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Asset pricing with uncertain betas: A long-term perspective

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Very preliminary and incomplete: Comments welcomed!

Abstract
How should one evaluate investment projects whose CCAPM betas are uncertain? This question is particularly crucial for projects yielding long-lasting impacts on the economy, as is the case for example for many green investment projects. We defined the notion of a certainty equivalent beta. We characterize it as a function of the characteristics of the uncertainties affecting the asset's beta and the economy as a whole. We show that its term structure is not constant and that, for short maturities, it equals the expected beta. If the expected beta is larger than a threshold (which is negative and large in absolute value in all realistic calibrations), the term structure of the certainty equivalent beta is increasing and tends to its largest plausible value. In the benchmark case in which the asset's beta is normally distributed, the certainty equivalent beta becomes infinite for finite maturities.

Keywords: asset prices, term structure, risk premium, certainty equivalent beta.

JEL Codes: G11, G12, E43, Q54.

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1. *Introduction*

Our ancestors have devoted much effort to innovate and to invest, from the domestication of fire, to the invention of the wheel, iron, the agriculture and the associated genetic selection of seeds and species, and to the accumulation of productive capital. When they contemplated the costs and benefits of these efforts, did they take into consideration the huge impacts of these actions over the entire time span of humanity? For example, when deciding to incur the huge costs to build an efficient roads network covering their empire, did the Romans consider the large benefits of their investment that was eventually enjoyed by the European citizens over the next two millennia? In the stagnant and unsustainable Roman economy, we know from the Ramsey rule that it would have been efficient to use a discount rate for safe investment projects that would be close to zero, so that the safe long term benefits should have mattered in the decision to invest. Projects with a negative beta should have been particularly valuable to serve as an insurance against the potential decline of the Roman economy over centuries. The problem is that it is particularly complex to determine the projects' betas for such long time horizons. In this paper, we take seriously the problem of the uncertainty affecting projects' betas in their normative evaluation.

These retrospective considerations serve as an illustration of more contemporary concerns about how one should evaluate our own efforts in favor of future generations. Indeed, the same questions arise when contemplating fighting climate change, investing in biotechnologies, or depleting non-renewable resources, for example. Economists have promoted the use of cost/benefit analysis to answer these questions. In particular, economic theory provides strong normative arguments in favor of using the Net Present Value criterion as a decision tool, with a discount rate that reflects both the opportunity cost of capital and the citizens' propensity to invest for the future. Under the standard assumptions of the Consumption-based Capital Asset Pricing Model (CCAPM, Lucas (1978)), this discount rate $r = r_f + \beta\pi_m$ is the sum of a risk-free rate r_f and a risk premium $\beta\pi_m$. Since Weitzman (1998), various authors have recommended to use a decreasing term structure for the risk-free discount rate, thereby putting more weight on long-term riskless impacts in the evaluation process.[2]

[2] See for example Weitzman (2001, 2007, 2009), Gollier (2002, 2008, 2012a), Newell and Pizer (2003), and Groom, Koundouri, Panopoulou and Pantelidis, (2007).

The development of this literature has mostly been devoted to the evaluation of safe projects. This is quite surprising, because most actions involving the distant future have highly uncertain impacts. For example, in spite of intense research efforts around the world over the last two decades, the socioeconomic impacts of climate change are still mostly unknown. We have learned from the normative version of the CCAPM that what matters to evaluate risky projects is their impact on the aggregate risk in the economy. This is evaluated by their beta, which measures the elasticity of the logarithm of their net benefits with respect to change in the logarithm of aggregate consumption $\ln c_t$. Projects with a larger beta will have a larger positive impact on the aggregate risk. They should be discounted at a larger rate. If two projects yield the same flow of expected benefits, the one with the smaller beta should have a larger social value.

An important problem is that socioeconomic betas are difficult to estimate. Large companies and assets funds tend to use them with parsimony. For example, Krueger, Landier and Thesmar (2012) demonstrate that conglomerates generally use a unique discount rate to evaluate different projects rather than project-specific ones. This tends low-beta conglomerates to overvalue high-beta projects. A more upsetting example is related to public policy evaluations in the western world. Up to our knowledge, all countries evaluate their actions using a unique discount rate independent of the uncertainty affecting their impacts. For example, a unique rate of 7% is used in the United States since 1992. It was argued at that occasion that the "*7% is an estimate of the average before-tax rate of return to private capital in the U.S. economy*" (OMB (2003)). In 2003, the OMB also recommended the use of a discount rate of *3%*, in addition to the *7%* mentioned above as a sensitivity. This new rate of *3%* was justified as follows: "*This simply means the rate at which society discounts future consumption flows to their present value. If we take the rate that the average saver uses to discount future consumption as our measure of the social rate of time preference, then the real rate of return on long-term government debt may provide a fair approximation*" (OMB, (2003)). In short, the OMB does not recommend evaluators to estimate the beta of the policy under scrutiny. Rather, it recommends estimating the policy's NPV using two discount rates, corresponding to a beta of zero or one, respectively. From our experience of advising public institutions in their evaluation of environmental policies, we believe that this is due to the complexity of estimating the beta of flows of (non-traded) socioeconomic benefits, often disseminated over a long period of time.

For an investment project whose cash flows share characteristics of those of some traded asset, one should use deleveraged market betas of these assets to compute the NPV of the project. This method is not without deficiencies. It is for example often the case that the resemblance between the cash flows of the project and those of the traded asset is weak, and that it is limited to a short period of time. We should also add to this picture the well-known failure of the CCAPM to predict market prices from the assets' betas. Finally, markets do not price the typical global, long-term externalities that motivated this paper, as those associated to climate change or genetic manipulations for example. For these reasons, the potential errors in the estimation of the project's beta should be taken into account when evaluating its social value.

In this paper, we propose to reconsider the CCAPM by explicitly recognizing that betas are uncertain. We show that the classical asset pricing formula of the CCAPM is robust to the introduction of an uncertain beta in the sense that this uncertainty does not affect the basic message of the CCAPM contained in the pricing formula $r = r_f + \beta\pi_m$. However, the uncertainty affecting the beta of the project necessitates the use of a Certainty Equivalent Beta (CEB) that must be used to replace the β contained in this formula. This paper is about the characterization of the CEB.

For short maturities, the uncertainty about the project's beta is shown to be irrelevant, so that the CCAPM formula should be used with a β equalling the expected beta of the project. The logic of compound interest plays a central role in the determination of the CEB for longer horizons. Two polar cases must be examined here. In the first polar case, the plausible betas are relatively large, so that it is likely that implementing the project raises the future aggregate risk in the economy. Because expected benefits increase with maturity t as c_t^β, the incremental aggregate risk generated by the project is magnified at longer maturities. Because of the compounding nature of the beta, the main engine of this magnification comes from the uncertainty about β much more than from the uncertainty coming from the growth of aggregate consumption. This implies that the term structure of the CEB must be increasing in this case. The second polar case arises when the plausible betas are negative and relatively large in absolute value. In that case, implementing the project reduces the future aggregate risk in the economy, and the entire argument is reversed. This yields a decreasing term structure of the CEB in that case. But in many instances, the support of the distribution of the project's beta will contain values compatible with the idea that the project will increase the aggregate risk, and others compatible with the idea that it will hedge the aggregate risk.

We show in Section 3 that a simple analytical solution to the CEB exists when the distribution of the beta is normal. In that case, the CEB and the associated discount rate using the CCAPM formula exist and are bounded only for relatively short maturities. The critical maturity is equal to the inverse of the product of the variance of the economic growth rate and of the beta. For example, if we assume a volatility of the economic growth rate of 4% per annum and a standard deviation of the beta equalling 1, this critical maturity above which the project's discount rate becomes infinite is equal to 625 years. Whether this is plus or minus infinity depends upon whether the expected beta is large enough. Suppose for example that relative risk aversion equals 2, and that the first two centered moments of the growth rate of consumption be identical. Then, the model shows that if the expected beta is positive, then the term structures of the CEB and of the discount rates tend to plus infinity at t=625 years. This means that all benefits occurring after this time horizon are completely irrelevant for the decision. This would be true independent of the potentially fabulous size of these benefits. Suppose alternatively that the expected beta of the project is negative. Then, the CEB and the discount rate tends to minus infinity at t=625 years. This means that the existence of any plausible benefit occurring after that critical time horizon should trigger the decision to invest, whatever the cost.

