{pmatrix} b-a & 1 \\ -b^2 & -b \end{pmatrix} \quad \text{and} \quad C \ := \ \begin{pmatrix} c \\ 0 \end{pmatrix} \tag{8}$$

The linearity of (7) and the fact that the initial condition is deterministic imply that at any time $t \geq 0$ the distribution of Z_t is Gaussian. As it will be checked in next section, this distribution converges for large time $t \geq 0$ toward μ, a normal distribution of mean 0 and whose variance matrix Σ is positive definite. Since the Markov process Z is Feller, μ is an invariant probability measure for Z. It is in fact the only one, because the generator L associated to the evolution equation (7) and given by

$$L \ := \ ((b-a)x + y)\partial_x - (b^2 x + by)\partial_y + \frac{c}{2}\partial_x^2 \tag{9}$$

is hypoelliptic (also implying that Σ is positive definite).

The study of the convergence to equilibrium of Z begins with the spectral resolution of A. Three situations occur:

• If $a > 4b$, A admits two real eigenvalues, $\lambda_\pm := (-a \pm \sqrt{a^2 - 4ab})/2$.

• If $a = 4b$, A is similar to the 2×2 Jordan matrix associated to the eigenvalue $-a/2$.

• If $a < 4b$, A admits two conjugate complex eigenvalues, $\lambda_\pm := (-a \pm i\sqrt{4ab - a^2})/2$.

But in all cases, let $l < 0$ be the largest real part of the eigenvalues, namely

$$l \ := \ \frac{-a + \sqrt{(a^2 - 4ab)_+}}{2} \tag{10}$$

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This quantity is the exponential rate of convergence of μ_t, the law of Z_t, toward μ, in the $\mathbb{L}^2$ sense: for $t > 0$, measure the discrepancy between μ_t and μ through

$$J(\mu_t, \mu) \ := \ \sqrt{\int \left(\frac{d\mu_t}{d\mu} - 1\right)^2 d\mu} \tag{11}$$

Since X was our primary object of interest, let us also denote by ν and ν_t the first marginal distributions of μ and μ_t respectively.

Proposition 3 *We have*

$$\lim_{t \to +\infty} \frac{1}{t} \ln(J(\mu_t, \mu)) \ = \ 2l \ = \ \lim_{t \to +\infty} \frac{1}{t} \ln(J(\nu_t, \nu))$$

These convergences can be extended to other measures of discrepancy, such as the square root of the relative entropy, or to initial distributions μ_0 of Z_0 more general than the Dirac measure at 0, at least under the assumption that $J(\mu_0, \mu) < +\infty$. Thus if we look at $X_{r/|l|}$ for large $r > 0$, it has almost forgotten that it started from 0 and its law is close to the Gaussian distribution ν, up to an error $\exp(-(1 + \circ(1))r)$.

Nevertheless, the periodicity features we are looking for appear only for $a < 4b$, as it can be guessed from the existence of non-real eigenvalues, which suggests $2\pi/\omega$ as period, where

$$\omega \ := \ \sqrt{ab - \frac{a^2}{4}} \tag{12}$$

In the regime where $b \gg a$, we have $\omega \gg 2|l|$: a lot of periods has to alternate before stationarity is approached. This phenomenon is often encountered in the study of ergodic Markov processes which are far from being reversible, e.g. a diffusion on a circle with a strong constant drift (for instance turning clockwise).

In order to quantify this behavior, we are interested in the return time τ to zero for X, which is of primary interest in the economic interpretation given at the beginning of the introduction (on the contrary to the relaxation time to equilibrium, which seems very far away in the future):

$$\tau \ := \ \inf\{t \geq 0 : X_t = 0\} \tag{13}$$

Of course it is no longer relevant to assume that $Z_0 = 0$ and instead we assume that $(X_0, Y_0) = (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}$. In practice, τ appears through a temporal shift: we are at time $s > 0$ which is such that $X_s > 0$ and we are wondering when in the future X will return to its equilibrium position 0. Up to the knowledge of (X_s, Y_s), the time left before this return has the same law as τ if (x_0, y_0) is initialized with the value (X_s, Y_s).

The next result shows that up to universal factors, the exponential rate of concentration of τ is given by $1/\omega$, confirming that when $b \gg a$, the return to zero happens much before the process reaches equilibrium.

Theorem 4 *For any $0 < a < 4b$, $c > 0$, $x_0 > 0$ and $y_0 \in \mathbb{R}$, we have*

$$\mathbb{P}_{(x_0, y_0)}[\tau > t] \ \leq \ 2 \exp\left(-\frac{\ln(2)}{\pi} \omega t\right).$$

Furthermore, if $(1 + \frac{1}{\sqrt{2}})a \leq b$, there exists a quantity $\epsilon(x_0, y_0) > 0$ (which in addition to x_0 and y_0, depends on the parameters a, b, c) such that

$$\forall \, t \geq 0, \qquad \mathbb{P}_{(x_0, y_0)}[\tau > t] \ \geq \ \epsilon(x_0, y_0) \exp\left(-4\omega t\right).$$

More generally, to any initial distribution m_0 on $D := \{(x, y) \in \mathbb{R}^2 : x > 0\}$, we can associate a quantity $\epsilon(m_0)$ such that

$$\forall \, t \geq 0, \qquad \mathbb{P}_{m_0}[\tau > t] \ \geq \ \epsilon(m_0) \exp\left(-4\omega t\right).$$

Surprisingly, we found the lower bound more difficult to obtain than the upper bound, while in reversible situations it is often the opposite which is experienced.

Remark 5 In the first appendix it will be shown that a quasi-stationary probability ν^D and a corresponding rate $\lambda_0(D) > 0$ can be associated to D: the support of ν^D is the closure of D and under $\mathbb{P}_{\nu^D}$, τ is distributed as an exponential random variable of parameter $\lambda_0(D)$:

$$\forall\, t \geq 0, \qquad \mathbb{P}_{\nu^D}[\tau \geq t] \;=\; \exp(-\lambda_0(D)t) \tag{14}$$

In the sequel, the quantity $\lambda_0(D)$ will be called the persistence rate of D. It can be seen as the smallest eigenvalue (in modulus) of the underlying Markov generator with a Dirichlet condition on the boundary of the domain D, when it is interpreted as acting on $\mathbb{L}^2(\mu^D)$, where μ^D is the restriction of μ on D. The above theorem then provides lower and upper bounds on $\lambda_0(D)$, essentially proportional to ω: at least for $0 < (1 + \frac{1}{\sqrt{2}})a \leq b$,

$$\frac{\ln(2)}{\pi}\omega \;\leq\; \lambda_0(D) \;\leq\; 4\omega \tag{15}$$

According to figure 2, starting from other initial distributions on D, the law of τ will no longer be exponential, nevertheless we believe that for any $(x_0, y_0) \in D$, the following limit takes place

$$\lim_{t \to +\infty} \frac{1}{t} \ln(\mathbb{P}_{x_0,y_0}[\tau > t]) \;=\; -\lambda_0(D)$$

The difficulty in obtaining this convergence stems from the non-reversibility of the process under consideration. In the literature, it is the reversible and elliptic situations which are the most thoroughly investigated. For a general reference on quasi-stationarity, see e.g. the book [6] of Collet, Martínez and San Martín, as well as the bibliography therein. □

The previous result provides a good picture for large values of τ, but is there a precursor sign that τ will be much shorter than expected? Indeed we cannot miss it, because in this situation of a precocious return to zero, the system has a strong tendency to first explode! To give a rigorous meaning of this statement, we need to introduce the bridges associated to Z. For $z, z' \in \mathbb{R}^2$ and $T > 0$, denote by $\mathbb{P}_{z,z'}^{(T)}$ the law of the process Z evolving according to (7), conditioned by the event $\{Z_0 = z,\ Z_T = z'\}$. Note that there is no difficulty to condition by this negligible set, because the process Z starting from z is Gaussian and the law of Z_T is non-degenerate.

For fixed $z, z' \in \mathbb{R}^2$ and $T > 0$ small, we are interested in the behavior of $\xi^{(T)} := (\xi_t^{(T)})_{t \in [0,1]}$, the process defined by

$$\forall\, t \in [0,1], \qquad \xi_t^{(T)} \;:=\; TZ_{Tt}$$

Let us define the trajectory $\varphi_{z,z'} : [0,1] \to \mathbb{R}^2$ by

$$\forall\, t \in [0,1], \qquad \varphi_{z,z'}(t) \;:=\; \begin{pmatrix} \frac{6\omega}{b^2} t(1-t)(y-y') \\ 0 \end{pmatrix} \tag{16}$$

where $z = (x,y)$ and $z' = (x', y')$.

Theorem 6 *For fixed $z, z' \in \mathbb{R}^2$, as T goes to 0_+, $\xi^{(T)}$ converges in probability (under $\mathbb{P}_{z,z'}^{(T)}$) toward the deterministic trajectory $\varphi_{z,z'}$, with respect to the uniform norm on $\mathcal{C}([0,1], \mathbb{R}^2)$.*
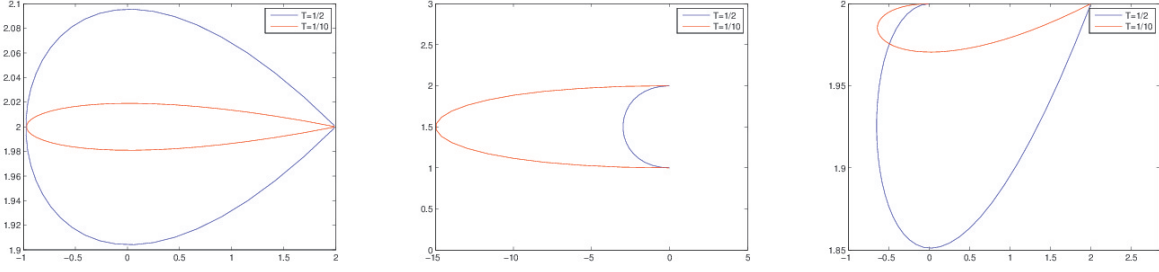
Figure 3: Expected trajectories of the hypo-elliptic bridge from z to z' within time $1/2$ and within small time $1/10$. Left: Non explosion when $z' = z$. Middle: explosion when $\Re(z) = \Re(z')$ and $\Im(z) \neq \Im(z')$. Right: Non explosion when $\Im(z) = \Im(z')$ and $\Re(z) \neq \Re(z')$.

In particular, if $z, z' \in \mathbb{R}^2$ are such that $\Re(z) > 0$, $\Re(z') \leq 0$ and $\Im(z) \neq \Im(z')$, the bridge $(Z_t)_{t \in [0,T]}$ relying z to z' for small $T > 0$ explodes as $1/T$. From the definition of $\varphi_{z,z'}$ given in (16), we can see that the explosion is in the x-direction, toward $+\infty$ or $-\infty$, depending on the sign of $\Im(z) - \Im(z')$, as it is illustrated by the pictures of Figure 3.

Remark 7 Note that the sharp behavior of the bridge when $T \to 0$ leads in Section 4 to a probabilistic proof of a lower-bound for $\mathbb{P}(\tau > t)$ (see Proposition 35). The interest of this alternative proof is that the approach is maybe more intuitive. However, we have not been able to provide some explicit constants following this method.

$\square$

The paper is constructed on the following plan. In next section we present the preliminaries on Z, especially its Gaussian features which enable to obtain Proposition 3. We will also see how to parametrize the process Z under a simpler form. The exit time defined in (13) is investigated in Section 3, where Theorem 4 is obtained. Section 4 is devoted to the study of bridges and to the proof of Theorem 6. At last, we shortly discuss in Section 5 on a numerical and statistical estimation problems related to the estimation of ω.

2 Preliminaries and simplifications

This section contains some basic results about the Ornstein-Uhlenbeck diffusion Z described by (7) and whose coefficients are given by (8).

2.1 Gaussian computations

Our main goal here is to prove Proposition 3.

We begin by checking that the process Z is Gaussian. Indeed, considering the process $\widetilde{Z}$ defined by

$$\forall\, t \geq 0, \qquad \widetilde{Z}_t \;:=\; \exp(-At)Z_t$$

we get that

$$\forall\, t \geq 0, \qquad d\widetilde{Z}_t \;=\; \exp(-At)(-AZ_t\,dt + dZ_t) \;=\; \exp(-At)C\,dB_t$$

It follows that

$$\forall\, t \geq 0, \qquad Z_t \;=\; \exp(At)Z_0 + \int_0^t \exp(A(t-s))C\,dB_s \tag{17}$$

$$=\; \int_0^t \exp(A(t-s))C\,dB_s$$

since we assumed that $Z_0 = 0$. It appears on this expression that for any $t \geq 0$, the law of Z_t is a Gaussian distribution of mean 0 and variance matrix Σ_t given by

$$\Sigma_t \;\coloneqq\; \int_0^t \exp(A(t-s))CC^* \exp(A^*(t-s))\,ds$$

$$=\; \int_0^t \exp(As)CC^* \exp(A^*s)\,ds \tag{18}$$

For $a, b > 0$, the eigenvalues of A have negative real parts, so that the above rhs converges as t goes to infinity toward a symmetric positive definite matrix Σ. As announced in the introduction, the Gaussian distribution μ of mean 0 and variance Σ is then an invariant measure for the evolution (7). It is a consequence of the fact that the underlying semi-group is Fellerian (i.e. it preserves the space of bounded continuous functions), as it can be seen from (17), where Z_t depends continuously on Z_0, for any fixed $t \geq 0$. Note furthermore that the above computations show that for any initial law of Z_0, the law of Z_t converges toward μ for large t, because $\exp(At)Z_0$ converges almost surely toward 0. It follows that μ is the unique invariant measure associated to (7). To obtain more explicit expressions for the above variances, we need the spectral decomposition of A. The characteristic polynomial of A being $X^2 + aX + ab$, we immediately obtain the results presented in the beginning of Subsection 1.2 about the eigenvalues of A. Let us treat in detail the case $a < 4b$, which is the most interesting for us: there are two conjugate eigenvalues, $\lambda_\pm = l \pm \omega i$, where $l = -a/2$ (see (10)) and ω is defined in (12).

Lemma 8 *If $a < 4b$, there exist two angles $\alpha \in (\pi/2, 3\pi/2)$ and $\beta \in [0, 2\pi)$ such that for any $t \geq 0$,*

$$\Sigma_t \;=\; R_0 - \exp(-at)R_t$$

where

$$R_t \;\coloneqq\; \frac{c^2}{4ab - a^2} \begin{pmatrix} \frac{b^2}{a}\left(2 + \sqrt{\frac{a}{b}}\cos(2\beta - \alpha - 2\omega t)\right) & \frac{b}{a}\left(2\cos(\beta) + \cos(\beta - \alpha - 2\omega t)\right) \\ \frac{b}{a}\left(2\cos(\beta) + \cos(\beta - \alpha - 2\omega t)\right) & \frac{b^4}{a}\left(2 + \sqrt{\frac{a}{b}}\cos(-\alpha - 2\omega t)\right) \end{pmatrix}$$

Passing to the limit as $t \to +\infty$, we get $\Sigma = R_0$ and we deduce more precisely that

$$\Sigma \;=\; \frac{c^2}{2a^2} \begin{pmatrix} a+b & -b^2 \\ -b^2 & b^3 \end{pmatrix}.$$

Proof

From the first line of the matrix A, we deduce that an eigenvector associated to $\lambda_\pm$ is $(1, \lambda_\pm + a - b)^*$. So writing

$$\triangle \;\coloneqq\; \begin{pmatrix} \lambda_- & 0 \\ 0 & \lambda_+ \end{pmatrix} \quad \text{and} \quad M \;\coloneqq\; \begin{pmatrix} 1 & 1 \\ \lambda_- + a - b & \lambda_+ + a - b \end{pmatrix}$$

we have that $A = M\triangle M^{-1}$, where

$$M^{-1} \;=\; \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} \lambda_+ + a - b & -1 \\ -\lambda_- + b - a & 1 \end{pmatrix}$$

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In view of (18), we need to compute for any $s \geq 0$,

$$\exp(As)CC^* \exp(A^*s) = M \exp(s\triangle)M^{-1}CC^*(M^*)^{-1}\exp(s\triangle^*)M^*$$

where $*$ is now the conjugate transpose operation. A direct computation leads to

$$\frac{\lambda_+ - \lambda_-}{c} M\exp(s\triangle)M^{-1}C = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} := \begin{pmatrix} z\exp(s\lambda_-) - \bar{z}\exp(s\lambda_+) \\ |z|^2(\exp(s\lambda_-) - \exp(s\lambda_+)) \end{pmatrix}$$

where $z := \lambda_+ + a - b = a/2 - b + i\omega$ and $\bar{z} := a - b + \lambda_- = a/2 - b - i\omega$. So we get that

$$\exp(As)CC^*\exp(A^*s) = \frac{c^2}{|\lambda_+ - \lambda_-|^2} \begin{pmatrix} |\sigma_1|^2 & \sigma_1\overline{\sigma_2} \\ \sigma_2\overline{\sigma_1} & |\sigma_2|^2 \end{pmatrix}$$

$$= \frac{c^2}{4\omega^2} \begin{pmatrix} 2|z|^2 e^{2ls} - 2\Re(z^2 e^{2\lambda_- s}) & 2\Re(z)e^{2ls} - 2\Re(ze^{2\lambda_- s}) \\ 2\Re(z)e^{2ls} - 2\Re(ze^{2\lambda_- s}) & 2|z|^4(e^{2ls} - \Re(e^{2\lambda_- s})) \end{pmatrix}$$

Integrating this expression with respect to s, we obtain, first for any $t \geq 0$,

$$\Sigma_t = \frac{c^2}{4\omega^2} \begin{pmatrix} |z|^2\frac{e^{2lt}-1}{l} - \Re(z^2\frac{e^{2\lambda_- t}-1}{\lambda_-}) & \Re(z)\frac{e^{2lt}-1}{l} - \Re(z\frac{e^{2\lambda_- t}-1}{\lambda_-}) \\ \Re(z)\frac{e^{2lt}-1}{l} - \Re(z\frac{e^{2\lambda_- t}-1}{\lambda_-}) & |z|^4(\frac{e^{2lt}-1}{l} - \Re(\frac{e^{2\lambda_- t}-1}{\lambda_-})) \end{pmatrix}$$

and next, recalling that $\Re(\lambda_-) = \Re(\lambda_+) = l < 0$,

$$\Sigma := \lim_{t\to+\infty} \Sigma_t = \frac{c^2}{4\omega^2} \begin{pmatrix} -|z|^2\frac{1}{l} + \Re(z^2\frac{1}{\lambda_-}) & -\Re(z)\frac{1}{l} + \Re(z\frac{1}{\lambda_-}) \\ -\Re(z)\frac{1}{l} + \Re(z\frac{1}{\lambda_-}) & -|z|^4(\frac{1}{l} - \Re(\frac{1}{\lambda_-})) \end{pmatrix}$$

Thus it appears that

$$\forall\, t \geq 0, \qquad \Sigma_t = \Sigma - e^{-at}R_t$$

where the last term is the sinusoidal matrix defined by

$$R_t := \frac{c^2}{4\omega^2} \begin{pmatrix} -|z|^2\frac{1}{l} + \Re(z^2\frac{e^{-2\omega it}}{\lambda_-}) & -\Re(z)\frac{1}{l} + \Re(z\frac{e^{-2\omega it}}{\lambda_-}) \\ -\Re(z)\frac{1}{l} + \Re(z\frac{e^{-2\omega it}}{\lambda_-}) & -|z|^4(\frac{1}{l} - \Re(\frac{e^{-2\omega it}}{\lambda_-})) \end{pmatrix}$$

Note that $\Sigma = R_0$ (this can also be deduced from $\Sigma_0 = 0$). To recover the matrices given in the statement of the lemma, we remark that $|\lambda_-|^2 = ab$ and $|z|^2 = b^2$, so there exist angles $\alpha, \beta \in [0, 2\pi)$ such that

$$\lambda_- = \sqrt{ab}\exp(i\alpha) \quad \text{and} \quad z = b\exp(i\beta)$$

Since $\Re(\lambda_-) < 0$, we have $\alpha \in (\pi/2, 3\pi/2)$ and the first announced results follow at once. Concerning the more explicit computation of Σ, just take into account that

$$\cos(\alpha) = -\frac{\sqrt{a}}{2\sqrt{b}} \quad \text{and} \quad \sin(\alpha) = -\frac{\omega}{\sqrt{ab}}$$

$$\cos(\beta) = \frac{a-2b}{2b} \quad \text{and} \quad \sin(\beta) = \frac{\omega}{b}$$

and expand the matrix

$$R_0 = \frac{c^2 b}{(4b-a)a^2} \begin{pmatrix} b(2 + \sqrt{\frac{a}{b}}\cos(2\beta - \alpha)) & (2\cos(\beta) + \cos(\beta - \alpha)) \\ (2\cos(\beta) + \cos(\beta - \alpha)) & b^3(2 + \sqrt{\frac{a}{b}}\cos(\alpha)) \end{pmatrix}$$

∎

Remark 9 In particular, it appears that for large time $t \geq 0$, X_t converges in law toward the centered Gaussian distribution ν of variance $c^2(b+a)/(2a^2)$.

