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Abstract

When faced with multiple inputs $X \in \mathbb{R}_+^p$ and outputs $Y \in \mathbb{R}_+^q$, traditional quantile regression of Y conditional on $X = x$ for measuring economic efficiency in the output (input) direction is thwarted by the absence of a natural ordering of Euclidean space for dimensions q (p) greater than one. Daouia and Simar (2007) used nonstandard conditional quantiles to address this problem, conditioning on $Y \geq y$ $(X \leq x)$ in the output (input) orientation, but the resulting quantiles depend on the a priori chosen direction. This paper uses a dimensionless transformation of the $(p+q)$-dimensional production process to develop an alternative formulation of distance from a realization of (X, Y) to the efficient support boundary, motivating a new, *unconditional* quantile frontier lying inside the joint support of (X, Y), but near the full, efficient frontier. The interpretation is analogous to univariate quantiles and corrects some of the disappointing properties of the conditional quantile-based approach. By contrast with the latter, our approach determines a unique partial-quantile frontier independent of the chosen orientation (input, output, hyperbolic or directional distance). We prove that both the resulting efficiency score and its estimator share desirable monotonicity properties. Simple arguments from extreme-value theory are used to derive the asymptotic distributional properties of the corresponding empirical efficiency scores (both full and partial). The usefulness of the quantile-type estimator is shown from an infinitesimal and global robustness theory viewpoints via a comparison with the previous conditional quantile-based approach. A diagnostic tool is developed to find the appropriate quantile-order; in the literature to date, this trimming order has been fixed *a priori*. The methodology is used to analyze the performance of U.S. credit unions, where outliers are likely to affect traditional approaches.

1 Introduction

In production theory and efficiency analysis, interest lies in estimating the boundary of the set of feasible combinations of inputs and outputs; with multiple inputs and multiple outputs, this is a surface in a multivariate space. Using nonparametric methods to measure firm performance has several advantages, especially when applying robust quantile regression approaches which are not overly influenced by extremes and outliers. As Hendricks and Koenker (1992, p. 58) stated, “In the econometric literature on the estimation of production technologies, there has been considerable interest in estimating so-called frontier production models that correspond closely to models for extreme quantiles of a stochastic production surface.” Landajo et al. (2008) and Daouia et al. (2013) review the basic features of quantile modeling for estimation of firms’ performance and provide some arguments for the usefulness of quantile regression for such purposes.

Unfortunately, generalization of traditional quantile regression methods to the full multivariate framework, where firms transform a vector of input quantities $X \in \mathbb{R}^p_+$ into a vector of output quantities $Y \in \mathbb{R}^q_+$, is thwarted by the absence of a natural ordering of Euclidean space for dimensions p, q greater than one. In applications where $p > 1$ and $q > 1$, if the production of a firm is y and its input usage is x, then its relative economic efficiency can be measured via a distance from the point (x, y) to the efficient frontier of the production set, i.e., the upper support boundary of (X, Y). While output- or input-oriented nonparametric methods based on ideas of Debreu (1951), Farrell (1957), and Shephard (1970) consider maximization of production along radial paths while holding inputs fixed, or minimization of inputs along radial paths while holding outputs fixed, Färe et al. (1985) suggest an hyperbolic distance function that measures the maximum feasible reduction in input quantities and simultaneous feasible expansion of output quantities along a hyperbolic path to the efficient frontier. Both radial and hyperbolic efficiency measures are multiplicative, and hence require that input and output quantities be nonnegative. Chambers et al. (1996) introduce a directional distance function that measures distance in an arbitrary, linear direction toward the frontier. The directional distance function can be viewed as an additive measure of efficiency, and thus is able to accommodate negative input or output quantities. All these cases are covered by our approach.

Most of the nonparametric approaches are based on envelopment estimators that are very sensitive to extreme data points and outliers since they envelop the cloud of sample observations. Quantile regression offers an attractive tool to build frontier estimates that are robust to these extreme data points, but, as pointed above, traditional quantile regression cannot be used due to the absence of a natural ordering of Euclidean space for dimensions p, q greater than one. Daouia and Simar (2007) implemented the idea in the output orientation by using quantiles of the conditional distribution of Y given $X \leq x$. A similar idea can be adapted to the input orientation, using the conditional distribution of X given $Y \geq y$. The use of such a non-standard conditional distribution is motivated by econometric considerations of tail monotonicity but the resulting estimators may exhibit disappointing behavior (see below).

In this paper, we provide a simple and promising procedure for measuring efficiency in the full multivariate case by exploiting *unconditional* quantiles and their attractive statistical and computational properties, without recourse to regression or dimension-reduction techniques, while overcoming some limitations of the Daouia and Simar (2007) approach based on *conditional* quantiles. We propose an alternative formulation of the distance from a point (x, y) to the optimal production surface by considering a dimensionless transformation of the $(p+q)$-dimensional production process. This motivates a new concept of partial frontiers inside the joint support of (X, Y) but lying close to its efficient full frontier, by using large unconditional quantiles of the transformed variable. By doing so, we recover also the concepts of quantile frontiers obtained by Wheelock and Wilson (2008) for the hyperbolic orientation and by Simar and Vanhems (2012) for the directional distance case. We show that, contrary to the conditional approach, the resulting α-th quantile frontier is uniquely determined regardless the chosen orientation. We also derive desirable monotonicity properties of the resulting efficiency scores and their nonparametric estimators. We provide the asymptotic distributional behavior of the resulting empirical efficiency scores (both full and partial) by using simple arguments from extreme-value theory. The usefulness of the quantile estimator is also established from infinitesimal and a global robustness theory points of view via a comparison with the properties of the conditional quantile-based approach of Daouia and Simar (2007). In addition, diagnostic tool suggested by robustness theory is presented to find the adequate quantile order α. Finally, we illustrate the approach with an application in the US Bank Industry.

The next section introduces the basic notation and summarizes the previous approaches in this field. Section 3 describes the new formulation of the model and the dimensionless transformation of (X, Y) that permits generalization of the asymptotic properties of the free-disposal hull (FDH) estimator of Daouia et al. (2010) to the full multivariate setup and extension these properties to hyperbolic and directional distances cases. Using this transformation, Section 3.2 introduces the new quantile-frontier concept, its nonparametric estimator, and provides its asymptotic properties. The links with the (conditional) order-α quantile frontier introduced by Daouia and Simar (2007) are discussed. Section 3.5 analyzes the properties of the new estimator from a robustness point of view by deriving its gross-error sensitivity and its finite sample breakdown point, and compares these with properties of the Daouia and Simar order-α conditional quantile frontier. The theoretical robustness properties suggest a diagnostic tool for choosing the quantile order α in applications. As demonstrated in Section 4 this methodology is particularly useful for measuring the performance of U.S. credit unions, where outliers outliers distort efficiency estimates obtained with more traditional methods.

2 Basic Notations and Usual Approaches

2.1 Previous work on efficiency analysis

Formally, let $x \in \mathbb{R}_+^p$ denote a vector of input quantities and let $y \in \mathbb{R}_+^q$ denote a vector of output quantities. The attainable set, i.e., the set of feasible combinations of inputs and outputs is

$$\Psi = \{(x, y) \in \mathbb{R}_+^p \times \mathbb{R}_+^q \mid y \text{ can be produced by } x\}. \tag{2.1}$$

The efficient frontier is defined by

$$\Psi^\partial = \left\{(x, y) \in \Psi \mid \left(\gamma\, x, \gamma^{-1}\, y\right) \notin \Psi \text{ for any } \gamma < 1\right\}. \tag{2.2}$$

A typical, minimal assumption on Ψ is free disposability of both inputs and outputs; i.e., if $(x, y) \in \Psi$, then $(x', y') \in \Psi$ for any (x', y') such that $x' \geq x$ and $y' \leq y$. This implies a monotonicity property of the frontier Ψ^∂. Sometimes convexity of Ψ is also assumed, but we will not use this assumption in our presentation (see, for example, Shephard, 1970 for a comprehensive presentation of production theory).

The efficiency of a firm operating at level $(x, y) \in \Psi$ is characterized by the distance to its projection on the efficient frontier. As noted in Section 1, there are several possible directions in which (x, y) might be projected onto Ψ^{∂} or in which efficiency might be measured. In the input direction, efficiency is measured by

$$\theta(x, y) = \inf\{\theta > 0 \mid (\theta x, y) \in \Psi\}, \tag{2.3}$$

while in the output direction efficiency is measured by

$$\lambda(x, y) = \sup\{\lambda > 0 \mid (x, \lambda y) \in \Psi\}. \tag{2.4}$$

The hyperbolic measure of efficiency is given by

$$\gamma(x, y) = \sup\{\gamma > 0 \mid (\gamma^{-1} x, \gamma y) \in \Psi\} \tag{2.5}$$

and the directional measure is given by

$$\delta(x, y \mid g_x, g_y) = \sup\{\delta \geq 0 \mid (x - \delta g_x,\ y + \delta g_y) \in \Psi\}, \tag{2.6}$$

where $g_x \in \mathbb{R}_+^p$ and $g_y \in \mathbb{R}_+^q$ give the direction in which (x, y) is projected onto Ψ^{∂}.

Recent work by Daraio and Simar (2005, 2007), Wheelock and Wilson (2008), and Simar and Vanhems (2012) has extended the probabilistic interpretation of these measures by Cazals et al. (2002). Assuming that the random pair (X, Y) is drawn from a density $f(x, y)$ with support over Ψ, the joint distribution of (X, Y) can be described by

$$H_{XY}(x, y) = \Pr(X \leq x,\ Y \geq y), \tag{2.7}$$

which gives the probability of the firm at (x, y) being dominated by another firm producing at least as much output as y but using no more input than x. Under the assumption of free disposability of inputs and outputs, the efficiency scores defined in (2.3)–(2.6) can be defined equivalently as

$$\theta(x, y) = \inf\{\theta > 0 \mid H_{XY}(\theta x, y) > 0\}, \tag{2.8}$$

$$\lambda(x, y) = \sup\{\lambda > 0 \mid H_{XY}(x, \lambda y) > 0\}, \tag{2.9}$$

$$\gamma(x, y) = \sup\{\gamma > 0 \mid H_{XY}(\gamma^{-1}x, \gamma y) > 0\}, \tag{2.10}$$

and $\delta(x, y) = \log(\delta^*)$, where

$$\delta^* = \sup\{\gamma > 0 \mid H_{X^*Y^*}(\gamma^{-1}x^*, \gamma y^*) > 0\}, \tag{2.11}$$

and (x^*, y^*) is a simple monotonic transformation of (x, y), (X^*, Y^*) denotes the corresponding transformed random vector, and $H_{X^*Y^*}(x, y)$ denotes the corresponding distribution function, analogous to (2.7). If all elements of g_x and g_y are strictly greater than zero, then $x^* = \exp(x./g_x)$ and $y^* = \exp(y./g_y)$, where ./ denotes the element-wise division of two vectors having the same dimension.[1]

In practice, Ψ is unknown and must be estimated from a sample of iid observations $\mathcal{X}_n = \{(X_i, Y_i)\}_{i=1}^n$. The familiar FDH estimator proposed by Deprins et al. (1984) estimates Ψ by the smallest monotone set enveloping the data in $\mathcal{X}_n$, and hence relies only on the free disposability assumption. The resulting estimators of the efficiency scores defined in (2.3)–(2.6) are obtained by replacing Ψ with the FDH estimator of Ψ, or equivalently, by replacing $H_{XY}(x, y)$ in (2.8)–(2.10) with its empirical analog,

$$\widehat{H}_{n,XY}(x, y) = n^{-1} \sum_{i=1}^{n} \mathbb{1}(X_i \leq x,\ Y_i \geq y), \tag{2.12}$$

where $\mathbb{1}(\cdot)$ is the indicator function. For directional distances, the similarly-defined empirical analog of $H_{X^*Y^*}(x^*, y^*)$ can be used to replace $H_{X^*Y^*}(x^*, y^*)$ in (2.11).

In the input and output oriented cases, asymptotic properties of the FDH estimators have been established by Park et al. (2000) for the case where the joint density of (X, Y) is strictly positive and finite on the frontier Ψ^∂. These properties have been extended, under similar conditions, to hyperbolic measures by Wheelock and Wilson (2008) and to directional distances by Simar and Vanhems (2012). Recently Daouia et al. (2010) extended the result for the input and the output oriented measures to more general settings (e.g., where the density $f(x, y)$ either tends smoothly to zero or explodes to infinity when approaching the frontier) using results from extreme value theory. To date, however, these extensions are limited to univariate output (in the output direction) or univariate input (in the input direction), with no such results available for the hyperbolic or directional cases.

[1] Note that some elements of (g_x, g_y) could be defined as zero for non-discretionary inputs or outputs (see Simar and Vanhems, 2012 for details on handling such situations).

2.2 A Quantile-based benchmark

In many empirical applications, it would be dubious to assume all observations are measured accurately, raising doubts about the meaningfulness of envelopment estimators such as FDH or data envelopment analysis (DEA).[2] In many applications, some observations may appear so isolated that they hardly seem related to the sample. Indeed, they may not be related when outliers result from data corruption due to by reporting, transcription, or other errors. Rather than estimating the full frontier Ψ^∂ or distance from a point (x, y) to Ψ^∂, it may be more sensible to use instead as a benchmark for gauging efficiency a partial frontier lying inside the FDH of the sample. This is the idea of benchmarking relative to quantile frontiers for large values of the quantile order. In the following presentation, we summarize the basic idea for the output orientation; extension to the input, hyperbolic, and directional orientations is trivial.