The normality assumption for the distribution of the beta is quite unrealistic because it makes plausible very large betas in absolute value, either positive or negative. In Section 4, we relax the normality assumption by allowing any possible distribution for the beta of the project. We determine the asymptotic properties of the term structure. We show in particular that the CEB most often tends to the largest or to the smallest plausible beta for very long maturities. When the support of the beta is bounded, this asymptotic CEB depends upon the position of the center of this support, rather than the mean as in the Gaussian specification. This indicates that the skewness of the distribution of the beta plays a crucial role for the evaluation of distant benefits. In particular the plausibility of a very negative beta will drive the determination of the discount rate. These results are in line with the observation by Martin (2012) that the value of long-term assets is mostly driven by the possibility of extreme events.

In Section 5, we apply these theoretical results to different contexts. We first show that the long-term beta of an environmental asset is equal to the inverse of the elasticity of substitution between this asset and consumption. We use time series data to estimate the elasticity of the demand for residential land in the United States. We show that the beta to be used for projects whose social benefit is to expand residential land should be increasing with maturity. We also measure the degree of uncertainty affecting socioeconomic and financial betas for different sectors of the economy in France and in the United States.

We provide two other interpretations of our model in Section 6. One of them allows us to use our results when investment projects have different components, each one with its own (certain) beta. This portfolio approach has recently been considered by Weitzman (2012) in the particular case of a $\beta = 0$ component and a $\beta = 1$ component. The structure of the model considered by Weitzman (2012) is the same than in this paper, except that Weitzman assumes that the portfolio is dynamically rebalanced in favour of the $\beta = 0$ component. We explain in Section 6 that this is the driving reason for why the Weitzman's certainty equivalent beta is always decreasing, contrary to what we obtain in this paper.

2. *The model*

We consider an asset whose social value V evolves stochastically through time in such a way that $V_t = V_{t-1}e^{R_t}$ and

$$R_t = a + \beta g_t + \varepsilon_t. \qquad (1)$$

In this equation, $g_t = \ln c_t / c_{t-1}$ denotes the change in log consumption between dates $t-1$ and t. We assume that ε_t has a zero mean and is independent of g_t. Moreover, we assume that $(\varepsilon_1, \varepsilon_2, ...)$ are i.i.d. random variables. Thus, β in equation (1) is the CCAPM beta of the asset. Another way to characterize this asset is to make explicit the relationship between the value of the asset and consumption. This can be done by rewriting equation (1) as follows:

$$V_t \middle| c_t = f_t \xi_t \left(\frac{c_t}{c_0} \right)^{\beta}, \qquad (2)$$

where ξ_t has a unit mean and is independent of aggregate consumption c_t. Finally, f_t is a free parameter in this model. In this paper, we generalize the CCAPM framework by allowing the beta of the asset to be uncertain. More specifically we consider the following model:

$$V_t \left| c_t = f_t \xi_t E\left(\frac{c_t}{c_0}\right)^{\tilde{\beta}_t}\right., \tag{3}$$

where the expectation is taken with respect to some cumulative distribution Q_t of $\tilde{\beta}_t$. In this paper, we alternatively examine the more natural family of projects in which, conditional to information at date 0, the distribution of $\tilde{\beta}_t = \tilde{\beta}$ is constant across maturities: $Q_t = Q$.[3]

Except for the uncertainty of the beta, our model duplicates the classical CCAPM model. We assume that relative risk aversion is a constant $\gamma > 0$, so that the utility function of the representative agent is $u(c) = c^{1-\gamma}/(1-\gamma)$. The intertemporal welfare of the representative agent is the prent value of the flow of future expected utility discounted at constant rate δ. We also assume that the growth rate of consumption defined as $g_t = \ln c_t / c_{t-1}$ follows a stationary random walk, so that $(g_1, g_2, ...)$ is an i.i.d. process. Finally, we assume that the growth rate g_t of consumption is normally distributed with mean μ_g and volatility σ_g.

3. *The Certainty Equivalent Beta in the Gaussian case*

In this section, we compute the Present Value (PV) at date 0 of the future benefit V_t characterized by (3) with an uncertain $\tilde{\beta}$. The PV of this future cash flow is the sure increase of current consumption that has the same effect on intertemporal welfare as the future increase in consumption V_t at date *t*. Because the project is marginal, we have that

$$PV_t(\tilde{\beta}) = e^{-\delta t} \frac{EV_t u'(c_t)}{u'(c_0)} = e^{-\delta t} f_t E\left(\frac{c_t}{c_0}\right)^{\tilde{\beta}-\gamma} = e^{-\delta t} f_t E\left(E\left[\exp((\tilde{\beta}-\gamma)g)\middle|\tilde{\beta}\right]\right)^t. \tag{4}$$

[3] Of course, there is learning going on in this model through the observation of the cash flows. In the long run, the ambiguity about $\tilde{\beta}$ vanishes.

Following the tradition in finance, let us define the discount rate r_t associated to future benefit V_t as the rate at which the expected benefit EV_t is discounted to compute its present value. This implies that $r_t(\tilde{\beta})$ is implicitly defined by the following equation:

$$PV_t(\tilde{\beta}) = e^{-r_t(\tilde{\beta})} EF_t, \tag{5}$$

with

$$EV_t = f_t E\left(E\left[\exp(\tilde{\beta}g)\middle|\tilde{\beta}\right]\right)^t \tag{6}$$

Define the Log of the Moment-Generating Function (LGMF) associated to random variable z as follows:

$$m_z(t) = \ln E \exp(zt). \tag{7}$$

From equations (4), (5) and (6), we obtain that the efficient discount rate is characterized by the following equation:

$$r_t(\tilde{\beta}) = \delta + \frac{1}{t} \ln \frac{E \exp(m_g(\tilde{\beta})t)}{E \exp(m_g(\tilde{\beta}-\gamma)t)}, \tag{8}$$

This can be rewritten in the following way:

$$r_t(\beta) = \delta + \frac{1}{t} m_{m_g(\tilde{\beta})}(t) - \frac{1}{t} m_{m_g(\tilde{\beta}-\gamma)}(t). \tag{9}$$

Observe that the each of the last two terms in the right-hand side of this equation compounds two moment-generating functions, the first one being associated to the randomness of the growth rate, and the second one being associated to the randomness of the beta. In this paper, we will often use the following technical result, which is proved in the Appendix 1.

Lemma 1: *Suppose that random variable z is normally distributed with mean μ_z and standard deviation σ_z. Consider any pair $(a,b) \in \mathbb{R}^2$ such that $b < 1/(2\sigma_z^2)$. Then, we have that*

$$E \exp(az + bz^2) = \exp(m_{az+bz^2}(1)) = \left(1 - 2b\sigma_z^2\right)^{-1/2} \exp\left(\frac{a\mu_z + 0.5a^2\sigma_z^2 + b\mu_z^2}{1 - 2b\sigma_z^2}\right). \tag{10}$$

This lemma is well-known in the special case corresponding to $b=0$, which states that the LGMF of $N(\mu,\sigma^2)$ evaluated at a is just $a\mu+0.5a^2\sigma^2$. The Arrow-Pratt approximation for the risk premium is exact when the utility function is exponential and the risk is normal. Lemma 1 shows that an analytical solution to such problems can be extended to the sum of a normal and a Chi-2.

We assume in this paper that the growth rate of consumption is normally distributed with mean μ_g and volatility σ_g. Using Lemma 1 with respectively $(a=\beta,b=0)$ and $(a=\beta-\gamma,b=0)$ yields

$$m_g(\beta)=\beta\mu_g+0.5\beta^2\sigma_g^2 \tag{11}$$

and

$$m_g(\beta-\gamma)=(\beta-\gamma)\mu_g+0.5(\beta-\gamma)^2\sigma_g^2. \tag{12}$$

Suppose now that $\tilde{\beta}$ is also normally distributed with mean μ_β and variance σ_β^2. Because $m_g(\beta)$ and $m_g(\beta-\gamma)$ are quadratic in β, we can again use Lemma 1 again to obtain the following Theorem, which represents the benchmark result of this paper. Its proof is provided in Appendix 2. It relies on two well-known ingredients of the CCAPM, the risk-free rate r_f and the aggregate risk premium π_m, which are defined as follows:

$$r_f=\delta+\gamma\mu_g-0.5\gamma^2\sigma_g^2, \tag{13}$$

and

$$\pi_m=\gamma\sigma_g^2. \tag{14}$$

Here is the benchmark result of this paper.