$\square$

We will need another basic ingredient, valid in any dimension, about the functional J defined in (11).

Lemma 10 *Let μ and $\widetilde{\mu}$ be two Gaussian distributions in $\mathbb{R}^d$, $d \geq 1$, of mean 0 and respective variance matrices Σ and $\widetilde{\Sigma}$, assumed to be positive definite. If $\widetilde{\Sigma}^{-1} - \Sigma^{-1}/2$ is positive definite, we have*

$$J(\widetilde{\mu}, \mu) = \sqrt{\frac{1}{\sqrt{\det(\mathrm{Id} - S^2)}} - 1}$$

where $S := \Sigma^{-1}\widetilde{\Sigma} - \mathrm{Id}$ and $J(\widetilde{\mu}, \mu) = +\infty$ otherwise.

Proof

From the above assumptions, we have

$$\forall \, x \in \mathbb{R}^d, \qquad \frac{d\widetilde{\mu}}{d\mu}(x) = \sqrt{\frac{\det(\Sigma)}{\det(\widetilde{\Sigma})}} \exp\left(-x^* \frac{(\widetilde{\Sigma}^{-1} - \Sigma^{-1})}{2} x\right)$$

Thus the function $\frac{d\widetilde{\mu}}{d\mu}$ belongs to $\mathbb{L}^2(\mu)$ (property itself equivalent to the finiteness of $J(\widetilde{\mu}, \mu)$), if and only if the symmetric matrix $\widetilde{\Sigma}^{-1} - \Sigma^{-1} + \Sigma^{-1}/2$ is positive definite. In this case, we have

$$
\begin{aligned}
\int \left(\frac{d\widetilde{\mu}}{d\mu}\right)^2 d\mu &= \frac{\det(\Sigma)}{\det(\widetilde{\Sigma})} \sqrt{\frac{\det((2\widetilde{\Sigma}^{-1} - \Sigma^{-1})^{-1})}{\det(\Sigma)}} \\
&= \frac{\sqrt{\det(\Sigma)}}{\det(\widetilde{\Sigma})} \sqrt{\det(\widetilde{\Sigma}) \det((2\mathrm{Id} - \Sigma^{-1}\widetilde{\Sigma})^{-1})} \\
&= \sqrt{\frac{\det(\Sigma)}{\det(\widetilde{\Sigma})}} \sqrt{\det((\mathrm{Id} - S)^{-1})} \\
&= \frac{1}{\sqrt{\det(\mathrm{Id} + S) \det(\mathrm{Id} - S)}}
\end{aligned}
$$

where we used that $\widetilde{\Sigma} = \Sigma(\mathrm{Id} + S)$. It remains to note that

$$J^2(\widetilde{\mu}, \mu) = \int \left[\left(\frac{d\widetilde{\mu}}{d\mu}\right)^2 - 2\frac{d\widetilde{\mu}}{d\mu} + 1\right] d\mu = \int \left(\frac{d\widetilde{\mu}}{d\mu}\right)^2 d\mu - 1$$

$\blacksquare$

Still in the case $4b > a$, we can now proceed to the

Proof of Proposition 3

In view of Lemma 8, we want to apply Lemma 10 with $\Sigma = R_0$ and $\widetilde{\Sigma} = R_0 - \exp(-at)R_t$, for $t \geq 0$. This amounts to take $S := -\exp(-at)R_0^{-1}R_t$, matrix converging to zero exponentially fast as t goes to $+\infty$. It follows that for t large enough, $\widetilde{\Sigma}^{-1} - \Sigma^{-1}/2$ is positive definite and we get

$$J(\mu_t, \mu) = \sqrt{\frac{1}{\sqrt{\det(\mathrm{Id} - S^2)}} - 1}$$

13

Taking into account that the matrices R_t are bounded uniformly over $t \in \mathbb{R}_+$, an expansion for large t gives

$$
\begin{aligned}
\frac{1}{\sqrt{\det(\mathrm{Id} - S^2)}} &= \frac{1}{\sqrt{1 - \mathrm{tr}(S^2) + \mathcal{O}(\|S^2\|_{\mathrm{HS}}^2)}} \\
&= 1 + \frac{1}{2}\mathrm{tr}(R_0^{-1}R_t R_0^{-1}R_t)\exp(-2at) + \mathcal{O}(\exp(-4at))
\end{aligned}
$$

(where $\|\cdot\|_{\mathrm{HS}}$ stands for the Hilbert-Schmidt norm, i.e. the square root of the sum of the squares of the entries of the matrix). We will be able to conclude to

$$
\lim_{t\to+\infty} \frac{1}{t}\ln(J(\mu_t, \mu)) = -a \tag{19}
$$

if we can show that

$$
\liminf_{t\to+\infty} \mathrm{tr}(R_0^{-1}R_t R_0^{-1}R_t) > 0 \tag{20}
$$

(since it is clear that $\limsup_{t\to+\infty} \mathrm{tr}(R_0^{-1}R_t R_0^{-1}R_t) < +\infty$). Taking advantage of the fact that R_0 is a symmetric and positive definite matrix, we consider for $t \geq 0$, $\widehat{R}_t := R_0^{-1/2}R_t R_0^{-1/2}$, which is also a symmetric matrix. Since $\mathrm{tr}(R_0^{-1}R_t R_0^{-1}R_t) = \mathrm{tr}(\widehat{R}_t^2) = \left\|\widehat{R}_t\right\|_{\mathrm{HS}}^2$, this quantity is nonnegative and can only vanish if $\widehat{R}_t$, or equivalently R_t, is the null matrix. This never happens, because the first entry of R_t, namely $(cb/a)^2(2 + \sqrt{a/b}\cos(2\beta - \alpha - 2\omega t))/(4b - a)$, is positive. The continuity and the periodicity of the mapping $\mathbb{R}_+ \ni t \mapsto R_t$ enables to check the validity of (20) and next of (19).

The corresponding result for the first marginal ν_t (the law of X_t) is obtained in the same way. Indeed, from Lemma 8, for any $t \geq 0$, ν_t is the real Gaussian law of mean 0 and variance $r_0 - \exp(-at)r_t$, where

$$
\forall\, t \geq 0, \qquad r_t := \frac{(cb)^2}{4a^2b - a^3}\left(2 + \sqrt{\frac{a}{b}}\cos(2\beta - \alpha - 2\omega t)\right)
$$

with the angles $\alpha \in (\pi/2, 3\pi/2)$ and $\beta \in [0, 2\pi)$ described in the proof of Lemma 8. In particular $\nu := \lim_{t\to+\infty} \nu_t$ is the real Gaussian law of mean 0 and variance r_0. Lemma 10 applied with $d = 1$ leads at once to

$$
\lim_{t\to+\infty} \frac{1}{t}\ln(J(\nu_t, \nu)) = -a
$$

$\blacksquare$

The remaining situations $a = 4b$ and $a > 4b$ can be treated in the same way. In view of the previous arguments, it is sufficient to check that it is possible to write

$$
\forall\, t \geq 0, \qquad \Sigma_t = \Sigma - \exp(-2lt)R_t
$$

where l is defined in (10) and where the family $(R_t)_{t\geq 0}$ is such that

$$
\lim_{t\to+\infty} \frac{1}{t}\ln(\|R_t\|) = 0
$$

for any chosen norm $\|\cdot\|$ on the space of 2×2 real matrices, due to their mutual equivalence. The obtention of the family $(R_t)_{t\geq 0}$ also relies on the spectral decomposition of A, with R_t converging for large times t if $a > 4b$ and exploding like t^2 if $a = 4b$.

2.2 Simplifications with the view to Theorem 4

Let us begin this subsection by emphasizing two important properties of Theorem 4:
- The result does not depend on the variance coefficient c.
- The exponent is proportional to $\omega = \sqrt{ab - a^2/4}$ which denotes the mean angular speed of the deterministic system $\dot{z} = Az$.

These properties can be understood through some linear and scaling transformations of the process $(Z_t)_{t\geq 0}$. More precisely, these transformations will be used in the sequel to reduce the problem to the study of a process with mean constant angular speed and a normalized diffusion component.

We choose to first give the idea in a general case and then, apply it to our model.

Let $A \in GL_2(\mathbb{R})$ with complex eigenvalues given by $\lambda_{\pm} = -\rho \pm i\omega$ where $\rho \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$. Let us consider the two-dimensional Gaussian differential system given by

$$d\zeta_t = A\zeta_t dt + \Sigma dB_t \tag{21}$$

where $\Sigma \in \mathbb{M}_2(\mathbb{R})$ and $(B_t)_{t\geq 0}$ is a standard two dimensional Brownian motion. For such a process, the precise transformation is given in Proposition 12. This proposition is based on the following lemma.

Lemma 11 *Let $A \in GL_2(\mathbb{R})$ with complex eigenvalues given by $\lambda_{\pm} = -\rho \pm i\omega$ where $\rho \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$. There exists $P \in GL_2(\mathbb{R})$ such that*

$$A = P(-\rho I_2 + \omega J_2)P^{-1}$$

where

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad and \qquad J_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{22}$$

Furthermore, for every $v \in \mathbb{R}^2 \backslash \{0\}$, $P := P_v$ given by $P_v = (v, \frac{D+\rho I_2}{\omega}v)$ is an admissible choice.

Proof

Set $B = \frac{D - \rho I_2}{\omega}$. The eigenvalues of B are $\pm i$ so that $B^2 = -I_2$. For any $v \in \mathbb{R}^2$, set $P_v = (v, Bv)$. The matrix P_v is clearly invertible and using that $B^2v = -v$, one obtains that $B = P_v J_2 P_v^{-1}$. The result follows.

∎

Proposition 12 *Let $(\zeta_t)_{t\geq 0}$ be a solution to (21) where $A \in GL_2(\mathbb{R})$ with complex eigenvalues given by $\lambda_{\pm} = -\rho \pm i\omega$ (with $\rho \in \mathbb{R}$ and $\omega \in \mathbb{R}_+^*$). For any $\alpha \in \mathbb{R}^*$ and $v \in \mathbb{R}^2 \backslash \{0\}$, set $\hat{\zeta}_t = \sqrt{\omega}\alpha P_v^{-1}\zeta_{\frac{t}{\omega}}$. The process $(\hat{\zeta}_t)_{t\geq 0}$ is a solution to*

$$d\hat{\zeta}_t = -\frac{\rho}{\omega}\hat{\zeta}_t + J_2\hat{\zeta}_t + \alpha P_v^{-1}\Sigma dW_t \tag{23}$$

where (W_t) is a standard two-dimensional Brownian motion.

Proof

First, set $\tilde{\zeta}_t^v = P_v^{-1}\zeta_t$. Owing to the preceding lemma, $(\tilde{\zeta}_t^v)_{t\geq 0}$ is a solution to

$$d\tilde{\zeta}_t^v = -\rho\tilde{\zeta}_t^v + \omega J_2\tilde{\zeta}_t^v + P_v^{-1}\Sigma dB_t.$$

For any $\alpha \in \mathbb{R}^*$, set $\hat{\zeta}_t = \sqrt{\omega}\alpha\tilde{\zeta}^v_{\frac{t}{\omega}}$. Setting $W_t = \sqrt{\omega}B_{\frac{t}{\omega}}$ (which is a Brownian motion), one checks that

$$d\hat{\zeta}_t = -\frac{\rho}{\omega}\hat{\zeta}_t + J_2\hat{\zeta}_t + \alpha P_v^{-1}\Sigma dW_t.$$

∎

We now apply this proposition to our problem.

Corollary 13 *Let $(Z_t)_{t\geq 0}$ be a solution to (7) and assume that $a < 4b$. Set $\omega = \sqrt{ab - \frac{a^2}{4}}$. Let $v \in \mathbb{R}^2 \setminus \{0\}$ and set $P_v = (v, Bv)$ with $B = \frac{1}{\omega}(A + \frac{a}{2}I_2)$. Then for any $\alpha \in \mathbb{R} \setminus \{0\}$, the process $(\hat{Z}_t)_{t\geq 0}$ defined by $\hat{Z}_t = \sqrt{\omega}\alpha P_v^{-1}Z_{\frac{t}{\omega}}$ is a solution to*

$$d\hat{Z}_t = -\frac{a}{2\omega}\hat{Z}_t + J_2\hat{Z}_t + \alpha c P_v^{-1}\Sigma dW_t \quad \text{with} \quad \Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{24}$$

where W is a standard two-dimensional Brownian motion. In particular, if $v = (\frac{1}{b^2}(\frac{a}{2} - b), 1)^*$ and $\alpha = \frac{\sqrt{2}\omega}{cb^2}$, then $(\hat{Z}_t) := (U_t, V_t)$ is a solution to

$$\begin{cases} dU_t = -\frac{a}{2\omega}U_t - V_t dt \\ dV_t = -\frac{a}{2\omega}V_t + U_t + \sqrt{2}dW_t. \end{cases} \tag{25}$$

where W is now a standard one-dimensional Brownian motion.

Remark 14 In the second part, of the corollary, one remarks that one chooses v in order that the transformed process has only a (normalized) diffusive component on the second coordinate.

Furthermore, if Z has $(x, y) \in \mathbb{R}^2$ for initial deterministic condition, then $\hat{Z}$ starts from the point $\sqrt{\omega}c^{-1}b^{-2}(\omega y, b^2 x + (b - a/2)y)$. The images of $(1, 0)^*$ and $(0, 1)^*$ by P_v^{-1} are particularly important for our purposes, since they enable to see that the half-plane $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ for Z is transformed into the half-plane $\{(u, v) \in \mathbb{R}^2 : v > \frac{2b-a}{2\omega}u\}$ for $\hat{Z}$. Note that in the setting $a << b$, the latter half-plane is quite similar to the former one, since $\omega \sim \sqrt{ab} << b$.

□

Proof

We recall that $(Z_t) = (X_t, Y_t)$ is a solution to

$$dZ_t = AZ_t + c\Sigma dB_t \quad \text{where} \quad A = \begin{pmatrix} b - a & 1 \\ -b^2 & -b \end{pmatrix} \quad \text{and}$$

and Σ is defined in (24). When $a < 4b$, the eigenvalues of A are given by

$$\lambda_{\pm} = -\frac{a}{2} \pm i\omega.$$

For any $v \in \mathbb{R}^2$, set $P_v := (v, Bv)$ with

$$B = \frac{1}{\omega}(A + \frac{a}{2}I_2).$$

Applying the previous proposition, we deduce that for any $\alpha \in \mathbb{R}^*$, $(\hat{Z}_t)_{t\geq 0} := (\alpha\sqrt{\omega}P_v^{-1}Z_{\frac{t}{\omega}})_{t\geq 0}$ is a solution to

$$d\hat{Z}_t = -\frac{a}{2\omega}\hat{Z}_t + J_2\hat{Z}_t + \alpha c P_v^{-1}\Sigma d\hat{W}_t$$

16

where $(\hat{W}_t)$ is a standard two-dimensional Brownian motion. For the second part, it remains to choose v and α so that

$$\alpha c P_v^{-1} \Sigma = \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}. \tag{26}$$

If $v = (u_1, u_2)^*$, then

$$P_v = \begin{pmatrix} u_1 & \frac{1}{\omega}\left((b-\frac{a}{2})u_1 + u_2\right) \\ u_2 & -\frac{1}{\omega}\left(b^2 u_1 + (b-\frac{a}{2})u_2\right) \end{pmatrix} \tag{27}$$

One deduces that condition (26) (or more precisely the fact that $(P_v^{-1}\Sigma)_{1,1} = 0$) implies $b^2 u_1 + (b-\frac{a}{2})u_2 = 0$. Setting $v = (\frac{1}{b^2}(\frac{a}{2}-b), 1)^*$, we have

$$P_v = \begin{pmatrix} \frac{1}{b^2}\left(\frac{a}{2}-b\right) & \frac{\omega}{b^2} \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad P_v^{-1} = \begin{pmatrix} 0 & 1 \\ \frac{b^2}{\omega} & \frac{1}{\omega}\left(\frac{a}{2}-b\right) \end{pmatrix}.$$

Condition (26) is then satisfied when $\alpha = \frac{\sqrt{2}\omega}{cb^2}$.

∎

3 Dirichlet eigenvalues estimates

This section is devoted to the proof of Theorem 4. More precisely, the aim is to obtain successively upper and lower bounds for $\mathbb{P}_{(x_0,y_0)}(\tau > t)$ where $\tau := \inf\{t \geq 0, X_t \leq 0\}$. In fact, some of the results will be stated for exit times of more general domains. For a given (open) domain $\mathcal{S}$ of $\mathbb{R}^2$, we will thus denote by

$$\tau_{\mathcal{S}} := \inf\{t \geq 0, (X_t, Y_t) \in \mathcal{S}^c\}.$$

3.1 Upper-bound for the exit time of an angular sector $\mathcal{S}$

3.1.1 The case $\mathcal{S} = \{(x,y), x > 0\}$

In this part, we focus on the particular stopping time τ of Theorem 4 which corresponds to the exit time of $D = \{(x,y), x > 0\}$. We have the following result:

Proposition 15 *Let $(Z_t)_{t\geq 0}$ be a solution to (7) with $a < 4b$. Then, for every $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0 > 0$,*

$$\mathbb{P}_{(x_0,y_0)}(\tau > t) \leq 2\exp\left(-\frac{\log 2}{\pi}\omega t\right).$$

Proof

Set $z(t) = (x(t), y(t)) = (\mathbb{E}[X_t], \mathbb{E}[Y_t])$. The function $(z(t))_{t\geq 0}$ being a solution to $\dot{z} = Az$, we deduce in particular that

$$\forall t \geq 0, \quad \ddot{x}(t) + a\dot{x} + abx(t) = 0.$$

Since $a < 4b$, the roots of the characteristic equation associated with the previous equation are: $\lambda_\pm = -\frac{a}{2} \pm \omega$ where $\omega = \sqrt{ab - a^2/4}$. As a consequence, there exists $C > 0$ and $\varphi_0 \in (-\pi, \pi]$ such that

$$x(t) = C\cos(\omega t + \varphi_0), \quad t \geq 0.$$

Reminding that $x_0 > 0$, we deduce that $\varphi_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Thus, at time $T_\omega = \frac{\pi}{\omega}$,

$$\forall x_0 > 0, \quad \omega T_\omega + \phi_0 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \implies x(T_\omega) < 0.$$

But $x(T_\omega) = \mathbb{E}[X_{T_\omega}]$ and X_{T_ω} is Gaussian and has a symmetric distribution. Thus, we deduce from what precedes that

$$\forall x_0 > 0, \ y_0 \in \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(X_{T_\omega} < 0) \geq \frac{1}{2}$$

which in turn implies that

$$\forall x_0 > 0, \ y_0 \in \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(\tau \geq T_\omega) \leq \frac{1}{2}. \tag{28}$$

Thus, we have a upper-bound at time T_ω which does not depend on the initial value (x_0, y_0). As a consequence, we can use a Markov argument. More precisely, owing to the Markov property and to (28), we have for every integer $k \geq 1$:

$$\mathbb{P}(\tau > kT_\omega | \tau > (k-1)T_\omega) = \frac{\mathbb{E}[\mathbb{P}_{(X_{(k-1)T_\omega}, Y_{(k-1)T_\omega})}(\tau > T_\omega)\mathbf{1}_{\tau > (k-1)T_\omega}]}{\mathbb{P}(\tau > (k-1)T_\omega)} \leq \sup_{x_0 > 0, y_0 \in \mathbb{R}} \mathbb{P}_{(x_0,y_0)}(\tau > T_\omega).$$

An iteration of this property yields

$$\forall n \in \mathbb{N}, \ \forall (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(\tau > nT) \leq \left(\frac{1}{2}\right)^n.$$

It follows that

$$\forall t \geq 0, \ \forall (x_0, y_0) \in \mathbb{R}_+^* \times \mathbb{R}, \quad \mathbb{P}_{(x_0,y_0)}(\tau > t) \leq \left(\frac{1}{2}\right)^{\lfloor \frac{t}{T_\omega} \rfloor} \leq 2 \exp\left(-\frac{\log 2}{T_\omega} t\right).$$

This concludes the proof.

