Endogenous Retirement and Monetary Cycles[1]

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Abstract

In a model of overlapping generations with a continuum of finitely-lived individuals, the aggregate price dynamics is characterized by a functional differential equation of mixed-type. Delays and advances are exogenous when the retirement age is mandatory while they become state-dependent when individuals are allowed to choose their age at retirement. Using Hopf bifurcation theorem, periodic solutions in the neighborhood of the monetary steady-state appearing with a mandatory retirement age, vanish with a chosen age.

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1 Introduction

In this article, the relationship between retirement age and macroeconomic fluctuations is analyzed. Our purpose is to show that allowing individuals to choose their retirement age reduces the aggregate prices volatility. The demonstration hinges on two main assumptions. First, the economy produces a single non storable good. Second, the demographic structure is in overlapping generations (OLG) with a continuum of finitely-lived individuals.

We study a simple economy with no capital, which produces a non storable good using a linear technology with respect to aggregate labor. We also restrict to monetary equilibria, for which the real value of the aggregate asset holdings remains positive forever. Under the mandatory retirement age scheme, the framework reduces to an exchange economy. Prices fluctuations have then only nominal and distributional effects. They modify the real consumption at the individual level but not at the aggregate one. Alternatively, when individuals choose their retirement age, the economy is of the 'yeoman farmer' type. Prices fluctuations then influence the real aggregate output, and consequently the aggregate consumption.

Our second assumption is about the age structure of the population. We consider an OLG model that is well-known to be able to generate cycling dynamics. These dynamics are appealing to economists because they can be seen as business cycles and are potentially linked to sunspot equilibria as described in Cass and Shell (1983). However, the initial proofs of the existence of such cycles, which were proposed by Gale (1973), Benhabib and Day (1982) and Grandmont (1985), used an OLG model composed, at each point of time, of only two generations. Reichlin (1986), Jullien (1988) and Benhabib and Laroque (1988), who extended the proofs to production economies, made the same assumption. This is worrying because the cycles these models may generate have periods greater than or equal to the individuals' life-span. Moreover, it has been conjectured by Sims (1986), that increasing the frequency of trade among generations, would allow the individuals to smooth the strong revenue effects which are necessary for the existence of cycles. Hence, the initial existence theorem has recently been extended to OLG models with either many generations (Aiyagari, 1989; Reichlin, 1992; Swanson, 1998; Simonovits, 1999; Bhattacharya and Russell, 2003; d'Albis and Augeraud-Véron, 2007) or, using the equivalence argument developed by Balasko et al. (1980), with many commodities (Kehoe and Levine, 1984; Kehoe et al. 1991; Ghiglino and Tvede, 1995). Among these studies, Ghiglino and Tvede (2004) notably propose a proof of existence for a general model with many generations and many commodities. However, all these extensions do not include labor supply decisions as it is crucially the case in Grandmont (1985). Hence, by introducing a retirement age choice, our purpose is to test the robustness of these recent results.

We use a continuous-time OLG model initially developed by Cass and Yaari (1967), modified to allow for individual retirement decisions as in Boucekkine et al. (2002) and (2004). The inter-temporal equilibrium is shown to be the solution of a non-linear functional differential equation of mixed-type (MFDE). The dynamics is indeed affected by discrete delays and advances. Delays are generated by the vintage structure of the population while advances rely on the expectations of the individuals. Moreover, when retirement is endogenous, some of the delays and advances are state-dependent. We successively characterize the monetary steady-state of our economy and study the existence of cycles in the neighborhood of the steady-state. To prove their existence, we follow Rustichini (1989) by looking for solutions of the linearized MFDE that may have Hopf bifurcation values. We find there exist sets of parameters for which a cycle exists when retirement is exogenous while there is no cycle when retirement is endogenous. This means that the strong revenue effects which still may yield cycles when the frequency of trade is high, are less operative when individuals choose their retirement age.

The sketch of the article is as follows. In section 2, we present an OLG model with continuous trading and characterize, in section 3, the inter-temporal equilibrium of the economy. In section 4, we study the linearized dynamics in the neighborhood of the monetary steady state and conclude in section 5.

2 The model

In this section, the basic framework of the model is presented. We successively describe the individual choices and the aggregation procedure. Time is assumed to be continuous and to have a finite starting point; let $t \geq 0$ denote the time index.