Theorem 1: *Suppose that the beta of the project is normally distributed with mean μ_β and variance σ_β^2. Then, the discount rate of the project is equal to*

$$r_t(\tilde{\beta})=r_f+\hat{\beta}_t(\tilde{\beta})\pi_m, \tag{15}$$

where, for all maturities $t < T = 1/\sigma_g^2\sigma_\beta^2$, *the Certainty Equivalent Beta* $\hat{\beta}_t(\tilde{\beta})$ *is defined as follows:*

$$\hat{\beta}_t(\tilde{\beta}) = \frac{\mu_\beta + t\sigma_\beta^2(\mu_g - 0.5\gamma\sigma_g^2)}{1 - t\sigma_g^2\sigma_\beta^2}. \tag{16}$$

Observe first that Theorem 1 generalizes the CCAPM. Indeed, suppose that the distribution of $\tilde{\beta}$ is degenerated at some β. This is the case by assuming $\mu_\beta = \beta$ and $\sigma_\beta = 0$. Theorem 1 implies that $\hat{\beta}_t(\tilde{\beta}) = \beta$ and $r_t(\tilde{\beta}) = r_f + \beta\pi_m$ in that case. In this case, the term structure of the discount rate is flat and well defined for all maturities ($T = +\infty$).

When beta is uncertain, Theorem 1 defines a certainty equivalent beta $\hat{\beta}_t(\tilde{\beta})$ that should be used in the standard CCAPM formula (15) to compute the discount rate of the project. This CEB defined by equation (16) has its own term structure. In the remainder of this section, we analyze the properties of this term structure of the CEB.

We first characterize the CEB for short maturities.

Corollary 1: *Under the assumptions of Theorem 1, the CEB is approximately equal to the mean beta for small maturities:*

$$\lim_{t\to 0} \hat{\beta}_t(\tilde{\beta}) = \mu_\beta. \tag{17}$$

The uncertainty affecting the beta of the project has no effect on the rate at which cash flows with short maturities should be discounted. For short maturities, in a fashion similar to diversifiable risks, this uncertainty should not be priced.

We now turn to the shape of the term structure of the CEB in the domain of maturities $t \in [0, T[$ where it is well defined by Theorem 1.

Corollary 2: *Under the assumptions of Theorem 1, the CEB is increasing with the maturity* $t \in [0, T[$ *if the mean beta is large enough:*

$$\frac{\partial}{\partial t}\hat{\beta}_t(\tilde{\beta}) \geq (<) 0 \quad \forall t \in [0,T[\quad \Leftrightarrow \quad \mu_\beta \geq (<) \bar{\mu}_\beta = 0.5\gamma - \frac{\mu_g}{\sigma_g^2}. \tag{18}$$

The term structure of the CEB is always monotone. It is increasing (decreasing) if the expected beta is larger (smaller) than a threshold $\bar{\mu}_\beta$ equals to the difference between half the relative risk aversion and the ratio of the mean and the variance of the growth rate of consumption. Notice that the term structure of the CEB is flat if the expected beta equals the threshold $\bar{\mu}_\beta$. It is useful to compute the order of magnitude of this threshold. A relative risk aversion of $\gamma = 2$ is usually considered as reasonable in the macro and finance literature. The average growth rate of consumption in the western world over the last two centuries has been around $\mu_g = 2\%$, whereas its mean volatility can be approximated at $\sigma_g = 4\%$ (see for example Maddison (1991)). Under this simple calibration that at least passes the laugh test, we obtain that the term structure of the CEB is increasing if and only if the expected beta is larger than $\bar{\mu}_\beta =$ $0.5 \times 2 - (0.02/0.04^2) = -11.5$. We believe that the vast majority of projects will satisfy this condition. An increase in risk aversion would increase this threshold. For example, the threshold would vanishes if relative risk aversion would increase to $\gamma = 25$. Observe that the shape of the term structure is quite sensitive to the trend of growth. For example, a reduction in the trend of growth from $\mu_g = 2\%$ to $\mu_g = 1\%$ would reduce the threshold to $\bar{\mu}_\beta =$-5.25, and it would vanish if the expected growth would be reduced to $\mu_g =$0.16%. It should also be noticed that the threshold is independent of the degree of uncertainty affecting the beta, which is measured by σ_β.

The following corollary shows that the comparative statics of an increase in uncertainty about the beta of the project is symmetric to the one of an increase in maturity. It is a trivial consequence of equation (16).

Corollary 3: *Under the assumptions of Theorem 1, the CEB is increasing with the degree of uncertainty affecting* $\tilde{\beta}$ *for the relevant maturities* $t \in [0,T[$ *if the mean beta is large enough:*

$$\frac{\partial}{\partial \sigma_\beta^2}\hat{\beta}_t(\tilde{\beta}) \geq (<) 0 \quad \forall t \in [0,T[\quad \Leftrightarrow \quad \mu_\beta \geq (<) \bar{\mu}_\beta. \tag{19}$$

In other words, the CEB is increasing in the uncertainty affecting the beta of the project if and only if the term structure of the CEB is increasing. In Figure 1, we draw the term structures of the CEB for different levels of uncertainty affecting the beta of the project, using a calibration with $\mu_\beta > \overline{\mu}_\beta$.

The CEB is defined for maturities below an upper limit $T = 1/\sigma_g^2\sigma_\beta^2$. In fact, for maturities approaching this upper limit from below, the CEB and the associated discount rate tend to infinity. Under the plausible assumption $\mu_\beta \geq \overline{\mu}_\beta$, they tend to plus infinity. In that case, T can be interpreted as a critical maturity above which cash flows become completely irrelevant for the cost-benefit analysis. Under the opposite assumption $\mu_\beta < \overline{\mu}_\beta$, the CEB tends to minus infinity. In this alternative case, T defines a critical maturity so that if some positive expected benefit (f_t) are generated by the project above this maturity T, then the project should be implemented at *any* cost. This critical maturity is equal to the inverse of the product of the variances of the consumption growth and of the beta. If we retain the calibration with $\sigma_g = 4\%$ per annum as above, it equals 625 times the precision of $\tilde{\beta}$. For a standard deviation of $\tilde{\beta}$ between 1% and 100%, we obtain a critical maturity between *T=625* years and *T=6 250 000* years. Thus, this critical maturity is well above the typical maturities for assets that are traded on financial markets. However, it is well in the range of some of the environmental projects currently debated in different countries, as those associated to climate change or to the management of nuclear waste for example.

4. The Certainty Equivalent Beta in the non-Gaussian case

In Section 3, we have seen that the normality assumption to describe the uncertainty about $\tilde{\beta}$ led to some extreme consequences, as for example infinite CEBs and discount rates. This is due to the fact that the normality assumption allows for extremely large and extremely low betas, which is quite unrealistic. In this section, we reexamine this problem by allowing any probability distribution for the beta of the project. Because we still assume that the growth of log consumption is normally distributed, we can use equations (11) and (12) to rewrite pricing formula (8) as follows:

$$\begin{aligned} r_t(\tilde{\beta}) &= \delta + \frac{1}{t}\ln\frac{E\exp((\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2)t)}{E\exp(((\tilde{\beta}-\gamma)\mu_g + 0.5(\tilde{\beta}-\gamma)^2\sigma_g^2)t)} \\ &= r_f + \frac{1}{t}\ln\frac{E\exp((\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2)t)}{E\exp((\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2 - \tilde{\beta}\pi_m)t)}. \end{aligned} \tag{20}$$

This can be rewritten as the CCAPM equation (15), but with a CEB that is defined in the following proposition.

Proposition 1: *The discount rate for a project with an uncertain beta satisfies equation (15) with a certainty equivalent beta which is defined as follows:*

$$\hat{\beta}_t(\tilde{\beta}) = -\frac{1}{\pi_m t}\ln\frac{E\exp(x(\tilde{\beta})t)\exp(-\tilde{\beta}\pi_m t)}{E\exp(x(\tilde{\beta})t)}, \tag{21}$$

with

$$x(\beta) = \beta\mu_g + 0.5\beta^2\sigma_g^2. \tag{22}$$

This means that the certainty equivalent risk premium $\hat{\beta}_t\pi_m$ is the LMGF of a distorted distribution of the ambiguous risk premium $\tilde{\beta}\pi_m$. This implies that the certainty equivalent beta is in the support of $\tilde{\beta}$. The probability of $\tilde{\beta} = \beta$ is distorted by multiplying it by a factor proportional to $\exp(x(\beta)t)$, which is nothing else than $E(c_t / c_0)^\beta$, i.e., the expected net benefit of the project at date t conditional to β. Equation (21) thus means that the certainty equivalent risk factor $\exp - \hat{\beta}_t(\tilde{\beta})\pi_m t$ for project $\tilde{\beta}$ is equal to the distorted expectation of the risk factor $\exp - \tilde{\beta}\pi_m t$. The probability associated to each β is distorted by the relative expected value $\exp x(\beta)t$ of the net benefit conditional to β.