3.1.2 Extension to general angular sectors

We now consider an angular sector $\mathcal{S}_{\alpha_1,\alpha_2}$ defined as

$$\mathcal{S}_{\alpha_1,\alpha_2} = \{(x,y) \in \mathbb{R}^2, \ x > 0, \ \alpha_1 x < y < \alpha_2 x\} \tag{29}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 < \alpha_2$. The set $\mathcal{S}_{\alpha_1,\alpha_2}$ can also be written $\mathcal{S}_{\alpha_1,\alpha_2} = \{(r\cos\theta, r\sin\theta), r > 0, \theta_1 < \theta < \theta_2\}$ with $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$. Note that for the sake of simplicity, we only consider angular sectors which are included in $\{(x,y), x > 0\}$. The results below can be extended to any angular sectors for which the angular size is lower than π. For such domains, we first give a result when the model has a constant (mean) angular speed even if such a result does not apply to the solutions of (7) for sake of completeness. This is the purpose of Lemma 16 below.

Concerning now our initial motivation, we also derive an extension of Proposition 15 for any general angular sector, and this result is stated in Proposition 17.

Lemma 16 *Let $(Z_t)_{t \geq 0}$ be a solution of*

$$dZ_t = -\rho Z_t + \omega J_2 Z_t + \Sigma dW_t$$

where $\rho \in \mathbb{R}$, $\omega \in \mathbb{R}_+^*$, $\Sigma \in \mathbb{M}_2(\mathbb{R})$ and W is a two-dimensional Brownian motion. Let $\mathcal{S}_{\alpha_1,\alpha_2}$ be defined by (29) where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 < \alpha_2$. Then, for any $(x_0, y_0) \in \mathcal{S}_{\alpha_1,\alpha_2}$,

$$\mathbb{P}(\tau_{\mathcal{S}_{\alpha_1,\alpha_2}} \geq t) \leq 2 \exp\left(-\frac{\ln(2)}{\theta_2 - \theta_1} \omega t\right)$$

with $\theta_1 = \text{Arctan}(\alpha_1)$ and $\theta_2 = \text{Arctan}(\alpha_2)$.

Proof

Let $z(t) = (\mathbb{E}[X_t], \mathbb{E}[Y_t])$ and define $(u(t))_{t \geq 0} := (e^{\rho t} z(t))_{t \geq 0}$, u is a solution of

$$\dot{u} = \omega J_2 u.$$

We deduce that

$$e^{\rho t} z(t) = (A \cos(\omega t + \varphi), A\omega \sin(\omega t + \varphi))$$

where $A \geq 0$ and $\varphi \in [-\pi, \pi)$. This implies that the angular rate of $(z(t))_{t \geq 0}$ is constant and is equal to ω. Thus, it follows that for every starting point $(x, y) \in \mathcal{S}_{\alpha_1, \alpha_2}$,

$$z(T_\omega) \in \mathcal{S}_{\alpha_1, \alpha_2}^c \quad \text{with} \quad T_\omega = \omega(\theta_2 - \theta_1).$$

One can then find a line passing through 0 and dividing $\mathbb{R}^2$ into two half-planes D^+ and D^- such that $\mathcal{S}_{\alpha_1, \alpha_2}$ is included in D^- and $z(T_\omega) \in D^+$. Owing to the symmetry of a one-dimensional centered Gaussian distribution, we have

$$\mathbb{P}((X_{T_\omega}, Y_{T_\omega}) \in D^-) = \mathbb{P}((X_{T_\omega}, Y_{T_\omega}) \in D^+) = \frac{1}{2}.$$

One finally deduces that for every $(x, y) \in \mathcal{S}_{\alpha_1, \alpha_2}$,

$$\mathbb{P}((X_{T_\omega}, Y_{T_\omega}) \in \mathcal{S}_{\alpha_1, \alpha_2}^c) \geq \frac{1}{2}.$$

and thus that

$$\forall (x, y) \in \mathcal{S}_{\alpha_1, \alpha_2}, \quad \mathbb{P}_{(x,y)}(\tau_{\mathcal{S}_{\alpha_1, \alpha_2}} > T_\omega) \leq \frac{1}{2}.$$

The end of the proof is then identical to that of Proposition 15.

∎

We now consider our initial bubble process $(Z_t)_{t \geq 0}$ which is solution of Equation (7). We have the following result.

Proposition 17 *Let $(Z_t)_{t \geq 0}$ be a solution to (7). Let $\mathcal{S}_{\alpha_1, \alpha_2}$ be defined by (29) where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 < \alpha_2$. Then, for any $(x_0, y_0) \in \mathcal{S}_{\alpha_1, \alpha_2}$,*

$$\mathbb{P}(\tau_{\mathcal{S}_{\alpha_1, \alpha_2}} \geq t) \leq 2 \exp\left(-\frac{\ln 2}{\tilde{\theta}_1 - \tilde{\theta}_2} \omega t\right)$$

with $\tilde{\theta}_i = \text{Arctan}\left(\frac{a/2 - b - \alpha_i}{\omega}\right)$, $i = 1, 2$.

Remark 18 Taking $\alpha_1 = -\infty$ and $\alpha_2 = +\infty$, we retrieve Proposition 15 since $\mathcal{S}_{-\infty, +\infty}$ then corresponds to the half-plane $\{x > 0\}$. Note that contrary to Lemma 16, the exponential rate is not directly proportional to ω. More precisely, due to the non constant angular speed, $\tilde{\theta}_1$ and $\tilde{\theta}_2$ depend on ω. For the particular domain of Proposition 15 this dependence does not appear since, even if the the angular rate is not constant, the time to do a *U-turn* is still proportional to ω.

□

Proof

By Corollary 13, for any v of $\mathbb{R}^2 \setminus \{0\}$, $(\tilde{Z}_t)_{t \geq 0} = (P_v^{-1} Z_{\frac{t}{\omega}})_{t \geq 0}$ (where $P_v = (v, Bv)$) is a solution of

$$d\tilde{Z}_t = (-\rho I + J_2)\tilde{Z}_t + \tilde{\Sigma} dW_t$$

where $\tilde{\Sigma}$ is a constant real matrix (whose exact expression is not so important) and where $\rho = a/(2\omega)$. In the new basis $\tilde{\mathcal{B}} = (v, Bv)$,

$$\mathcal{S}_{\alpha_1, \alpha_2} = \{\tilde{z} = (\tilde{x}, \tilde{y})_{\tilde{\mathcal{B}}} \in \mathbb{R}^2, \alpha_1 (P_v \tilde{z})_1 < (P_v \tilde{z})_2 < \alpha_2 (P_v \tilde{z})_1\}.$$

Setting $v = (1, \frac{a}{2} - b)$, we deduce from Equation (27) that

$$P_v = \begin{pmatrix} 1 & 0 \\ (\frac{a}{2} - b) & -\omega \end{pmatrix}$$

In such a case

$$P_v \tilde{z} = \begin{pmatrix} \tilde{x} \\ (\frac{a}{2} - b)\tilde{x} - \omega \tilde{y} \end{pmatrix}$$

so that

$$\mathcal{S}_{\alpha_1, \alpha_2} = \{\tilde{z} = (\tilde{x}, \tilde{y})_{\tilde{\mathcal{B}}} \in \mathbb{R}^2, \left(\frac{a}{2} - b - \alpha_2\right) \tilde{x} < \omega \tilde{y} < \left(\frac{a}{2} - b - \alpha_1\right) \tilde{x}\}.$$

Thus, we deduce from Lemma 16 that

$$\mathbb{P}_{\tilde{x}, \tilde{y}}(\tilde{\tau}_{\mathcal{S}_{\alpha_1, \alpha_2}} \geq t) \leq 2 \exp\left(-\frac{\ln 2}{\tilde{\theta}_1 - \tilde{\theta}_2} t\right)$$

where for a given domain A, $\tilde{\tau}_A := \inf\{t \geq 0, \tilde{Z}_t \in A^c\}$ and $\tilde{\theta}_i = \text{Arctan}\left(\frac{a/2 - b - \alpha_i}{\omega}\right)$, $i = 1, 2$. The result follows.

■

3.2   Lower-bound

3.2.1   General tool

In this second part, our aim is to obtain the lower-bound part of Theorem 4, in particular we want to derive a upper-bound on:

$$\bar{\lambda} \;:=\; \limsup_{t \to +\infty} \; -\frac{1}{t} \log(\mathbb{P}(\tau \geq t)).$$

The results of this section are based on the following (classical) proposition.

Proposition 19 *Let $(X_t)_{t \geq 0}$ be a $\mathbb{R}^d$-valued Markov process with infinitesimal generator L and initial distribution m_0. Let $\mathcal{S}$ be an (open) domain of $\mathbb{R}^d$ and assume that $m_0(\mathcal{S}) = 1$. Let $\tau := \inf\{t > 0, X_t \in \mathcal{S}^c\}$. Then, if there exists a bounded function $f : \mathbb{R}^d \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that*

$$\begin{cases} f_{/\partial \mathcal{S}} = 0 & \text{and} \quad f_{/\mathcal{S}} > 0 \\ \forall x \in \mathcal{S}, & Lf(x) \geq -\lambda f(x) \end{cases} \tag{30}$$

then, $\mathbb{E}_{m_0}[e^{\lambda \tau}] = +\infty$. As a consequence,

$$\limsup_{t \to +\infty} \; -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau \geq t)) \;\; \leq \;\; \lambda.$$

Proof

Owing to the Dynkin formula, we have for every $t \geq 0$

$$e^{\lambda(t\wedge\tau)}f(X_{t\wedge\tau}) = f(X_0) + \int_0^{t\wedge\tau} (\lambda f(X_{s\wedge\tau}) + Lf(X_{s\wedge\tau}))e^{\lambda(s\wedge\tau)}ds + M_{t\wedge\tau}$$

where (M_t) is a local martingale. Owing to a localization argument and to Fatou's Lemma, we deduce that

$$\mathbb{E}_{m_0}[e^{\lambda(t\wedge\tau)}f(X_{t\wedge\tau})] = \int f(x)m_0(dx) + \mathbb{E}_{m_0}\left[\int_0^{t\wedge\tau} (\lambda f(X_{s\wedge\tau}) + Lf(X_{s\wedge\tau}))e^{\lambda(s\wedge\tau)}ds\right].$$

Suppose that $\mathbb{E}_{m_0}[e^{\lambda\tau}] < +\infty$. Then, using the dominated convergence theorem and the fact that $f_{/\partial\mathcal{S}} = 0$, we have

$$\lim_{t\to+\infty} \mathbb{E}_{m_0}[e^{\lambda(t\wedge\tau)}f(X_{t\wedge\tau})] = 0$$

whereas the fact that $Lf \geq -\lambda f$ implies that the right-hand term is uniformly lower-bounded by $\int f(x)m_0(dx)$ which is (strictly) positive. This yields a contradiction. Thus, $\mathbb{E}_{m_0}[e^{\lambda\tau}] = +\infty$. The second assertion then classically follows from the equality

$$\mathbb{E}_{m_0}[e^{\lambda\tau}] = \int_{\mathbb{R}} \lambda e^{\lambda t}\mathbb{P}_{m_0}(\tau > t)dt.$$

$\blacksquare$

The end of Section 3.2 is devoted to the construction of a function f satisfying (30). In fact, for this part, the degeneracy of the process described by Equation (7) implies a significant amount of difficulties. That is why we propose in the next subsection to first focus on the elliptic case which can be handled more easily. Some of the ideas developed in this framework will then be extended to the initial hypoelliptic setting.

3.2.2 The elliptic case

By Corollary 13, we know that we can reduce the problem to the study of a process (U_t, V_t) solution to (25). In this part we focus on its elliptic counterpart: we consider a two dimensional process $(\xi_t)_{t\geq 0}$ solution of the following stochastic differential equation

$$d\xi_t = (-\rho\xi_t + J_2\xi_t)dt + \sqrt{2}dW_t \tag{31}$$

where ρ is a real number and W is a standard two-dimensional Brownian motion.
We define $\mathcal{S}_0 = \{(x,y), x > 0\}$. In order to build a suitable function f satisfying (30) with $D = \mathcal{S}_0$ and relatively to the infinitesimal generator L_ρ associated to (31), we first switch to polar coordinates: Proposition 41 (stated in the second appendix) shows that L_ρ is given on $\mathcal{C}^2(\mathbb{R}_+^* \times \mathbb{R})$ by

$$L_\rho = -\rho r\partial_r + \partial_\theta + \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2. \tag{32}$$

To justify the approach developed below, let us forget formally the derivatives with respect to r by fixing $r > 0$ and by considering unknown angular functions $G_r : [-\pi/2, \pi/2] \to \mathbb{R}$. The problem of finding G_r and λ_r such that (30) holds (with $D = (-\pi/2, \pi/2)$) reduces to solve the second order ordinary differential equation

$$r^{-2}G_r''(\theta) + G_r'(\theta) = -\lambda_r G_r(\theta), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

with $G_r(\pi/2) = G_r(-\pi/2) = 0$. The solutions are given by exponential functions $G_r(\theta) = \alpha_1 e^{\rho_1 \theta} + \alpha_2 e^{\rho_2 \theta}$ where (ρ_1, ρ_2) are the complex roots of the quadratic characteristic equation $X^2/r^2 + X + \lambda_r = 0$ associated to the linear ODE given above. One can easily check that $\Re(\rho_1) = \Re(\rho_2) = -r^2/2$ and the boundary conditions imply in particular to choose λ_r such that $\Im(\rho_1) = -\Im(\rho_2) = 1$ otherwise the function G_r cannot remain positive on $]-\pi/2; \pi/2[$. This is possible if and only if $\lambda_r = \frac{1}{r^2} + \frac{r^2}{4}$ and the solutions of this simplified spectral problem are then proportional to the function defined by

$$\forall \, \theta \in [-\pi/2, \pi/2], \qquad G_r(\theta) \; := \; e^{-\frac{r^2}{2}\theta} \cos \theta.$$

The previous construction cannot really be extended to the exact initial problem $L_\rho g = -\lambda g$. Nevertheless, this suggests to search potential solutions g to the problem $L_\rho g \geq -\lambda g$ under the following form

$$g(r, \theta) = r e^{\beta(\theta) r^2} \cos(\theta), \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \; r \geq 0 \tag{33}$$

where $\beta : [-\pi/2, \pi/2] \to \mathbb{R}$ is a $\mathcal{C}^2$-function which must be non-positive (due to the boundedness condition in Proposition 19). The action of the generator L_ρ described in (32) is given as follows (the proof is postponed to the second appendix).

Proposition 20 *For any $g \in \mathcal{C}^2 \left(\mathbb{R}_+ \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], \mathbb{R}\right)$ given by (33), one has*

$$\forall (r, \theta) \in \mathbb{R}_+^* \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad L_\rho g(r, \theta) = \left[\psi_1(\theta) r^2 + \psi_2(\theta)\right] g(r, \theta)$$

where

$$\psi_1(\theta) = -2\rho\beta(\theta) + \beta'(\theta) + \left(4\beta^2(\theta) + (\beta'(\theta))^2\right)$$
$$\psi_2(\theta) = -\rho + 8\beta(\theta) - (1 + 2\beta'(\theta)) \tan \theta + \beta''(\theta).$$

In order to apply Proposition 19, the problem is now reduced to find a non-positive angular function β such that $\frac{L_\rho g}{g}$ is lower-bounded on $\mathbb{R}_+^* \times]-\frac{\pi}{2}, \frac{\pi}{2}[$. The previous computation shows that we mainly need to satisfy the following constraint

$$\forall \theta \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[, \quad \psi_1(\theta) \geq 0.$$

An admissible value of λ_ρ for the spectral inequality $L_\rho g \geq -\lambda_\rho g$ will then be obtained by

$$\lambda_\rho := \inf_{\theta \in]-\frac{\pi}{2}, \frac{\pi}{2}[} \psi_2(\theta) > -\infty.$$

Note that this implies in particular that

$$\limsup_{\theta \to \frac{\pi}{2}} 1 + 2\beta'(\theta) \leq 0 \quad \text{and} \quad \liminf_{\theta \to -\frac{\pi}{2}} 1 + 2\beta'(\theta) \geq 0.$$

A solution of the problem is given in the next proposition.

Proposition 21 *(i) Let $\rho \geq 0$ and let g be given by (33) with*

$$\beta(\theta) = \begin{cases} \frac{1}{4}(1 - \sqrt{3}) & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{4}) \\ \frac{1}{4}(\sin(2\theta) - \sqrt{3}) & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}]. \end{cases} \tag{34}$$

Then, for every $r > 0$ and $\theta \in [-\pi/2, \pi/2]$ such that $\theta \neq \pi/4$,

$$L_\rho g(r, \theta) \geq -\lambda_\rho g(r, \theta) \quad \text{with } \lambda_\rho = 2\sqrt{3} + \rho.$$

(ii) Let $\rho \geq 0$ and consider $(\xi_t)_{t \geq 0}$ solution to (31). Then, for every $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0 > 0$,

$$\limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}_{(x_0, y_0)}(\tau \geq t)) \leq \lambda_\rho$$

where $\tau := \inf\{t \geq 0, (\xi_t)_1 < 0\}$.

Remark 22 Note that by the scaling and linear transformations previously described, this result can be transferred to general elliptic two-dimensional Ornstein-Uhlenbeck evolutions whose linear drift is given via a matrix admitting complex conjugate eigenvalues (and whose trajectories have thus a tendency to turn around $(0,0)$).

The function g is a $\mathcal{C}^1$-function but only a piecewise $\mathcal{C}^2$-function. However, since these functions still belong to the domain of L_ρ, the conclusions of Proposition 19 still hold. Also note that if we now switch to cartesian coordinates, the counterpart of g has the following (nice) form:

$$f(x,y) = xe^{-\frac{\sqrt{3}-1}{4}(x^2+y^2)}e^{-\frac{(x-y)^2}{4}}1_{\{y \geq x\}}.$$

$\square$

Proof

(i) First, assume that $\rho = 0$ and consider the function β stated in the former statement. One checks that β is a piecewise $\mathcal{C}^2$-function on $[-\pi/2, \pi/2]$. Note that we can use this function since Itô's formula is still available in this case. Furthermore, we check that

$$\psi_1(\theta) = \begin{cases} 1 - \frac{\sqrt{3}}{2} & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{4}) \\ 1 + \cos(\frac{\pi}{3} + 2\theta) & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases} \tag{35}$$

so that ψ_1 is non-negative on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. As well, easy computations yield:

$$\psi_2(\theta) = \begin{cases} 2 - 2\sqrt{3} - \tan\theta & \text{if } \theta \in [-\frac{\pi}{2}, \frac{\pi}{4}) \\ -2\sqrt{3} & \text{if } \theta \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases} \tag{36}$$

It follows that ψ_2 is lower-bounded by $-2\sqrt{3}$ and the result follows when $\rho = 0$. The extension to the case $\rho > 0$ is obvious using that $-\rho\beta$ is a non-negative function.

(ii) This statement follows from Proposition 19, since g is $\mathcal{C}^1$ and piecewise $\mathcal{C}^2$.

$\blacksquare$

Remark 23 Figure 4 represents the partition of the state space $\mathbb{R}^2$ (seen as $\mathbb{R}_+ \times [0, 2\pi)$ in the second picture) for the construction of the function β (and g) as well as the function $g(r, \theta)$ for several values of r. We should understand the function g as follows: $g(r, \theta)$ must be large when the dynamical system is suspected to take long time to exit the set $\mathcal{S}_0 = \mathbb{R}_+^* \times (-\pi/2, \pi/2)$ from (r, θ). Conversely, it should be small in the region where the vector field of the underlying determinist dynamical system push the trajectories out of $\mathcal{S}_0$. As pointed out by Figure 4, we do not need to consider sub-domain of $\mathcal{S}_0$: the action of the Brownian motion is elliptic and we can always build some trajectories starting from any point of $\mathcal{S}_0$ and staying an arbitrarily long time in $\mathcal{S}_0$. Note that when r is small, the starting point is near the origin, whatever the value of θ is and hence, the function $g(r, \theta)$ is small (see the right side of Figure 4).