Instead of estimating the full efficiency measure $\lambda(x, y)$ defined in (2.9), Daouia and Simar (2007) extend the "univariate" ideas of Aragon et al. (2005) to estimate the order-α partial efficiency score

$$\lambda_\alpha(x, y) = \sup\left\{\lambda > 0 \mid \frac{H_{XY}(x, \lambda y)}{H_{XY}(x, 0)} > 1 - \alpha\right\}, \tag{2.13}$$

where $\alpha \in (0, 1]$ and $H_{XY}(x, 0) = F_X(x)$ is the marginal distribution function of X. Note that this is related to a quantile of the non-standard conditional survival function $S_{Y|X}(y \mid x) = \Pr(Y \geq y \mid X \leq x)$ of Y given $X \leq x$ since we can write equivalently

$$\lambda_\alpha(x, y) = \sup\left\{\lambda > 0 \mid S_{Y|X}(\lambda y \mid x) > 1 - \alpha\right\}. \tag{2.14}$$

Here, for any level of input x such that $F_X(x) > 0$, the order-α output oriented frontier could be described by the surface $y^\partial_\alpha(x) = y\, \lambda_\alpha(x, y)$.

Note that for fixed sample size n, $\lambda_\alpha(x, y) \to \lambda(x, y)$ as $\alpha \to 1$; i..e, when α is close to 1, $\lambda_\alpha(x, y)$ is close to the full measure $\lambda(x, y)$. The empirical analog of (2.13) is

$$\widehat{\lambda}_\alpha(x, y) = \sup\left\{\lambda > 0 \mid \frac{\widehat{H}_{n,XY}(x, \lambda y)}{\widehat{H}_{n,XY}(x, 0)} > 1 - \alpha\right\}, \tag{2.15}$$

[2] DEA estimators are based on using either the conical or convex hulls of the FDH of sample observations; see Simar and Wilson (2013) for a recent survey and discussion.

and it is easy to see that for fixed n, $\widehat{\lambda}_\alpha(x, y) \to \widehat{\lambda}_\alpha(x, y)$ as $\alpha \to 1$, where

$$\widehat{\lambda}(x, y) = \sup\left\{\lambda > 0 \mid \widehat{H}_{n,XY}(x, \lambda y) > 0\right\} \tag{2.16}$$

is the FDH estimator of $\lambda(x, y)$. Daouia and Simar (2007) establish that if the order $\alpha = \alpha(n) > 0$ approaches 1 at the rate $n^{(p+q+1)/(p+q)}(1 - \alpha(n)) \to 0$ as $n \to \infty$, then $\widehat{\lambda}_\alpha(x, y)$ provides alternative estimator of $\lambda(x, y)$ with asymptotic properties similar to those of the FDH estimator (i.e., with Weibull limiting distribution and convergence rate $n^{1/(p+q)}$). For finite n, $\alpha(n) < 1$ and so the corresponding order-α frontier surface (which converges to the full frontier surface Ψ^∂) will not envelop all the data points, and so will be more robust to extreme points and outliers than the FDH estimators. The properties of $\widehat{\lambda}_\alpha(x, y)$ from the viewpoint of robustness theory have been investigated by Daouia and Ruiz-Gazen (2006) and Daouia and Gijbels (2011).

Daouia and Simar (2007, Proposition 2.5) establish that $\lambda_\alpha(x, y)$ is monotone nondecreasing with x for all α if and only if tail monotonicity of the conditional distribution $F_{Y|X}(y \mid x) = \Pr(Y \leq y \mid X \leq x)$ holds. The latter is formalized as follows. Let $\mathcal{S}_X \subset \mathbb{R}^p_+$ and $\mathcal{S}_Y \subset \mathbb{R}^q_+$ denote the supports of X and Y, respectively. Then tail monotonicity of $F_{Y|X}(y \mid x)$ holds if

$$F_{Y|X}(y \mid x) \leq F_{Y|X}(y|x') \;\forall\, y \in \mathcal{S}_Y,\; x \geq x',\; x, x' \in \mathcal{S}_X. \tag{2.17}$$

The hypothesis (2.17) is natural in production theory; it implies that the chance of producing less than some value y decreases as firms use more inputs (see, e.g., Cazals et al., 2002). However, in finite samples, and as illustrated in a simple example below, the estimator $\widehat{\lambda}_\alpha(x, y)$ does not share this property.[3] It will be seen later that both the unconditional quantile-based benchmark introduced below in Section 3.2 and its estimator share this desirable monotonicity property, even without the assumption (2.17).

The adaptation of the conditional quantile approach for other orientations (input, hyperbolic, directional distance) is straightforward (see the references given above, or the survey provided by Simar and Wilson, 2013). It is important to remember, however, that the resulting order-α frontiers are different depending on the chosen orientation, except in the trivial case where $\alpha = 1$ (see, e.g., Figure 1 in Wheelock and Wilson, 2008). We will see in Section

[3] This drawback has been addressed by Daouia and Simar (2005) for the case $q = 1$ by isotonizing the resulting estimate of the production function.

3 that the unconditional order-α quantile frontier defined therein is uniquely determined regardless the chosen orientation.[4]

3 New Model Formulation and Results

3.1 Traditional full efficiency measures

Let $(X,Y),\ (X_1,Y_1),\ (X_2,Y_2),\ \ldots$ be a sequence of independent observations in $\mathbb{R}_+^p \times \mathbb{R}_+^q$ with a continuous, common distribution, and let (x,y) to be input-output pair of the production unit of interest. The following discussion establishes an important connection between traditional efficiency measurement, its empirical estimation, and extreme-value theory. We focus on the output-orientation, but the results extend trivially to other directions. We begin by considering a new formulation of $\lambda(x,y)$ and its FDH estimator $\widehat{\lambda}(x,y)$. First, define the random variable $Z^{xy}(X,Y)$ by writing

$$Z^{xy}(X,Y) = \min_{1\le j\le q} \frac{Y^{[j]}}{y^{[j]}}\, \mathbb{1}(X \le x), \tag{3.1}$$

where superscripts j denote the jth elements of the vectors Y and y. It is then easily seen that the survival function for the transformed random variable $Z^{xy}(X,Y)$ is given by

$$S_{xy}(z) = 1 - F_{xy}(z) = \begin{cases} H_{XY}(x, zy) & \text{if } z \ge 0; \\ 1 & \text{if } z < 0, \end{cases} \tag{3.2}$$

and that its right endpoint $F_{xy}^{\leftarrow}(1)$ coincides with the technical efficiency of interest $\lambda(x,y)$. Here, $F_{xy}^{\leftarrow}(1)$ is the smallest value of z such that $F_{xy}(z) = 1$, where $F_{xy}(z)$ is defined implicitly in (3.2). More generally, let

$$F_W^{\leftarrow}(\alpha) = \inf\{w \mid F_W(w) \ge \alpha\} \tag{3.3}$$

denote the quantile of order $\alpha \in (0,1]$ of a random variable W with distribution function F_W.

Now let $Z^{xy}(X_i,Y_i) = \min_{1\le j\le q} \frac{Y_i^{[j]}}{y^{[j]}} \mathbb{1}(X_i \le x)$ for $i = 1,\ \ldots,\ n$. This leads to a sample of n independent draws of $Z^{xy}(X,Y)$. By denoting the order statistics of the transformed sample $\{Z^{xy}(X_1,Y_1),\ \ldots,\ Z^{xy}(X_n,Y_n)\}$, by $Z_{(1)}^{xy} \le \ldots \le Z_{(n)}^{xy}$, it is clear that the maximum

[4] Of course, the distance to the unconditional order-α frontier will depend on the chosen orientation.

value $Z^{xy}_{(n)}$ coincides with the FDH estimator $\widehat{\lambda}(x, y)$. As shown below, the general asymptotic distributional behavior of the latter follows immediately from classical extreme-value theory.

Before proceeding, consider briefly the input, hyperbolic, and directional orientations. In the input-orientation, redefine $Z^{xy}(X, Y)$ as

$$Z^{xy}(X, Y) = -\max_{1\le j\le p} \frac{X^{[j]}}{x^{[j]}} / \mathbb{1}(Y \ge y), \tag{3.4}$$

where division of a nonnegative real number by 0 is defined to yield infinity. Then $Z^{xy}(X, Y)$ has support $[-\infty, 0]$ and survival function

$$1 - F_{xy}(z) = \begin{cases} H_{XY}(-zx, y) & \text{if } z > -\infty; \\ 1 & \text{if } z = -\infty \end{cases} \tag{3.5}$$

(note that here, the support of $Z^{xy}(X, Y)$ is the affinely extended set of nonpositive real numbers $\mathbb{R}_- \cup \{-\infty\}$, indulging some abuse due to the possibility that an element $y^{[j]}$ might equal zero). It is easy to see that $\widehat{\theta}(x, y) = -\max_{1\le i\le n} Z^{xy}(X_i, Y_i)$ and $\theta(x, y) = -F^{\leftarrow}_{xy}(1)$.

Likewise, for hyperbolic paths, it is not hard to verify that $\widehat{\gamma}(x, y) = \max_{1\le i\le n} Z^{xy}(X_i, Y_i)$ and $\gamma(x, y) = F^{\leftarrow}_{xy}(1)$, where the transformed random variable is

$$Z^{xy}(X_i, Y_i) = \min\left\{ \min_{1\le j\le p} \frac{x^{[j]}}{X_i^{[j]}}, \min_{1\le j\le q} \frac{Y_i^{[j]}}{y^{[j]}} \right\}, \tag{3.6}$$

with survival function given by

$$S_{xy}(z) = 1 - F_{xy}(z) = H_{XY}(z^{-1}x, zy) \tag{3.7}$$

for $z \ge 0$.

By using the monotonic transformation described in Simar and Vanhems (2012), the directional distance case, for any direction $(g_x, g_y) > 0$, is covered by the hyperbolic case, with the modifications given in Simar and Vanhems if some of the components of (g_x, g_y) are equal to zero. Note that efficiency measurement is not a symmetric concept, and thus the three directions related to the distributions $H_{XY}(x, zy)$, $H_{XY}(-zx, y)$ and $H_{XY}(z^{-1}x, zy)$ have to be treated separately. For the sake of conciseness, the presentation below is only in terms of the output-orientation; similar considerations apply for the other directions.

The following proposition provides a necessary and sufficient condition under which the FDH estimator converges in distribution and characterizes the limit distribution with the convergence rate.

Proposition 3.1. *There exist constants $b_n(x,y)$ and a non-degenerate distribution G_{xy} such that as $n \to \infty$,*

$$b_n^{-1}(x,y)[\widehat{\lambda}(x,y) - \lambda(x,y)] \xrightarrow{\mathcal{L}} G_{xy} \tag{3.8}$$

if and only if

$$H_{X,Y}(x,zy) = [\lambda(x,y) - z]^{\rho_{xy}} \ L_{xy}\left([\lambda(x,y) - z]^{-1}\right) \tag{3.9}$$

for some $\rho_{xy} > 0$, where L_{xy} is a slowly varying function ($L_{xy} \in RV_0$); i.e.,

$$\lim_{t \uparrow \infty} \frac{L_{xy}(tw)}{L_{xy}(t)} = 1 \ \forall \ w > 0. \tag{3.10}$$

In addition, the only possible limit distribution is the Weibull with parameter ρ_{xy}, i.e.,

$$G_{xy}(w) = \begin{cases} \exp\left(-(-w)^{\rho_{xy}}\right), & w < 0 \\ 1, & w \geq 0. \end{cases} \tag{3.11}$$

The normalizing constants may be chosen as $b_n(x,y) = \lambda(x,y) - F_{xy}^{\leftarrow}(1 - 1/n)$.

Proof: Following, e.g., Resnick (1987, Proposition 0.3, p.9), if there exist $b_n > 0$ and a non-degenerate distribution G such that $b_n^{-1}\left(Z_{(n)}^{xy} - F_{xy}^{\leftarrow}(1)\right) \xrightarrow{\mathcal{L}} G$ as $n \to \infty$, then G is of the type of extreme-value distribution described in the proposition. By Proposition 1.13 in Resnick (1987, p.59), there exists $b_n > 0$ such that $b_n^{-1}\left(Z_{(n)}^{xy} - F_{xy}^{\leftarrow}(1)\right) \xrightarrow{\mathcal{L}} G_{xy}$ if and only if the function $U(t) = 1 - F_{xy}\left(F_{xy}^{\leftarrow}(1) - 1/t\right)$ is regularly varying at ∞ with index ρ_{xy}, that is

$$\lim_{t \uparrow \infty} \frac{U(tw)}{U(t)} = w^{\rho_{xy}} \ \forall \ w > 0. \tag{3.12}$$

In this case, we may set $b_n = F_{xy}^{\leftarrow}(1) - F_{xy}^{\leftarrow}(1 - 1/n)$. It is easily seen that the necessary and sufficient condition is equivalent to (3.9), which completes the proof given that $\lambda(x,y) = F_{xy}^{\leftarrow}(1)$ and $\widehat{\lambda}(x,y) = Z_{(n)}^{xy}$. ∎

In the particular class of slowly varying functions L_{xy} such that $L_{xy}(t) = \ell_{xy} > 0$ as $t \to \infty$, or equivalently, $L_{xy}\left([\lambda(x,y) - z]^{-1}\right) = \ell_{xy}$ when $z \uparrow \lambda(x,y)$, we recover the standard assumption in the statistical literature on frontier modeling that the joint distribution of (X,Y) is an algebraic function of the distance from its support boundary ; see, *e.g.*, Härdle et al. (1995), Hall et al. (1997), Hall et al. (1998), Gijbels and Peng (2000), Hwang et al. (2002), and Daouia et al. (2010). This translates in our context into the property

$$H_{XY}(x,zy) = \ell_{xy}[\lambda(x,y) - z]^{\rho_{xy}} \quad \text{as} \quad z \uparrow \lambda(x,y). \tag{3.13}$$

The condition (3.13) turns out to have an intuitive interpretation in terms of the data dimension $(p+q)$ and of the shape of joint density

$$f(x,y) = (-1)^q \frac{\partial^{p+q}}{\partial_{x_1}\dots\partial_{x_p}\partial_{y_1}\dots\partial_{y_q}} H_{XY}(x,y)$$

near the upper support boundary of (X,Y). Indeed, assuming that $\rho_{xy} > p+q-1$ and that ρ_{xy}, ℓ_{xy} and $\lambda(x,y)$ are differentiable with positive first partial derivatives of $\lambda(x,y)$ with respect to x, and negative first partial derivatives with respect to y, it is not hard to verify that

$$f(x,zy) = c_{xyz}\, [\lambda(x,y)-z]^{\beta_{xy}} + o\left([\lambda(x,y)-z]^{\beta_{xy}}\right), \text{ as } z \uparrow \lambda(x,y), \tag{3.14}$$

where $\beta_{xy} = \rho_{xy} - (p+q) > -1$ and c_{xyz} is a positive constant.