2.1 The individual behavior

Individuals live for an interval of time of length $\omega > 1$. They derive utility from consumption and from the length of their retirement. Moreover, isoelastic preferences and no time discount are assumed. The inter-temporal utility of an individual who born at time $s \geq 0$, denoted as $u(s)$, is given by:

$$u(s) = \int_s^{s+\omega} \frac{c(s,t)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} dt + \frac{(\omega - (z(s))^{\alpha})^{1-\frac{1}{\eta}}}{1-\frac{1}{\eta}}, \tag{1}$$

where $c(s,t) \geq 0$ denotes the real consumption of an individual who born at time s as of time t and $z(s) \in [0, \omega]$ the age of retirement. Moreover, $\sigma > 0$ stands for the elasticity of inter-temporal substitution, $\eta > 0$ and $\alpha \in \{0, 1\}$. Observe that $\alpha = 0$ corresponds to the exogenous retirement case, for which the retirement age is mandatory and normalized to 1. Alternatively, when $\alpha = 1$, the retirement age is endogenous.

During a lifetime, a wage is received provided that the individual is working; the real wage is denoted $e(s,t)$ and satisfies:

$$e(s,t) = \begin{cases} 1 \text{ if } t \in [s, s + (z(s))^{\alpha}], \\ 0 \text{ otherwise.} \end{cases} \tag{2}$$

Individuals have access to a competitive asset market that yields $r(t)$, the interest rate, which also represents the deflation rate. Let $a(s,t)$ denotes the real wealth of an individual who born at time s as of time t. The instantaneous budget constraint is:

$$\frac{\partial a(s,t)}{\partial t} = r(t) a(s,t) + e(s,t) - c(s,t). \tag{3}$$

Individuals are born with no financial assets and cannot die indebted. Initial and terminal conditions thus write $a(s,s) = 0$ and $a(s, s+\omega) \geq 0$. It is assumed that $a(s,t)$ and $c(s,t)$ are piecewise $\mathcal{C}^1(\mathbb{R}_+^2)$, that $r(t)$ is continuous for all $t \in [s, s+\omega]$ and that $e(s,t)$ is $L^2(\mathbb{R}_+^2)$. Finally, it will be convenient to define the relative price between time t and time 0, such that: $R(t) = \exp\left(-\int_0^t r(u)\, du\right)$.

The individual program is to maximize Eq. (1) subject to Eq. (3), and the initial and terminal conditions. The separability of the objective allows to solve the program in two steps. First, a family of optimal consumption profiles parametrized by $(z(s))^{\alpha}$ is derived. Then, the optimal retirement age is computed.

Lemma 1 *The optimal consumption profile satisfies:*

$$c(s,t) = \frac{\int_s^{s+(z(s))^{\alpha}} R(v)\,dv}{\int_s^{s+\omega} (R(v))^{(1-\sigma)}\,dv} (R(t))^{-\sigma}. \tag{4}$$

Proof: It is standard and available upon request. □

Lemma 2 *There exists an optimal age of retirement that belongs to* $(0,\omega)$ *and satisfies:*

$$\alpha R(s+(z(s))^{\alpha}) \left(\frac{\int_s^{s+(z(s))^{\alpha}} R(v)\,dv}{\int_s^{s+\omega} (R(v))^{1-\sigma}\,dv} \right)^{-\frac{1}{\sigma}} - \alpha(\omega - (z(s))^{\alpha})^{-\frac{1}{\eta}} = 0. \tag{5}$$

The optimal age of retirement is unique if:

$$-r(s+(z(s))^{\alpha}) - \frac{1}{\sigma} \frac{R(s+(z(s))^{\alpha})}{\int_s^{s+(z(s))^{\alpha}} R(v)\,dv} - \frac{1}{\eta}(\omega - (z(s))^{\alpha})^{-1} < 0. \tag{6}$$