Equation (21) describes the two opposite effects of the uncertain beta on the CEB. The first effect is isolated by considering a binary distribution of the beta with β_1 and β_2 such that $x(\beta_1) = x(\beta_2)$. This means that the two plausible betas yield the same expected growth of net benefits. In that special case, the CEB is equal to the LMGF with the non-distorted distribution of $\tilde{\beta}$:

$$\hat{\beta}_t(\tilde{\beta}) = -\frac{1}{\pi_m t}\ln E\exp(-\tilde{\beta}\pi_m t) \tag{23}$$

This CEB, together with the corresponding discount rate, have a term structure that is unambiguously decreasing. It tends to the smallest plausible beta for large maturities. This result à la Weitzman (1998, 2012) is based on the intuition made more precise in Gollier and Weitzman (2010): When the discount rate is constant but uncertain, compounding these rates in the expected NPV magnifies the role of the smallest plausible discount rate at large maturities.

But in general, different betas generate different expected growth rates $x(\beta)$ of the net benefit. This generates a differential-growth effect which takes the form in equation (21) of a distortion in the probability distribution of the discount factor $\exp(-\tilde{\beta}\pi_m t)$. More weight is given at large maturities to betas associated to faster growing components of the expected net benefit. Notice that the expected growth $x(\beta)$ of the net benefit is increasing in β for all $\beta \geq -0.5\mu_g / \sigma_g^2$, which is in general negative and large in absolute value. This implies that, when contemplating longer maturities, more weight is given to larger betas. This tends make the term structure of the CEB increasing. Thus, in general, the compounding effect and the differential-growth effect go into opposite directions to determine the shape of the term structure of the CEB. This is only when most of the plausible betas are smaller than $-0.5\mu_g / \sigma_g^2$ that the two effects go together to make it unambiguously decreasing.

We have seen in the previous section in which we assumed that $\tilde{\beta}$ is normally distributed that the CEB is equal to the expected beta at small maturities, and is increasing if the expected beta is large enough. We now show that these properties are robust to the relaxation of the normality assumption. The first corollary generalizes Corollary 1.

Corollary 4: *The CEB is approximately equal to the mean beta for small maturities:*

$$\lim_{t\to 0}\hat{\beta}_t(\tilde{\beta}) = E\tilde{\beta}. \tag{24}$$

Proof: Using L'Hospital's rule, equation (21) implies that

$$\hat{\beta}_0(\tilde{\beta})\pi_m = \lim_{t\to 0}\hat{\beta}_t(\tilde{\beta})\pi_m = \lim_{t\to 0}\left[\frac{E\tilde{x}\exp(\tilde{x}t)}{E\exp(\tilde{x}t)} - \frac{E\tilde{y}\exp(\tilde{y}t)}{E\exp(\tilde{y}t)}\right] \\ = E\tilde{x} - E\tilde{y}, \tag{25}$$

where $\tilde{x} = x(\tilde{\beta})$ and $\tilde{y} = \tilde{x} - \tilde{\beta}\pi_m$. This directly implies that $\hat{\beta}_0(\tilde{\beta}) = E\tilde{\beta}$. □

We now examine the slope of the term structure of the certainty equivalent beta. When there is no uncertainty about the beta of project, we know that this term structure is flat. When it is uncertain, the full differentiation of equation (21) with respect to maturity t yields

$$\frac{\partial\hat{\beta}_t(\tilde{\beta})}{\partial t}\pi_m = H_{x(\tilde{\beta})}(t) - H_{x(\tilde{\beta})-\tilde{\beta}\pi_m}(t), \tag{26}$$

where function $H_{\tilde{z}}$ is defined as follows:

$$H_{\tilde{z}}(t) = \frac{\partial}{\partial t}\left[\frac{1}{t}m_{\tilde{z}}(t)\right]. \tag{27}$$

In the next corollary, we suppose that the uncertainty of the beta of the project is limited, so that the risks on $x(\tilde{\beta}) = m_g(\tilde{\beta})$ and $x(\tilde{\beta}) - \tilde{\beta}\pi_m$ are small. Now, the standard properties of moment-generating functions imply that

$$\begin{aligned} H_{\tilde{z}}(t) &= \frac{\partial}{\partial t}\left[\frac{1}{t}m_{\tilde{z}-E\tilde{z}}(t)\right] \\ &= \frac{\partial}{\partial t}\left[\frac{1}{t}\ln\left(1+0.5t^2E\left(\tilde{z}-E\tilde{z}\right)^2+...+\frac{1}{k!}t^kE\left(\tilde{z}-E\tilde{z}\right)^k+...\right)\right] \\ &\simeq \frac{\partial}{\partial t}\left[0.5tVar(\tilde{z})\right] = 0.5Var(\tilde{z}). \end{aligned} \tag{28}$$

This approximation is more precise for small maturities. It implies that

$$\frac{\partial\hat{\beta}_t(\tilde{\beta})}{\partial t}\pi_m \simeq 0.5\left(Var\left(x(\tilde{\beta})\right) - Var\left(x(\tilde{\beta}) - \tilde{\beta}\pi_m\right)\right). \tag{29}$$

This simplifies to

$$\begin{aligned}\frac{\partial \hat{\beta}_t(\tilde{\beta})}{\partial t} &\simeq Cov(\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2, \tilde{\beta}) - 0.5\gamma\sigma_g^2 Var(\tilde{\beta}) \\ &= \left(\mu_g - 0.5\gamma\sigma_g^2\right)Var(\tilde{\beta}) + 0.5\sigma_g^2 Cov(\tilde{\beta}^2, \tilde{\beta})\end{aligned} \tag{30}$$

If $\tilde{\beta}$ is symmetrically distributed around its mean, we also know that

$$Cov(\tilde{\beta}^2, \tilde{\beta}) = 2E\tilde{\beta}\, Var(\tilde{\beta}). \tag{31}$$

Combining these results with equation (30) yields the following corollary.

Corollary 5: *Suppose that the uncertainty about the beta of the project is small and symmetrically distributed around its mean. Then, the CEB satisfies the following property:*

$$\frac{\partial \hat{\beta}_t(\tilde{\beta})}{\partial t} \simeq Var(\tilde{\beta})\sigma_g^2 \left[E\tilde{\beta} - \left(0.5\gamma - \frac{\mu_g}{\sigma_g^2} \right) \right]. \tag{32}$$

It implies that the term structure is increasing if $E\tilde{\beta} \geq 0.5\gamma - (\mu_g / \sigma_g^2)$.

Corollary 5 is thus symmetric to Corollary 2, where the normality assumption on $\tilde{\beta}$ has been replaced by the assumption that the risk on $\tilde{\beta}$ is small and symmetric. When the distribution of the beta is asymmetric, equation (32) must be generalized as follows:

$$\frac{\partial \hat{\beta}_t(\tilde{\beta})}{\partial t} \simeq \sigma_g^2 Var(\tilde{\beta}) \left[E\tilde{\beta} - \left(0.5\gamma - \frac{\mu_g}{\sigma_g^2} \right) \right] + 0.5\sigma_g^2 E\left[\tilde{\beta} - E\tilde{\beta} \right]^3. \tag{33}$$

A negative skewness in the distribution of the beta tends to reduce the slope of term structure of the CEB.