$\square$

3.2.3 The hypoelliptic case

We now come back to the study of the lower-bound of Theorem 4. This result is proved in Proposition 29 stated below. We know from Corollary 13 that up to linear changes of variables in time and space, the initial dynamic may be reduced to the simplified stochastic evolution described
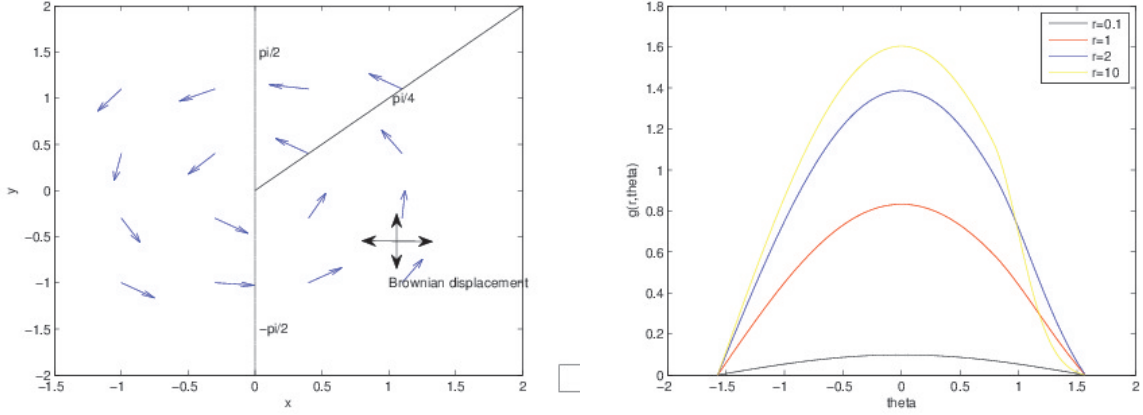
Figure 4: Left: the domain to avoid is $\theta \in [\pi/2, 3\pi/2]$. The elliptic situation is illustrated by the full rank black double arrow: the Brownian motion always move in all directions. In blue: rotation + homothety vector field. Right: function $\theta \mapsto g(r,\theta)$ for several values of r.

by Equation (25). Again let us write down the corresponding infinitesimal generator $\mathcal{L}_\rho$ in polar coordinates (see Proposition 41 given in the second appendix):

$$\mathcal{L}_\rho = -\rho r \partial_r + \partial_\theta + \frac{\sin^2 \theta}{2} \partial_{rr}^2 - \frac{\sin \theta \cos \theta}{r^2} \partial_\theta + \frac{\sin \theta \cos \theta}{r} \partial_{r\theta}^2 + \frac{\cos^2 \theta}{2r} \partial_r + \frac{\cos^2 \theta}{2r^2} \partial_\theta^2 \qquad (37)$$

with $\rho = -\frac{a}{2\omega}$. As mentioned before, we would like to use a strategy similar to the one considered in the elliptic case. However, the hypoelliptic problem is more involved. Roughly speaking, the degeneracy of the diffusive component implies that in the neighbourhood of $\pi/2$, the paths of the solutions to (25) can not be strongly slowed down by the action of the Brownian motion (see Remark 23 and the study on Brownian bridges below). In other words, we are not able (and it seems indeed impossible) to build a function β such that the function ψ_2 defined in the previous subsection is lower-bounded. Thus, the idea is to reduce the domain to a smaller angular sector $\mathcal{S}$ included in $\{(x,y), x > 0\}$ where the diffusive action of the Brownian motion is more likely to keep the process in $\mathcal{S}$.

Consequently, we consider a more general class of functions g (which must be calibrated in the sequel) and define

$$g(r,\theta) = r^n \gamma(\theta) e^{\beta(\theta) r^2} \qquad (38)$$

where n is a positive integer and γ and β are some sufficiently smooth functions. Now β should be bounded above by a negative constant for g to have a chance to be bounded. The new function γ will be chosen in order that g is positive in the interior of the angular sector and vanishes on the boundary of $\mathcal{S}$. We first describe the effect of $\mathcal{L}_\rho$ on such a function g (the computations are deferred to the second appendix).

Proposition 24 *For any $g \in \mathcal{C}^2 \left(\mathbb{R}_+ \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \mathbb{R} \right)$ given by (38), one has*

$$\forall (r,\theta) \in \mathbb{R}_+^* \times \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \qquad L_\rho g(r,\theta) = \left[\varphi_1(\theta) r^2 + \varphi_2(\theta) + \frac{\varphi_3(\theta)}{r^2} \right] g(r,\theta)$$

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where

$$\varphi_1(\theta) = -2\rho\beta(\theta) + \beta'(\theta) + \left(2\sin\theta\beta(\theta) + \cos\theta\beta'(\theta)\right)^2,$$

$$\varphi_2(\theta) = -n\rho + \beta(\theta)\left((4n+2)\sin^2\theta + 2\cos^2\theta\right)$$

$$+ (1 + 2\beta'(\theta)\cos^2\theta + 4\beta(\theta)\sin\theta\cos\theta)\frac{\gamma'}{\gamma}(\theta) + \beta''(\theta)\cos^2(\theta) + 2(n+1)\cos\theta\sin\theta\beta'(\theta),$$

$$\varphi_3(\theta) = (n^2 - n)\sin^2\theta + \cos^2\theta(n + \frac{\gamma''(\theta)}{\gamma(\theta)}) + 2(n-1)\sin\theta\cos\theta\frac{\gamma'(\theta)}{\gamma(\theta)}.$$

We now need to find an (open) angular sector $\mathcal{S} = \{(r\cos\theta, r\sin\theta), \theta_1 < \theta < \theta_2\}$, a positive integer n, some functions γ and β such that

1. $\gamma(\theta) > 0$ on (θ_1, θ_2), $\gamma(\theta_1) = \gamma(\theta_2) = 0$, $\beta(\theta) \leq 0$ on $[\theta_1, \theta_2]$,

2. φ_1 and φ_3 are non-negative on $\mathcal{S}$,

3. φ_2 is lower-bounded.

4. β is bounded above by a negative constant.

This is the purpose of the next proposition.

Proposition 25 *Let $\rho \geq 0$.*
(i) Let g be defined by (38) with $n = 2$,

$$\gamma(\theta) = \begin{cases} -\sin(2\theta) & \text{if } \theta \in [-\frac{\pi}{2}, -\frac{\pi}{4}] \\ \cos^2(\pi/4 + \theta) & \text{if } \theta \in [-\frac{\pi}{4}, \frac{\pi}{4}] \end{cases} \tag{39}$$

and $\beta(\theta) = -\frac{1}{2}$. Then, for every $r > 0$ and $\theta \in]-\frac{\pi}{2}, \frac{\pi}{4}[$ with $\theta \neq -\frac{\pi}{4}$,

$$\mathcal{L}_\rho g(r, \theta) \geq -(3 + 2\rho)g(r, \theta).$$

(ii) As a consequence, for any open half-plane H such that $\mathcal{S} := \{(r\cos\theta, r\sin\theta), r > 0, \theta \in]-\frac{\pi}{2}, \frac{\pi}{4}[\} \subset H$, for any probability measure m_0 on $\mathbb{R}^2$ such that $m_0(H) = 1$, we have

$$\limsup_{t \to +\infty} -\frac{1}{t}\log(\mathbb{P}_{m_0}(\tau_H \geq t)) \leq 3 + 2\rho.$$

Remark 26 The vector field corresponding to the drift part of the stochastic evolution under study, as well as the most favorable positions (which are expected to be the points where g is large) for the starting point in order to keep the process in $\mathcal{S}$ for large times are illustrated in Figure 5. As pointed out above, the angular sector $[\pi/4, \pi/2]$ is now avoided to keep the process in the half-plane $x > 0$. Moreover, the right side of Figure 5 shows that excessive values of r (too large or too small ones) are also prohibited: small values are unfavourable since it corresponds to starting positions very close to the origin (and naturally close to the axis $x = 0$). Large values of r are also disadvantageous owing to the large norm of the drift vector field against which the Brownian motion has to fight to keep the process in $\mathcal{S}$. □

Proof

With the proposed choices of n and γ, one checks that

$$\varphi_3(\theta) = \begin{cases} 0 & \text{if } \theta \in [-\frac{\pi}{2}, -\frac{\pi}{4}) \\ \frac{2}{1-\sin(2\theta)} & \text{if } \theta \in (-\frac{\pi}{4}, \frac{\pi}{4}) \end{cases}$$
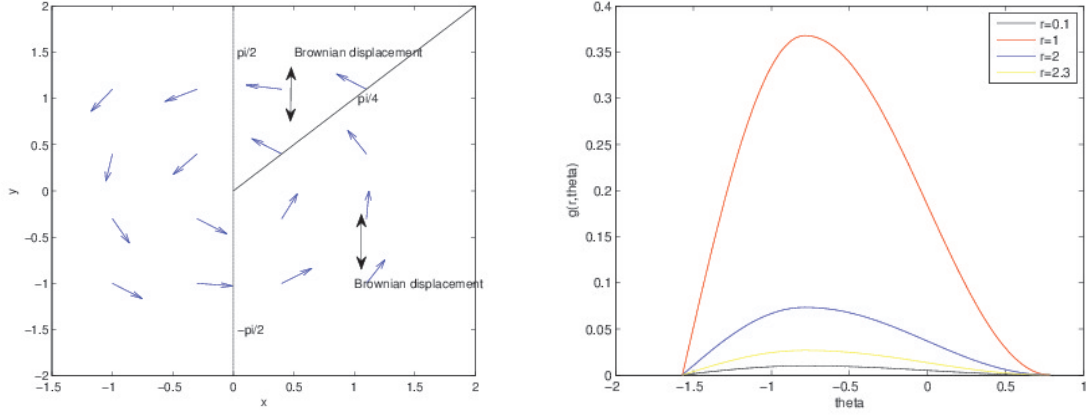
Figure 5: Left: the domain to avoid is $\theta \in [\pi/4, 3\pi/2]$. The hypo-elliptic situation is illustrated by the rank 1 double arrow: the Brownian motion can only move in vertical directions. In blue: rotation + homothety vector field. Right: function $\theta \mapsto g(r, \theta)$ for several values of r.

so that φ_3 is non-negative. Since β is constant and ρ is non-negative, the fact that φ_1 is non-negative is obvious. Thus, it remains to focus on φ_2. In fact, easy computations show that $\varphi_2(\theta) = -(3 + 2\rho)$ on $[-\frac{\pi}{4}, \frac{\pi}{4})$ whereas

$$\forall \theta \in (-\frac{\pi}{2}, -\frac{\pi}{4}], \quad \varphi_2(\theta) = -(3 + 2\rho) + \frac{2}{\tan(2\theta)}.$$

The conclusion of the first assertion follows.

(ii) By Proposition 19 and what precedes, for any probability measure $m_{\mathcal{S}}$ on $\mathbb{R}^2$ such that $m_{\mathcal{S}}(\mathcal{S}) = 1$,

$$\limsup_{t \to +\infty} -\frac{1}{t} \log \left(\mathbb{P}_{m_{\mathcal{S}}}(\tau_{\mathcal{S}} \geq t) \right) \leq 3 + 2\rho. \tag{40}$$

Now, consider the general case. Let m_0 be a probability such that $m_0(H) = 1$. Then, for every $t > 0$, for every $a.s.$ finite stopping time T,

$$\mathbb{P}_{m_0}(\tau_H \geq t) \geq \mathbb{P}_{m_0}(\tau_H \geq T + t) \geq \mathbb{P}_{m_0}(\tau_H > T, Z_s \in \mathcal{S} \ \forall s \in [T, T + t]).$$

Thus,

$$\mathbb{P}_{m_0}(\tau_H \geq t) \geq \mathbb{E}_{m_0} \left[1_{\{\tau_H > T, Z_T \in \mathcal{S}\}} \mathbb{P}(Z_{s+T} \in \mathcal{S}, \forall s \in [0, t] | \mathcal{F}_T) \right].$$

and it follows from the Markov property that

$$\mathbb{P}_{m_0}(\tau_H \geq t) \geq \mathbb{E}_{m_0}[1_{\{\tau_H > T, Z_T \in \mathcal{S}\}} \mathbb{P}_{Z_T}(\tau_{\mathcal{S}} \geq t)].$$

If we assume for a moment that T is such that

$$\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S}) > 0, \tag{41}$$

then,

$$-\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau_H \geq t)) \leq -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S})) - \frac{1}{t} \log \left(\mathbb{P}_{m_{\mathcal{S}}}(\tau_{\mathcal{S}} \geq t) \right),$$

where $m_{\mathcal{S}}$ is the probability measure defined for every bounded measurable function $h : \mathbb{R}^2 \to \mathbb{R}$ by

$$m_{\mathcal{S}}(h) = \frac{1}{\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S})} \mathbb{E}_{m_0}[h(Z_T) 1_{\{\tau_H > T, Z_T \in \mathcal{S}\}}].$$

26

By (40) and the (strict) positivity of $\mathbb{P}_{m_0}(\tau_H > T, Z_T \in \mathcal{S})$, we obtain that

$$\limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau_H \geq t)) \leq 3 + 2\rho.$$

Thus, it remains to prove (41). It is certainly enough to show that for every $(x_0, y_0) \in H$, there exists a deterministic positive $T(x, y)$ such that

$$\mathbb{P}_{(x_0, y_0)}(\tau_H > T(x_0, y_0), Z_{T(x_0, y_0)} \in \mathcal{S}) > 0.$$

The idea is to build some "good" controlled trajectories: let $\varphi \in L^{2,\mathrm{loc}}(\mathbb{R}_+, \mathbb{R})$ and denote by $(z_\varphi(t))_{t \geq 0}$ the solution of the controlled system

$$\begin{cases} \dot{x}(t) = -\rho x(t) - y(t) \\ \dot{y}(t) = -\rho y(t) + x(t) + \varphi(t) \end{cases}$$

starting from $z_0 = (x_0, y_0) \in H$. The classical Support Theorem (see [17]) can be applied since the coefficients of the diffusion are Lipschitz continuous. This implies that (41) is true as soon as there exists such a φ for which the solution $(z_\varphi(t))_{t \in [0, T(x_0, y_0)]}$ belongs to H and such that $z_\varphi(T(x_0, y_0))$ belongs to $\mathcal{S}$. Such a controlled trajectory can be built through the following lemma. ∎

Lemma 27 *Let $\kappa \in (0, +\infty]$ and set $H_\kappa = \{(x, y), y < \kappa x\}$ and $H_\infty = D \ (= \{(x, y), x > 0\})$.*

(i) Let $(x_0, y_0) \in H_\kappa$ with $y_0 \geq 0$. Then, for every $v \in (-\infty, y_0]$, there exists a controlled trajectory $(x_\varphi(t), y_\varphi(t))_{t \geq 0}$ starting from (x_0, y_0) and a positive T_v such that $\{z_\varphi(t) : t \geq 0\} \subset H_\kappa \cap H_\infty$, $x_\varphi(T_v) > 0$ and $y_\varphi(T_v) = v$.

(ii) Let $(x_0, y_0) \in H_\kappa$ with $y_0 \leq 0$ and consider $(x(t), y(t))_{t \geq 0}$ the solution to the free dynamical system (i.e. the controlled trajectory with $\varphi \equiv 0$) starting from (x_0, y_0). Then, there exists $T > 0$ such that $(x(t), y(t))_{t \in [0, T]} \subset H_\kappa$ and such that $(x(T), y(T)) = (a_T, 0)$ with $a_T > 0$. Furthermore, writing $(x_0, y_0) = (r_0 \cos(-\theta_0), r_0 \sin(-\theta_0))$ (with $r_0 > 0$ and $\theta_0 \in (\pi - \mathrm{Arctan}(\kappa), 0])$, this property holds with $T = \theta_0$ and $a_T = r_0 e^{-\rho \theta_0}$.

Remark 28 Note that this lemma will be also used in the proof of Proposition 35 (see Step 3). This is the reason why its statements are a little sharper than what we need for the proof of the previous proposition. □

Proof

(i) Without loss of generality, we only prove the result when $\kappa < +\infty$. The idea is to build φ such that the derivative of the second component is large enough. More precisely, for every $M > 0$,

$$\begin{cases} \dot{x}_M(t) = -\rho x_M(t) - y_M(t) \\ \dot{y}_M(t) = -M \end{cases}$$

is certainly an equation of a controlled trajectory (by setting $\varphi(t) = -M + \rho y_M(t) + x_M(t)$). Furthermore, denoting by $z_0 = (x_0, y_0)$ its starting point, we have

$$y_M(t) = -Mt + y_0 \quad \text{and} \quad x_M(t) = \left(x_0 + \frac{M}{\rho^2} + \frac{y_0}{\rho}\right) e^{-\rho t} + \frac{M}{\rho} t - \frac{M}{\rho^2} - \frac{y_0}{\rho}.$$

First, let us choose M large enough in order that for all $t \geq 0$, $x_M(t) > 0$ and $(x_M(t), y_M(t)) \in H_\kappa$, i.e. such that $x_M(t) > 0$ and $\kappa x_M(t) - y_M(t) \geq 0$ for all $t \geq 0$. A simple study of the derivative of $t \to x_M(t)$ yields

$$\forall t \geq 0, \quad x_M(t) \geq x_M(t_M^*) \quad \text{with} \quad t_M^* = \frac{1}{\rho} \log\left(1 + \frac{\rho}{M}(y_0 + \rho x_0)\right) \quad \text{and}$$

$$x_M(t_M^*) = \frac{M}{\rho^2} \log\left(1 + \frac{\rho}{M}(y_0 + \rho x_0)\right) - \frac{y_0}{\rho} \xrightarrow{M \to +\infty} x_0.$$

Thus, for every $\varepsilon > 0$, there exists M_ε large enough such that $x(t_{M_\varepsilon}^*) \geq \kappa x_0 - \varepsilon$. Using that for any $M > 0$ and $t \geq 0$, $y_M(t) \leq y_0$ and setting $\varepsilon = \frac{\kappa x_0 - y_0}{2}$, we obtain that

$$\forall t \geq 0, \quad x_{M_\varepsilon}(t) > 0 \quad \text{and} \quad \kappa x_{M_\varepsilon}(t) - y_{M_\varepsilon}(t) > 0.$$

Since y_{M_ε} is a continuous function such that $y_{M_\varepsilon}(t) \to -\infty$ as $t \to +\infty$, it follows that for every $v \in (-\infty, y_0]$, there exists $T_v > 0$ such that $y_{M_\varepsilon}(T_v) = v$.
(ii) The result is obvious since the solution to the free dynamical system satisfies

$$(x(t), y(t)) = r_0 e^{-\rho t}(\cos(t - \theta_0), \sin(t - \theta_0)), \quad t \geq 0.$$

■

We are now able to prove the lower-bound of Theorem 4.

Proposition 29 *Let $(Z_t)_{t \geq 0}$ be a solution of (7) with $(1 + \frac{1}{\sqrt{2}})a \leq b$ and let $\tau = \inf\{t > 0, X_t = 0\}$. Then, for every probability measure m_0 on $\mathbb{R}^2$ such that $m_0(\{(x, y), x > 0\}) = 1$,*

$$\limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau \geq t)) \leq \left(3 + \frac{a}{\omega}\right)\omega.$$

Remark 30 Since $\frac{a}{\omega} = (\frac{b}{a} - \frac{1}{4})^{-\frac{1}{2}}$,

$$\sup_{(a,b),0 < (1 + \frac{1}{\sqrt{2}})a \leq b} \left(3 + \frac{a}{\omega}\right) = 3 + (\frac{3}{4} + \frac{1}{\sqrt{2}})^{-\frac{1}{2}} \leq 4.$$

This corresponds to the bound given in Theorem 4. However, the reader can remark that the above result yields some sharper bounds. In particular, when a tends to 0, $3 + a/\omega$ tends to 3.