Proof: Replacing Eq. (4) in Eq. (1) yields $\hat{u}(z(s))$. Observe first that $z(s) = 0$ is not a solution because $\lim_{z(s)\to 0} \hat{u}(z(s)) = -\infty$ if $\sigma \in (0,1]$ and $\lim_{z(s)\to 0} \hat{u}'(z(s)) = +\infty$ if $\sigma > 1$. Moreover, $z(s) = \omega$ is not a solution because $\lim_{z(s)\to\omega} \hat{u}(z(s)) = -\infty$ if $\eta \in (0,1]$ and $\lim_{z(s)\to\omega} \hat{u}'(z(s)) = -\infty$ if $\eta > 1$. Hence, there exists $z(s) \in (0,\omega)$ which is a solution to the problem. This optimal solution satisfies $d\hat{u}(z(s))/dz(s) = 0$ or equivalently Eq. (5). Then, a sufficient condition for a global maximum, is that

$d^2\hat{u}\left(z\left(s\right)\right)/d\left(z\left(s\right)\right)^2 < 0$ at the optimal point. This condition is given by Eq. (6). $\square$

The optimal age of retirement is given by a standard consumption-leisure arbitrage. Eq. (4) shows that a longer retirement period implies a lower level of consumption at each age and Eq. (5) says that the optimum is obtained when the marginal utility yield by a supplementary unit of leisure equals the marginal desutility yield by the decrease in consumption. The optimal age is necessarily an interior solution of the individual program but one should not exclude multiple local maximum. If the condition given in Eq. (6) is not satisfied, meaning that $r\left(t\right)$ is negative, it may indeed exist multiple solutions to Eq. (5). Nevertheless, it will be shown in section 4 that the optimal age of retirement is unique in the neighborhood of the monetary steady-state.

2.2 Aggregation

The demographic structure is in OLG. Each cohort, whose size is normalized to 1, is composed of identical individuals. There is no population growth and, at each point of time, a new cohort enters the economy while the oldest one leaves it. The size of the population is consequently equal to ω.

Assume there exists a single non storable good which is produced using a linear technology with respect to aggregate labour. Total output hence equals the size of the active population, denoted $P\left(t\right)$, which solves:

$$P\left(t\right) = \int_{t-\omega}^{t} e\left(t-s\right) ds, \tag{7}$$

where $e\left(t-s\right)$ is defined in Eq. (2). The aggregate real consumption, denoted $C\left(t\right)$, is obtained by integrating over the birth date individual consumptions, $c(s,t)$. Replacing Eq. (4) yields:

$$C(t) = \int_{t-\omega}^{t} \frac{\int_{s}^{s+(z(s))^{\alpha}} R(v)\,dv}{\int_{s}^{s+\omega} (R(v))^{(1-\sigma)}\,dv} (R(t))^{-\sigma}\,ds. \tag{8}$$

Similarly, the aggregate real wealth is denoted $A(t)$. Money is available in this economy: it is a non perishable and non consumable bond that may constitute the counterpart of individual assets. It is assumed that a given quantity of money was distributed at time $t = 0$ and that there were no further emission since then.

3 The monetary equilibrium

In this section, the price dynamics around the monetary steady-state is characterized. We prove its existence and provide some comparative statics. An inter-temporal equilibrium is defined as follows:

Definition 1 An inter-temporal equilibrium with perfect foresight is a function $F(t) = (C(t), A(t), R(t), P(t), z(t))$, $F : \mathbb{R}_+ \to \mathbb{R}_+^4 \times [0,\omega]$, $F(t) \in \left(\mathcal{C}^1(\mathbb{R}_+)\right)^3 \times \left(L^2(\mathbb{R}_+)\right)^2$ such that (i) individuals maximize their utility subject to the budget constraints, (ii) the aggregate consumption equals the output: $C(t) = P(t)$, and (iii) the aggregate wealth is non negative: $A(t) \geq 0$.

The existence of money allows for a positive aggregate wealth at the equilibrium. Consider now the following particular equilibrium:

Definition 2 A monetary steady-state is an inter-temporal equilibrium with perfect foresight such that the aggregate wealth is a positive constant: $A > 0$.