Thus the regular variation exponent ρ_{xy} turns into a parameter with an intuitive interpretation. We see that the case $\rho_{xy} = p+q$ corresponds to a joint density having a jump at the frontier (i.e. $\beta_{xy} = 0$).[5] In this case we easily recover the standard rate of convergence $n^{1/(p+q)}$ for the FDH estimator $\widehat{\lambda}(x,y)$ as established by Park et al. (2000), where the rate was obtained after a rather complicated proof under some restrictive conditions. The case $\rho_{xy} > p+q$ (respectively: $\rho_{xy} < p+q$) corresponds to a joint density which decays to zero smoothly (respectively: rises up to infinity) as it approaches the support boundary.

The mean-square error of $\widehat{\lambda}(x,y)$ follows from the next proposition.

Proposition 3.2. *If* $b_n^{-1}(x,y)\left(\widehat{\lambda}(x,y)-\lambda(x,y)\right) \xrightarrow{\mathcal{L}} G_{xy}$ *with* $b_n(x,y) = \lambda(x,y) - F_{xy}^{\leftarrow}(1-1/n)$*, then for any integer* $m \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}\left(b_n^{-1}(x,y)\left[\widehat{\lambda}(x,y)-\lambda(x,y)\right]\right)^m = (-1)^m \Gamma\left(1+\frac{m}{\rho_{xy}}\right), \tag{3.15}$$

where $\Gamma(\cdot)$ *denotes the gamma function.*

Proof: Since $\mathbb{E}(|Z^{xy}|^m) \leq \lambda^m(x,y) < \infty$ for any integer $m \geq 1$, we have by Proposition 2.1 in Resnick (1987, p.77) that

$$\mathbb{E}\left[\frac{\left[Z_{(n)}^{xy} - F_{xy}^{\leftarrow}(1)\right]}{\left[F_{xy}^{\leftarrow}(1) - F_{xy}^{\leftarrow}(1-1/n)\right]}\right]^m \longrightarrow (-1)^m\Gamma\left(1+\frac{m}{\rho_{xy}}\right), \tag{3.16}$$

provided that the convergence in distribution to G_{xy} holds. ■

[5] In the econometric literature on efficiency analysis it is common to assume that the density of (X,Y) has a jump at the frontier; e.g., see Park et al. (2000), Kneip et al. (1998), Kneip et al. (2008) for nonparametric models and Aigner et al. (1977), Meeusen and van den Broeck (1977), Battese and Corra (1977), and Stevenson (1980) for parametric models.

3.2 New quantile-based partial efficiency scores

Section 3.1 demonstrates the that transformation to the multivariate, random variables (X, Y) to the univariate random variable $Z^{xy}(X, Y)$ introduced in (3.1) allows derivation of asymptotic properties of the FDH efficiency estimator in a much simpler fashion than was previously known. Of course, applying the transformation in (3.1) to each of the sample observations does not get rid of any effects of outliers in the sample. However, the transformation allows us to derive *unconditional* output- and input-oriented quantile estimators that are robust with respect to outliers.

Motivated by the arguments and the transformation introduced above, we first propose the following alternative formulation of a quantile-type efficiency score and its estimator, in place of $\lambda_\alpha(x, y)$ and $\widehat{\lambda}_\alpha(x, y)$ defined in (2.13)–(2.15). For any $\alpha \in (0, 1)$, define

$$\lambda^\star_\alpha(x, y) = F^{\leftarrow}_{xy}(\alpha) \tag{3.17}$$

and

$$\widehat{\lambda}^\star_\alpha(x, y) = \widehat{F}^{\leftarrow}_{xy}(\alpha), \tag{3.18}$$

where $\widehat{F}_{xy}(z) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(Z^{xy}(X_i, Y_i) \leq z)$ and $\widehat{F}^{\leftarrow}_{xy}(\alpha)$ is the quantile of order α corresponding to the empirical distribution function $\widehat{F}_{xy}(z)$. Hence $\widehat{\lambda}^\star_\alpha(x, y) = Z^{xy}_{([\alpha n]+1)}$, where $[\alpha n]$ denotes the integer part of αn.

The new α-score $\lambda^\star_\alpha(x, y)$ is closely related to the quantile efficiency score $\lambda_\alpha(x, y)$. On one hand we have

$$\begin{aligned} \lambda^\star_\alpha(x, y) &= \inf\{z \mid F_{xy}(z) \geq \alpha\} = \sup\{z \mid F_{xy}(z) < \alpha\} \\ &= \sup\{z \mid S_{xy}(z) > 1 - \alpha\} = \sup\{z > 0 \mid H_{XY}(x, zy) > 1 - \alpha\}. \end{aligned} \tag{3.19}$$

On the other hand, the efficiency score of Daouia and Simar (2007) is

$$\lambda_\beta(x, y) = \sup\{z > 0 \mid H_{XY}(x, zy) > F_X(x)(1 - \beta)\}. \tag{3.20}$$

Clearly, the former is equal to the latter, i.e. $\lambda^\star_\alpha(x, y) = \lambda_\beta(x, y)$, if and only if $F_X(x)(1-\beta) = 1 - \alpha$, implying

$$\lambda^\star_\alpha(x, y) = \begin{cases} 0 & \text{if } \alpha \leq 1 - F_X(x) \\ \lambda_{1-\frac{1-\alpha}{F_X(x)}}(x, y) & \text{otherwise.} \end{cases} \tag{3.21}$$

This provides a natural lower bound for the choice of the order α for our new measure: for a particular firm operating at the level (x, y), we shall use the efficiency score $\lambda^\star_\alpha(x, y)$ of orders $\alpha > 1 - F_X(x)$, i.e. orders exceeding the probability of observing firms with larger inputs than the level x. We will discuss below in detail how one might choose α in practice, when robustness of the estimates is of concern.

Note also that the empirical order-α efficiency score in (2.15) has no guarantee of being monotone even if the population counterpart in (2.13) is so. However, both $\lambda^\star_\alpha(x, y)$ and its estimator $\widehat{\lambda}^\star_\alpha(x, y)$ enjoy the desirable monotonicity property established in the next result.

Proposition 3.3. *The quantile score function $x \mapsto \lambda^\star_\alpha(x, y)$ is monotone nondecreasing on the support of X, for every $y \in \mathbb{R}^q$ and $\alpha \in (0, 1]$. The same property holds for the function $x \mapsto \widehat{\lambda}^\star_\alpha(x, y)$.*

Proof: We have seen that

$$1 - F_{xy}(z) = \begin{cases} H_{XY}(x, zy) & \text{if } z \geq 0; \\ 1 & \text{if } z < 0. \end{cases} \tag{3.22}$$

From this, it is easily seen that the function $x \mapsto F_{xy}(z)$ is monotone nonincreasing with x, for every $y \in \mathbb{R}^q$ and $z \in \mathbb{R}$. Recall that $\lambda^\star_\alpha(x, y) = F^{\leftarrow}_{xy}(\alpha)$, i.e., the α-quantile of F_{xy}. Let $\alpha \in (0, 1]$ and $y \in \mathbb{R}^q$. If $x_1 \leq x_2$, we have

$$F_{x_1y}\left(F^{\leftarrow}_{x_2y}(\alpha)\right) \geq F_{x_2y}\left(F^{\leftarrow}_{x_2y}(\alpha)\right) \geq \alpha, \tag{3.23}$$

where the last inequality holds by the definition of quantiles. It follows that $\lambda^\star_\alpha(x_2, y) = F^{\leftarrow}_{x_2y}(\alpha) \in \{z \mid F_{x_1y}(z) \geq \alpha\}$. Therefore

$$\lambda^\star_\alpha(x_2, y) \geq \inf\{z \mid F_{x_1y}(z) \geq \alpha\} = \lambda^\star_\alpha(x_1, y), \tag{3.24}$$

which completes the proof. The same argument can be applied for the estimator once we realize that

$$1 - \widehat{F}_{xy}(z) = \begin{cases} \widehat{H}_{n,XY}(x, zy) & \text{if } z \geq 0; \\ 1 & \text{otherwise}, \end{cases} \tag{3.25}$$

and using the fact that $\widehat{\lambda}^\star_\alpha(x, y)$ is the α-quantile of $\widehat{F}_{xy}$. ■

By contrast, and as observed in Section 2, the usual conditional quantile score $\lambda_\alpha(x, y)$ shares this monotonicity property if and only if (2.17) holds, whereas, the nonparametric estimator $\widehat{\lambda}_\alpha(x,y)$ does not share this property in finite samples, leading often to disappointing results. It is also clear that this monotonicity property holds for any chosen orientation (input, hyperbolic or directional distance).

As discussed by Daouia and Simar (2007) and illustrated by Wheelock and Wilson (2008) and the example below in Section 3.3, the conditional approach in Daouia and Simar yields partial frontiers of the same order that are different, depending on whether an input- or output-orientation is used. As noted above, the (unconditional) quantile frontier developed here is unique for a given order α; i.e., it does not depend on the chosen direction. Consider the set

$$\Psi_\alpha^\partial = \{(x,y) \in \Psi \mid \lambda_\alpha^\star(x,y) = 1\}\,. \tag{3.26}$$

This frontier has a natural economic interpretation as the locus of production plans having probability $(1-\alpha)$ of being dominated. Clearly it is straightforward to adapt the notations for the other orientations, defining the order-α frontiers by the set of points satisfying $\theta_\alpha^\star(x,y) = 1$, $\gamma_\alpha^\star(x,y) = 1$ or $\delta_\alpha^\star(x,y) = 0$, respectively. The interesting feature of the unconditional approach is that the order-α frontier is uniquely determined, keeping the same economic interpretation regardless the chosen orientation. Of course distance to the frontier, measured by the order-α efficiency scores, will differ depending on the orientation.

3.3 A simple example

To illustrate the ideas presented so far, consider a simple example where $p = q = 1$ and Ψ is the triangle with vertices at $(0,0)$, $(1,0)$, and $(1,1)$ with the joint density of (X,Y) given by

$$f(x,y) = \begin{cases} 2 & \forall\, x \in [0,1],\ y \in [0,x], \\ 0 & \text{otherwise.} \end{cases} \tag{3.27}$$

For $\alpha \in (0,1]$ and $0 < y \le x \le 1$, it is easy to see that the input- and output-oriented conditional order-α efficiency measures suggested by Daouia and Simar (2007) are given by

$$\theta_\alpha(x,y) := \inf\left\{\theta > 0 | F_{X|Y}(\theta x|y) > 1-\alpha\right\} = [y + (1-y)\sqrt{1-\alpha}]/x \tag{3.28}$$

and

$$\lambda_\alpha(x,y) := \sup\left\{\lambda > 0 | S_{Y|X}(\lambda y|x) > 1-\alpha\right\} = x(1-\sqrt{1-\alpha})/y. \tag{3.29}$$

It is also not hard to verify that the hyperbolic order-α efficiency measure introduced by Wheelock and Wilson (2008) is given by

$$\begin{aligned}\gamma_\alpha(x,y) &:= \sup\{\gamma > 0 | H_{XY}(\gamma^{-1}x, \gamma y) > 1-\alpha\} \\ &= \begin{cases} \left(\sqrt{1-\alpha+4xy} - \sqrt{1-\alpha}\right)/2y & \text{if } (1-\alpha) \leq (1-xy)^2 \\ \left(1-\sqrt{1-\alpha}\right)/y & \text{otherwise.} \end{cases}\end{aligned} \tag{3.30}$$

We have in particular, $\lambda(x,y) = \theta^{-1}(x,y) = \gamma^2(x,y) = x/y$. The corresponding partial α-frontiers are defined by the sets $\{(\theta_\alpha(x,y)x, y) \mid (x,y) \in \Psi\}$ in the input direction, $\{(x, \lambda_\alpha(x,y)y) \mid (x,y) \in \Psi\}$ in the output direction, and $\{(\gamma_\alpha(x,y)^{-1}x, \gamma_\alpha(x,y)y) \mid (x,y) \in \Psi\}$ in the hyperbolic direction.