□

Proof

Let $z_0 = (x_0, y_0) \in \mathbb{R}^2$ such that $x_0 > 0$. Owing to the symmetry of the Brownian motion, one can check that

$$\mathbb{P}_{z_0}(\tau \geq t) = \mathbb{P}_{-z_0}(\tau_{D_-} \geq t)$$

where $z_0 = (x_0, y_0)^*$, $D_- = \{(x, y), x < 0\}$ and $\tau_{D_-} = \inf\{t \geq 0, Z_t \in D_-^c\}$.
Second, set $v = (\frac{1}{b^2}(\frac{a}{2} - b), 1)^*$ and $P_v = (v, Bv)$ with $B = \frac{1}{\omega}(A + \frac{a}{2}I_2)$. By Corollary 13, there exists $\alpha > 0$ such that $(\tilde{Z}_t)_{t \geq 0} := (\sqrt{\omega}\alpha P_v^{-1} Z_{\frac{t}{\omega}})_{t \geq 0}$ is a solution of (25). Denote respectively by (x, y) and by $(\tilde{x}, \tilde{y})$, the coordinates in the canonical basis and in the basis $\tilde{\mathcal{B}} = (v, Bv)$. Computing $P_v(\tilde{x}, \tilde{y})^*$, one checks that in the new basis, the set D_- corresponds to the half-plane H_κ defined by

$$H_\kappa = \{(\tilde{x}, \tilde{y}), \tilde{y} < \kappa\tilde{x}\} \quad \text{with} \quad \kappa = \frac{1}{\omega}\left(b - \frac{a}{2}\right).$$

Furthermore, from the very definition of $(\tilde{Z}_t)_{t \geq 0}$, we have

$$\tau_{D_-}^{-z_0} = \tilde{\tau}_{H_\kappa}^{-\tilde{z}_0} \quad \text{with } \tilde{\tau}_{H_\kappa} = \inf\{t \geq 0, \tilde{Z}_t \in H_\kappa^c\} \text{ and } \tilde{z}_0 = \sqrt{\omega}\alpha P_v^{-1} z_0.$$

In particular, $\mathbb{P}_{-z_0}(\tau_{D_-} \geq t) = \mathbb{P}_{-\tilde{z}_0}(\tilde{\tau}_{H_\kappa} \geq \omega t)$ so that for any probability m_0 on $\mathbb{R}^2$ such that $m_0(\{(x,y), x > 0\}) = 1$,

$$\mathbb{P}_{m_0}(\tau_{D_-} \geq t) = \mathbb{P}_{\tilde{m}_0}(\tilde{\tau}_{H_\kappa} \geq \omega t)$$

where $\tilde{m}_0 := m_0 \circ (z \mapsto -\sqrt{\omega}\alpha P_v^{-1} z)$ satisfies $\tilde{m}_0(H_\kappa) = 1$. Now, when $(1 + \frac{1}{\sqrt{2}})a \leq b$, one checks that $\kappa \geq 1$ so that H_κ contains the set $\mathcal{S} = \{((\tilde{x}, \tilde{y}), \tilde{x} > 0, \tilde{y} < \tilde{x}\}$ of Proposition 25 (written in polar coordinates). Applying the second item of this proposition with $\rho = a/(2\omega)$, we finally obtain

$$\limsup_{t \to +\infty} -\frac{1}{t} \log(\mathbb{P}_{m_0}(\tau \geq t)) = \omega \limsup_{t \to +\infty} -\frac{1}{\omega t} \log(\mathbb{P}_{\tilde{m}_0}(\tilde{\tau}_{H_\kappa} \geq \omega t)) \leq \omega \left(3 + \frac{a}{\omega}\right).$$

∎

4   Bridges at small times and persistence rate

In this section, we study the diffusion bridge associated to our dynamical system. We then use it to establish some lower-bounds for $\mathbb{P}_{(x_0, y_0)}(\tau \geq t)$ (where $\tau := \inf\{t \geq 0, Z_t \in \{(x,y), x < 0\}\}$).

4.1   Explosion of bridges at small times

Our objective here is to prove Theorem 6 and to discuss some related results.

Since we are mainly to take advantage of the Gaussian features of the problem, we could have worked directly with the process Z whose evolution is given by (7). Nevertheless the computations presented in Subsection 2.1 suggest that it is more advisable to first consider the simplifications made in Subsection 2.2. So we begin by considering the two-dimensional Ornstein-Uhlenbeck process $(Z_t)_{t \geq 0} := (X_t, Y_t)_{t \geq 0}$ whose evolution is dictated by

$$\begin{cases} dX_t &= (-\rho X_t - Y_t)\, dt \\ dY_t &= (-\rho Y_t + X_t)\, dt + \sqrt{2} dW_t \end{cases} \tag{42}$$

where $\rho \in \mathbb{R}$ and $(W_t)_{t \geq 0}$ is a standard real Brownian motion. Let us assume furthermore that the initial condition of Z is a deterministic point $z_0 = (x_0, y_0)^* \in \mathbb{R}^2$. The arguments of Subsection 2.1 show that Z is Gaussian and more precisely we have:

Lemma 31 *For any $t \geq 0$, Z_t is distributed as a Gaussian law of mean $m_t(z_0)$ and variance Σ_t, with*

$$m_t(z_0) := \exp(-\rho t) \begin{pmatrix} x_0 \cos(t) - y_0 \sin(t) \\ x_0 \sin(t) + y_0 \cos(t) \end{pmatrix}$$

$$\Sigma_t(1,1) := \frac{1 - e^{-2\rho t}}{2\rho} - \frac{e^{-2\rho t}}{2(1+\rho^2)}(\sin(2t) - \rho\cos(2t)) - \frac{\rho}{2(1+\rho^2)}$$

$$\Sigma_t(1,2) = \Sigma_t(2,1) := \frac{e^{-2\rho t}}{2(1+\rho^2)}(\cos(2t) + \rho\sin(2t)) - \frac{1}{2(1+\rho^2)}$$

$$\Sigma_t(2,2) := \frac{1 - e^{-2\rho t}}{2\rho} + \frac{e^{-2\rho t}}{2(1+\rho^2)}(\sin(2t) - \rho\cos(2t)) + \frac{\rho}{2(1+\rho^2)}$$

Proof

Let us denote

$$A := \begin{pmatrix} -\rho & -1 \\ 1 & -\rho \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

From the beginning of Subsection 2.1, we get that for any $t \geq 0$, on one hand

$$\begin{aligned} m_t(z_0) &= \exp(At)z_0 \\ &= \exp(-\rho t) \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \end{aligned}$$

and on the other hand, the validity of (18). We compute that for any $s \geq 0$,

$$\begin{aligned} \exp(As)CC^* \exp(A^*s) &= 2\exp(-2\rho s) \begin{pmatrix} \cos(s) & -\sin(s) \\ \sin(s) & \cos(s) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \\ &= 2\exp(-2\rho s) \begin{pmatrix} \sin^2(s) & -\cos(s)\sin(s) \\ -\cos(s)\sin(s) & \cos^2(s) \end{pmatrix} \\ &= \exp(-2\rho s) \begin{pmatrix} 1 - \cos(2s) & -\sin(2s) \\ -\sin(2s) & 1 + \cos(2s) \end{pmatrix} \end{aligned}$$

The announced expressions for the entries of Σ_t follow from immediate integrations. For instance for $\Sigma_t(1,1)$, we have

$$\begin{aligned} \Sigma_t(1,1) &= \int_0^t \exp(-2\rho s)(1 - \cos(2s))\, ds \\ &= \frac{1 - \exp(-2\rho t)}{2\rho} - \Re\left(\int_0^t \exp(2(i-\rho)s)\, ds\right) \\ &= \frac{1 - \exp(-2\rho t)}{2\rho} - \Re\left(\frac{\exp(2(i-\rho)t) - 1}{2(i-\rho)}\right) \\ &= \frac{1 - \exp(-2\rho t)}{2\rho} + \frac{1}{2(1+\rho^2)}\Re\left((\rho + i)(\exp(2(i-\rho)t) - 1)\right) \\ &= \frac{1 - \exp(-2\rho t)}{2\rho} + \frac{1}{2(1+\rho^2)}(\exp(-2\rho t)(\rho\cos(2t) - \sin(2t)) - \rho) \end{aligned}$$

■

For $t > 0$, let us denote by $p_t(z_0, z)\, dz$ the law of Z_t knowing that $Z_0 = z_0$. With the notations of the above lemma we have

$$\forall\, t > 0,\ \forall\, z_0, z \in \mathbb{R}^2, \qquad p_t(z_0, z) = \frac{1}{2\pi\det(\Sigma_t)} \exp(-(z - m_t(z_0))^*(2\Sigma_t)^{-1}(z - m_t(z_0)))$$

Using Bayes' formula, we get that for $0 < t < T$ and $z_0, z_T \in \mathbb{R}^2$, the law of Z_t conditioned by $Z_T = z_T$ (and still by $Z_0 = z_0$) admits a density proportional to $z \mapsto p_t(z_0, z)p_{T-t}(z, z_T)$ where $z \in \mathbb{R}^2$. It is a non-degenerate Gaussian law, let $\eta_t^{(T)}(z_0, z_T)$ (resp. $\sigma_t^{(T)}$) be its mean vector (resp. its covariance matrix), formula (44) below will show that the covariance matrix does not depend on z_0 and z_T. We furthermore define

$$\forall\, u \in [0,1], \qquad \varphi_{z_0, z_T}(u) := \begin{pmatrix} 0 \\ 6u(1-u)(x_0 - x_T) \end{pmatrix}$$

where $z_0 := (x_0, y_0)^*$ and $z_T := (x_T, y_T)^*$. The next result contains all the required technicalities we will need.

30

Proposition 32 *For all $z_0, z_T \in \mathbb{R}^2$ and $u \in (0,1)$, we have*

$$
\lim_{T \to 0_+} T \eta_{uT}^{(T)}(z_0, z_T) = \varphi_{z_0, z_T}(u)
$$

$$
\lim_{T \to 0_+} \sigma_{uT}^{(T)}(z_0, z_T) = 0
$$

Proof

To simplify notations, for $u \in (0,1)$, we denote $v := 1 - u$, $\eta_u := \eta_{uT}^{(T)}(z_0, z_T)$ and $\sigma_u := \sigma_{uT}^{(T)}(z_0, z_T)$. With the notations of Lemma 31, the vector η_u and the matrix σ_u are such that for any $z \in \mathbb{R}^2$,

$$
(z - \eta_u)^* \sigma_u^{-1} (z - \eta_u) = (z - m_{uT}(z_0))^* \Sigma_{uT}^{-1}(z - m_{uT}(z_0)) + (z_T - m_{vT}(z))^* \Sigma_{vT}^{-1}(z_T - m_{vT}(z))
$$
$$
+ C(z_0, z_T),
$$

where $C(z_0, z_T)$ is a normalizing term which is independent of z. It follows that

$$
\eta_u = \sigma_u(e^{-\rho u T} S_u B_u z_0 + e^{-\rho v T} B_v^* S_v z_T) \tag{43}
$$

$$
\sigma_u = \left(S_u + e^{-2\rho v T} B_v^* S_v B_v\right)^{-1} \tag{44}
$$

where for any $w \geq 0$,

$$
B_w := \begin{pmatrix} \cos(wT) & -\sin(wT) \\ \sin(wT) & \cos(wT) \end{pmatrix}
$$

$$
S_w := \Sigma_{wT}^{-1}
$$

$$
= \frac{1}{D_w} \begin{pmatrix} \Sigma_{wT}(2,2) & -\Sigma_{wT}(1,2) \\ -\Sigma_{wT}(1,2) & \Sigma_{wT}(1,1) \end{pmatrix}
$$

$$
D_w := \det(\Sigma_{wT})
$$

$$
= \Sigma_{wT}(1,1)\Sigma_{wT}(2,2) - (\Sigma_{wT}(1,2))^2
$$

All these expressions depend on $T > 0$ and the announced convergences will be obtained by expanding them for small $T > 0$. Indeed, simple computations show that for $w \in (0,1)$, as $T \to 0_+$,

$$
\begin{pmatrix} \Sigma_{wT}(1,1) & \Sigma_{wT}(1,2) \\ \Sigma_{wT}(1,2) & \Sigma_{wT}(2,2) \end{pmatrix} = \begin{pmatrix} \frac{2(wT)^3}{3} + \mathcal{O}((wT)^4) & -(wT)^2 + \mathcal{O}((wT)^3) \\ -(wT)^2 + \mathcal{O}((wT)^3) & 2wT + \mathcal{O}((wT)^2) \end{pmatrix}
$$

where $\mathcal{O}((wT)^p)$, for $p \in \mathbb{R}$, stands for a quantity bounded by $A(wT)^p$, uniformly over $\rho \in [-1,1]$ and for wT small enough. It follows that

$$
D_w = \frac{(wT)^4}{3} + \mathcal{O}((wT)^4)
$$

$$
S_w = \begin{pmatrix} \frac{6}{(wT)^3} + \mathcal{O}((wT)^{-2}) & \frac{3}{(wT)^2} + \mathcal{O}((wT)^{-1}) \\ \frac{3}{(wT)^2} + \mathcal{O}((wT)^{-1}) & \frac{2}{wT} + \mathcal{O}(1) \end{pmatrix}
$$

Using furthermore that for $v \in (0,1)$, we have $e^{-2\rho v T} = 1 + \mathcal{O}(vT)$ and that

$$
B_v = \begin{pmatrix} 1 + \mathcal{O}((vT)^2) & -vT + \mathcal{O}((vT)^3) \\ vT + \mathcal{O}((vT)^3) & 1 + \mathcal{O}((vT)^2) \end{pmatrix}
$$

we deduce that

$$
e^{-2\rho v T} B_v^* S_v B_v = \begin{pmatrix} \frac{6}{(vT)^3} + \mathcal{O}((vT)^{-2}) & -\frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) \\ -\frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) & \frac{2}{vT} + \mathcal{O}(1) \end{pmatrix}
$$

Thus from (44), we get that for $u \in (0,1)$,

$$\sigma_u = \frac{1}{d(u,v)} \begin{pmatrix} 2(u+v)(uv)^3T^3 + \mathcal{O}(T^4) & 3(u^2 - v^2)(uv)^2T^2 + \mathcal{O}(T^3) \\ 3(u^2 - v^2)(uv)^2T^2 + \mathcal{O}(T^3) & 6(u^3 + v^3)uvT + \mathcal{O}(T^2) \end{pmatrix}$$

with

$$d(u,v) = 12(u^3 + v^3)(u+v) - 9(u^2 - v^2)^2 + \mathcal{O}(T^{-1})$$

Recalling that $v = 1 - u$, it appears that $d(u,v) = 12 + \mathcal{O}(T^{-1})$, so we obtain

$$\sigma_u = \begin{pmatrix} \frac{2}{3}(uv)^3T^3 + \mathcal{O}(T^4) & (u - v)(uv)^2T^2 + \mathcal{O}(T^3) \\ (u - v)(uv)^2T^2 + \mathcal{O}(T^3) & 2(u^3 + v^3)uvT + \mathcal{O}(T^2) \end{pmatrix} \tag{45}$$

The second convergence announced in the proposition follows at once. To deduce the first one, we begin by checking that for $u \in (0,1)$,

$$e^{-\rho u T} S_u B_u = \begin{pmatrix} \frac{6}{(uT)^3} + \mathcal{O}((uT)^{-2}) & -\frac{3}{(uT)^2} + \mathcal{O}((uT)^{-1}) \\ \frac{3}{(uT)^2} + \mathcal{O}((uT)^{-1}) & -\frac{1}{uT} + \mathcal{O}(1) \end{pmatrix}$$

$$e^{-\rho v T} B_v^* S_v = \begin{pmatrix} \frac{6}{(vT)^3} + \mathcal{O}((vT)^{-2}) & \frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) \\ -\frac{3}{(vT)^2} + \mathcal{O}((vT)^{-1}) & -\frac{1}{vT} + \mathcal{O}(1) \end{pmatrix}$$

In conjunction with (45), we get

$$\sigma_u e^{-\rho u T} S_u B_u = \begin{pmatrix} 1 - 3u^2 + 2u^3 + \mathcal{O}(T) & -u(1-u)^2T + \mathcal{O}(T^2) \\ \frac{6u(1-u)}{T} + \mathcal{O}(1) & 1 - 4u + 3u^2 + \mathcal{O}(T) \end{pmatrix}$$

$$\sigma_u e^{-\rho v T} B_v^* S_v = \begin{pmatrix} 3u^2 - 2u^3 + \mathcal{O}(T) & u^2(1-u)T + \mathcal{O}(T^2) \\ -\frac{6u(1-u)}{T} + \mathcal{O}(1) & -2u + 3u^2 + \mathcal{O}(T) \end{pmatrix}$$

In these expression, the $(2,1)$-entries explode as $T \to 0_+$, it explains the renormalisation by T considered in the above proposition for $\eta_{uT}^{(T)}(z_0, z_T)$ and resulting convergence.

∎

Remark 33 Note that when $x_0 = x_T$ (namely if z_0 and z_T are on the same vertical line), it is simpler for the underlying vertical Brownian motion to put them in relation. Hence, the second component of φ_{z_0, z_T} is equal to 0. In this case, we thus expect the second component of $(\eta_{uT}^{(T)})$ to be convergent when $T \to 0$. Pushing further the previous developments yields

$$(\sigma_u e^{-\rho u T} S_u B_u)_{2,1} = \frac{6u(1-u)}{T} - 2\rho u(1-u)(2-u) + \mathcal{O}(T) \tag{46}$$

and

$$(\sigma_u e^{-\rho v T} B_v^* S_v)_{2,1} = -\frac{6u(1-u)}{T} - 2\rho u(1-u^2) + \mathcal{O}(T). \tag{47}$$

Combined with the computations of the end of the previous proof, we deduce that if $x_0 = x_T$, then no renormalisation is needed for the mean vector and we get

$$\lim_{T \to 0_+} \eta_{uT}^{(T)}(z_0, z_T) = \begin{pmatrix} x_0 \\ (1 - 4u + 3u^2)y_0 - (2u - 3u^2)y_T - 6\rho u(1-u)x_0 \end{pmatrix}$$

In particular even in the case when $z_0 = z_T$, the asymptotical bridge doesn't stay still (except if $y_0 = 0$), since

$$\lim_{T \to 0_+} \eta_{uT}^{(T)}(z_0, z_0) = \begin{pmatrix} x_0 \\ (1 - 6u + 6u^2)y_0 - 6\rho u(1-u)x_0 \end{pmatrix}.$$

$\square$

Similarly to the notational conventions endorsed in the introduction, for $T > 0$ and $z, z' \in \mathbb{R}^2$, let $\mathbb{P}_{z,z'}^{(T)}$ be the law of the process Z evolving according to (42), conditioned by the event $\{Z_0 = z, \ Z_T = z'\}$ and consider the process $\xi^{(T)} := (\xi_u^{(T)})_{u \in [0,1]}$ defined by

$$\forall \, u \in [0,1], \qquad \xi_u^{(T)} \ := \ TZ_{Tu}$$

Under $\mathbb{P}_{z,z'}^{(T)}$ this process is Gaussian and Proposition 32 enables to see that for fixed $z, z' \in \mathbb{R}^2$, as T goes to 0_+, $\xi^{(T)}$ converges in probability (under $\mathbb{P}_{z,z'}^{(T)}$) toward the deterministic trajectory $\varphi_{z,z'}$, with respect to the uniform norm on $\mathcal{C}([0,1], \mathbb{R}^2)$). Indeed, $\lim_{T \to 0_+} T^2 \sigma_{uT}^{(T)}(z_0, z_T) = 0$ would even have been sufficient for this behavior. Using the linear space-time transformation described in Subsection 2.2, this result can be retranscripted under the form of Theorem 6.

Remark 34 Following Remark 33, if $z = (x, y)$ and $z' = (x', y')$ are such that $x = x'$, then the process $\widetilde{\xi}^{(T)} := (\widetilde{\xi}_u^{(T)})_{u \in [0,1]}$, defined by

$$\forall \, u \in [0,1], \qquad \widetilde{\xi}_u^{(T)} \ := \ Z_{Tu}$$

converges in probability (under $\mathbb{P}_{z,z'}^{(T)}$) toward the deterministic trajectory $\widetilde{\varphi}_{z,z'}$, with respect to the uniform norm on $\mathcal{C}([0,1], \mathbb{R}^2)$), where

$$\forall \, u \in [0,1], \qquad \widetilde{\varphi}_{z,z'}(u) \ := \ \begin{pmatrix} x \\ (1 - 4u + 3u^2)y - (2u - 3u^2)y' - 6\rho u(1-u)x \end{pmatrix}$$

Using the linear space-time transformation described in Subsection 2.2, this result can also rewritten in the original setting of the Introduction.

$\square$

4.2 A probabilistic proof of a persistence rate upper-bound

The previous developments on the diffusion bridge associated to (42) enable us to retrieve a lower-bound of $\mathbb{P}(\tau \geq t)$, for τ defined in (13).

Proposition 35 *(i) Let $(Z_t)_{t \geq 0}$ be a solution of (42) with $\rho \geq 0$. For $\kappa \geq 1$, let $H_\kappa = \{(x, y), y < \kappa x\}$. Then, for any positive ρ_0, there exists a constant $\tilde{\lambda} > 0$ such that for any $\rho \in [0, \rho_0]$ satisfying $\kappa \geq 3\rho$ and any $z_0 \in H_\kappa$, one can find a constant C (which depends on z_0 as well as on the parameters ρ_0 and κ) such that*

$$\mathbb{P}_{z_0}(\tau_{H_\kappa} \geq t) \geq C \exp(-\tilde{\lambda} t), \quad t > 0.$$

(ii) Let (Z_t) be a solution of (7). There exists $\tilde{\lambda} > 0$ such that if $0 < 2a \leq b$, we have for every $z_0 \in D = \{(x, y), x > 0\}$

$$\mathbb{P}_{z_0}(\tau \geq t) \geq C \exp(-\tilde{\lambda}\omega t), \quad t > 0,$$

where $\omega = \sqrt{ab - a^2/4}$ and C is a constant which depends on z_0, a, b and c.

