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Balanced cycles in an OLG model with a continuum of finitely-lived individuals

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Abstract This paper studies the intertemporal equilibrium of a barter economy populated with a continuum of finitely-lived overlapping generations. Assuming isoelastic preferences and zero endowments at the beginning and the end of the individuals' life-span, it proves the existence of an Hopf bifurcation and provides sufficient conditions on parameters for its occurrence.

Keywords: Overlapping-generations Models, Mixed-type Functional Differential Equations, Endogenous Fluctuations, Hopf Bifurcation.

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1 Introduction

This paper investigates the long-run fluctuations that may emerge in stationary OLG economies. To this end, we follow Demichelis and Polemarchakis [5] who study a continuous-time model with a continuum of finitely-lived individuals[1]. The intertemporal equilibrium is shown to be the solution of a functional differential equation of mixed-type (MFDE). By extending this model to isoelastic preferences, the MFDE is nonlinear. The proof of the existence of long-run fluctuations then relies on Rustichini [10] and Benhabib and Rustichini [2]: it uses the linearized MFDE that characterizes the local motion of the economy and looks for solutions which have Hopf bifurcation values. Assuming zero discounting and a specific endowment distribution, we show that there exist some sets of parameters such that a barter economy exhibits a cycle on the neighborhood of a steady state. An elasticity of intertemporal substitution lower than one and zero endowments at beginning and end of the individuals' life-span are sufficient conditions to obtain this result. This paper consequently shows that an increase in the frequency of trade during the individual' life-span does not eliminate the possibility of endogenous fluctuations in OLG economies[2].

The sketch of the paper is as follows. Section 2 presents an overlapping-generations model with continuous trading and finitely-lived individuals. Sec-

[1] The pioneer OLG model of this kind is due to Cass and Yaari [4].

[2] Note that this issue is usually debated in discrete time environment: see notably Aiyagari [1] and Ghiglino and Tvede [6] in a framework with many generations and [8], [7] and [9] in models with two generations.

tion 3 characterizes the intertemporal equilibrium of a barter economy and analyses the existence and uniqueness properties of the steady state. Section 4 studies the spectral decomposition of the linearized dynamics in the neighborhood of a steady state and gives sufficient conditions for the existence of an Hopf bifurcation.

2 The model

Time is continuous and has a finite starting point; let $t \geq 0$ denote the time index. Individuals live for an interval of time of length 1. They only derive utility from consumption and have isoelastic preferences and no time discount. Let $c(s,t) \geq 0$ denotes the consumption of an individual who born at time s as of time t. Hence, the intertemporal utility of an individual who born at time $s \geq 0$, denoted as $u(s)$, is:

$$u(s) = \int_s^{s+1} \frac{c(s,t)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} dt \tag{1}$$

where $\sigma > 0$ stands for the elasticity of intertemporal substitution. During a lifetime, an age-dependent endowment is received; it is denoted $w(t-s)$ and satisfies:

$$w(t-s) = \begin{cases} w(t) \ \text{ if } t \in [s+\alpha, s+\beta] \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

with $0 \leq \alpha < \beta \leq 1$. Individuals have access to a competitive asset market that yields the interest rate $r(t)$. Let $a(s,t)$ denotes the real wealth of an individual who born at time s as of time t. The instantaneous budget constraint is therefore:

$$\frac{\partial a(s,t)}{\partial t} = r(t)\, a(s,t) + w(t-s) - c(s,t) \tag{3}$$

Individuals are born with no financial assets and cannot die indebted. Therefore, initial and terminal conditions write:

$$a(s,s) = 0 \tag{4}$$

$$a(s,s+1) \geq 0 \tag{5}$$

It is assumed that $a(s,t)$ and $c(s,t)$ are $\mathcal{C}^1\left(\mathbb{R}_+^2\right)$ and that $r(t)$ and $w(t-s)$ are continuous for all $t \in [s, s+1]$. The individual program is to maximize (1) subject to (3), (4) and (5).

Lemma 1 *The optimal consumption profile satisfies:*

$$c(s,t) = \frac{\int_{s+\alpha}^{s+\beta} w(z)\, e^{-\int_s^z r(u)du} dz}{\int_s^{s+1} e^{-(1-\sigma)\int_s^z r(u)du} dz} e^{\sigma \int_s^t r(u)du} \tag{6}$$

Proof: The first order conditions are:

$$\frac{\partial c(s,t)}{\partial t} = \sigma r(t)\, c(s,t) \tag{7}$$

and $a(s,s+1) = 0$. Integrating forward condition (3), and using the second optimal condition yields:

$$\int_s^{s+1} c(s,z)\, e^{-\int_s^z r(u)du} dz = \int_{s+\alpha}^{s+\beta} w(z)\, e^{-\int_s^z r(u)du} dz \tag{8}$$