Figure 1 shows the full frontier Ψ^∂ corresponding to the density in (3.27) as a dashed, 45-degree line. Setting $\alpha = 0.95$ and using the equations above to plot the conditional order-α quantiles produces the two dash-dot-dash lines in Figure 1; the less-steeply sloped line corresponds to the output orientation, and the line with greater slope corresponds to the input orientation. For the same value of α, the two quantiles are different; as input level increases, the output-oriented conditional order-α quantile diverges from Ψ^∂, while the input-oriented conditional order-α quantile approaches Ψ^∂ as input level increases.

Now turn to the new measure $\lambda^\star_\alpha(x,y)$ given by (3.19). The corresponding partial frontier is the set $\{(x, \lambda^\star_\alpha(x,y)y) : (x,y) \in \Psi\}$, with the associated unconditional quantile function being $\lambda^\star_\alpha(x,y)y = \max\{0, x - \sqrt{1-\alpha}\}$. This is plotted as a solid line in Figure 1, again with $\alpha = 0.95$. It is clear from the plot as well as the previous expression that the unconditional quantile is parallel to Ψ^∂, due to the uniform density in (3.27). Turning to the input orientation, we have

$$\theta^\star_\alpha(x,y) := \inf\left\{\theta > 0 | H_{XY}(\theta x, y) > 1-\alpha\right\}, \tag{3.31}$$

with the unconditional quantile frontier function $\theta^\star_\alpha(x,y)x = \min\{y + \sqrt{1-\alpha},\ 1\}$. It is apparent that the two quantiles are the same; i.e., the unconditional order-α quantile is determined uniquely for a given value of α, regardless of the orientation. Similar results obtain for the hyperbolic and directional cases, where there is no conditioning on either x or y.

Now consider a random sample of size $n = 100$ drawn from the density in (3.27). Panels (a)–(b) in Figure 2 show such a sample. In both panels, the full frontier Ψ^∂ is depicted by a dotted line. In panel (a), the unconditional order-α quantile for $\alpha = 0.95$ is shown by a dashed line; the corresponding estimate of this quantile is shown by the solid curve, which is clearly monotonic. For comparison, the FDH estimate of Ψ^{∂} is shown by a dash-dot-dash pattern, and is also monotonic. Panel (a) shows that, for this particular draw of data, the unconditional quantile estimator tracks the true quantile rather well; by contrast, the FDH frontier estimate deviates widely from Ψ^{∂} in the neighborhood where $x \approx 0.4$, and is clearly biased. Panel (b) of Figure 2 shows, for the same data and the same value of α, the estimated output-oriented conditional order-α quantile as a solid curve. The estimate appears to be more variable in this example, and is clearly not monotonic.

Panel (a) of Figure 3 shows the same data as in Figure 2, but with two additional observations, (0.2, 0.5) and (0.4, 0.9) that are outliers. The dash-dot-dash pattern shows the FDH frontier estimate, while the solid curve shows the unconditional order-α quantile estimate for $\alpha = 0.95$ (the dashed line again depicts the true quantile for $\alpha = 0.95$). Comparing this with panel (a) in Figure 2, it is apparent that the outliers have a large effect on the FDH estimate, but almost no discernible effect on the quantile estimate, demonstrating the robustness of the quantile method.

As in panel (b) of Figure 2, panel (b) of Figure 3 shows the estimated conditional output-oriented α-quantile for $\alpha = 0.95$. The estimated quantile is less affected by the outliers than the FDH estimate in panel (a) of Figure 3, but nonetheless is still affected because the outliers lie toward the left end of the range of inputs. The lack of monotonicity for the conditional quantile estimate is even more apparent with the outliers that have been added.

We will return to this example, and to panels (c)–(d) in Figures 2–3 later, in Section 3.5 after discussing asymptotic properties of the new estimators below in Section 3.4.

3.4 Asymptotic properties of $\widehat{\lambda}^{\star}_{\alpha}(x,y)$

Continuing the focus on the output-oriented case, we first establish some basic asymptotic properties of $\widehat{\lambda}^{\star}_{\alpha}(x,y)$ for cases where α is fixed.

Proposition 3.4. *For a fixed order $\alpha \in (0,1)$, suppose that the derivatives $f_{xy} = F'_{xy}$ and f'_{xy} exist in a neighborhood of $\lambda^{\star}_{\alpha}(x,y)$ with $f_{xy}(\lambda^{\star}_{\alpha}(x,y)) > 0$. Then*

$$\widehat{\lambda}^{\star}_{\alpha}(x,y) = \lambda^{\star}_{\alpha}(x,y) + \frac{\alpha - \widehat{F}_{xy}(\lambda^{\star}_{\alpha}(x,y))}{f_{xy}(\lambda^{\star}_{\alpha}(x,y))} + R_n, \tag{3.32}$$

where $R_n = O\left(n^{-3/4}(\log n)^{3/4}\right)$ *as* $n \to \infty$ *with probability one.*

Proof: Using the fact that $\lambda^\star_\alpha(x,y)$ and $\widehat{\lambda}^\star_\alpha(x,y)$ are identical to the population and sample quantiles $F^{\leftarrow}_{xy}(\alpha)$ and $\widehat{F}^{\leftarrow}_{xy}(\alpha)$, respectively, it follows immediately that (3.32) corresponds to the well known Bahadur-Kiefer representation for central quantiles. A proof can be found, e.g. in Serfling, 1980, p.91). ■

Important limiting properties of the estimator $\widehat{\lambda}^\star_\alpha(x,y)$ can be obtained from the Bahadur-Kiefer-type representation in (3.32), such as asymptotic normality which follows immediately from the central limit theorem applied to $n^{1/2}\left[\alpha - \widehat{F}_{xy}(\lambda^\star_\alpha(x,y))\right]$. The conclusion stated in the proposition goes much farther, however, and may alternatively expressed as follows: *the difference between the random variable* $n^{1/2}\left[\widehat{\lambda}^\star_\alpha(x,y) - \lambda^\star_\alpha(x,y)\right]$ *and the random variable* $n^{1/2}\left[\alpha - \widehat{F}_{xy}(\lambda^\star_\alpha(x,y))\right]/f_{xy}(\lambda^\star_\alpha(x,y))$ *tends to zero as* $n \to \infty$ *almost surely with rate* $n^{1/4}(\log n)^{-3/4}$.

Asymptotic normality of $\widehat{\lambda}^\star_\alpha(x,y)$ is established in the next result, and does not depend on the extra requirement on f'_{xy} needed in Proposition 3.4.

Proposition 3.5. *Let* $0 < \alpha < 1$. *If* F_{xy} *is differentiable at* $\lambda^\star_\alpha(x,y)$ *with* $f_{xy}(\lambda^\star_\alpha(x,y)) > 0$, *then*

$$\frac{\sqrt{n}}{\sqrt{\alpha(1-\alpha)}} f_{xy}(\lambda^\star_\alpha(x,y))\left[\widehat{\lambda}^\star_\alpha(x,y) - \lambda^\star_\alpha(x,y)\right] \xrightarrow{\mathcal{L}} N(0,1), \ \text{as } n \to \infty. \tag{3.33}$$

Proof: By Serfling (1980, Theorem A, p.77),

$$\sqrt{n}\left[\widehat{F}^{\leftarrow}_{xy}(\alpha) - F^{\leftarrow}_{xy}(\alpha)\right] \xrightarrow{\mathcal{L}} N\left(0, \frac{\alpha(1-\alpha)}{\left[f_{xy}\left(F^{\leftarrow}_{xy}(\alpha)\right)\right]^2}\right) \tag{3.34}$$

as $n \to \infty$. Then the desired result holds automatically since $\sqrt{n}\left[\widehat{F}^{\leftarrow}_{xy}(\alpha) - F^{\leftarrow}_{xy}(\alpha)\right]$ coincides with $\sqrt{n}\left[\widehat{\lambda}^\star_\alpha(x,y) - \lambda^\star_\alpha(x,y)\right]$. ■

By the asymptotic normality of $\widehat{\lambda}^\star_\alpha(x,y)$, the interval $I_{Q_n}(z) = \left]\widehat{\lambda}^\star_\alpha(x,y) \pm \frac{z[\alpha(1-\alpha)]^{1/2}}{\sqrt{n} f_{xy}(\lambda^\star_\alpha(x,y))}\right[$ satisfies $\lim_{n\to\infty} \Pr[\lambda^\star_\alpha(x,y) \in I_{Q_n}(z)] = 2\Phi(z) - 1$ for all $z > 0$, where Φ denotes the standard normal distribution function. However, this asymptotic confidence interval depends on the density function $f_{xy}(\lambda^\star_\alpha(x,y))$, which is difficult to estimate.[6] The next result provides simple

[6] For example, one might use a kernel density estimator, but this would introduce a nonparametric rate ($n^{1/5}$) of convergence. Moreover, standard kernel density estimators, without some modification, are biased and inconsistent near support boundaries.

and asymptotically valid confidence intervals for $\lambda^\star_\alpha(x,y)$ that do not depend on $f_{xy}(\lambda^\star_\alpha(x,y))$ and that are easy to compute.

Proposition 3.6. *Assume the conditions of Proposition 3.5. Define for any $z>0$ the interval*

$$I_{S_n}(z) = \left]\widehat{\lambda}^\star_{\alpha_{n1}}(x,y),\ \widehat{\lambda}^\star_{\alpha_{n2}}(x,y)\right[, \tag{3.35}$$

where $\alpha_{n1} = \alpha - z(\alpha(1-\alpha)/n)^{1/2}$ and $\alpha_{n2} = \alpha + z(\alpha(1-\alpha)/n)^{1/2}$. Then as $n \to \infty$,

$$\Pr\left[\lambda^\star_\alpha(x,y) \in I_{S_n}(z)\right] \to 2\Phi(z) - 1 \tag{3.36}$$

and

$$\sqrt{n}\,|length\,(I_{S_n}(z)) - length\,(I_{Q_n}(z))| \xrightarrow{a.s.} 0. \tag{3.37}$$

Proof: The result follows after applying the asymptotic approach of Serfling (1980, Section 2.6.3, pp. 103–104) in conjunction with the identities $I_{Q_n}(z) = \left]\widehat{F}^{\leftarrow}_{xy}(\alpha) \pm \frac{z[\alpha(1-\alpha)]^{1/2}}{\sqrt{n} f_{xy}(F^{\leftarrow}_{xy}(\alpha))}\right[$ and $I_{S_n}(z) = \left]\widehat{F}^{\leftarrow}_{xy}(\alpha_{n1}),\ \widehat{F}^{\leftarrow}_{xy}(\alpha_{n2})\right[$. ■

Note that the interval $I_{S_n}(z)$ does not require the value of the density function f_{xy} at $\lambda^\star_\alpha(x,y)$ to be known, and is asymptotically equivalent to I_{Q_n} in the sense that their lengths coincide asymptotically at the rate $n^{-1/2}$. The value of z can be chosen by the researcher to obtain confidence intervals of some specified coverage; for example, to obtain a 95-percent confidence interval, $z = \Phi^{-1}(0.975) \approx 1.959964$.

Next, we establish an analogous asymptotic representation for $\alpha = \alpha_n \to 1$ with $n(1-\alpha_n) \to \infty$ as $n \to \infty$.

Proposition 3.7. *Suppose F_{xy} is twice differentiable in a left neighborhood of $\lambda(x,y)$ with f'_{xy} bounded, and $\lim_{z\uparrow\lambda(x,y)} f_{xy}(z)$ exists and is positive. Let $\alpha_n = 1 - k_n/n$ such that $k_n/n \to 0$ and $k_n/(\log n)^3 \to \infty$. Then (3.32) holds for $\alpha = \alpha_n$, where $R_n = O\left(n^{-1}k_n^{1/4}(\log n)^{3/4}\right)$ as $n \to \infty$ with probability one.*

Proof: Given that $\lambda^\star_\alpha(x,y) = F^{\leftarrow}_{xy}(\alpha)$ and $\widehat{\lambda}^\star_\alpha(x,y) = \widehat{F}^{\leftarrow}_{xy}(\alpha)$, the proof for the case $\alpha_n = k_n/n \to 0$ as $n \to \infty$ can be found in Watts (1980, Theorem 1). It is not hard to verify that the case $\alpha_n = 1 - k_n/n$ is similar, so the proof is omitted here. ■

The next result establishes asymptotic normality of $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ for $\alpha_n \to 1$ at a suitable rate, so that $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ also converges to the FDH estimator $\widehat{\lambda}(x,y)$. The proposition uses a sufficient condition that is standard in extreme value theory, i.e., the von Mises condition.

Proposition 3.8. *If*
$$\lim_{z\uparrow\lambda(x,y)} \frac{\lambda(x,y)-z}{1-F_{xy}(z)} f_{xy}(z) = \rho_{xy}, \tag{3.38}$$
for some $\rho_{xy} > 0$, then for $\alpha_n = 1 - k_n/n$ with $k_n \to \infty$ and $k_n/n \to 0$ as $n \to \infty$,
$$\frac{\sqrt{n}}{\sqrt{\alpha_n(1-\alpha_n)}} f_{xy}(\lambda^\star_{\alpha_n}(x,y)) \left[\widehat{\lambda}^\star_{\alpha_n}(x,y) - \lambda^\star_{\alpha_n}(x,y)\right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1). \tag{3.39}$$

Proof: This elegant result is due to Falk (1989), who proved under the von Mises condition (3.38) that
$$\frac{n}{\sqrt{k_n}} f_{xy}\big(F^{\leftarrow}_{xy}(\alpha_n)\big) \left[\widehat{F}^{\leftarrow}_{xy}(\alpha_n) - F^{\leftarrow}_{xy}(\alpha_n)\right] \xrightarrow{\mathcal{L}} \mathcal{N}(0,1) \text{ as } n \to \infty, \tag{3.40}$$
from which the desired result follows immediately. ■

Proposition 3.8 indicates that the conclusion of Proposition 3.5 remains valid for $\alpha = \alpha_n \to 1$ with $n(1-\alpha_n) \to \infty$ as $n \to \infty$.