Remark 36 It is possible to be more precise, the same proof showing that for all ε, $\varepsilon' > 0$, one can find a corresponding $\widetilde{\lambda}(\varepsilon, \varepsilon') > 0$ such that (i) is satisfied if $\kappa \geq 3(1 + \varepsilon)\rho/2$, $\kappa \geq \varepsilon'$ and if $\widetilde{\lambda}$ is replaced by $\widetilde{\lambda}(\varepsilon, \varepsilon')$ (the constant C has then also to depend on ε and ε'). It follows that in (ii) the condition $b \geq 2a$ can be replaced by $b \geq (5/4 + \varepsilon)a$ with the price that $\widetilde{\lambda}$ (and C) must depend on

$\varepsilon > 0$. But we believe that even these results are not optimal (e.g. one could hope for the condition $b > (1/4 + \varepsilon)a$ in (ii)), so we won't detail them. Proposition 35 is just an illustration of how the bridges could be used further.

$\square$

Proof

(i) The proof is divided into three steps. In the first one, we show that we can build a subset $\mathcal{S}$ of H_κ for which any bridge associated to (42), starting and ending in $\mathcal{S}$ (at a time T which will be chosen small) stays in H_κ with a high probability. Then, in the second one, we use a Markov-type argument close to the one used in the proof of Proposition 15 to obtain the announced result when the starting point of $(Z_t)_{t \geq 0}$ is in $\mathcal{S}$. Finally, we extend the result to any initial point in H_κ.

Step 1. *Lower-bound for* $\inf_{z,z' \in \mathcal{S}} \mathbb{P}_{z,z'}^{(T)}(\tau_{H_\kappa} > T)$ *for a particular $T > 0$.* Let $z_0 = (x_0, y_0)$ and $z_T = (x_T, y_T)$ belong to $\mathbb{R}^2$. Denote respectively by $\eta_{uT}^1(z_0, z_T)$ and $\eta_{uT}^2(z_0, z_T)$ the first and second coordinate of $\eta_{uT}^{(T)}(z_0, z_T)$. First, owing to Proposition 32 (and to the more precise developments stated in its proof) and to (46) and (47), one checks that

$$\eta_{uT}^1(z_0, z_T) = \left(x_0 + (x_T - x_0)(3u^2 - 2u^3)\right) + \gamma_1(z_0, z_T, T) \tag{48}$$

and

$$\eta_{uT}^2(z_0, z_T) = \frac{6u(1 - u)(x_0 - x_T)}{T} + 2\rho u(1 - u)\left((2 - u)x_0 + (1 + u)x_T\right) + \gamma_2(z_0, z_T, T) \tag{49}$$

where γ_1 and γ_2 satisfy: there exists $T_0 > 0$ and a positive constant C such that for every $T \in (0, T_0]$, for any z_0 and z_T and for any $\rho \in [0, \rho_0]$ (due to the uniformity of $\mathcal{O}(T)$ in (46) and (47) with respect to ρ in a compact set of $\mathbb{R}_+$).

$$|\gamma_1(z_0, z_T, T)| \leq C(|z_0| + |z_T|)T \quad \text{and} \quad |\gamma_2(z_0, z_T, T)| \leq C(|y_0| + |y_T| + (|z_0| + |z_T|)T).$$

Second, by Theorem V.5.3 of [1] (applied with $\alpha = 1$ and $K = T_0$), there exists a universal constant C such that

$$\forall T \geq 0, \ \forall h \geq 1, \quad \mathbb{P}_{z_0, z_T}^{(T)}\left(\sup_{u \in [0,T]} |Z_{uT} - \eta_{uT}^{(T)}(z_0, z_T)| > h\right) \leq Ch \exp\left(-\frac{h^2}{2\bar{\sigma}_T}\right)$$

where $\bar{\sigma}_T = \sup_{u \in [0,1]} |(\sigma_{uT}^{(T)})_{1,1}| + \sup_{u \in [0,1]} |(\sigma_{uT}^{(T)})_{2,2}|$. By Proposition 32, for every $u \in [0,1]$, $\sigma_{uT}^{(T)}(z_0, z_T) \to 0$ as $T \to 0$. Note that this convergence is uniform in z_0 and z_T since the covariance matrix does not depend of them. By (45), it appears that the convergence is also uniform in u and a little sharper study of the dependence of $\sigma_{(uT)}^{(T)}$ in ρ yields in fact that for every $\rho_0 > 0$,

$$\sup_{\rho \in [0, \rho_0], z_0, z_T \in \mathbb{R}^2} \bar{\sigma}_T \xrightarrow{T \to 0} 0.$$

Applying the previous inequality with $h = \sqrt{\sigma_T}$, we deduce that there exists $T_1 \in (0, T_0]$ such that for every $T \in (0, T_1]$, for every $z_0, z_T \in \mathbb{R}^2$ and every $\rho \in [0, \rho_0]$,

$$\mathbb{P}_{z_0, z_T}^{(T)}\left(\sup_{u \in [0,T]} |Z_{uT} - \eta_{uT}^{(T)}(z_0, z_T)| \leq \sqrt{\sigma_T}\right) \geq \frac{1}{2}.$$

We shall now build a box $\mathcal{S} = [1, 1 + h_1] \times [-h_2, h_2]$ (where h_1 and h_2 are positive numbers) for which there exists a positive T such that when $(z_0, z_T) \in \mathcal{S}^2$, the mean $\eta_{uT}^{(T)}(z_0, z_T)$ stays at a

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distance greater than 1 of the boundary of H_κ. Checking that for a point (x, y) of $\mathbb{R}^2$, the distance from (x, y) to the boundary ∂H_κ is equal to $|\kappa x - y|/\sqrt{\kappa^2 + 1}$, we thus need to find h_1, h_2 and T in order that

$$\inf_{z_0, z_T \in \mathcal{S}} \inf_{u \in [0,1]} \kappa \eta_{uT}^1 - \eta_{uT}^2 \; > \; \sqrt{\kappa^2 + 1}\sqrt{\sigma_T}$$
$$\geq \; \sqrt{2}\kappa\sqrt{\sigma_T}. \tag{50}$$

Note that T, h_1 and h_2 will depend on ρ_0 but not on $\rho \in [0, \rho_0]$ and $\kappa \geq 1$ satisfying $\kappa \geq 3\rho$. Using (48) and (49), we obtain that there exists $C > 0$ such that for every $\rho \in [0, \rho_0]$, for every $T \in (0, T_1]$ and every $(z_0, z_T) \in \mathcal{S}$

$$\eta_{uT}^1 \geq 1 - C(1 + h_1 + h_2)T \quad \text{and} \quad \eta_{uT}^2 \leq \frac{3}{2T}h_1 + \frac{3\rho}{2} + Ch_2 + C(1 + h_1 + h_2)T. \tag{51}$$

Equation (51) shows that (50) is fulfilled as soon as

$$\kappa - \frac{3\rho}{2} - C(h_1 + h_2)(1 + \kappa)T - \frac{3}{2T}h_1 - Ch_2 \; > \; \sqrt{2}\kappa\sqrt{\sigma_T}.$$

Taking into account that $\kappa \geq 1$ and $\kappa \geq 3\rho$, the above inequality is satisfied if

$$\frac{1}{2} \; > \; 2C(h_1 + h_2)T + \frac{3}{2T}h_1 + Ch_2 + \sqrt{2}\sqrt{\sigma_T}. \tag{52}$$

relation which no longer depends on κ. We can now set for instance $h_1 := T^2$ and $h_2 := T$ and choose $T \in (0, T_1]$ small enough so that (52) is satisfied. As a consequence, the subset $\mathcal{S} = [1, 1 + h_1] \times [-h_2, h_2]$ of H_κ is such that for every $\rho \in [0, \rho_0]$ and $\kappa \geq 1$ verifying $\kappa \geq 3\rho$, we have

$$\inf_{z, z' \in \mathcal{S}} \mathbb{P}_{z, z'}^{(T)}(\tau_{H_\kappa} > T) \; \geq \; \frac{1}{2}. \tag{53}$$

Step 2. _Lower-bound for $\mathbb{P}_{z_0}(\tau_{H_\kappa} > t)$ when $z_0 \in \mathcal{S}$._ We consider a time $T > 0$ and a subset $\mathcal{S} = [1, 1 + h_1] \times [-h_2, h_2]$ of H_κ (depending only on ρ_0) for which (53) holds. For every $\ell \geq 1$, we have

$$\mathbb{P}_{z_0}(\tau_{H_\kappa} > \ell T, Z_{\ell T} \in \mathcal{S}) \geq$$
$$\mathbb{P}_{z_0}(\tau_{H_\kappa} > \ell T, Z_{\ell T} \in \mathcal{S} | \tau_{H_\kappa} > (\ell - 1)T, Z_{(\ell-1)T} \in \mathcal{S}) \mathbb{P}_{z_0}(\tau_{H_\kappa} > (\ell - 1)T, Z_{(\ell-1)T} \in \mathcal{S}).$$

By the Markov property,

$$\mathbb{P}_{z_0}(\tau_{H_\kappa} > \ell T, Z_{\ell T} \in \mathcal{S} | \tau_{H_\kappa} > (\ell - 1)T, Z_{(\ell-1)T} \in \mathcal{S}) \; = \; \int H(z)\, \mu_{z_0, (\ell-1)T}(dz)$$

where

$$\forall\, z \in \mathcal{S}, \qquad H(z) \; := \; \mathbb{P}_z(Z_T \in \mathcal{S}) \int_{\mathcal{S}} \mathbb{P}_{z, z'}^{(T)}(\tau_{H_\kappa} > T)\, \mu_{z, T}(dz').$$

and $\mu_{z_0, (\ell-1)T}$ is the conditional law (under $\mathbb{P}_{z_0}$) of $Z_{(\ell-1)T}$ on $\{\tau_{H_\kappa} > (\ell - 1)T, Z_{(\ell-1)T} \in \mathcal{S}\}$. Owing to (53) and to the fact that the support of $\mu_{z_0, \ell-1)T}$ is included in $\mathcal{S}$, we get

$$\mathbb{P}(\tau_{H_\kappa} > \ell T, Z_{\ell T} \in \mathcal{S}) \geq \varsigma \mathbb{P}(\tau_{H_\kappa} > (\ell - 1)T, Z_{(\ell-1)T} \in \mathcal{S}) \quad \text{with} \quad \varsigma := \frac{1}{2} \inf_{z \in \mathcal{S}} \mathbb{P}_z(Z_T \in \mathcal{S}).$$

Note that for $T > 0$, the transition density $\mathbb{R}^2 \ni z' \mapsto f_{z,T}(z') := \frac{dP_z(Z_T \in dz')}{dz'}$ is positive and continuous with respect to $(z, z') \in (\mathbb{R}^2)^2$. It follows that the coefficient ς is positive by compactness

of $\mathcal{S}$. Furthermore it is uniform over κ, ρ satisfying the conditions of Proposition 35 (but a priori ς depends on ρ_0, as T and $\mathcal{S}$ do). Then, since for all $t > 0$,

$$\mathbb{P}_z(\tau_{H_\kappa} > t) \geq \mathbb{P}_z(\tau_{H_\kappa} > k_T T, Z_{k_T T} \in \mathcal{S}),$$

where $k_T = \lfloor t/T \rfloor + 1$, we deduce from an induction that for every $z \in \mathcal{S}$,

$$\mathbb{P}_z(\tau_{H_\kappa} > t) \geq \varsigma^{k_T} \geq C \exp(-\tilde{\lambda} t),$$

where $\tilde{\lambda} = -\log(\varsigma)/T$ (depending only on ρ_0).

Step 3. *Lower-bound for $\mathbb{P}_z(\tau_{H_\kappa} > t)$ when $z \in H_\kappa$.* The idea of this step is identical to that of the proof of Proposition 25(ii). More precisely, to extend the lower-bound obtained above to any $z_0 = (x_0, y_0) \in H_\kappa$ (up to a constant C which depends to z_0), it is enough to build a controlled trajectory $(z_\varphi(t))_{t \geq 0}$ such that $z_\varphi(0) = z_0$, $z_\varphi(t_0)$ belongs to $\mathcal{S}$ and such that $z_\varphi(t) \in H_\kappa$ for every $t \in [0, t_0]$.

Owing to Lemma 27(ii), it is enough to consider the case $y_0 = 0$. We treat successively the cases $x_0 \leq 1$ and $x_0 \geq 1 + h_1$. If $x_0 \leq 1$, the idea is to join a point of a path of the free dynamical system which passes through $z_\mathcal{S} = (1 + h_1/2, 0)$. More precisely, by Lemma 27(ii), we know that the solution $(x(t), y(t))_{t \geq 0}$ to the free dynamical system starting from $(0, -(1 + h_1/2)e^{\frac{\rho\pi}{2}})$ passes through $z_\mathcal{S}$ at time $\pi/2$ and that $\mathcal{C} = \{(x(t), y(t)), 0 < t < \pi/2\}$ is a curve included in $(0, +\infty) \times (-(1 + h_1/2)e^{\frac{\rho\pi}{2}}, 0)$. It remains to join a point of $\mathcal{C}$ (without leaving out H_κ). This can be done applying Lemma 27(i) with $v = -(1 + h_1/2)e^{\frac{\rho\pi}{2}}$.

Suppose now that $x_0 \geq 1 + h_1$. The construction is slightly different in the cases $\rho > 0$ and $\rho = 0$ (by the assumptions of Proposition 35, the situation $\rho < 0$ is excluded). If $\rho > 0$, we join $z_\mathcal{S}$ by crossing the segment $[z_0, z_\mathcal{S}]$ with a controlled trajectory. Set $(x(t), y(t))_{t \geq 0} = (x_0 e^{-\rho t}, 0)_{t \geq 0}$. This trajectory can be viewed as the solution starting from z_0 of

$$\begin{cases} \dot{x}(t) = -\rho x(t) - y(t) \\ \dot{y}(t) = -\rho y(t) + x(t) + \varphi(t) \end{cases}$$

with $\varphi(t) = -x(t)$. Hence, this is a controlled trajectory which clearly crosses $[z_0, z_\mathcal{S}]$ in a finite time. Finally, if $\rho = 0$, we join a point $z_1 = (x_1, -h_2/2)$ using Lemma 27(i). If $x_1 \leq 1$, we are reduced to the first considered case. So we can assume that $x_1 \geq 1 + h_1$ and we join the point $\tilde{z}_\mathcal{S} = (1 + h_1/2, -h_2/2)$ by crossing the segment $[z_1, \tilde{z}_\mathcal{S}]$ with a controlled trajectory (more precisely, $(x_1 - h_2 t/2, -h_2/2)_{t \geq 0}$ is a controlled trajectory). This ends the proof of (i).

(ii) By Corollary 13, there exists α such that $(\hat{Z}_t)_{t \geq 0} := \left(\sqrt{\omega} \alpha P_v^{-1} Z_{\frac{t}{\omega}}\right)_{t \geq 0}$ is a solution of (25).

Remind that $\tau = \inf\{t > 0, \Re(Z_t) < 0\}$ and for a subset C of $\mathbb{R}^2$, set $\tau_C^{\hat{Z}} = \inf\{t > 0, \hat{Z}_t \in C^c\}$. Similarly to the proof of Proposition 29, one checks that for every $z \in \{(u, v), u > 0\}$,

$$\mathbb{P}_z(\tau > t) = \mathbb{P}_{\hat{z}}(\tau_{H_\kappa}^{\hat{Z}} > \omega t)$$

where $\omega = \sqrt{ab - a^2/4}$, $\kappa = \frac{1}{\omega}(b - \frac{a}{2})$, $H_\kappa = \{(u, v), v < \kappa u\}$ and $\hat{z}$ belongs to H_κ. The result follows by applying the first part of this proposition with $\rho = a/(2\omega)$. Indeed, the assumption $\kappa \geq 3\rho$ amounts to $b \geq 2a$. For the other condition, $\kappa \geq 1$, taking into account that $\omega < \sqrt{ab}$, it is sufficient that $b - a/2 \geq \sqrt{ab}$, namely $\frac{b}{a} \geq \left(\frac{1+\sqrt{3}}{2}\right)^2$ and note that $\left(\frac{1+\sqrt{3}}{2}\right)^2 \leq 2$.

$\blacksquare$

5 Simulations and statistical considerations

In this section, we shortly focus on some statistical problems related to the estimation of the real parameters which govern the trajectories solutions of (1). We denote the unknown underlying parameters (a^*, b^*, c^*) and aim at developing some statistical estimation methods of these parameters. We are also interested in the average time for the process $(X_t)_{t \geq 0}$ to return to the equilibrium price. When c^* is small, it can be shown that such an average time is close to $T^*/2$ (see the results of [11] and [2] for small noise asymptotics of random dynamical system) where T^* is the period of the deterministic process associated to the model:

$$T^* := \frac{2\pi}{\omega^*} \quad \text{where} \quad \omega^* := \sqrt{a^*(b^* - \frac{a^*}{4})}. \tag{54}$$

With a slight abuse of language, T^* will be then called *pseudo-period* of the process.

It is natural to wonder if it is efficient to first estimate (a^*, b^*) and then, to plug these estimates $(\hat{a}, \hat{b})$ in the analytical formula given above. We describe in the next paragraph how one can use the maximum likelihood estimator to approximate (a^*, b^*). Then, a short simulation study exhibit rather different behaviours of the estimator of the pseudo-period $\hat{T}$ derived from the values $(\hat{a}_{ML}, \hat{b}_{ML})$ plugged into (54). When c is small, we compare such an estimation with a more natural estimator derived from hitting times of the level $X = 0$ and show that in some cases, this last estimation can perform better for the recovery of T^*. In our short study, we will assume that the process starts from its equilibrium, that is $X_0 = 0$. This assumption slightly simplifies the MLE derived below.

5.1 Maximum likelihood estimator for parameters a^* and b^*

Statistical settings In this paragraph, we first detail the computation of the MLE of the real parameters denoted (a^*, b^*) when one observes the whole trajectory of the price X between 0 and t. This is of course an idealization and a dramatical simplification of the true statistical problem since in practical situations, we can only handle some values of X in a discreted observation grid $(k\Delta)_{0 \leq k \leq t/\Delta}$.

Even if the relative size of Δ compared to the time length of observation t is of first interest for some real statistical applications, we simplify this short study and consider only continuous observation times. We leave this important question of the statistical balance between Δ and t to a future work. In such a situation, it is easy to recover the parameter c^* by considering the normalized quadratic variation of the trajectory $(X_s)_{0 \leq s \leq t}$.

$$c^* = \frac{\langle X \rangle_t}{t}$$

Hence, in the sequel, we only consider the problem of the estimation of a^* and b^* and we assume the knowledge of c^* (we fix it to 1 for sake of convenience).

Change of measure formula In order to estimate (a^*, b^*), we can only handle the process X since Y depends on the unobserved parameter b^* through the relation

$$\forall \, t \geq 0, \qquad Y_t = b^* \int_0^t \exp(b^*(s - t))dX_s - b^* X_t.$$

For any choice of $(a, b) \in \mathbb{R}_+^2$, we consider the two processes defined by

$$\forall \, t \geq 0, \qquad Y_t^b = b \int_0^t \exp(b(s - t))dX_s - bX_t, \tag{55}$$

and
$$H_t^{a,b} = (b-a)X_t + Y_t^b.$$

In Equation (55), Y^b depends on the increments dX_s. A simple integration by part yields the equivalent expression:
$$\forall\, s \geq 0, \qquad Y_s^b = e^{-bt}\left(-b^2 \int_0^t e^{bs} X_s ds\right).$$

The fact that $Y = Y^{b^*}$ follows from the definition of the process Y. Thus, $(X_t)_{t\geq 0}$ satisfies
$$dX_t = H_t^{a^*,b^*} dt + dB_t.$$

Now, we can apply the Girsanov formula: if we denote by $\mathbb{P}_{a^*,b^*}$ the law of the process, we then obtain the change of measure formula:
$$\frac{d\mathbb{P}_{a^*,b^*}}{d\mathbb{Q}_0}(X) = \exp\left(\int_0^t H_s^{a^*,b^*}(X_s)dX_s - \frac{1}{2}\int_0^t H_s^{a^*,b^*}(X_s)^2 ds\right).$$

Maximum likelihood Given any trajectory X, we can then define L_t, the log-likelihood of the parameters (a,b) as follows:
$$L_t(a,b) = \int_0^t H_s^{a,b}(X_s)dX_s - \frac{1}{2}\int_0^t H_s^{a,b}(X_s)^2 ds.$$

The expression above can be modified using an integration by part. We then obtain the "robust" formulation:
$$
\begin{aligned}
L_t(a,b) \;=\; & \frac{b-a}{2}\left[X_t^2 - t\right] + X_t Y_t^b \\
& + \int_0^t b^2 X_s^2 + b X_s Y_s^b - \frac{1}{2}\left[(b-a)X_s + Y_s^b\right]^2 ds. \quad\quad (56)
\end{aligned}
$$

The maximum likelihood estimator is then formally defined by
$$(\hat{a}_t^{ML}, \hat{b}_t^{ML}) := \arg\max_{(a,b)\in\mathbb{R}_+^2} L_t(a,b).$$

For any $b \geq 0$, $a \longmapsto L_t(a,b)$ is a concave function and thus the optimal value of a given any b is
$$a_b = b + \frac{\int_0^t X_s Y_s^b ds + \frac{t-X_t^2}{2}}{\int_0^t X_s^2 ds}.$$

Hence, $\hat{b}_t^{ML}$ is obtained by maximizing $b \longmapsto L_t(a_b, b)$. Unfortunately, we did not find any explicit formula regarding the relation $b \longmapsto Y^b$. Hence, to estimate b^*, we use an exhaustive numerical search of the optimal value of b and we obtain $\hat{b}_t^{ML}$.