Let us now examine the CEB for large maturities in the non-Gaussian case. We hereafter assume that the support of $\tilde{\beta}$ bounded, so that the CEB $\hat{\beta}_t(\tilde{\beta})$ defined by equation (21) exists for all maturities. Let this support be $[\beta_{\min}, \beta_{\max}]$. Using L'Hospital's rule again, equation (21) implies that

$$\hat{\beta}_{\infty}(\tilde{\beta})\pi_m = \lim_{t\to\infty} \hat{\beta}_t(\tilde{\beta})\pi_m = \lim_{t\to\infty}\left[\frac{E\tilde{x}\exp(\tilde{x}t)}{E\exp(\tilde{x}t)} - \frac{E\tilde{y}\exp(\tilde{y}t)}{E\exp(\tilde{y}t)}\right] \tag{34}$$
$$= x_{\max} - y_{\max},$$

with $\tilde{x} = x(\tilde{\beta})$, $\tilde{y} = \tilde{x} - \tilde{\beta}\pi_m$, and $x_{\max}$ and $y_{\max}$ are the upper bounds of the supports of respectively $\tilde{x}$ and $\tilde{y}$. Because $x(\beta) = \beta\mu_g + 0.5\beta^2\sigma_g^2$, it is easy to check that

$$x_{\max} = \begin{cases} \beta_{\min}\mu_g + 0.5\beta_{\min}^2\sigma_g^2 & if \quad 0.5(\beta_{\min} + \beta_{\max}) \le -\mu_g / \sigma_g^2 \\ \beta_{\max}\mu_g + 0.5\beta_{\max}^2\sigma_g^2 & if \quad 0.5(\beta_{\min} + \beta_{\max}) > -\mu_g / \sigma_g^2. \end{cases} \tag{35}$$

Similarly, we obtain that

$$y_{\max} = \begin{cases} \beta_{\min}\mu_g + 0.5\beta_{\min}^2\sigma_g^2 - \beta_{\min}\pi_m & if \quad 0.5(\beta_{\min} + \beta_{\max}) \le \gamma - (\mu_g / \sigma_g^2) \\ \beta_{\max}\mu_g + 0.5\beta_{\max}^2\sigma_g^2 - \beta_{\max}\pi_m & if \quad 0.5(\beta_{\min} + \beta_{\max}) > \gamma - (\mu_g / \sigma_g^2). \end{cases} \tag{36}$$

We can thus conclude that

$$\hat{\beta}_t(\tilde{\beta}) = \begin{cases} \beta_{\min} & if \quad \beta^* \le -\mu_g / \sigma_g^2 \\ \beta_{\min} + \left(\beta_{\max} - \beta_{\min}\right)\left(\dfrac{\mu_g + \beta^*\sigma_g^2}{\pi_m}\right) & if \quad -\mu_g / \sigma_g^2 < \beta^* \le \gamma - (\mu_g / \sigma_g^2) \\ \beta_{\max} & if \quad \beta^* > \gamma - (\mu_g / \sigma_g^2), \end{cases} \tag{37}$$

where $\beta^* = 0.5(\beta_{\min} + \beta_{\max})$ is the center of the support of $\tilde{\beta}$. If β^* is larger than $\gamma - (\mu_g / \sigma_g^2)$, the asymptotic certainty equivalent beta is the upper bound of the support of $\tilde{\beta}$. On the contrary, if β^* is smaller than $-(\mu_g / \sigma_g^2)$, the asymptotic certainty equivalent beta is the lower bound of the support of $\tilde{\beta}$. In between, the CEB converges toward a linear interpolation of the two bounds of the support of the plausible betas. If we consider the same calibration than in the previous section with $\mu_g = 2\%$, $\sigma_g = 4\%$ and $\gamma = 2$, the critical values for the center of the support of $\tilde{\beta}$ are -12.5 and -10.5. In the Gaussian case, this interval collapses to the singleton -11.5.

An interesting feature of the asymptotic properties of the CEB is that it is the position of the center of the support of $\tilde{\beta}$ rather than its mean (as in the Gaussian case) that determines the CEB for long maturities. If the distribution of beta is negatively skewed, it may be the case that

$\beta^* < -\mu_g / \sigma_g^2$ in spite that the mean beta is large. The CEB would tend to $\beta_{\min}$ in that case. This is an important difference with respect to the Gaussian case. We illustrate this point in Figure 2. Suppose that $\mu_g = 0.5\%$, $\sigma_g = 4\%$, and $\gamma = 2$. Suppose first that $\tilde{\beta}$ is normally distributed with mean $\mu_\beta = 0.5$ and standard deviation $\sigma_\beta = 2$. From Theorem 1, the CEB equals 0.5 for short maturities, is increasing, and tends to infinity for maturities tending to $\mathrm{T} \simeq 156$ years. Let us alternatively assume that the normal distribution of $\tilde{\beta}$ is truncated to interval $[\beta_{\min}, \beta_{\max}]$, with $\beta_{\max} = 3$. Figure 1 depicts the term structure of the CEB for $\beta_{\min} = -20, -10, -9, ..., -6.$ In spite of the fact that the truncations only affect the long tails of the distribution of the beta, they have radical effects on the CEB for very long maturities. First, they make the CEB bounded at all maturities. Second, for very asymmetric truncations, the term structure of the DEB is decreasing at long maturities. In spite of the fact that the beta of the project is very unlikely to be negative and large in absolute value, the mere plausibility of this hypothesis drives the choice of the discount rate for long maturities.

5. *Measuring the uncertainty affecting beta*

In this section, we show how our methodology can be used in different contexts.

5.1. The beta of environmental assets

Guesnerie (2004), Hoel and Sterner (2007), Sterner and Persson (2008), Gollier (2010) and Traeger (2011) have shown that the evolution of relative prices and substituability are crucial in the evaluation of environmental policies. Environmental assets that cannot be substituted by other goods in the economy and whose supply is constant over time have a social value which will be highly sensitive to economic growth. Their beta will thus be relatively large. Our objective in this subsection is to clarify the link between the beta of environmental assets and their degree of substitutability, and to illustrate this relation with an example.

Consider an economy with 2 goods, a numeraire good *c*, and an environmental asset *x*. The investment project under scrutiny is aimed at increasing the quantity of x. Following the authors mentioned above, the utility function of the representative consumer is assumed to belong to the CES family, with

$$U(x,c)=\frac{1}{1-\gamma}y^{1-\gamma},\ \text{ with } y=\left[\alpha x^{1-\beta}+(1-\alpha)c^{1-\beta}\right]^{\frac{1}{1-\beta}} \tag{38}$$

where y is an aggregate good, γ is the aversion to risk on this aggregate good, and $\alpha \neq 1$ and $\beta \in \mathbb{R}^{+}$ are two scalars.[4] Parameter β is the inverse of the elasticity of substitution. The marginal benefit of increasing the consumption of good x expressed in the numeraire is equal to

$$V=-\left.\frac{dc}{dx}\right|_{U}=\frac{\alpha}{1-\alpha}\left(\frac{c}{x}\right)^{\beta} \tag{39}$$

If we assume that the endowment in good x in the economy is constant, equation (39) for the sensitivity of the cash flow to aggregate consumption is equivalent to equation (2), where the beta of the project is equal to the inverse of the elasticity of substitution between good x and the numeraire.

The simplest method to estimate the beta in this context is to observe that the relative value V of x satisfies the following relationship:

$$g_V=\beta(g_c-g_x). \tag{40}$$

In other words, the beta of the project under scrutiny is equal to the ratio of the growth rate of the relative price of good x to the difference between the growth rates of c and x. Inspired by Hoel and Sterner (2007), one can illustrate this method by applying to residential land. Suppose that the supply of residential land is fixed ($g_x=0$). Davis and Heathcote (2007) provide data on the real

price of residential land in the United States over the period 1975Q1-2012Q1. Using the yearly version of their data, one can estimate the following linear regression, which is equivalent to equation (1):

$$g_V=a+\beta g_c+\varepsilon. \tag{41}$$

[4] When $\beta=1$, we get a Cobb-Douglas function with $y=c^{\alpha}x^{1-\alpha}$

The OLS estimator of b equals $\mu_\beta = 2.84$, with a large standard error $\sigma_\beta = 1.27$. This suggests a small elasticity of substitution of residential land and other goods in the economy. Observe also that the standard deviation of the beta is large. Under the normality assumption, there is a 1% probability that the true beta be in fact negative. Suppose also that $\mu_g = 2\%$, $\sigma_g = 4\%$ and $\gamma = 2.$ Because $\mu_\beta = 2.87 > -11.5 = 0.5\gamma - (\mu_g / \sigma_g^2),$ Corollary 2 tells us that the term structure of the CEB is increasing. Moreover, under the assumption that $\tilde{\beta} \sim N(\mu_\beta, \sigma_\beta^2)$, the CEB tends to plus infinity for finite maturities ($T = 387$ years). The CEB equals 8 (18) for a maturities around 100 (200) years.