The last result in this section establishes an important connection between $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ and the true, full efficiency measure $\lambda(x,y)$ as α_n approaches 1. When $\alpha_n = 1 - k/n$ with k **fixed**, the estimator $\widehat{\lambda}^\star_{\alpha}(x,y)$ converges to the true efficiency $\lambda(x,y)$, with the same scaling as the FDH estimator but with a different limiting distribution.

Proposition 3.9. *If $b_n^{-1}(x,y)[\widehat{\lambda}(x,y) - \lambda(x,y)] \xrightarrow{\mathcal{L}} G_{xy}$, then for $\alpha_n = 1 - k/n$ with $k \geq 0$ being any fixed integer,*
$$b_n^{-1}(x,y)[\widehat{\lambda}^\star_{\alpha_n}(x,y) - \lambda(x,y)] \xrightarrow{\mathcal{L}} \mathcal{H}_{xy}, \tag{3.41}$$
where the distribution function
$$\mathcal{H}_{xy}(z) = G_{xy}(z) \sum_{j=0}^{k} \frac{\big(-\log G_{xy}(z)\big)^j}{j!}. \tag{3.42}$$

Proof: By van der Vaart (1998, Theorem 21.18, p. 313), if $b_n^{-1}[Z^{xy}_{(n)} - F^{\leftarrow}_{xy}(1)] \xrightarrow{\mathcal{L}} G_{xy}$, then $b_n^{-1}[Z^{xy}_{(n-k)} - F^{\leftarrow}_{xy}(1)] \xrightarrow{\mathcal{L}} \mathcal{H}_{xy}$ for $k \geq 0$. ■

3.5 Robustness and Tuning Parameter Selection

Here we first demonstrate the superiority of our benchmark statistic
$$\widehat{\lambda}^\star_{\alpha_n}(x,y) = \inf\{z > 0 | \widehat{H}_{XY}(x,zy) \leq 1 - \alpha_n\} := T^{\alpha_n}_{xy}(\widehat{H}_{XY}) \tag{3.43}$$

over the conditional quantile-based version $\widehat{\lambda}_{\alpha_n}(x,y)$ from a robustness theory point of view. Based on the information provided by the influence curve and the sample breakdown point, we then introduce a diagnostic tool to facilitate choice between using as a benchmark either (i) to favor the use of either the full support frontier (*i.e.* $\alpha_n = 1$), or (ii) a partial frontier of order $\alpha_n = 1 - k_n/n < 1$ (with k_n to be determined) for measuring production performance via $\widehat{\lambda}^{\star}_{\alpha_n}$.

Given that both $\lambda^{\star}_{\alpha_n}(x,y)$ and $\widehat{\lambda}^{\star}_{\alpha_n}(x,y)$ are represented as a functional $T^{\alpha_n}_{xy}$ of the probability distributions H_{XY} and $\widehat{H}_{XY}$, respectively, the corresponding influence function $(x_0,y_0) \mapsto \mathrm{IC}((x_0,y_0); T^{\alpha_n}_{xy}, H_{XY})$ is defined as the first Gâteaux derivative of $T^{\alpha_n}_{xy}$ at H_{XY}, where the point (x_0,y_0) plays the role of the coordinate in the infinite-dimensional space of probability distributions (see Hampel *et al.*, 1986, Definition 1, p. 84). The IC describes on one hand the effect of an infinitesimal contamination at the point (x_0,y_0) on the estimate, standardized by the mass of the contamination, and allows on the other hand assessment of the relative influence of individual observations (X_i,Y_i) on the value of $\widehat{\lambda}^{\star}_{\alpha_n}(x,y)$. If unbounded, an outlier can cause trouble.

Proposition 3.10. *Suppose F_{xy} is differentiable at $\lambda^{\star}_{\alpha_n}(x,y)$ for a given α_n, with derivative $f_{xy}(\lambda^{\star}_{\alpha_n}(x,y)) > 0$. Then, for any $(x_0,y_0) \in \mathbb{R}^p_+ \times \mathbb{R}^q_+$,*

$$IC((x_0,y_0); T^{\alpha_n}_{xy}, H_{XY}) = \frac{\alpha_n - \mathbb{1}(Z^{xy}(x_0,y_0) \le \lambda^{\star}_{\alpha_n}(x,y))}{f_{xy}(\lambda^{\star}_{\alpha_n}(x,y))}. \tag{3.44}$$

Proof: By using the identities $\lambda^{\star}_{\alpha_n}(x,y) = F^{\leftarrow}_{xy}(\alpha_n) := S^{\alpha_n}(F_{xy})$ and $\widehat{\lambda}^{\star}_{\alpha_n}(x,y) = \widehat{F}^{\leftarrow}_{xy}(\alpha_n) := S^{\alpha_n}(\widehat{F}_{xy})$, which implicitly define the functional S^{α_n}, all of the quantitative robustness characteristics of univariate sample quantiles carry over automatically to the theoretical α_n-quantile $\lambda^{\star}_{\alpha_n}(x,y)$ as well as its empirical version $\widehat{\lambda}^{\star}_{\alpha_n}(x,y)$. In particular, we have $\mathrm{IC}((x_0,y_0); T^{\alpha_n}_{xy}, H_{XY}) = \mathrm{IC}(Z^{xy}(x_0,y_0); S^{\alpha_n}, F_{xy})$, which establishes the result. ■

As the trimming order α_n exceeds 1/2, the maximum absolute value

$$\mathrm{GES}\left(\widehat{\lambda}^{\star}_{\alpha_n}(x,y)\right) := \sup_{(x_0,y_0)\in\mathbb{R}^{p+q}_+} |\mathrm{IC}((x_0,y_0); T^{\alpha_n}_{xy}, H_{XY})| = \frac{\alpha_n}{f_{xy}(\lambda^{\star}_{\alpha_n}(x,y))} \tag{3.45}$$

defines the worst case scenario, termed the gross-error sensitivity. The influence of an outlier (X_i,Y_i) on the estimator $\widehat{\lambda}^{\star}_{\alpha_n}(x,y)$ cannot be unbounded if its gross-error sensitivity is finite. This important robustness requirement, which corresponds to a finite GES, is known as

B-robustness (Rousseeuw, 1981). Compared to the conditional variant $\widehat{\lambda}_{\alpha_n}(x,y)$ introduced in Daouia and Simar (2007), it is not hard to check from Daouia and Ruiz-Gazen (2006, Theorem 3.0.3) that

$$\mathrm{GES}\left(\widehat{\lambda}_{\alpha_n}(x,y)\right) = \frac{\alpha_n}{f_{xy}(\lambda_{\alpha_n}(x,y))} \tag{3.46}$$

for $\alpha_n > 1/2$. In particular, when $\alpha_n > \frac{1}{2} \vee (1 - F_X(x))$, we have $\lambda^\star_{\alpha_n}(x,y) = \lambda_{\beta_n}(x,y)$ with $\beta_n = 1 - \frac{1-\alpha_n}{F_X(x)}$, and so $\mathrm{GES}\left(\widehat{\lambda}^\star_{\alpha_n}(x,y)\right) = \frac{\alpha_n}{\beta_n}\,\mathrm{GES}\left(\widehat{\lambda}_{\beta_n}(x,y)\right)$. Thus

$$\mathrm{GES}\left(\widehat{\lambda}^\star_{\alpha_n}(x,y)\right) \sim \mathrm{GES}\left(\widehat{\lambda}_{\beta_n}(x,y)\right) \tag{3.47}$$

as $\alpha_n \to 1$. Since $\beta_n < \alpha_n$, the conditional β_n-quantile $\widehat{\lambda}_{\beta_n}(x,y)$ is more resistant than the α_n-quantile $\widehat{\lambda}_{\alpha_n}(x,y)$. Therefore $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ can be viewed as infinitesimally more robust than $\widehat{\lambda}_{\alpha_n}(x,y)$ for α_n large enough, in view of (3.47).

Note, however, that $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ can be B-robust and yet still highly sensitive to small, finite perturbations. To measure its global robustness, the richest quantitative information is provided by the finite sample breakdown point as shown by Donoho and Huber (1983). It measures the smallest fraction of contamination of an initial sample $(X,Y)^n = \{(X_i,Y_i)\}_{i=1}^n$ that can cause the estimator $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ to take values arbitrarily far from its value at the initial sample:

$$\begin{aligned}\mathrm{RB}\left(\widehat{\lambda}^\star_{\alpha_n}(x,y),(X,Y)^n\right) &= \mathrm{RB}\left(T^{\alpha_n}_{xy},(X,Y)^n\right)\\ &:= \min\left\{\frac{k}{n} \mid k=1,\ \ldots,\ n,\quad \sup_{(X,Y)^n_k}\left|T^{\alpha_n}_{xy}\left((X,Y)^n_k\right) - T^{\alpha_n}_{xy}\left((X,Y)^n\right)\right| = \infty\right\},\end{aligned} \tag{3.48}$$

where $(X,Y)^n_k$ denotes the contaminated sample by replacing k points of $(X,Y)^n$ with arbitrary values.

Proposition 3.11. *For $(x,y) \in \mathbb{R}^{p+q}_+$ and $\alpha_n \in (0,1)$,*

$$RB\left(\widehat{\lambda}^\star_{\alpha_n}(x,y),(X,Y)^n\right) = \begin{cases}\left(n(1-\alpha_n)+1\right)/n & \text{if } n\alpha_n = [n\alpha_n];\\ \left(n - [n\alpha_n]\right)/n & \text{otherwise,}\end{cases} \tag{3.49}$$

where $[n\alpha_n]$ denotes the integer part of $n\alpha_n$.

Proof: Our replacement breakdown value can be recovered immediately from the breakdown point of univariate quantiles by using $\mathrm{RB}\left(T^{\alpha_n}_{xy},(X,Y)^n\right) = \mathrm{RB}\left(S^{\alpha_n},(Z^{xy}(X,Y))^n\right)$. ■

Note that the conditional quantile-based version $\widehat{\lambda}_{\alpha_n}(x,y)$ achieves the sample breakdown point

$$\mathrm{RB}\left(\widehat{\lambda}_{\alpha_n}(x,y),(X,Y)^n\right)=\begin{cases}(n\widehat{F}_X(x)(1-\alpha_n)+1)/n & \text{if } n\widehat{F}_X(x)\alpha_n=[n\widehat{F}_X(x)\alpha_n];\\(n\widehat{F}_X(x)-[n\widehat{F}_X(x)\alpha_n])/n & \text{otherwise,}\end{cases}\tag{3.50}$$

as can be seen from Daouia and Gijbels (2011, Theorem 2.2) and where $\widehat{F}_X$ denotes the empirical marginal distribution function of X. In the limiting case, for an (intermediate) sequence $\alpha_n \to 1$ such that $n(1-\alpha_n)\to\infty$, we have $\mathrm{RB}\left(\widehat{\lambda}^\star_{\alpha_n}(x,y),(X,Y)^n\right)\sim(1-\alpha_n)$ and $\mathrm{RB}\left(\widehat{\lambda}_{\alpha_n}(x,y),(X,Y)^n\right)\sim(1-\alpha_n)F_X(x)$ with probability one, assuming the distribution function $F_X(x)>0$. Thus, the fraction of bad outliers the efficiency score $\widehat{\lambda}_{\alpha_n}(x,y)$ can cope with depends heavily on the input usage x, while the global robustness of our alternative measure $\widehat{\lambda}^\star_{\alpha_n}(x,y)$ attains a higher breakdown value that only depends on the sample size n and the trimming order α_n.

Consequently, when considering the 'robustified' unconditional quantile-type efficiency measure $\widehat{\lambda}^\star_{\alpha_n}(x,y)$, a common value for α_n can be used for all production units (x,y). As such, we suggest the heuristic statistic

$$D_n(\alpha):=\max_{1\le i\le n}\left\{\widehat{\lambda}^\star_1(X_i,Y_i)-\widehat{\lambda}^\star_\alpha(X_i,Y_i)\right\}\tag{3.51}$$

which measures the maximal distance between the non-robust FDH frontier related to $\widehat{\lambda}^\star_1(x,y)\equiv\widehat{\lambda}(x,y)$ and the resistant partial surface corresponding to $\widehat{\lambda}^\star_\alpha(x,y)$, uniformly in (x,y). The idea is to look at the evolution of the distance $D_n(\alpha)$ as a function of $\alpha:=\alpha(k)=1-k/n$ for $k=0,\ldots,n-1$. In practical applications, it should be sufficient to examine this diagnostic for values of k ranging from 0 up to perhaps 50–100 to avoid excessive computational burden. In absence of anomalous data, the maximum distance $D_n(\alpha(k))$ should decrease smoothly as a 'staircase' function with the discrete order $\alpha(k)$. In this case, it is most efficient to use the full efficiency scores related to the extreme order $\alpha_n=\alpha(0)=1$. In contrast, if the distance curve exhibits a clearly severe jump at some large value, say $\alpha(k_0)$, this would indicate the presence of potential outliers and that the estimates $\widehat{\lambda}^\star_\alpha(X_i,Y_i)$ remain globally robust for orders $\alpha\le\alpha(k_0+1)$ before breaking down at $\alpha(k_0)$. In this case, it is prudent to seek robustness by choosing the limit value $\alpha_n=\alpha(k_0+1)$ for which $\widehat{\lambda}^\star_{\alpha_n}(\cdot,\cdot)$ is sensitive to the magnitude of valuable extreme firms but, at the same time, remains resistant to the influence of isolated outliers.[7]

Returning to the example described in Section 3.3, panel (c) in Figures 2–3 show values of the diagnostic in (3.51) plotted against values of $k = 0,\ 1,\ \ldots,\ 100$. Both panels show a jump between $k = 1$ and $k = 2$, but the jump is larger in Figure 3. There are other jumps corresponding to larger values of k, but these are much smaller than the initial jump. With $n = 100$ (or $n = 102$ with the outliers in Figure 3), $k = 2$ corresponds to $\alpha = 0.98$ (or $\alpha \approx 0.9804$, which gives the same results as $\alpha = 0.98$ since in either case $[\alpha n] = 98$). Panel (d) in Figures 2–3 show the full frontier Ψ^∂ as dotted lines, the true, unconditional order-α quantile for $\alpha = 0.98$ as dashed lines, the corresponding estimate of the quantile by the solid curve, and the FDH estimate of Ψ^∂ by the dash-dot-dash pattern.