5.2 Estimation of the mean pseudo-period T^* with hitting times strategy

It may be possible to estimate T^* using the maximum likelihood estimators $(\hat{a}_t^{ML}, \hat{b}_t^{ML})$ defined above with and plug them in the relation (54) (which is supposed to be true only for $\mathcal{T}^*$ with a vanishing noise level)
$$\hat{T}_t^{ML} := \frac{2\pi}{\sqrt{\hat{a}_t^{ML}\left(\hat{b}_t^{ML} - \hat{a}_t^{ML}/4\right)}}.$$

Of course, the ability of this estimator to well approximate T^* highly depends on the asymptotic behaviour of $(\hat{a}_t^{ML}, \hat{b}_t^{ML})$. We will discuss on several statistical questions related to this study in the next section.

We can also compare $\hat{T}_t^{ML}$ with a more nature way to estimate the mean return time all along the trajectories by considering the sequence of crossing times of level 0 of $(X_s)_{0 \leq s \leq t}$. In this view, let us consider $\epsilon > 0$, and define a skeleton chain associated to the trajectory $(X_s)_{0 \leq s \leq t}$. The sequences $(\tau_k)_{k \geq 0}$ and $(r_k)_{k \geq 0}$ are initialized with:

$$\tau_0 := 0 \qquad \text{and} \qquad r_0 := \inf\{s \geq \tau_0 | |X_s| \geq \epsilon\}$$

and recursively built as follows:

$$\forall k \geq 0 \qquad \tau_{k+1} := \inf\{s \geq r_k | X_s = 0\} \qquad \text{and} \qquad r_{k+1} := \inf\{s \geq \tau_k | |X_s| \geq \epsilon\}.$$

When c^* is small, we can now define a quite natural estimator of the mean pseudo-period using this construction. If we set $N_t := \sup\{k \geq 0 | \tau_k \leq t\}$, we set

$$\hat{T}_t^\epsilon := 2 \frac{\sum_{k=1}^{N_t}(\tau_{k+1} - \tau_k)}{N_t} = 2 \frac{\tau_{N_t} - \tau_0}{N_t}.$$

5.3 Statistical performances and open problems

To establish the numerical performances of $\hat{T}_t^\epsilon$ and $\hat{T}_t^{ML}$, we use the following statistical setting: several trajectories defined on $[0, T]$ are observed on some discrete times $(t_k)_{0 \leq k \leq K}$. The observation time are equally sampled with a constant step size Δ such that $t_k - t_{k-1} = \Delta$. We are interested in the behaviour of our two estimators in the two distinct asymptotic settings:

- High resolution sampling scheme $\Delta \longrightarrow 0$
- Long time observation $T \longrightarrow +\infty$.

For our purpose, the threshold is defined empirically after several runs of the estimator. It should be carefully chosen since ϵ is the parameter which enables to distinguish real crossings of $X = 0$ from the natural volatility of the model carried out by the brownian noise. Thus, the calibration of ϵ should be related to the noise level contained in c^*. In our simulation, we have chosen $\epsilon = c^* \sqrt{\Delta}$. We use for each estimator $N = 10^3$ Monte-Carlo simulations to obtain the repartition of $\hat{T}_t^\epsilon$ and $\hat{T}_t^{ML}$ around T^*.

Moreover, we use a discretized version of the stochastic differential equation with several step size. We show in Figure 6 the performances of $\hat{T}_t^{ML}$ for several size of discretization step Δ as well as the performances of $\hat{T}_t^\epsilon$ in Figure 7.

One may instantaneously remark that the step size Δ has an important influence on the ability of $\hat{T}_t^{ML}$ to recover T^* although this parameter does not seem so important for the estimator $\hat{T}_t^\epsilon$. Simulations show that the smaller Δ, the smaller the bias of $\hat{T}_t^{ML}$. Moreover, the variance of the estimator is mainly determined by the length of simulation (time t). Hence, for a fixed step size of simulation, Figures 6 and 7 demonstrate that it seems better to use $\hat{T}_t^\epsilon$ to infer T^*. When one may let $\Delta \longmapsto 0$, the maximum likelihood estimator $\hat{T}_t^{ML}$ seems more convenient.

We shortly describe several problems of interest which concern the estimation of T^*. First, the influence of Δ as well as the influence of t needs to be understood for the estimation of T^* using the MLE. This question may be faced with a careful understanding of the natural score function defined by the log-likelihood L_t. Our simulations tend to show that a special asymptotic behaviour of $(\Delta, t) \longrightarrow (0, +\infty)$ should be considered to obtain optimal estimations.

Second, the size of the threshold ϵ in the definition of $\hat{T}_t^\epsilon$ has not been theoretically investigated, although it has a great influence on the ability of $\hat{T}_t^\epsilon$ to nicely recover T^*. Hence, there should also exist a precise asymptotic regime of $(\epsilon, t) \longrightarrow (0, +\infty)$ which may permit to obtain statistical reconstruction properties. Such a last result should be derived from a careful inspection of the
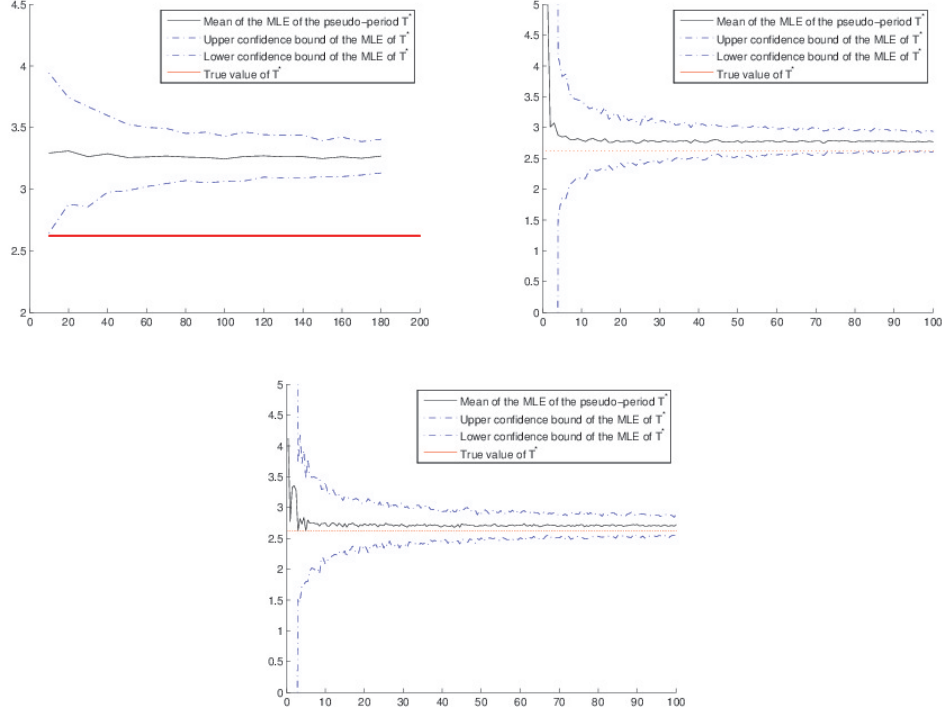
Figure 6: Estimation of T^* using $\hat{T}_t^{ML}$ with respect to the observation time t for $a = 1, b = 6, c = 1$. (top left: $\Delta = 10^{-2}, c = 1$, top right: $\Delta = 5.10^{-3}$, bottom left: $\Delta = 10^{-3}$, bottom right: $\Delta = 5.10^{-4}$).

local times spent by $(X_t)_{t\geq 0}$ around the level 0 (to fix ϵ) as well as the concentration rate of the hitting times which may be obtained using Theorem 4.

At last, the link between T^* and the expected time needed for $(X_t)_{t\geq 0}$ to return to its equilibrium price is still mysterious. We only identify this link in the small noise asymptotics and even though such a relation seems to be true in more general situations for T^*, a theoretical proof is missing.

These three questions are far beyond the scope of this study, and we let them open for future works.

6   Conclusion

In this paper a model of speculative bubble evolution was proposed. The dynamics has to be at least of second order, to have a chance to display a weak periodic behavior typical of this kind of phenomena. This second order is induced by the way the process under consideration weights its past evolution to infer its future behavior (increase/decrease in the close past favoring an immediate tendency to follow the same trend). Dynamics of all orders (including non-integer ones) could be obtained in the same fashion, by modifying the weights. At the "microscopic level", the latter are related to the distribution of the backward time windows used by a multitude of agents in order to speculate on the future evolution.

But we restricted ourselves to second order dynamics: it is the simplest one and in some sense it mimics Newtonian mecanics, which are also of second order, the forces impacting directly on the acceleration. A main difference is the noise entering our modeling, which is required to maintain some stability of the system under study. Nevertheless and informally, this analogy with the physics law of motion enables to unmask some misleading arguments used by the real estate agencies and the mass media: they mainly explain the evolution of prices by making an inventory of the forces
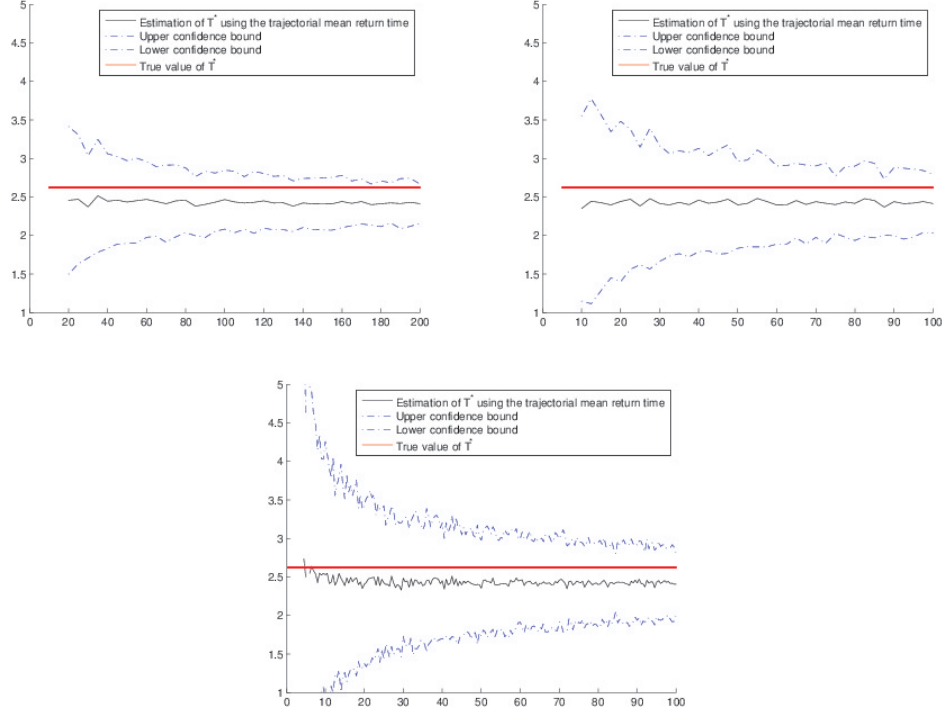
Figure 7: Estimation of T^* using $\hat{T}_t^\epsilon$ with respect to the observation time t for $a = 1, b = 6, c = 1$. (top left: $\Delta = 10^{-2}, c = 1$, top right: $\Delta = 5.10^{-3}$, bottom left: $\Delta = 10^{-3}$, bottom right: $\Delta = 5.10^{-4}$).

in the housing market, such as loan interest rates, the growth of population, etc. (all these factors, as well as opposite leverages, are summed up in our parameter a), forgetting the order of the evolution equation (induced by the parameter b). Transposed in the astronomy field, it would amount to make the observation that the main interaction between the Sun and the Earth goes through gravitation, so that we should conclude that our planet would soon end up in the Sun. Luckily, we are essentially saved by the second order of the kinetics law which enables the Earth to turn around the Sun!

We have also assumed that the repelling force to the equilibrium level $X = 0$ is linear and traduced in (1) by the dritf term $-aX_t$ at any time t. This drift term mimics an economic repelling force which may be non-linear. Nevertheless, our linearization may be considered as a first order valid approximation at least near the equilibrium state $X = 0$.

From the mathematical point of view, our main interest was in the return time to the equilibrium "price" and we have shown that it is more concentrated than the relaxation to the equilibrium distribution of the prices. This feature explains the bubble/almost periodic aspect of the typical trajectories. We have obtained some lower and upper bounds of this concentration rate in Theorem 4. Even if there is still a gap of order around ten between our lower and upper bounds, numerous simulations (not shown in this paper) using the Fleming-Viot's type algorithm described in [7] lead to the conjecture that $\lambda_0(D) = \frac{\log(2)}{\pi}\omega$.

One failing of our framework is that the parameters a, b and c were assumed to be time-independent, hypothesis which is certainly wrong in practice. Our model should apply only to a few periods and a finer modeling would take into account the time-inhomogeneity of a, b and c. Such an extension remains Gaussian, but its investigation is out of the scope of this paper.

Nevertheless and heuristically, let us just consider the example of the home price index relative to disposable income per household in France from 1995 to 2013 shown in Figure 8. Consider for X the logarithm of the quantity displayed in this picture, since it is a ratio and not a difference as
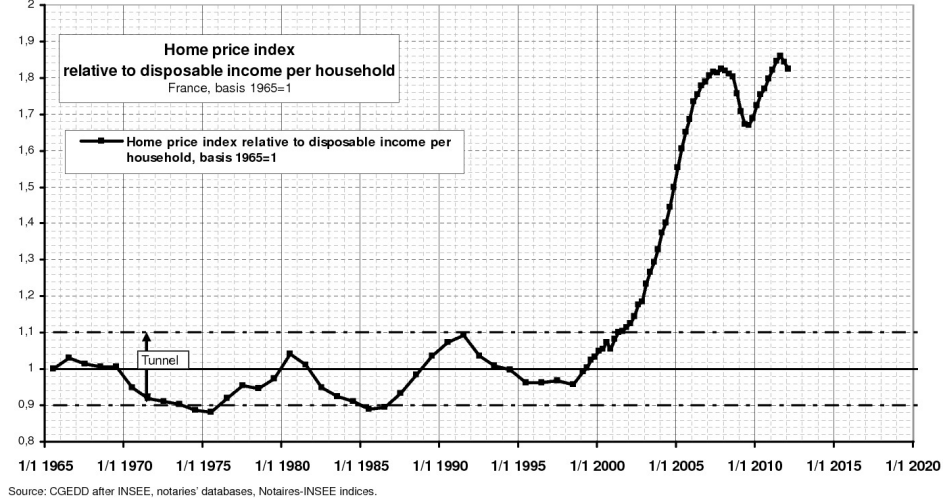
Figure 8: Friggit's curve [12] of the index of asset price relatively to the disposable income in France.

wanted in the introduction. If we assume that the modeling of speculation presented in Section 1.1 can be applied to this case, it seems that the coefficients a, b and c valid for the epoch 1965-1998 are not the same as those for 1998-2013. For simplicity, let us make the hypothesis that c did not change (the resolution is too coarse to check that) and call a_1, b_1 and a_2, b_2 the respective values of a and b. That $a_2 < a_1$ can be explained by the fact that the conditions were more favorable for buying in the second epoch, especially due to low interest rates. Furthermore one can argue that the advent of Internet has probably modified the extend to which the agents have access to the past evolution of the prices, both in short and long terms. So its effect on b is not a priori clear (recall that $1/b$ should be proportional to the mean length of the backward time window). To get a rough idea, we can proceed as follows. Let $t_0 := 1965$, $t_1 := 1999$ and $t_2 := 2008$. Figure 8 suggests that between t_0 and t_1 there were 3 periods under the coefficients a_1, b_1 and c and that between t_1 and t_2 there was one quarter of a period under the coefficients a_2, b_2 and c (except if the equilibrium price has itself changed and that the evolution between 2006 and 2012 is interpreted as one period and half of a new epoch). It follows that $1/\omega_1$ and $1/\omega_2$ should respectively be proportional to $(t_1 - t_0)/3$ and $4(t_2 - t_1)$. Since we do not plan to be very precise, let us make the assumption that $a_1 \ll b_1$ and that $a_2 \ll b_2$, so that $\omega_1 \approx \sqrt{a_1 b_1}$, $\omega_2 \approx \sqrt{a_2 b_2}$ and

$$a_1 b_1 \approx \chi a_2 b_2$$

where

$$\chi := \left(\frac{12(t_2 - t_1)}{t_1 - t_0} \right)^2$$

To deduce another equation, let us believe that an ergodic theorem takes place very rapidly (permitted by the non reversibility of the process). Thus according to Remark 9, we would get

$$\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} X_s^2 \, ds \approx c^2 \frac{b_1 + a_1}{2a_1^2} \approx c^2 \frac{b_1}{2a_1^2}$$

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} X_s^2 \, ds \approx c^2 \frac{b_2 + a_2}{2a_2^2} \approx c^2 \frac{b_2}{2a_2^2}$$

(more carefully, an empirical variance should be computed), and it follows that

$$\frac{b_1}{a_1^2} \approx \tilde{\chi} \frac{b_2}{a_2^2}$$

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where

$$\widetilde{\chi} \ := \ \frac{(t_2 - t_1) \int_{t_0}^{t_1} X_s^2 \, ds}{(t_1 - t_0) \int_{t_1}^{t_2} X_s^2 \, ds}$$

can be computed numerically on Figure 8. We deduce that

$$\begin{aligned} a_1 & \approx & \chi^{5/3} \widetilde{\chi}^{1/3} a_2 \\ b_1 & \approx & \chi^{2/3} \widetilde{\chi}^{1/3} b_2 \end{aligned}$$

Numerically, we obtain that $\chi^{2/3} \widetilde{\chi}^{1/3} \approx 3.73 > 1$, so it would seem that the advent of Internet has led people to rather use more recent trend of the housing market to make their speculation.

At last, using the lower bound obtained in Theorem 4, we may postulate that the probability that the index price X hits the equilibrium level 1 (see Figure 8) before year 2017 (which corresponds to an average annual loss of around 13%) is at least 50%. We can thus wonder if the famous *kiss landing* generally announced by estate agents may not more probably end in a crash ...

A   On the persistence rate

Our goal here is to prove the existence of the quasi-stationary distribution and its persistence rate, as alluded to in Remark 5. It is based on general considerations relegated in this appendix because they do not lead to explicit estimates such as (15), which are more important from a practical point of view than the mere existence of $\lambda_0(D)$. Furthermore, these a priori bounds will be useful in the development to follow.