5.2. The socioeconomic and financial betas in various economic sectors of the economy

In this subsection, we examine investment projects that are aimed to contribute to the development of a specific sector of the economy. This could for example take the form of an expansion of the electricity sector by using the current technology mix. If we assume that the economies of scale are approximately constant, one can use macroeconomic data measuring the creation of social value of the electricity sector to estimate the social benefit of such an investment. The French INSEE provides yearly data about the real value added produced by different sectors of the French economy.[5] The value added of a sector is defined as the value of production minus intermediate consumption. Table 1 summarizes the OLS estimation of equation (41) for of subset of the sectors listed in this data set for period 1975-2011, where g_V is the yearly growth rate of real value added of the sector under scrutiny.

The standard error of the estimator of the beta lies between a low σ_β=0.15 for the education sector and a relatively large σ_β =0.81 for the agricultural sector. If we suppose as before that $\mu_g = 2\%$, $\sigma_g = 4\%$ and $\gamma = 2,$ we obtain that the OLS estimator μ_β is always larger than the threshold $0.5\gamma - (\mu_g / \sigma_g^2) = -11.5$ defined in Corollary 2, so that the term structure of the CEB to

[5] See data set « 6.202 Valeur ajoutée brute par branche en volume aux prix de l'année précédente chaînés » on the INSEE website http://www.insee.fr/fr/themes/comptes-nationaux/tableau.asp?sous_theme=5.2.2&xml=t_6202d. This approach is inspired from Pierre Fery’s appendix of Gollier (2011), which is a report to the French government on the economic evaluation of public policies under uncertainty.

be used to evaluate such investment projects is increasing for all the sectors listed in Table 1. This table also provides the sectoral CEB for the 0, 50, 100 and 200 maturities.

The advantage of the value added approach is that it takes into account of the entire social value creation, with the exception of non-internalized externalities. Thus, the estimations described in Table 1 are about “socioeconomic” betas. One could alternatively examine the “financial” betas, in which one takes account of only the fraction of the value added accruing to investors. In Table 2, we report these financial betas for the two-digit Fama-French industry (FF48) of the U.S. economy.

TBC

These examples are illustrative of the difficulty to quantify CCAPM in a very precise manner. The problem is usually made more complex than described above because most investment projects have a risk profile that does not correspond to the risk profile of the economic sector in which these investments will be implemented. To illustrate, it would make little sense to use the beta of the electricity sector in France (which is heavily biased in favor of the nuclear technology) to evaluate an investment project in photovoltaic solar panels. In the same vein, this sectoral beta would not be useful to evaluate the project to build a high-voltage connection between Italy and France to make the two national electricity networks more resilient to asymmetric demand shocks. The evaluation of such an investment project would require estimating the elasticity of the demand for insurance against electricity outages to changes in GDP. The standard errors associated to such estimations are likely to be larger than those described in Tables 1 and 2 of this subsection.

6. *Alternative interpretations of the model and discussion*

The assumption of our model is that there exists a linear relationship between the social return of the investment project and the growth rate of the economy, as expressed in equation (1). But the β of this linear relationship is initially unknown to the evaluator. There exist two other possible interpretations to this model which are alternative to the uncertainty affecting the project’s beta.

The first reinterpretation is based on the following rewriting of equation (3), using the normalization $c_0 = 1$:

$$V_t \left| c_t = f_t \xi_t \int \exp\left(\beta \ln c_t\right) q(\beta) d\beta. \right. \tag{42}$$

The integral in the right-hand-side of this equality can be interpreted as the Laplace transform of function q evaluated at $\ln c_t$. Thus, our results can be used to evaluate any investment project whose cash flows are related to log consumption through a Laplace transform of a distribution function. The CCAPM is limited to the evaluation of cash flows that are linked to log consumption through an *exponential* function, as is implicitly stated in equation (2).

Let us alternatively rewrite equation (3) using a discrete distribution $(\beta_1, q_{1t};...;\beta_n, q_{nt})$ for $\tilde{\beta}$:

$$V_t \left| c_t = f_t \sum\nolimits_{\theta=1}^{n} q_{\theta t} \xi_{\theta t} \left(\frac{c_t}{c_0} \right)^{\beta_\theta} . \right. \tag{43}$$

It is easy to check that allowing the noise ξ_θ to be specific to the β_θ does not affect the pricing formula (8) of the model. Now, observe that V_t can be reinterpreted as the cash flow of a (time-varying) portfolio of n different assets indexed by $\theta = 1,...,n$. Asset θ has a sure constant beta equaling β_θ. Thus, our results are useful to evaluate conglomerates composed of different investments, each with each own beta. Krueger, Landier and Thesmar (2012) have examined the investment strategy of such conglomerates in the US over the last three decades.

Weitzman (2012) discusses the discount rate to evaluate a portfolio that contains two assets, the first being safe ($\beta_1 = 0$), and the other being the aggregate portfolio ($\beta_2 = 1$). Weitzman considers a portfolio of these two assets whose structure (q_{1t}, q_{2t}) evolves through time in a deterministic way. More precisely, Weitzman assumes that the share $q_{\theta t}$ of each component varies with maturity in opposition to its expected value $E\left(c_t / c_0\right)^{\beta_\theta}$:

$$q_{\theta t} = k_t \frac{\alpha_\theta}{E\left(c_t / c_0\right)^{\beta_\theta}} \tag{44}$$

where $\Sigma_\theta \alpha_\theta = 1$ and k_t is a normalizing constant, so that $\Sigma_\theta q_{\theta t} = 1$. This means that the expected share α_θ of the value of each component θ of the portfolio in the total expected value of the portfolio remains constant across maturities. Using this specific time-varying distribution Q_t for $\tilde{\beta}_t$ in equation (8) allows us to rewrite it as follows:

$$r_t(\tilde{\beta}_t) = \delta - \frac{1}{t} \ln \sum_{\theta=1}^{n} \alpha_\theta \frac{\exp(m_g(\beta_\theta - \gamma)t)}{\exp(m_g(\beta_\theta)t)}. \tag{45}$$

Assuming that g is normally distributed, this simplify to the standard CAPM formula (15) with a CEB equaling

$$\hat{\beta}_t(\tilde{\beta}_t) = -\frac{1}{\pi_m t} \ln \sum_{\theta=1}^{n} \alpha_\theta \exp(-\beta_\theta \pi_m t). \tag{46}$$

Equation (46) implies that the certainty-equivalent beta $\hat{\beta}_t(\tilde{\beta}_t)$ of this time-varying portfolio is decreasing with *t*, as shown by Weitzman (2012). The driving force behind this unambiguous property of the term structure of the CEB is the fact that the composition of the portfolio is continuously rebalanced towards the components with the smallest betas. To illustrate, suppose that there are two possible betas, $\beta_1 = 0$ and $\beta_2 = 1$ as in Weitzman (2012), and that $\alpha_1 = \alpha_2 = 0.5$. Suppose also that $\mu_g = 2\%$, $\sigma_g = 4\%$ and $\gamma = 2$. Under this calibration, the portfolio examined by Weitzman initially contains q_{10}=50% of the risk free component. According to equation (44), this share q_{1t} goes down to 88.9%, 98.5% and 99.8% respectively after t=100, 200 and 300 years. This time-varying portfolio should be discounted by using a beta that reflects this dynamic reallocation of the portfolio in favor of the $\beta_1 = 0$ component, as shown in Figure 3. This figure also shows the CEB that should be used when the composition of the portfolio is fixed, i.e., when $q_{1t} = q_{2t} = 0.5$ for all *t*. This term structure is increasing, as predicted by Corollary 5, since $E\tilde{\beta} = 0.5 > -11.5 = 0.5\gamma - (\mu_g / \sigma_g^2)$.

Our results are thus compatible with those of Weitzman (2012). The two models differ about the nature of the investment projects that must be evaluated. Whether real projects satisfy the constant-expected-value structure as in Weitzman (2012), or the constant-component structure as in this paper will have to be tested by the evaluators of these projects.