Comparing panel (d) with panel (a) in Figure 2, we see that use of the diagnostic function in (3.51) leads to estimation of a quantile closer to the full frontier than the arbitrary choice of $\alpha = 0.95$ in panel (a). Visual inspection in panel (d) suggests that the quantile estimate tracks the true quantile closer than the FDH estimate tracks Ψ^∂. Turning to panel (d) in Figure 3 where the sample has been contaminated with two outliers, and comparing with panel (d) in Figure 2, we see once again that the unconditional quantile estimator suffers almost no effect from the outliers, unlike the conditional quantile estimator in panel (b). The unconditional quantile estimator estimates a unique quantile, independent of the direction chosen a priori, and appears more robust than the FDH estimator as well as the conditional quantile estimator, while providing a monotonic estimate of the unconditional quantile, in contrast to the conditional version in panel (b).

3.6 A multivariate example

Before turning to our empirical application involving U.S. credit unions, we give here another brief example to illustrate the ideas that have been developed so far. Charnes et al. (1981, Tables 1–4, pp. 680–682) list input and output data for 70 schools in an application where they examine efficiency in educational production. These data serve to illustrate our new methods in a multivariate framework with $p = 5$ inputs, $q = 3$ outputs, and $n = 70$ observations. In addition, the Charnes et al. data can be used by the interested reader

[7] If one plots, in an application, $D_n(\alpha(k))$ for say, $k = 0,\ 1,\ \ldots,\ 100$ and finds a large jump near $k = 100$, the range of values of k over which $D_n(\alpha(k))$ is plotted might be increased.

to replicate this example. Moreover, these data have been used by Wilson (1993) and Simar (2003) to illustrate outlier-detection methods, and are known to contain several outliers.

Table 1 contains FDH efficiency estimates in the columns labeled $\widehat{\lambda}(x,y)$ for the 70 schools studied by Charnes et al.; the results are displayed in the same order as in the tables in their paper. With 8 dimensions and only 70 observations, it is not surprising that all but 6 of these estimates are equal to 1. Values of the diagnostic function defined in (3.51) corresponding to $k = i - 1$ are shown in the columns labeled $D_n(\alpha)$ (note that the column labeled i in Table 1 indexes observations in the case of the columns giving the FDH efficiency estimates, but here it serves to define k in determining $\alpha = 1 - \frac{k}{n}$ for purposes of computing $D_n(\alpha)$). Starting with $k = 69$ (i.e., $i = 70$) and working backward, the diagnostic is flat until k goes from 19 to 18 (i.e., i goes from 20 to 19), suggesting that one should set $\alpha = 1 - \frac{19}{70} \approx 0.7286$.

The columns in Table 1 show the transformation in (3.1) applied to each observation $i = 1, \ldots, n$ using the first observation as (x, y). In the columns labeled "Sorted," these values have been sorted by algebraic value. The largest value is 1.0000, giving the value for $\widehat{\lambda}(x,y)$ corresponding to $i = 1$ in the first row of the table. For $\alpha = 0.7286$ we have $[\alpha n] + 1 = 52$, and hence $\widehat{\lambda}^{\star}_{\alpha}(x,y) = 0.4334$, obtained from the row in Table 1 corresponding to $i = 52$ and the column labeled "Sorted." For the first observation in the Charnes et al. data, this is an estimate of the output-oriented, unconditional quantile-efficiency of order $\alpha = 0.7286$ defined in Section 3.2. Of course, to obtain similar estimates for observations 2–70, one would have to recompute the transformed variable $Z^{xy}(X,Y)$ for each observation.

In the next section, we apply our new estimator to examine the performance of U.S. credit unions.

4 Efficiency among U.S. Credit Unions

4.1 The Credit Union Industry

Wheelock and Wilson (2011) used nonparametric, local polynomial regression methods to estimate returns to scale among U.S. Credit Unions over the period 1989–2006, and found strong evidence of increasing returns to scale throughout the size-distribution of credit unions. Here, we analyze the technical efficiency of U.S. credit unions in 1989 and 2006 using the methods developed above.

Credit unions are an important component of the financial services industry in the U.S. A number of studies, including Smith (1984), Fried et al. (1993), Fried et al. (1999), Frame and Coelli (2001), Frame et al. (2003), and Bauer (2008) have previously examined credit unions' technical efficiency; these studies have typically either employed fully parametric (and consequently, perhaps misspecified) models, or have used non-parametric methods such as DEA or FDH estimators which are extremely sensitive to outliers. Credit unions are financial intermediaries, as are banks; excessive technical inefficiency among credit unions would reflect wasted capital and would presumably reflect foregone economic growth or productivity.

Over the past three decades, advances in information-processing and communications technology (IT) and changes in regulation have had a profound impact on the environment in which depository institutions operate. IT advances have enabled the development of new bank services (from automated teller machines to internet banking), financial instruments (such as various types of derivative securities), payments instruments (such as debit cards and automated clearinghouse payments), and credit evaluation and monitoring platforms.[8] The same period saw the deregulation of deposit interest rates and branch banking, the imposition of risk-based capital requirements, and numerous other regulatory changes affecting depository institutions.[9]

Most credit unions operate at small scale while specializing in "relationship" lending. Credit unions are mutual organizations that provide deposit, lending, and other financial services to members (i.e., depositors or borrowers) sharing a common occupational, fraternal or other bond. A common bond is advantageous because it can reduce the cost of assessing the credit-worthiness of potential borrowers, facilitating unsecured lending on reasonable terms to the credit union's members. However, as with other lenders, recent advances in information processing and communications technology have reduced costs of acquiring information about potential borrowers, and consequently have reduced some of the advantages of small scale and common bond that in the past enabled credit unions to provide financial services at low cost to their memberships.

Over the last three decades, membership in credit unions has grown at a faster rate than

[8] See Berger (2003) for details and analysis of the effects of new technology, including advances in IT, on productivity growth in the banking industry and on the structure of the banking industry.

[9] Spong (2000) provides a summary of current U.S. banking regulations.

U.S. population. Wheelock and Wilson (2011) note that credit unions served 52 million members in 1985, 80 million members in 2000, and 93 million members by October 2009. The increases in credit union membership correspond to a rapid increase in credit unions' share of total industry assets, which have increased from 3.3 percent in 1985 to 6.0 percent in 2005. Wheelock and Wilson further note that much of this gain came at the expense of savings and loan associations and savings banks, whose share of the industry's assets declined from 30.1 percent to 15.9 percent over the same period, while the share of industry assets held by commercial banks rose from 66.1 percent to 78.1 percent.

In addition, credit unions appear to have gained market share as a result of the recent financial crisis. For example, the share of home mortgages originated by credit unions rose from 3.6 percent in 2007 to 6.2 percent in 2008. Credit unions now hold roughly 10 percent of U.S. household savings deposits, 9 percent of all consumer loans, and 13.2 percent of non-revolving consumer loans. Wheelock and Wilson (2011) observe that credit unions are increasingly also a source of business loans, although current law caps credit unions' business loans at 12.25 percent of total assets (several attempts have been made in the U.S. Congress to increase this limit).

Large credit unions have experienced faster growth in total assets, membership, and earnings than small credit unions (Goddard et al., 2002). Wheelock and Wilson (2011) report that, after adjusting for inflation, the average credit union held 6.5 times more assets in 2006 than the average credit union in 1985.[10] As with banks and savings institutions, the number of credit unions has declined sharply due to consolidation within the industry. While a peak of 23,866 credit unions operated in 1969, by 2006 only 8,662 credit unions were in operation. Wheelock and Wilson (2011) note that the Credit Union Membership Access Act of 1998 facilitated this consolidation by weakening the common bond requirement, permitting credit unions to accept members from unrelated groups. Since passage of this act, the number of credit unions characterized by multiple common bonds has since increased rapidly.[11]

[10] Average assets held by U.S. credit unions amounted to $84.6 million in 2006, ($50.6 million in constant 1985 dollars) as opposed to $7.8 million in 1985.

[11] See Wheelock and Wilson (2011) and references cited therein for additional details on U.S. credit unions.

4.2 Empirical Analysis

Credit unions use a number of inputs to produce a wide range of services. In order to examine empirically the performance of credit unions, limited data and, in the case of non-parametric approaches, limits on the number of dimensions that can reasonably be examined, dictate use of simplified models. Our analysis is uses an input-output mapping similar to that used by Wheelock and Wilson (2011), which in turn is similar that employed by Frame et al. (2003) and Frame and Coelli (2001). Specifically, we model credit unions as service providers serving as financial intermediaries that borrow from cash-rich members and lend to cash-poor members, subject to constraints of the prevailing production technology.

We specify three output quantities ($q = 3$) (i) total loans (Y1); (ii) investments (Y2); (iii) average interest rate on deposits (Y2); and (iv) the inverse of average interest rate on loans (Y3). The first and second outputs reflect the lending function of credit unions, while the third and fourth outputs serve to capture the "service" provided to credit union members in terms of favorable rates on deposits and loans. In addition, we specify two inputs ($p = 2$): (i) total shares and deposits, reflecting borrowing by credit unions (X1); and (ii) labor, measured in full-time equivalents (X2). Further details, including credit union call-report variables used to construct our inputs and outputs, can be found in Wheelock and Wilson (2011, Table 1).[12]

Our input-output specification does not include a measure of risk. Consequently, our results should be interpreted with some caution. For a given level of deposits, a credit union that maximizes loans, investments, and the two price variables may find itself operating with a very thin capital margin as well as spread between interest rates on loans and deposits.

[12] Call report data for individual credit unions are available from the National Credit Union Administration (www.ncua.gov).

In Y1, Y2, and X1, we use the (constant) dollar amounts of loans, investments, and shares and deposits. Although one might wish to consider the number of credit union members that are served, data for the number of loans, investments, shares, or deposits are not available in the call report data.

Our specification of credit unions' inputs and outputs follows the lines of other studies such as Frame and Coelli (2001), Frame et al. (2003), Wheelock and Wilson (2011, 2012), and others. In particular, our input-output specification reflects the view that credit unions are similar to small community banks, with the additional mandate to provide "service" in the form of favorable interest rates to depositors and borrowers. We treat deposits as an input, as do the studies listed above, because credit unions necessarily must borrow from depositors in order to lend to borrowers. We estimate technical efficiency in the output direction so that our results can be interpreted as a measure of how well credit unions produce loans and other outputs given their observed level of deposits and other inputs.

Of course, this translates into risk. In our model, the "optimal" level of efficiency may be short of operating on the full frontier. While it is difficult to quantify risk from the data that are available to us, our results give an idea of the operating characteristics of the credit union industry as a whole. Consequently, in the discussion that follows, we focus on the distribution of estimated efficiencies, rather than on efficiencies of individual credit unions in our sample.

Our data include 13,223 and 8,161 observations for (year-end) 1989 and 2006, respectively.[13] Summary statistics for the input and output variables are given in Table 2, where the columns labeled "Q1,", "Q2," and "Q3" give the first, second (median), and third quartiles of the distributions of each variable. Comparison of the summary statistics between 1989 and 2006 reveals that the distributions of the two inputs as well as that of the loan and investment outputs shifted rightward during the period 1989–2006; the three quartiles as well as the means are larger for each variable X1, X2, Y1, and Y2 in 2006 than in 1989. This reflects the fact that credit unions have grown larger in terms of total assets over this period, in part through consolidation via merger activity. Table 2 also indicates that rates that credit unions paid on deposits (Y5) declined from 1989 to 2006, but loan rates also declined as indicated by the increase in the quartiles for Y6. The prime bank lending rate at the end of 1989 stood at 8.75 percent, compared to 8.25 percent at the end of 2006; 30-year conventional mortgage rates fell from 10.65 percent to 6.14 percent over the same period.[14]

Figure 4 shows the diagnostic function $D_n(\alpha)$ given in (3.51) with $k = 0,\ 1,\ \ldots, 200$. For 1989, the first panel in Figure 4 shows several large jumps in the diagnostic $D_n(\alpha(k))$ for small values of k, and a final jump when k is increased from 83 to 84. For 2006, moving from right to left in the second panel of Figure 4, we see a shallow decrease in the diagnostic function around k=150, and then a sharp decrease beginning when k goes from 40 to 39. With 13,223 and 8,161 observations in 1989 and 2006, respectively, setting $k = 84$ and 40 gives $\alpha(k) = 1 - \frac{k}{n} = 0.9936$ and 0.9951 for 1989 and 2006, respectively.[15]

[13] We omitted observations where either loans or investments were negative, interest rates were outside the range $(0, 1)$, or where inputs were negative. Such observations reflect obviously incorrect values.