Recall that $D := \{(x, y) \in \mathbb{R}^2 \ : \ x > 0\}$ and let ∂D be its boundary. We are interested in L_D, the realization on D of the differential operator L given by (9) with Dirichlet boundary condition on ∂D. From a probabilist point of view, it is constructed in the following way. For any $z \in \mathbb{R}^2$, let $(Z_t^z)_{t \geq 0}$ be a diffusion process whose evolution is dictated by L and whose initial condition is $Z_0^z = z$. Starting from z, $(Z_t^z)_{t \geq 0}$ can be obtained by solving the stochastic differential equation (7) with coefficients given by (8). Let τ be the stopping time defined by (13), namely

$$\tau \ := \ \inf\{t \geq 0 \ : \ Z_t^z \in \partial D\}$$

For any $t \geq 0$, any $z \in D$ and any measurable and bounded function f defined on D, consider

$$P_t^D[f](z) \ := \ \mathbb{E}[f(Z_t^z) \mathbb{1}_{t < \tau}] \tag{57}$$

Recall that μ is the invariant Gaussian probability measure of L and denote by μ^D its restriction to D. Then P_t^D can be extended into a contraction operator on $\mathbb{L}^2(\mu^D)$. Indeed, let P_t be the full operator associated to L: any $z \in D$ and any measurable and bounded function f defined on $\mathbb{R}^2$, we have

$$P_t[f](z) \ := \ \mathbb{E}[f(Z_t^z)] \tag{58}$$

Since μ is invariant for P_t, for any measurable and bounded function f defined on D (which can also be seen as a function on $\mathbb{R}^2$ by assuming that it vanishes outside D), we get by Cauchy-Schwarz inequality,

$$\begin{aligned} \mu^D[(P_t^D[f])^2] & \leq & \mu^D[P_t^D[f^2]] \\ & \leq & \mu^D[P_t[f^2]] \\ & \leq & \mu[P_t[f^2]] \\ & = & \mu[f^2] \\ & = & \mu^D[f^2] \end{aligned}$$

This bound enables to extend P_t^D as a contraction on $\mathbb{L}^2(\mu^D)$. The Markov property implies that $(P_t^D)_{t \geq 0}$ is a semi-group, which is easily seen to be continuous in $\mathbb{L}^2(\mu^D)$. The operator L^D is then defined as the generator of this semi-group (in the Hille-Yoshida sense): its domain $\mathcal{D}(L_T^D)$ is the dense subspace of $\mathbb{L}^2(\mu^D)$ consisting of functions f such that $(P_t^D[f] - f)/t$ converges in $\mathbb{L}^2(\mu^D)$ as t goes to 0_+ and the limit is $L^D[f]$ by definition.

The spectrum of $-L^D$ admits a smallest element (in modulus) $\lambda_0(D)$. It is a positive real number and the main objective of this appendix is to justify the assertions made in Remark 5. We begin by being more precise about the existence of $\lambda_0(D)$:

Proposition 37 *There exists a number $\lambda_0(D) > 0$ and two functions $\varphi, \varphi^* \in \mathcal{D}(L^D) \cap \bigcap_{r \geq 1} \mathbb{L}^r(\mu^D) \setminus \{0\}$, which are positive on D, such that*

$$
\begin{aligned}
L^D[\varphi] &= -\lambda_0(D)\varphi \\
L^{D*}[\varphi^*] &= -\lambda_0(D)\varphi^*
\end{aligned}
$$

where L^{D*} is operator adjoint of L^D in $\mathbb{L}^2(\mu^D)$.

Essentially, this result is a consequence of the Krein-Rutman theorem (which is an infinite version of the Perron-Frobenius theorem, see for instance the paper [8] of Du) and the fact that the eigenfunctions belong to $\mathbb{L}^p(\mu^D)$ instead of $\mathbb{L}^2(\mu^D)$ comes from the hyperboundedness of the underlying Dirichlet semi-group.

The rigorous proof relies on a simple technical lemma about the kernels of the operators P_t^D for $t > 0$. To check their existence, we first come back to P_t for a given $t > 0$: from the computations of Section 2, this operator is indeed given by a kernel

$$
\forall\, z \in \mathbb{R}^2,\ \forall\, f \in \mathbb{L}^2(\mu), \qquad P_t[f](z) \;=\; \int p_t(z, z') f(z')\, \mu(dz')
$$

where

$$
\forall\, z, z' \in \mathbb{R}^2, \quad p_t(z, z') \;:=\; \sqrt{\frac{\det(\Sigma)}{\det(\Sigma_t)}} \exp\left(-(z' - z_t)^* \Sigma_t (z' - z_t) + (z')^* \Sigma z'\right) \tag{59}
$$

with

$$
z_t \;:=\; \exp(At)z
$$

It follows easily from (57) and (58) that the same is true for P_t^D: there exists a function $D^2 \ni (z, z') \mapsto p_t^D(z, z') \geq 0$ such that

$$
\forall\, z \in D,\ \forall\, f \in \mathbb{L}^2(\mu^D), \qquad P_t^D[f](z) \;=\; \int p_t^D(z, z') f(z')\, \mu(dz')
$$

and satisfying

$$
\forall\, z, z' \in D, \qquad p_t^D(z, z') \;\leq\; p_t(z, z') \tag{60}
$$

More refined arguments based on the hypoellipticity of L^D enable to see that the mapping p_t^D is continuous and positive on D^2. We can now state a simple but crucial observation:

Lemma 38 *For any $r > 1$, there exists a time $T_r > 0$ such that*

$$
\forall\, t \geq T_r, \qquad \int (p_t^D(z, z'))^r\, \mu^D(dz)\, \mu^D(dz') \;<\; +\infty
$$

Proof

From (60), it is sufficient to prove that

$$\int (p_t(z, z'))^r \, \mu(dz) \, \mu(dz') \quad < \quad +\infty$$

and this can be obtained without difficulty from (59) and from the explicit computations of $\exp(tA)$ of Σ_t and of Σ presented in Section 2.

∎

We can now come to the

Proof of Proposition 37

We begin by applying Lemma 38 with $r = 2$ to find some $T_2 > 0$ such that for $t \geq T_2$ we have

$$\int (p_t^D(z, z'))^2 \, \mu^D(dz) \, \mu^D(dz') \quad < \quad +\infty$$

which implies that P_t^D is of Hilbert-Schmidt class and thus a compact operator. Note furthermore that the spectral radius of P_t^D is positive for all $t \geq 0$. Indeed, this feature can be deduced from the second bound of Theorem 4, which implies that for all $z \in D$, $P_t^D[\mathbb{1}_D](z) = \mathbb{P}_z[\tau > t] > 0$. Thus we are in position to apply Krein-Rutman theorem (see Theorems 1.1 and 1.2 of Du [8], where the abstract Banach X space should be $\mathbb{L}^2(\mu^D)$ and the cone K should consist of the nonnegative elements of $\mathbb{L}^2(\mu^D)$): if $\theta_t > 0$ is the spectrum radius of P_t^D, then there exists a positive function $\varphi_t \in \mathbb{L}^2(\mu^D) \setminus \{0\}$ such that $P_t[\varphi_t] = \theta_t \varphi_t$. This property characterizes θ_t and φ_t (up to a constant factor): if θ is a positive real and if $\varphi \in \mathbb{L}^2(\mu^D)$ is a positive function such that $P_t[\varphi] = \theta \varphi$ then is $\theta = \theta_t$ and φ is proportional to φ_t. This suggests to consider the renormalization $\mu^D[\varphi_t^2] = 1$, so that φ_t is uniquely determined (being positive). From the previous property, we deduce that for all $t \geq T_2$ and all $n \in \mathbb{N}$, $\varphi_{nt} = \varphi_t$ and $\theta_{nt} = \theta_t^n$. Indeed, it is sufficient to note that

$$\begin{aligned} P_{nt}^D[\varphi_t] &= (P_t^D)^n[\varphi_t] \\ &= \theta_t^n \varphi_t \end{aligned}$$

We deduce that for any $r \in \mathbb{Q} \cap [1, +\infty)$, $\varphi_{T_2 r} = \varphi_{T_2}$ and $\theta_{T_2 r} = \theta_{T_2}^r$: write $r = p/q$ with $p, q \in \mathbb{N}$ and note that $\varphi_{T_2} = \varphi_{pT_2} = \varphi_{qrT_2} = \varphi_{rT_2}$ and similarly $\theta_{T_2}^p = \theta_{rT_2}^q = \theta_{pT_2}$. Let us define $\varphi := P_{T_2}^D \varphi_{T_2} = \theta_{T_2} \varphi_{T_2}$. Since $T_2 > 0$ and $P_{T_2}^D(\mathbb{L}^2(\mu^D))$ is included in the domain of L^D, we have $\varphi \in \mathcal{D}(L^D)$. Furthermore from the general Hille-Yoshida theory we have in $\mathbb{L}^2(\mu^D)$,

$$\lim_{t \to 0_+} \frac{P_{T_2+t}^D[\varphi_{T_2}] - P_{T_2}^D[\varphi_{T_2}]}{t} \quad = \quad L^D[P_{T_2}[\varphi_{T_2}]]$$

Thus considering t of the form qT_2 with $q \in \mathbb{Q}_+$ going to zero, we deduce that

$$\begin{aligned} L^D[\varphi] &= \lim_{q \in \mathbb{Q}, \, q \to 0_+} \frac{\theta_{T_2}^{q+1} - \theta_{T_2}}{T_2 q} \varphi_{T_2} \\ &= \theta_{T_2} \frac{\ln(\theta_{T_2})}{T_2} \varphi_{T_2} \\ &= \frac{\ln(\theta_{T_2})}{T_2} \varphi \end{aligned}$$

It remains to set $\lambda_0(D) = -\ln(\theta_{T_2})/T_2$. Since θ_{T_2} is the spectral norm of the contraction operator P_{T_2}, it appears that $\lambda_0(D) \geq 0$. The first bound of Theorem 4 enables to check that $\lambda_0(D) > 0$: from Cauchy-Schwarz inequality, we get that for all $f \in \mathbb{L}^2(\mu^D)$ and all $z \in D$,

$$\begin{aligned} (P_{T_2}^D[f])^2(z) &\leq P_{T_2}^D[f^2](z) P_{T_2}^D[\mathbb{1}_D](z) \\ &\leq P_{T_2}^D[f^2](z) \sup_{z' \in D} P_{T_2}^D[\mathbb{1}_D](z') \end{aligned}$$

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it follows that

$$\mu^D[(P^D_{T_2}[f])^2] \leq \sup_{z \in D} P^D_{T_2}[\mathbb{1}_D](z)\mu^D[P^D_{T_2}[f^2]]$$
$$\leq \sup_{z \in D} P^D_{T_2}[\mathbb{1}_D](z)\mu^D[f^2]$$

So the norm operator of $P^D_{T_2}$ satisfies

$$\theta_{T_2} = \|P^D_{T_2}\|_{\mathbb{L}^2(\mu^D) \to \mathbb{L}^2(\mu^D)} \leq \sup_{z \in D} P^D_{T_2}[\mathbb{1}_D](z) = \sup_{z \in D} \mathbb{P}_z[\tau > T_2] \tag{61}$$

which itself is strictly less than 1 for T_2 large enough. Up to the choice of such a T_2 in the above arguments, we conclude that $\lambda_0(D) > 0$.

Let us now check that $\varphi \in \bigcap_{r \geq 1} \mathbb{L}^r(\mu^D)$, since a priori we only know that $\varphi \in \mathbb{L}^2(\mu^D) = \bigcap_{r \in [1,2]} \mathbb{L}^r(\mu^D)$. This is due to the hyperboundedness of $(P^D_t)_{t \geq 0}$. Let $r > 2$ be given and a corresponding $T_r > 0$ such that the conclusion of Lemma 38 is satisfied. Let $f \in \mathbb{L}^2(\mu^D)$ be given. Cauchy-Schwarz and Hölder inequalities imply that for all $z \in D$ and all $t \geq T_r$,

$$(P^D_t[f](z))^r = \left(\int f(z')p^D_t(z,z')\,\mu^D(dz')\right)^r$$
$$\leq \left(\int f^2(z')\,\mu^D(dz')\right)^{\frac{r}{2}}\left(\int (p^D_t(z,z'))^2\,\mu^D(dz')\right)^{\frac{r}{2}}$$
$$\leq \left(\int f^2(z')\,\mu^D(dz')\right)^{\frac{r}{2}}\left(\int (p^D_t(z,z'))^r\,\mu^D(dz')\right)$$

Integrating this bound with respect to $\mu^D(dz)$, it follows that

$$\left(\int (P^D_t[f])^r\,d\mu^D\right)^{\frac{1}{r}} \leq \left(\int (p^D_t(z,z'))^r\,\mu^D(dz)\mu^D(dz')\right)^{\frac{1}{r}}\left(\int f^2(z')\,\mu^D dz'\right)^{\frac{1}{2}}$$

namely P^D_t send continuously $\mathbb{L}^2(\mu^D)$ into $\mathbb{L}^r(\mu^D)$. If furthermore t is of the form T_2q with $q \in \mathbb{Q} \cap [1,+\infty)$, we get from $\varphi_{T_2q} = P^D_{T_2q}[\varphi_{T_2q}]/\theta_{T_2q}$ that $\varphi = \varphi_{T_2q}$ belongs to $\mathbb{L}^r(\mu^D)$.

The same arguments are also valid for the adjoint semigroup $(P^{D*}_t)_{t \geq 0}$. Its elements for $t > 0$ admit the kernels p^{D*}_t where

$$\forall\, t > 0,\, \forall\, z, z' \in D, \qquad p^{D*}_t(z,z') := \frac{\mu^D(z')p^D_t(z',z)}{\mu^D(z)} = \frac{\mu(z')p^D_t(z',z)}{\mu(z)}$$

We end up with the same quantity $\lambda_0(D)$, since for any $t > 0$ the operators P^D_t and P^{D*}_t have the same spectral radius.

∎

Let ν^D be the probability measure on D which admits $\varphi^*/\mu^D[\varphi^*]$ as density with respect to μ^D. Next result shows the validity of (14):

Proposition 39 *The probability measure ν^D is a quasi-stationary distribution for L^D and under $\mathbb{P}_{\nu^D}$, τ is distributed as an exponential law of parameter $\lambda_0(D)$.*

Proof
Let a test function $f \in \mathcal{D}(L^D)$ be given. We compute that for all $t \geq 0$,

$$\partial_t \nu^D[P^D_t[f]] = \nu^D[L^D P^D_t[f]]$$
$$= \mu^D[\varphi^* L^D P^D_t[f]]/\mu^D[\varphi^*]$$
$$= \mu^D[L^{D*}[\varphi^*]P^D_t[f]]/\mu^D[\varphi^*]$$
$$= -\lambda_0(D)\mu^D[[\varphi^*]P^D_t[f]]/\mu^D[\varphi^*]$$
$$= -\lambda_0(D)\nu^D[P^D_t[f]]$$

By integration it follows that

$$\nu^D[P_t^D[f]] \;=\; \exp(-\lambda_0(D)t)\nu^D[f]$$

at least for $f \in \mathcal{D}(L^D)$, but by usual approximation procedures, this can be extended to any f which is measurable and bounded (or nonnegative). This means that ν^D is a quasi-stationary distribution for L^D with rate $\lambda_0(D)$. In particular with $f = \mathbb{1}_D$, we get

$$
\begin{aligned}
\mathbb{P}_{\nu^D}[\tau > t] &= \nu^D[P_t^D[\mathbb{1}_D]] \\
&= \exp(-\lambda_0(D)t)\nu^D[\mathbb{1}_D] \\
&= \exp(-\lambda_0(D)t)
\end{aligned}
$$

which amounts to τ being distributed as an exponential law of parameter $\lambda_0(D)$ under $\mathbb{P}_{\nu^D}$.
∎

The bounds (15) are now easy to deduce. Indeed recalling the definition of $\lambda_0(D)$ in terms of θ_{T_2} given in the proof of Proposition 37 (and the fact that T_2 can be chosen arbitrary large), we get from the first bound of Theorem 4 that $\lambda_0(D) \geq \ln(2)\omega/\pi$.

The second bound of Theorem 4 applied with $m_0 = \nu^D$ gives that $\lambda_0(D) \leq 4$.

Remark 40 Is ν^D the unique quasi-stationary probability measure associated to L^D? A priori one has to be careful since this is wrong for the usual one-dimensional Ornstein-Uhlenbeck process with respect to a half-line. Nevertheless we believe there is uniqueness in our situation, because it is easy for the underlying process to get out of D uniformly over the starting point (as shown by the first bound of Theorem 4) and this should be a sufficient condition (in the spirit of Section 7.7 of the book [6] of Collet, Martínez and San Martín, which unfortunately only treat the case of one-dimensional diffusions). At least from the uniqueness statement included in Krein-Rutman theorem (cf. again Theorem 1.2 of Du [8]), we deduce that ν^D is the unique quasi-stationary measure admitting a density with respect to μ^D which is in $\mathbb{L}^2(\mu^D)$. By hyperboundedness of $(P_t^D)_{t \geq 0}$, the latter condition can be relaxed by only requiring that the density belongs to $\bigcap_{p>1} \mathbb{L}^p(\mu^D)$.
□

B Computations in polar coordinates

For the sake of completeness, we give below a series of elementary but tedious computations which are omitted in Section 3. We start with the proof of (32) and (37)

Proposition 41 *In the usual polar coordinates (r, θ), the infinitesimal generator L_ρ of the elliptic diffusion whose evolution is described by (31) is given by*

$$\forall g \in \mathcal{C}^2(\mathbb{R}_+^* \times \mathbb{R}), \qquad L_\rho g(r,\theta) = -\rho r \partial_r g(r,\theta) + \partial_\theta g(r,\theta) + \partial_r^2 g(r,\theta) + \frac{1}{r}\partial_r g(r,\theta) + \frac{\partial_\theta^2}{r^2}g(r,\theta).$$

In a similar way, the action infinitesimal generator $\mathcal{L}_\rho$ of the hypo-elliptic diffusion described by (25) is given by

$$\mathcal{L}_\rho = -\rho r \partial_r + \partial_\theta + \frac{\sin^2\theta}{2}\partial_{rr}^2 - \frac{\sin\theta\cos\theta}{r^2}\partial_\theta + \frac{\sin\theta\cos\theta}{r}\partial_{r\theta}^2 + \frac{\cos^2\theta}{2r}\partial_r + \frac{\cos^2\theta}{2r^2}\partial_\theta^2.$$

Proof

First, write $g(r, \theta) = f(r \cos \theta, r \sin \theta)$. Using that

$$\partial_x f = \cos \theta \partial_r g - \frac{\sin \theta}{r} \partial_\theta g, \quad \partial_y f = \sin \theta \partial_r g + \frac{\cos \theta}{r} \partial_\theta g,$$

one checks that,

$$-(\rho x + y)\partial_x f + (x - \rho y)\partial_y f = -\rho r \partial_r g + \partial_\theta g.$$

Second,

$$\partial_x^2 f = \cos^2 \theta \partial_r^2 g + 2 \frac{\sin \theta \cos \theta}{r^2} \partial_\theta g - 2 \frac{\sin \theta \cos \theta}{r} \partial_{r\theta}^2 g + \frac{\sin^2 \theta}{r} \partial_r g + \frac{\sin^2 \theta}{r^2} \partial_\theta^2 g,$$

$$\partial_y^2 f = \sin^2 \theta \partial_r^2 g - 2 \frac{\sin \theta \cos \theta}{r^2} \partial_\theta g + 2 \frac{\sin \theta \cos \theta}{r} \partial_{r\theta}^2 g + \frac{\cos^2 \theta}{r} \partial_r g + \frac{\cos^2 \theta}{r^2} \partial_\theta^2 g.$$

The expressions (32) and (37) follow.

∎

Proofs of Proposition 20 and Proposition 24 In Subsections 3.2.2 and 3.2.3, we need to compute $\frac{L_\rho g}{g}$ and $\frac{\mathcal{L}_\rho g}{g}$ where g has the following form

$$g(r, \theta) = r^n \gamma(\theta) e^{\beta(\theta) r^2}.$$

Note that in Subsection 3.2.2, $n = 1$ and $\gamma(\theta) = \cos \theta$. Then, $\frac{L_\rho g}{g}$ and $\frac{\mathcal{L}_\rho g}{g}$ are expressed in terms of some functions denoted by ψ_1, ψ_2, φ_i, $i = 1, 2, 3$. The computation of these functions follows from those of the derivatives of g given below:

$$\frac{\partial_r g}{g}(r, \theta) = \left(\frac{n}{r} + 2\beta(\theta) r \right),$$

$$\frac{\partial_r^2 g}{g}(r, \theta) = \frac{n^2 - n}{r^2} + 4r^2 \beta^2(\theta) + (4n + 2)\beta(\theta),$$

$$\frac{\partial_\theta g}{g}(r, \theta) = \beta'(\theta) r^2 + \frac{\gamma'(\theta)}{\gamma(\theta)},$$

$$\frac{\partial_{r\theta}^2 g}{rg}(r, \theta) = 2\beta(\theta)\beta'(\theta) r^2 + \left((2 + n)\beta'(\theta) + 2\frac{\gamma'(\theta)}{\gamma(\theta)}\beta(\theta) \right) + n\frac{\gamma'(\theta)}{r^2 \gamma(\theta)}$$

$$\frac{1}{r^2}\frac{\partial_\theta^2 g}{g}(r, \theta) = \beta'(\theta)^2 r^2 + \left(\beta''(\theta) + 2\beta'(\theta)\frac{\gamma'(\theta)}{\gamma(\theta)} \right) + \frac{1}{r^2}\frac{\gamma''(\theta)}{\gamma(\theta)}.$$

We can now use carefully the expressions of the elliptic (resp. hypo-elliptic) generator L_ρ (resp. $\mathcal{L}_\rho$) given by (32) (resp. (37)).

∎

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