7. *Conclusion*

The starting point of this research is that CCAPM betas are often difficult to estimate. This is likely to be the main reason why the standard toolbox for public investment and policy evaluation does not say much about how risk should be integrated. In fact, believe it or not, four decades after the discovery of the normatively-appealing CAPM, evaluators at U.S. Environmental Protection Agency or at the World Bank, to give two prominent examples, are still requested to use a single discount rate independent of the risk profile of the policy under scrutiny. This implies that we collectively invest too much in projects that raise the macroeconomic risk, and too little in projects that insure us against it. In this paper, we have taken seriously the origin of the problem by explaining how one should take into account of the potential errors in the estimation of the betas in cost-benefit analysis.

To each probability distribution describing the uncertainty associated to a project, we have defined and characterized a "certainty equivalent beta" that should be used to determine the rate at which this project should be discounted. When the mean beta of a project is positive, the potential error associated to it introduces uncertainties about future benefits that must be compounded over time. This means that the increment of systematic risk that this project generates is magnified at long horizons. This phenomenon explains the main result of the paper, which is that the certainty equivalent beta for such a project should have an increasing term structure. This penalizes long-dated projects with highly uncertain betas.

This research opens new paths for exploration. On the empirical dimension, it would be interesting to test the hypothesis that long-dated traded assets with a more uncertain beta have a smaller market value. On the theoretical dimension, we have often assumed in this paper that the growth rate of consumption follows an arithmetic Brownian motion. This implies that the risk free rate and the systematic risk premium have a flat term structure. It also implies that our results are subjects to the standard critiques of the risk free rate puzzle and of the equity premium puzzle. If we allow for parametric uncertainty about the stochastic process of economic growth, the risk free rate and the systematic risk premium will have respectively a decreasing and an increasing term structure, as shown by Gollier (2012b). It would be interesting to explore a model in which the parametric uncertainties about economic growth and about the project's beta are combined.

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Appendix 1: Proof of Lemma 1

We have that

$$E\exp(az+bz^2)=\frac{1}{\sigma_z\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(az+bz^2-\frac{(z-\mu_z)^2}{2\sigma_z^2}\right)dz. \tag{47}$$

After rearranging terms in the integrant, this is equivalent to

$$E\exp(az+bz^2)=\frac{\exp\left(-\frac{\mu_z^2}{2\sigma_z^2}-y\right)}{\sigma_z/\hat{\sigma}}\left[\frac{1}{\hat{\sigma}\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(-\frac{\left(z-\hat{\mu}\right)^2}{2\hat{\sigma}^2}\right)dz\right], \tag{48}$$

with

$$y=\frac{\left(a+(\mu_z/\sigma_z^2)\right)^2}{4b-(2/\sigma_z^2)},$$

$$\hat{\mu}=-\frac{a+(\mu_z/\sigma_z^2)}{2b-(1/\sigma_z^2)},$$

and

$$-\frac{1}{2\hat{\sigma}^2}=b-\frac{1}{2\sigma_z^2}.$$

Notice that $\hat{\sigma}$ exists only if we assume that $b<1/(2\sigma_z^2)$. Notice also that the bracketed term in equation (48) is the integral of the density function of the normal distribution with mean $\hat{\mu}$ and variance σ_z^2. This must be equal to unity. This equation can thus be rewritten as

$$\begin{aligned}E\exp(az+bz^2)&=\frac{\hat{\sigma}}{\sigma_z}\exp\left(-\frac{\mu_z^2}{2\sigma_z^2}-\frac{\left(a+(\mu_z/\sigma_z^2)\right)^2}{4b-(2/\sigma_z^2)}\right)\\&=\left(1-2b\sigma_z^2\right)^{-1/2}\exp\left(\frac{a\mu_z+0.5a^2\sigma_z^2+b\mu_z^2}{1-2b\sigma_z^2}\right).\end{aligned} \tag{49}$$

This concludes the proof of Lemma 1. □

Appendix 2: Proof of Theorem 1

Combining equations (8), (11) and (12) implies that

$$\begin{aligned} r_t(\tilde{\beta}) &= \delta + \frac{1}{t}\ln\frac{E\exp((\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2)t)}{E\exp(((\tilde{\beta}-\gamma)\mu_g + 0.5(\tilde{\beta}-\gamma)^2\sigma_g^2)t)} \\ &= \delta + \frac{1}{t}\ln\frac{E\exp((\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2)t)}{\exp((-\gamma\mu_g + 0.5\gamma^2\sigma_g^2)t)E\exp((\tilde{\beta}(\mu_g - \gamma\sigma_g^2) + 0.5\tilde{\beta}^2\sigma_g^2)t)} \\ &= r_f + \frac{1}{t}\ln\frac{E\exp((\tilde{\beta}\mu_g + 0.5\tilde{\beta}^2\sigma_g^2)t)}{E\exp((\tilde{\beta}(\mu_g - \pi_m) + 0.5\tilde{\beta}^2\sigma_g^2)t)}. \end{aligned} \tag{50}$$

If we assume that $0.5\sigma_g^2 t < 1/(2\sigma_\beta^2)$, i.e., $t < T$, we know from Lemma 1 that both expectations in this ratio are finite. Applying this lemma allows us to rewrite the above equality as follows:

$$\begin{aligned} r_t(\tilde{\beta}) &= r_f + \frac{1}{t}\ln\frac{\exp\left(\dfrac{\mu_g\mu_\beta t + 0.5\mu_g^2\sigma_\beta^2 t^2 + 0.5\sigma_g^2\mu_\beta^2 t}{1-\sigma_g^2\sigma_\beta^2 t}\right)}{\exp\left(\dfrac{(\mu_g - \pi_m)\mu_\beta t + 0.5(\mu_g - \pi_m)^2\sigma_\beta^2 t^2 + 0.5\sigma_g^2\mu_\beta^2 t}{1-\sigma_g^2\sigma_\beta^2 t}\right)} \\ &= r_f + \left[\frac{\mu_g\mu_\beta + 0.5\mu_g^2\sigma_\beta^2 t + 0.5\sigma_g^2\mu_\beta^2}{1-\sigma_g^2\sigma_\beta^2 t} - \frac{(\mu_g - \pi_m)\mu_\beta + 0.5(\mu_g - \pi_m)^2\sigma_\beta^2 t + 0.5\sigma_g^2\mu_\beta^2}{1-\sigma_g^2\sigma_\beta^2 t}\right] \\ &= r_f + \pi_m\left[\frac{\mu_\beta + \mu_g\sigma_\beta^2 t - 0.5\pi_m\sigma_\beta^2 t}{1-\sigma_g^2\sigma_\beta^2 t}\right]. \end{aligned} \tag{51}$$

This concludes the proof of Theorem 1. □

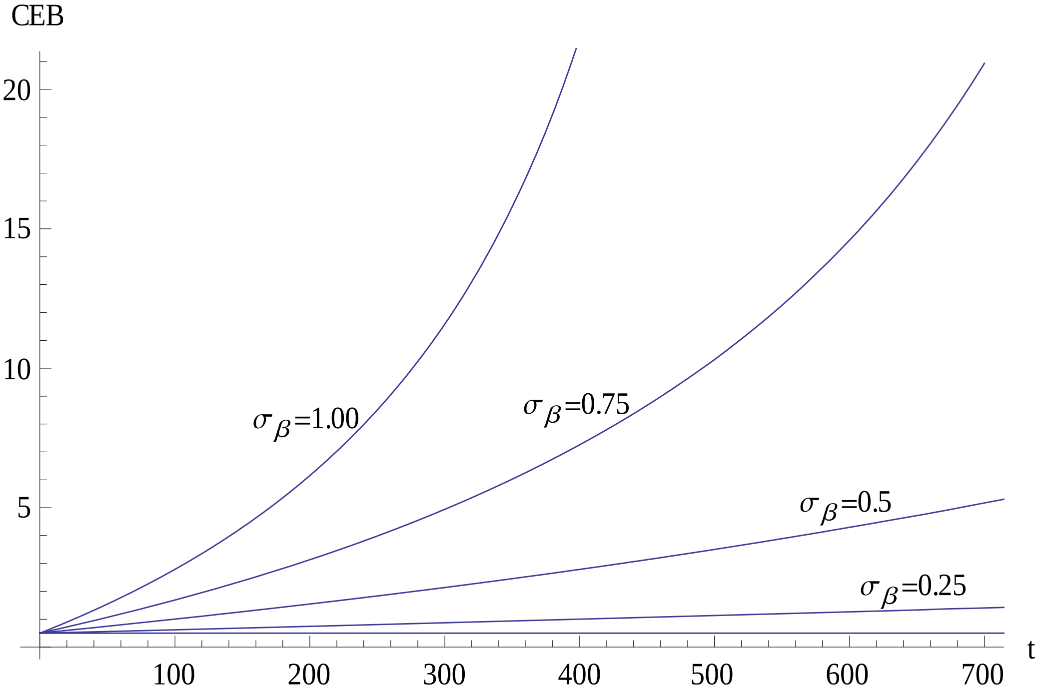

Figure 1: Term structure of the certainty equivalent beta, under the following specification: $g \sim N(2\%,(4\%)^2)$, $\tilde{\beta} \sim N(0.5,\sigma_\beta^2)$, and $\gamma = 2$.