[14] Interest rate data are from series MPRIME and MORTG, not seasonally adjusted, St. Louis Federal Reserve Bank FRED database, http://www.research.stouisfed.org/fred2/.

[15] Given the large sample sizes in both years, it is perhaps not surprising that the diagnostic $D_n(\alpha)$ would lead to choosing large values for α. Note, however, that the diagnostic does not return $\alpha = 1$, which would lead to estimation of the full-frontier. Instead, the diagnostic indicates that a quantile lying perhaps "very close" to the frontier should be the benchmark. To further examine the performance of the diagnostic in

Table 3 shows results for estimation of (output-oriented) technical efficiency in both 1989 and 2006; the columns labeled "Q1," "Q2,", and "Q3" give the first, second, and third quartiles of the estimated efficiency levels. The maximum among both the FDH and DEA estimates for 1989 are implausibly large; it is difficult to imagine, from a practical viewpoint, that there might be a credit union that could increase its output by a factor of more than 7, or even 3, while holding input levels fixed. Similarly for 2006, both the FDH and DEA estimates yield implausibly large values, though less so than for 1989. By contrast, the quantile estimates are somewhat smaller, even with $\alpha = 0.999$, where the quantile is arguably very close to the full frontier. For each value of α in the table, the median efficiency estimates are less than 1, indicating that more than half the observations lie above the corresponding quantile, and suggesting that the data are very disperse over the input-output space.

Comparing the unconditional quantile estimates across 1989 and 2006 (using $\alpha = 0.994$ and $\alpha = 0.995$, respectively in Table 3) suggests that median (Q2) efficiency decreased slightly. The first quartiles are almost the same, but the third quartile and the maximum values are somewhat smaller in 2006 than in 1989. This contrasts with the DEA estimates, where the median is 1.683 for 1989 and 1.781 for 2006, which would suggest that median inefficiency increased by about 0.098. Since the measures are multiplicative, this would suggest that the median credit union was about 10 percent less efficient in 2006 than in 1989, though the differences are smaller than with the DEA estimates. The FDH estimates also suggest that inefficiency may have increased between 1989 and 2006. By contrast, the quantile estimates for corresponding values of α across the two years indicate a very slight decrease in median efficiency over this period.

In order to gauge the precision of our estimates, we used the result in Proposition 3.6 to estimate 95-percent confidence intervals for each credit union in 2006. Using $\alpha = 0.9951$ as suggested by the diagnostic function $D_n(\alpha)$ as detailed above, we obtain non-zero estimates $\widehat{\lambda}^{\star}(x, y)$ 8,110 cases among 8,161 observations (recall from (3.21) that estimates equal to zero occur whenever $\alpha \leq 1 - \widehat{F}_X(x)$). Among the 8,110 observations with non-zero efficiency

(3.51), we repeated the exercise using the data for 2006, but with only the first 200, 500, 1,000, and then 2,000 observations. The corresponding values of α chosen by the diagnostic exercise were 0.9000, 0.9260, 0.9710, and 0.9755. This seems reasonable; in smaller samples, one is necessarily less certain than in larger samples whether a particular extreme observation should be classified as an outlier. Our diagnostic procedure reflects this, and chooses quantiles closer to the full frontier as the sample size becomes larger.

estimates, the widths of our estimated 95-percent confidence intervals range from 0.0163 to 0.5436, with a median value of 0.0632. Ninety percent of these estimated confidence intervals have width less than 0.0986, and 99-percent have width less than 0.1571. Of course, these results are specific to the sample of credit unions we have used, and depend on the density of the data over the production set. Nonetheless, given the wide variation in estimated efficiencies in Table 3, and the tightness of most of our estimated confidence intervals, it is apparent that our new estimator finds many significant differences in efficiency among the credit unions in our sample.

5 Summary and Conclusions

The (unconditional) hyperbolic order-α quantile and its estimator introduced by Wheelock and Wilson (2008) was motivated, in part, by the different (i.e., depending on a priori chosen direction) conditional order-α quantiles introduced by Daouia and Simar (2007). More recently, these ideas have been extended to directional measures by Simar and Vanhems (2012), where the resulting order-α quantile is also independent of the chosen direction, and is the same as in the hyperbolic case. The new quantile methods presented in this paper reconcile the input and output oriented measures; as noted in Section 3.2, the *unconditional* quantile frontier of order α is unique, and does not depend on on the chosen direction. This differs from the *conditional* quantiles of order α discussed by Daouia and Simar, which differ depending on whether an input or an output orientation is used. The new input- and output-oriented estimators have been shown to yield estimates of order-α quantiles that are monotonic. Estimates of the order-α quantiles for the hyperbolic and directional cases based on the ideas of Wheelock and Wilson and Simar and Vanhems are also shown to be monotonic.

In addition, this paper has established links to both extreme value theory and robustness theory. The link with extreme value theory permits much simpler proofs of statistical consistency, derivation of limiting distributions, and other important asymptotic results. In particular, extensions to cases where the density $f(x, y)$ approaches 0 at the frontier are handled easily using the new theory. The links with robustness theory permit more careful quantification of just how robust the estimators are than has apparently been possible in the past, and thereby has provided new understanding. Our new estimators are shown to be *more* robust than the conditional order-α quantile estimators. In addition, the link with robustness theory suggests a natural way to choose the quantile-order, using the diagnostic given in (3.51).

Finally, we have used the new results to examine the technical efficiency of U.S. credit unions in 1989 and 2006. The conclusions drawn form using the new methods are rather different from what would have been concluded using only the more traditional DEA and FDH full-envelopment estimators.

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Table 1: Example Using Charnes et al. (1981) Data

i	$\widehat{\lambda}(x,y)$	$D_n(\alpha)$	$Z^{xy}(X,Y)$	Sorted	i	$\widehat{\lambda}(x,y)$	$D_n(\alpha)$	$Z^{xy}(X,Y)$	Sorted
1	1.0000	0.0000	1.0000	0.0000	36	1.0000	1.1456	0.2113	0.2639
2	1.0000	0.0000	0.4528	0.0000	37	1.0797	1.1456	0.2927	0.2740
3	1.0000	1.0000	0.6677	0.0000	38	1.0000	1.1456	0.1817	0.2778
4	1.0204	1.0000	0.2740	0.0000	39	1.0000	1.1456	0.5582	0.2881
5	1.0000	1.0000	0.1177	0.0000	40	1.0000	1.1456	0.4150	0.2927
6	1.0000	1.0000	0.2309	0.0000	41	1.0000	1.1456	0.4476	0.3163
7	1.0000	1.0000	0.2881	0.0000	42	1.0000	1.1456	0.3844	0.3343
8	1.0000	1.0000	0.0000	0.0000	43	1.0000	1.1456	0.4001	0.3416
9	1.0000	1.0000	0.4792	0.0000	44	1.0000	1.1456	0.0000	0.3558
10	1.0000	1.0000	0.0000	0.0000	45	1.0000	1.1456	0.1737	0.3580
11	1.0000	1.0000	0.6397	0.0000	46	1.0000	1.1456	0.0000	0.3611
12	1.0000	1.0000	0.6643	0.0000	47	1.0000	1.1456	0.5395	0.3844
13	1.0339	1.0797	0.4856	0.0000	48	1.0000	1.1456	0.1412	0.4001
14	1.0000	1.0797	0.1841	0.0000	49	1.0000	1.1456	0.2232	0.4150
15	1.0000	1.0797	0.2639	0.0000	50	1.0000	1.1456	0.0000	0.4190
16	1.0000	1.0797	0.0000	0.0842	51	1.0000	1.1456	0.1827	0.4285
17	1.0000	1.0797	0.3163	0.1177	52	1.0000	1.1456	0.8284	0.4334
18	1.0000	1.0797	0.5052	0.1190	53	1.0672	1.1456	0.3343	0.4438
19	1.0000	1.0797	0.0000	0.1204	54	1.0000	1.1456	0.0000	0.4476
20	1.0000	1.1456	0.5397	0.1342	55	1.0000	1.1456	0.4438	0.4528
21	1.0000	1.1456	0.6870	0.1412	56	1.0000	1.1456	0.2481	0.4792
22	1.0000	1.1456	0.3558	0.1670	57	1.0000	1.1456	0.0000	0.4856
23	1.0000	1.1456	0.0000	0.1737	58	1.0000	1.1456	0.3580	0.5052
24	1.0000	1.1456	0.4334	0.1817	59	1.0000	1.1456	0.0000	0.5395
25	1.0000	1.1456	0.4285	0.1827	60	1.0000	1.1456	0.2778	0.5397
26	1.0000	1.1456	0.5799	0.1841	61	1.0000	1.1456	0.1204	0.5582
27	1.0000	1.1456	0.5742	0.2113	62	1.0000	1.1456	0.1670	0.5742
28	1.0000	1.1456	0.3611	0.2135	63	1.0000	1.1456	0.2272	0.5799
29	1.0000	1.1456	0.1190	0.2206	64	1.0000	1.1456	0.3416	0.5938
30	1.0000	1.1456	0.2135	0.2232	65	1.0000	1.1456	0.2206	0.6397
31	1.1456	1.1456	0.2307	0.2272	66	1.0021	1.1456	0.0000	0.6643
32	1.0000	1.1456	0.0842	0.2307	67	1.0000	1.1456	0.0000	0.6677
33	1.0000	1.1456	0.0000	0.2309	68	1.0000	1.1456	0.4190	0.6870
34	1.0000	1.1456	0.5938	0.2406	69	1.0000	1.1456	0.1342	0.8284
35	1.0000	1.1456	0.0000	0.2481	70	1.0000	1.1456	0.2406	1.0000

Table 2: Summary Statistics

	Min	Q1	Q2	Mean	Q3	Max
1989						
X1	5.000000	737.0000	2476.0000	13270.0000	8506.0000	3292000.0000
X2	0.500000	1.0000	2.5000	9.3660	6.5000	2160.0000
Y1	2.000000	604.0000	1965.0000	9684.0000	6401.0000	2959000.0000
Y2	0.000000	137.0000	567.0000	4208.0000	2362.0000	1232000.0000
Y3	0.000012	0.0548	0.0611	0.0605	0.0673	0.3333
Y4	1.667000	7.9890	8.6880	8.7290	9.3520	94.5500
2006						
X1	47.000000	3470.0000	11340.0000	74090.0000	40310.0000	20710000.0000
X2	0.500000	2.5000	6.0000	27.6500	20.0000	5282.0000
Y1	2.000000	2382.0000	7814.0000	60790.0000	30050.0000	20580000.0000
Y2	0.000000	1095.0000	3754.0000	20580.0000	12270.0000	4900000.0000
Y3	0.000321	0.0151	0.0202	0.0205	0.0253	0.1190
Y4	2.000000	13.0700	14.7000	14.5200	16.1400	47.0000

Table 3: Estimates of Output Efficiency for Credit Unions ($p = 2,\ q = 4$)

	Min	Q1	Q2	Mean	Q3	Max
			1989			
DEA	1.000	1.433	1.581	1.644	1.766	8.183
FDH	1.000	1.000	1.004	1.042	1.062	3.214
$\alpha = 0.999$	0.000	0.816	0.883	0.865	0.938	2.743
$\alpha = 0.998$	0.000	0.762	0.841	0.816	0.900	2.571
$\alpha = 0.997$	0.000	0.725	0.811	0.782	0.874	2.476
$\alpha = 0.996$	0.000	0.694	0.787	0.755	0.853	2.327
$\alpha = 0.995$	0.000	0.665	0.764	0.731	0.835	2.214
$\alpha = 0.994$	0.000	0.641	0.745	0.711	0.820	2.071
$\alpha = 0.993$	0.000	0.620	0.729	0.693	0.805	2.000
$\alpha = 0.992$	0.000	0.600	0.713	0.677	0.793	1.929
$\alpha = 0.991$	0.000	0.581	0.698	0.661	0.780	1.881
$\alpha = 0.990$	0.000	0.564	0.685	0.647	0.769	1.738
$\alpha = 0.980$	0.000	0.442	0.581	0.544	0.680	1.308
			2006			
DEA	1.000	1.449	1.673	1.685	1.902	3.291
FDH	1.000	1.000	1.000	1.049	1.069	1.874
$\alpha = 0.999$	0.000	0.778	0.861	0.841	0.925	1.610
$\alpha = 0.998$	0.000	0.719	0.811	0.786	0.879	1.540
$\alpha = 0.997$	0.000	0.680	0.778	0.751	0.849	1.473
$\alpha = 0.996$	0.000	0.648	0.752	0.723	0.825	1.426
$\alpha = 0.995$	0.000	0.623	0.731	0.700	0.806	1.378
$\alpha = 0.994$	0.000	0.602	0.713	0.680	0.790	1.358
$\alpha = 0.993$	0.000	0.580	0.694	0.660	0.773	1.335
$\alpha = 0.992$	0.000	0.562	0.679	0.644	0.759	1.322
$\alpha = 0.991$	0.000	0.548	0.665	0.630	0.747	1.286
$\alpha = 0.990$	0.000	0.532	0.653	0.617	0.736	1.279
$\alpha = 0.980$	0.000	0.423	0.558	0.519	0.647	1.151