Replacing (7) in (8) gives $c(s,s)$ and consequently (6). $\square$

The demographic structure is in overlapping generations. Each individual belongs to a cohort whose size is normalized to 1. There is no population growth and, at each point of time, a new cohort enters the economy since the oldest one leaves it. Hence, the aggregate counterpart, $x(t)$, of any individual variable, $x(s,t)$, is obtained by integrating over the birth date, such that:

$$x(t) = \int_{t-1}^{t} x(s,t)\,ds \tag{9}$$

Assume there exists a single non storable good and that the aggregate endowment equals the size of the population; then: $\int_{t-1}^{t} w(t-s)\,ds = 1$. Replacing the endowment distribution rule given by (2) yields: $w(t) = 1/(\beta-\alpha)$. Using (6), the aggregate consumption, denoted $c(t)$, hence satisfies:

$$c(t) = \frac{1}{(\beta-\alpha)} \int_{t-1}^{t} \frac{\int_{s+\alpha}^{s+\beta} e^{-\int_s^z r(u)du} dz}{\int_s^{s+1} e^{-(1-\sigma)\int_s^z r(u)du} dz} e^{\sigma \int_s^t r(u)du} ds \tag{10}$$

There is no money in this economy and assets are constituted by consumption loans. The aggregate wealth is denoted $a(t)$; using (3), (4) and the optimal condition: $a(s,s+1)=0$, its dynamics writes:

$$\frac{da(t)}{dt} = r(t)\,a(t) + 1 - c(t) \tag{11}$$

3 Equilibrium and steady states

Definition 1 An equilibrium with perfect foresight is a function $F(t) = (c(t), a(t), r(t))$, $F : \mathbb{R}_+ \to \mathbb{R}_+^2 \times \mathbb{R}$, $\mathcal{C}^1(\mathbb{R}_+)$ such that (i) individuals maximize their utility subject to the budget constraint, (ii) the aggregate consumption equals the aggregate endowment (i.e. $c(t) = 1$), and (iii) the aggregate asset is zero (i.e. $a(t) = 0$).

Using (3), it can be inferred from definition 1, that an equilibrium reduces to a function $r(t)$.

Definition 2 A steady state equilibrium is a triplet (c, a, r) that satisfies: (i) $c = \phi(r)$ and $a = (c-1)/r$, (ii) $c = 1$ and $a = 0$; where function $\phi : \mathbb{R} \to \mathbb{R}_{++}$ is such that:

$$\phi(r) = \frac{\int_0^1 e^{\sigma r z} dz}{(\beta - \alpha)} \frac{\int_\alpha^\beta e^{-rz} dz}{\int_0^1 e^{-(1-\sigma) rz} dz} \tag{12}$$

Property 1 A steady state interest rate is a r that satisfies $\phi(r) = 1$ if $r \neq 0$ and $\phi'(r) = 0$ if $r = 0$.

Proof: For $r \neq 0$, the property is an immediate implication of definition 2; for $r = 0$, just observe that $\phi(0) = 1$ and use l'Hôpital's Rule. □

Lemma 2 *$r = 0$ is a steady state interest rate if and only if $\alpha + \beta = 1$.*

Proof: Replace $r = 0$ in $\phi'(r)$ and use property 1 to conclude. □

Corollary 1 The individual consumption is age-independant if and only if $\alpha + \beta = 1$.

Lemma 3 *There exist (α, β, σ) such that there is no steady-state equilibrium.*

Proof: It can be easily shown that there is no solution $r \neq 0$ such that $\phi(r) = 1$ if $\alpha = 0$ and $\beta \in (0, 1-\sigma]$ or if $\beta = 1$ and $\alpha \in [\sigma, 1)$. □

Lemma 4 *There exist (α, β, σ) such that there are multiple steady-state equilibria.*

Proof: Suppose $\beta = 1 - \alpha$ and recall with lemma 2 that $r = 0$ is a steady state; it can be easily shown that there exist at least two other solutions $r \neq 0$ such that $\phi(r) = 1$ if $\sigma \in (\alpha; 2\alpha(1-\alpha))$. $\square$

4 Endogenous cycles

The equilibrium satisfies the following non linear dynamics:

$$\frac{1}{(\beta-\alpha)} \int_{t-1}^{t} \frac{\int_{s+\alpha}^{s+\beta} e^{-\int_s^z r(u)du} dz}{\int_s^{s+1} e^{-(1-\sigma)\int_s^z r(u)du} dz} e^{\sigma \int_s^t r(u)du} ds = 1 \tag{13}$$

In this section, it is the local dynamics around steady-state r^* which is studied; it is the one of $x(t)$ defined such that $r(t) = r^* + \epsilon x(t)$.