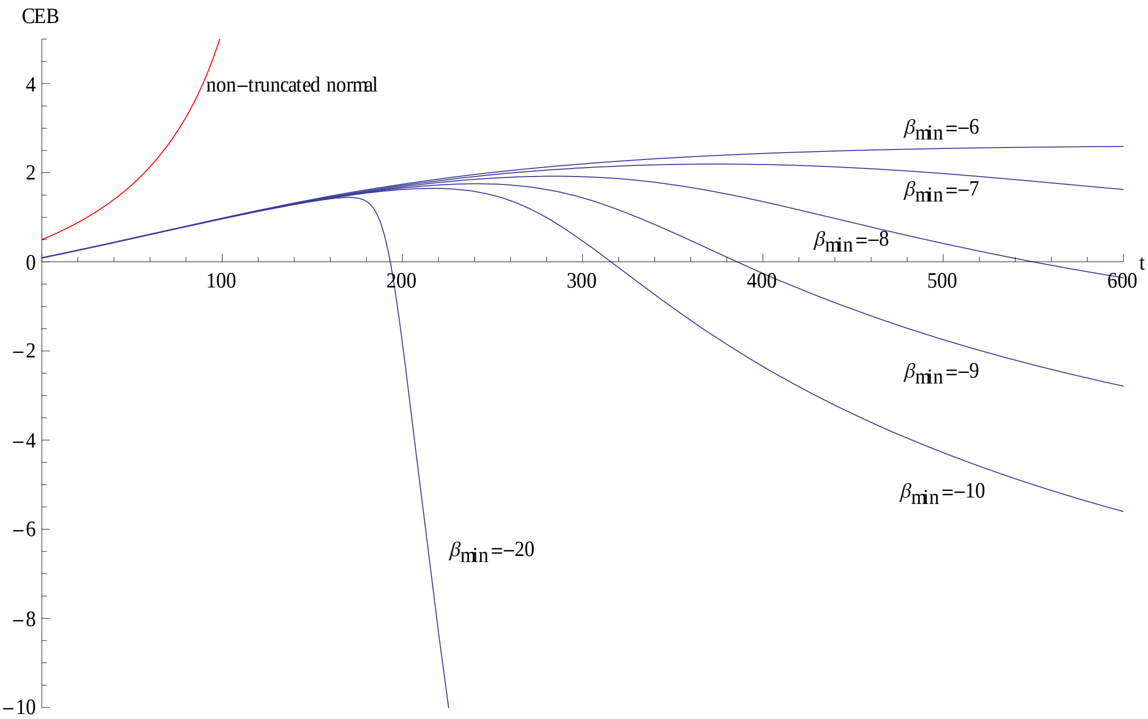

Figure 2: Term structure of the CEB with $\mu_g = 0.5\%$, $\sigma_g = 4\%$, and $\gamma = 2$. The red curve corresponds to $\tilde{\beta}$ being normally distributed with mean $\mu_\beta = 0.5$ and standard deviation $\sigma_\beta = 2$. The other curves correspond to the truncated version of this normal distribution with $\beta_{\max} = 3$ and various $\beta_{\min}$.

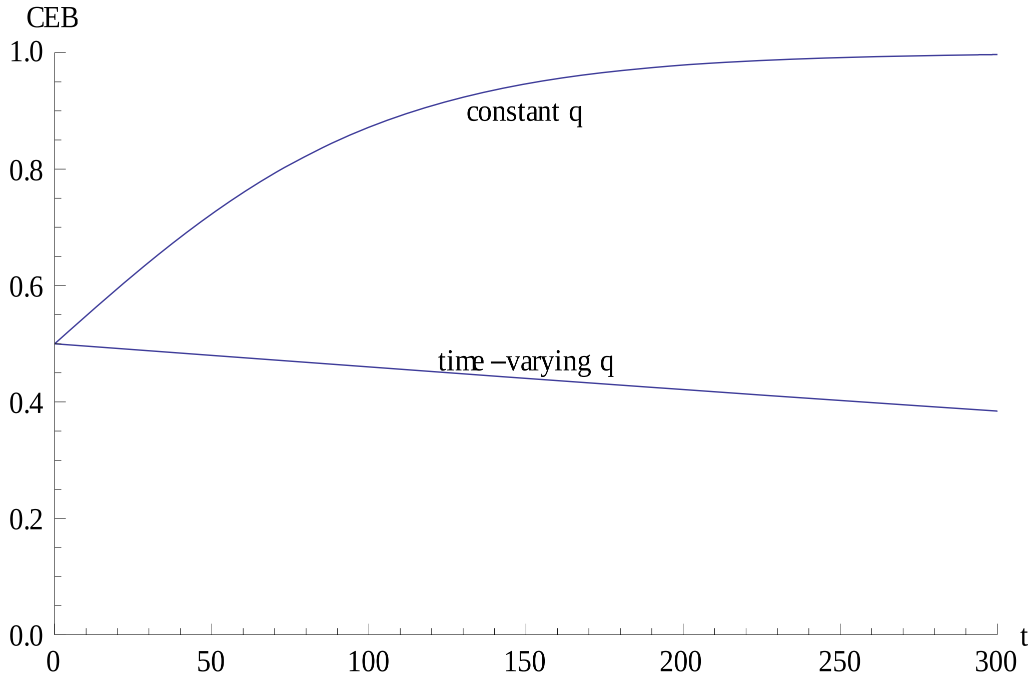

Figure 3: Certainty equivalent betas for two portfolios with a risk free asset $\beta_1 = 0$ and a risky asset $\beta_2 = 1$. The first portfolio has a constant composition with $q_{1t} = q_{2t} = 0.5$. The second portfolio has a time-varying composition as described by equation (44) with $q_{1t} = 0.5/(0.5+0.5(Ec_t/c_0)^{-1})$, as in Weitzman (2012). We assume that $\mu_g = 2\%$, $\sigma_g = 4\%$ and $\gamma = 2$.

Sector	σ_β	Certainty equivalent beta for different maturities			
		$\mu_\beta = \hat{\beta}_0$	$\hat{\beta}_{50}$	$\hat{\beta}_{100}$	$\hat{\beta}_{200}$
Agriculture	*0.81*	0.67	1.34	2.10	3.90
Electricity	*0.49*	1.93	2.19	2.47	3.05
Water management	*0.31*	0.41	0.50	0.60	0.79
Electronic equipment	*0.56*	1.93	2.28	2.64	3.42
Electrical equipment	*0.51*	2.81	3.11	3.43	4.11
Textiles	*0.39*	1.72	1.88	2.05	2.40
Paper and printing	*0.27*	0.89	0.96	1.04	1.19
Chemicals	*0.61*	0.93	1.31	1.72	2.61
Pharmaceutics	*0.54*	1.35	1.66	1.98	2.67
Steel works	*0.32*	1.25	1.36	1.46	1.68
Construction	*0.30*	1.28	1.37	1.47	1.66
Transportation	*0.23*	1.53	1.58	1.64	1.75
Restaurants, hotels	*0.25*	0.73	0.79	0.85	0.98
Communication	*0.55*	1.47	1.79	2.13	2.86
Finance and insurance	*0.37*	0.10	0.23	0.36	0.63
Real estate	*0.19*	0.64	0.68	0.71	0.78
R&D	*0.42*	0.02	0.18	0.35	0.71
Arts and entertainment	*0.28*	0.40	0.48	0.55	0.71
Education	*0.15*	0.51	0.53	0.55	0.60
Healthcare	*0.21*	0.14	0.18	0.22	0.31

Table 1: Standard error and mean of the OLS estimator of the β in equation (41), where g_V is the yearly growth rate of real added value of the sector. Data set: France, 1975-2011, INSEE 6.202.