Figure 1: Uniform Triangle Example ($p = q = 1$) — Truth

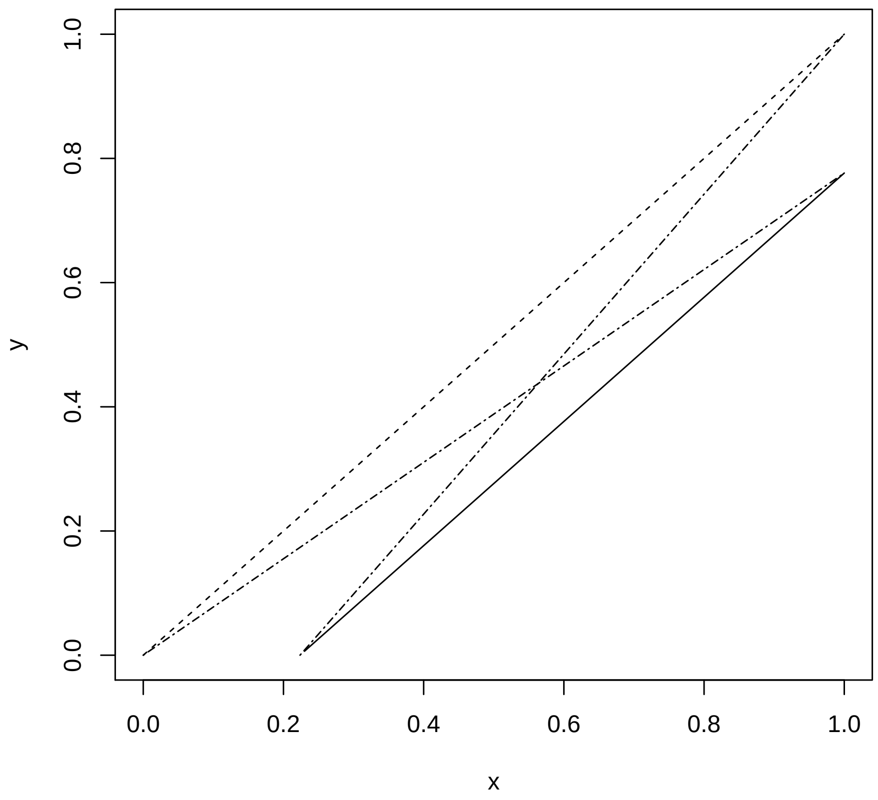
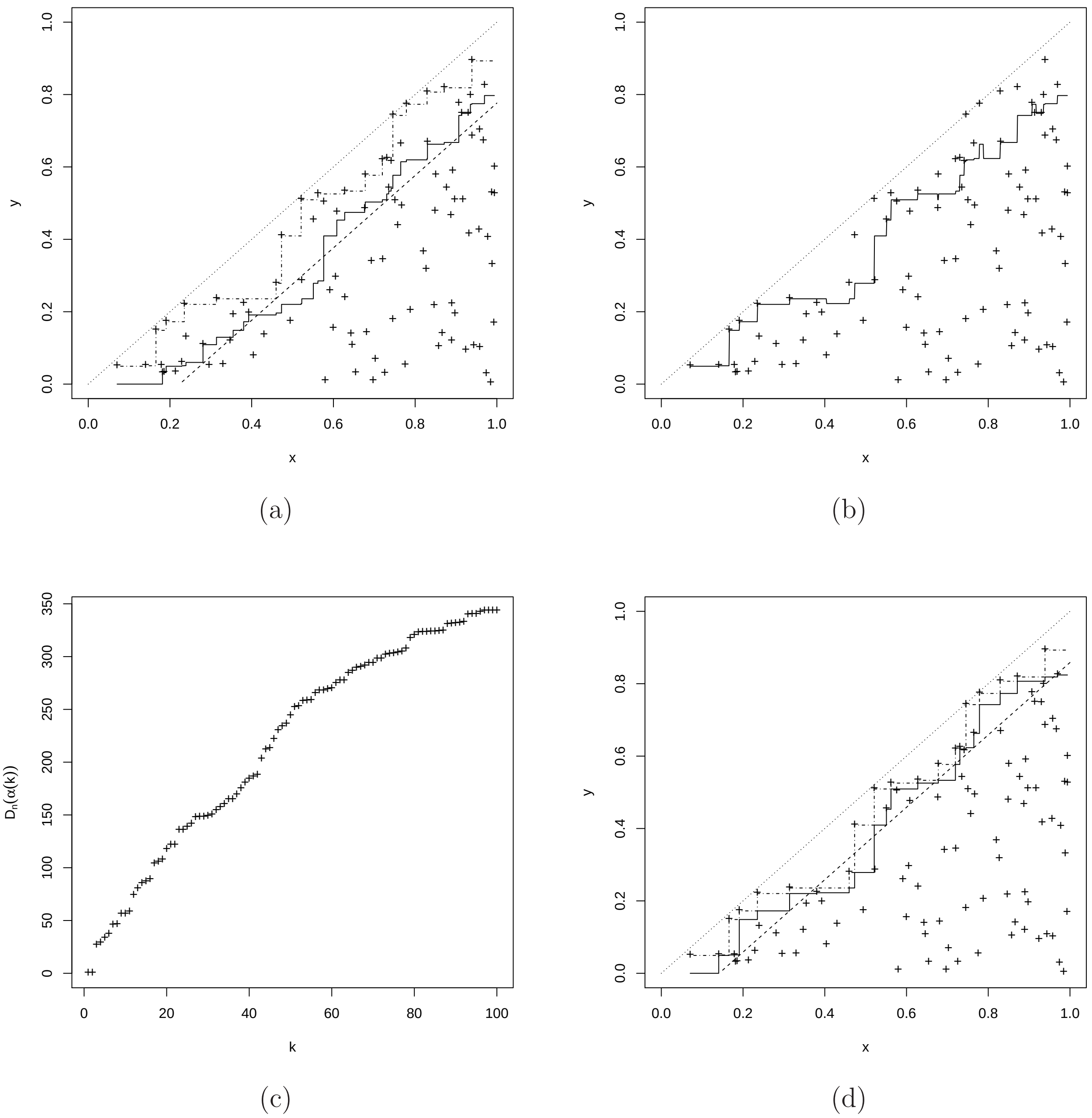


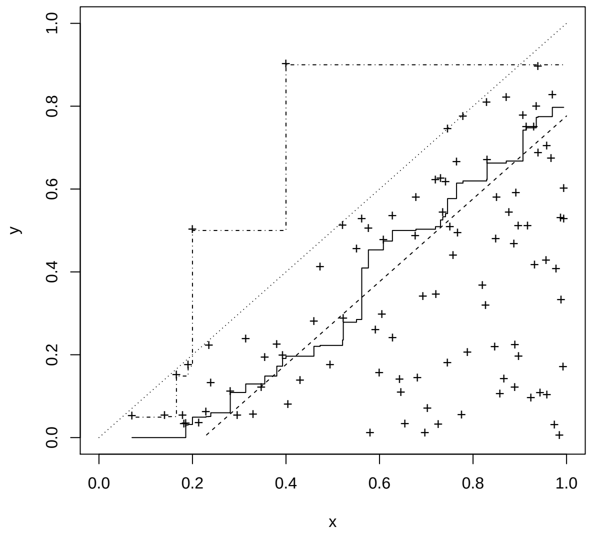
Figure 2: Uniform Triangle Example ($p = q = 1,\ n = 100$)



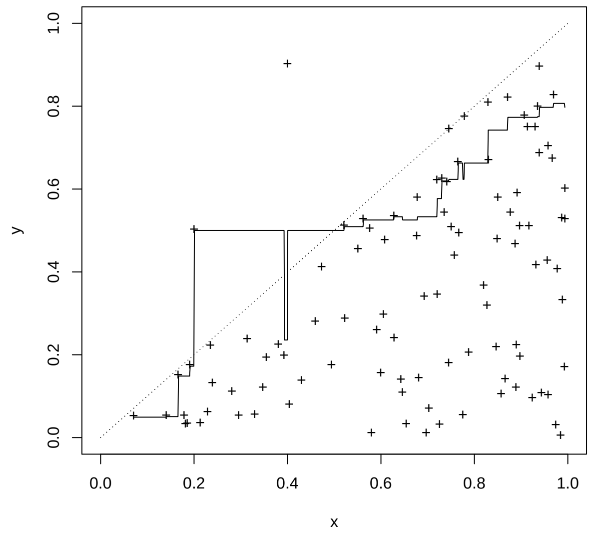
(a) (b)

(c) (d)

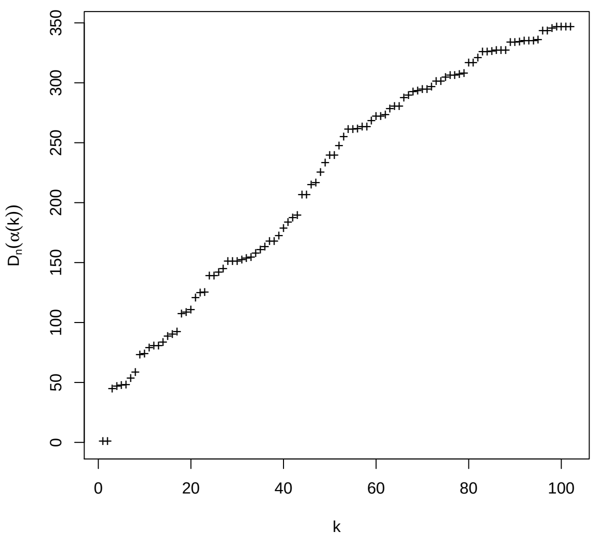
Figure 3: Uniform Triangle Example with Outliers ($p = q = 1,\ n = 102$)



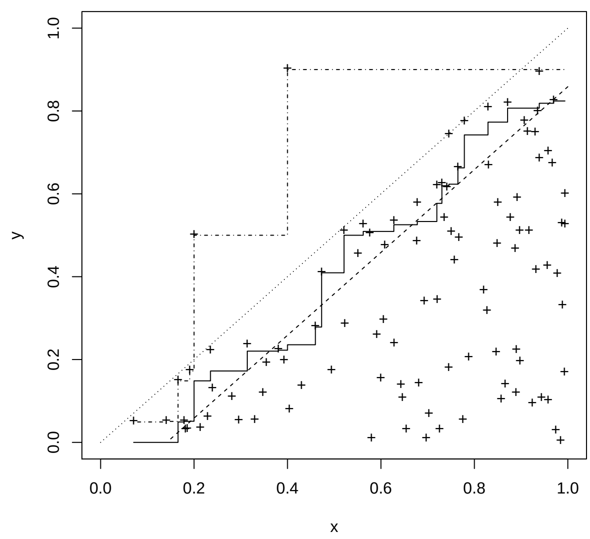
(a)



(b)

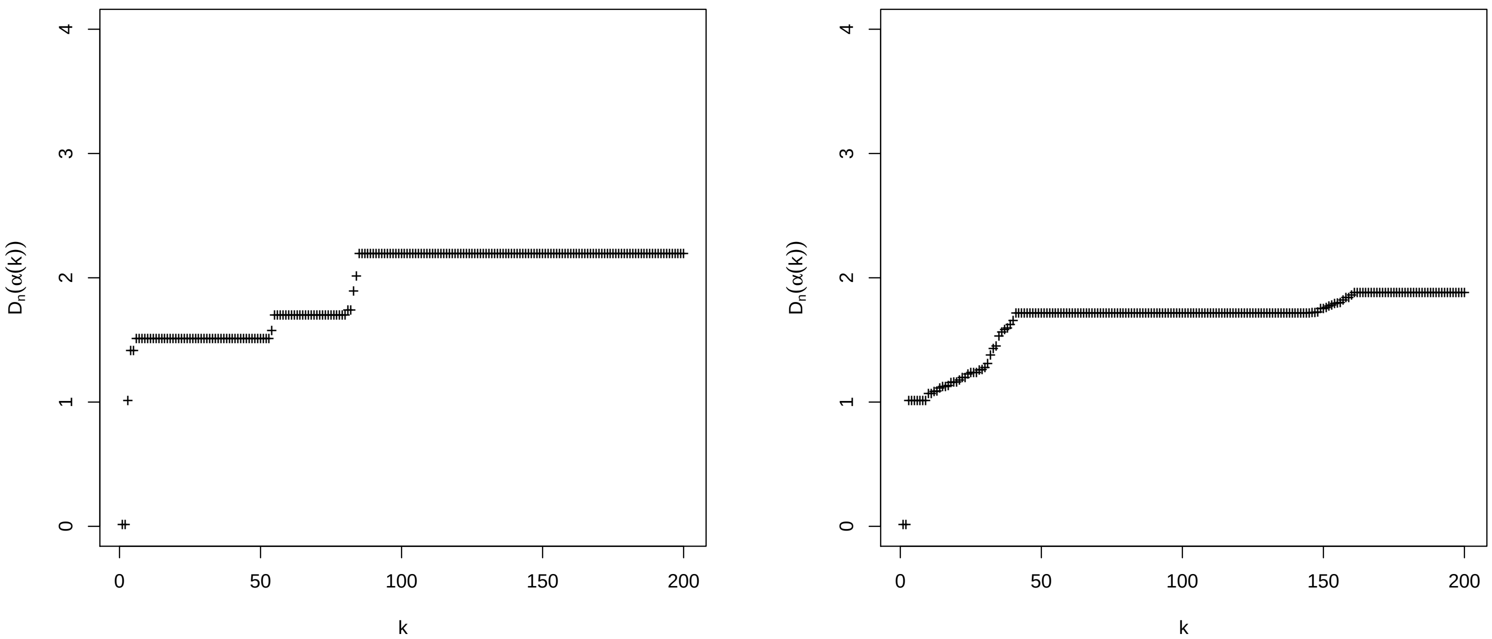


(c)



(d)

Figure 4: Diagnositc Functions $D_n(\alpha(k))$ for Credit Unions ($p = 2,\ q = 4$)



1989　　　　2006