Property 2 The characteristic function of $x(t)$ is denoted $Q(\lambda)$ and satisfies:

$$Q(\lambda) = -\frac{\int_\alpha^\beta e^{(\lambda - r^*)s} dz}{\int_\alpha^\beta e^{-r^* z} dz} + (1-\sigma)\frac{\int_0^1 e^{(\lambda-(1-\sigma)r^*)z} dz}{\int_0^1 e^{-(1-\sigma)r^* z} dz} + \sigma \frac{\int_0^1 e^{\sigma r^* z} dz}{\int_0^1 e^{(\sigma r^* - \lambda)z} dz} \tag{14}$$

Proof: Replace $r(t) = r^* + \epsilon x(t)$ in (13). Then do a Taylor expansion in the neighborhood of $\epsilon = 0$ and rearrange using $\phi(r) = 1$. Define $X(t) = \int_0^t x(u)\, du$. It yields:

$$X(t) = \frac{\int_{t-1}^{t} e^{\sigma r^*(t-s)} \left(\frac{\int_{s+\alpha}^{s+\beta} e^{-r^*(z-s)} X(z) dz}{(\beta-\alpha)} - \frac{(1-\sigma)\int_s^{s+1} e^{-(1-\sigma)r^*(z-s)} X(z) dz}{\int_0^1 e^{\sigma r^* z} dz} \right) ds}{\sigma \int_0^1 e^{-(1-\sigma) r^* z} dz} \tag{15}$$

Finally, $Q(\lambda)$ is obtained by the following change of variable: $x(t) = e^{\lambda t}$ and rearranging using $\phi(r) = 1$. $\square$

Lemma 5 *The characteristic function $Q(\lambda)$ has an infinity of complex roots with negative real parts and an infinity of complex roots with positive real parts.*

Proof: Roots of $Q(\lambda)$ are asymptotic to those of the following equations[3]:

$$P_1(\lambda) = -\lambda\left(\sigma\lambda^2\int_0^1 e^{\sigma r^* z}dz + \frac{(1-\sigma)\,e^{\lambda-(1-\sigma)r^*}}{\int_0^1 e^{-(1-\sigma)r^* z}dz}\right) \quad (16)$$

$$P_2(\lambda) = -\lambda\left(\sigma\lambda^2\int_0^1 e^{\sigma r^* z}dz + \frac{(1-\sigma)\,e^{-\lambda+\sigma r^*}}{\int_0^1 e^{-(1-\sigma)r^* z}dz}\right) \quad (17)$$

Asymptotically, roots of $P_1(\lambda)$ have a positive real part while those of $P_2(\lambda)$ have a negative real part. □

Corollary 2 The dynamics is generically characterized by oscillations that decrease in magnitude and eventually disappear.

Consider now the case $\beta = 1-\alpha$ and focus on the neighborhood of the steady state $r^* = 0$. The characteristic function (14) rewrites:

$$Q(\lambda) = -\frac{\int_\alpha^{1-\alpha} e^{\lambda z}dz}{(1-2\alpha)} + (1-\sigma)\left(\int_0^1 e^{\lambda z}dz\right) + \sigma\frac{1}{\left(\int_0^1 e^{-\lambda z}dz\right)} \quad (18)$$

Lemma 6 *There exist (α, β, σ) such that there are pure imaginary roots which are Hopf bifurcation values.*

Proof: Let $\lambda = p+iq$. The proof proceeds in two steps: (i) it supposes $\lambda = iq$ and proves that for $\sigma < 2\alpha(1-\alpha)$, there exists a $q > 0$ such that $Q(iq) = 0$; it hence defines $(\alpha_0, \sigma_0(\alpha_0))$ the pair of parameters for which this root does exist; (ii) it uses σ as a bifurcation parameter and shows that there exists a neighborhood of σ_0 such that $d\,\mathrm{Re}(Q(\lambda))/d\sigma$ is not equal to zero.

(i) Replace $\lambda = iq$ in (18) and define $\tilde{Q}(q)$ such that:

$$\tilde{Q}(q) = \frac{\cos(\alpha q) - \cos((1-\alpha)q)}{1-2\alpha} + (1-\sigma)[\cos(q) - 1] - \frac{\sigma q^2}{2} \quad (19)$$

[3] See Bellman and Cooke (1963) p. 410.

A Taylor expansion in the neighborhood of $q = 0$ yields:

$$\tilde{Q}(q) = \frac{q^4}{12}\left(\alpha(1-\alpha) - \frac{\sigma}{2}\right) + O\left(q^6\right) > 0 \tag{20}$$

Moreover, $\lim_{q \to +\infty} \tilde{Q}(q) = -\infty$. Therefore, there exists a $q > 0$ such that $\tilde{Q}(q) = 0$.

(ii) From (18), it yields:

$$\left.\frac{d\,\mathrm{Re}\,(Q(\lambda))}{d\sigma}\right|_{\sigma_0} = \frac{\sin(q)}{q}\left(\frac{q^2}{2(1-\cos(q))} - 1\right) \tag{21}$$

The roots of this latter function are those of $\sin(q)$ and $\cos(q) = 1 - q^2/2$. They are the $\{2\pi + k\pi, k \in \mathbb{Z}\} \cup \{q_1, q_2, q_3, q_4\}$, where q_1, q_2, q_3 and q_4 are the 4 roots of the following polynomial:

$$\left(1 - \frac{q^2}{2}\right)^2 + \left(1 - \frac{(q - \frac{\pi}{2})^2}{2}\right)^2 = 1 \tag{22}$$

Since elements of $\{2\pi + k\pi, k \in \mathbb{Z}\} \cup \{q_1, q_2, q_3, q_4\}$ are independent with respect to a_0, there exists a_0, such that $a_0 \notin \{2\pi + k\pi, k \in \mathbb{Z}\} \cup \{q_1, q_2, q_3, q_4\}$.

□

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