Lemma 3 *There exists a unique monetary steady-state which is characterized by the quintuple* (C, A, P, R, z) *that satisfies:* $R = 1$, $A = (\omega - z^{\alpha})\, z^{\alpha}/2$,

$P = C = z^{\alpha}$ *where* z^{α} *is such that:*

$$\alpha \left(\frac{z^{\alpha}}{\omega} \right)^{-\frac{1}{\sigma}} = \alpha \left(\omega - z^{\alpha} \right)^{-\frac{1}{\eta}}. \tag{9}$$

<u>Proof</u>: At the equilibrium, the aggregate consumption equals the output, the aggregate wealth writes $dA(t)/dt = r(t) A(t)$. This implies that $r = 0$, or equivalently, $R = 1$, is necessary to obtain a constant and positive aggregate wealth. For $r = 0$, the condition in Eq. (6) is satisfied and, consequently, there exists a unique age of retirement that is constant. The end of the computation is immediate. $\square$

It is well-known since Samuelson (1958) and Gale (1973), that the interest rate equals the demographic growth rate at the monetary steady-state. By having supposed a stationary population, we obtain constant steady-state prices and constant individual consumptions over the life-cycle. Interestingly, Lemma 3 states that money is always valuated. This is a direct consequence of the introduction of retirement: in OLG economies, the money has indeed a value if and only if the average age of the consumer is strictly greater than the average age of the worker. Then, workers finance the retired and the aggregate wealth is positive. Given our simple demographic structure, these ages are easy to compute and are worth $\omega/2$ and $z^{\alpha}/2$ respectively. With the statement of Lemma 2 such that $z^{\alpha} \in (0, \omega)$, we conclude that the monetary steady-state exists. In real terms, the value of money is equal to the product of the output and the difference between the average ages of consumption and production. At the limit such that $z^{\alpha} \to \omega$, the money has no value and the economy is autarkic.

Remark, moreover, the ambiguous effect of retirement age on aggregate wealth. Postponing retirement has a positive effect on wealth if and only if $z^{\alpha} < \omega/2$. Increasing z increases, on the one hand, the aggregate output and the individual savings while it increases, on the other hand, the average age of the worker which reduces the incentive to save. Another intuition is obtained by computing the age at which each individual begins to dissave. Simple algebra would lead to show that this age is always equal to the retirement age. Hence, an increase of the retirement age increases the aggregate wealth if the age at which the dissaving begins belongs to the first half of the life.

When the retirement age is exogenous, the monetary steady-state is simply characterized by the relative size of the active population. With Lemma 3, observe that the aggregate wealth increases with the life-span ω: this indeed increases the length of retirement and consequently creates an extra incentive for individual saving. Remark that the elasticity of inter-temporal substitution has no influence on the steady-state because we consider the particular case of a stationary population; this is no longer true when the retirement age is endogenous. Thus, the following proposition provides some comparative statics for the yeoman farmer economy:

Proposition 1 *When the retirement age is endogenous,*
(i) an increase of longevity increases the age of retirement and the aggregate wealth. Moreover, there exists $\bar{\omega} > 1$ such that $z \geq 1 \Leftrightarrow \omega \geq \bar{\omega}$.
(ii) an increase of the elasticity of inter-temporal substitution reduces the age of retirement and has an ambiguous effect on the aggregate wealth. Moreover, if $\omega < 2$, there exists $\bar{\sigma} > 0$ such that $z \geq 1 \Leftrightarrow \sigma \leq \bar{\sigma}$; if $\omega \geq 2$, $z > 1$.

<u>Proof</u>: Use, for $\alpha = 1$, Eq. (9) as an implicit equation. For part (i) observe

that $dz/d\omega \in (0,1)$ and that $dA/d\omega > 0$ and finally, observe that $\bar{\omega}$ can be computed explicitly by replacing $z = 1$ in the implicit equation. For part (ii) observe that because $\omega - z < 1$ is always true, one has $dz/d\sigma < 0$. Hence, the sign of $dA/d\sigma$ is the opposite of the one of dA/dz if and only if $z > \omega/2$. Finally, with the implicit equation, observe that $z \to \omega$ when $\sigma \to 0$ and, replacing $z = 1$ compute $\bar{\sigma}$ to conclude that $\bar{\sigma}$ is positive only if $\omega < 2$. □

The intuition for Proposition 1 is the following: because of the consumption/leisure arbitrage, an increase in longevity both increases the age of retirement and the length of retirement; hence, $dz/d\omega \in (0,1)$. The magnitude of the latter derivative crucially depends on the parameter η that characterizes the curvature of the utility function with respect to leisure. A lower η means a utility more concave and consequently a higher $dz/d\omega$. At the limit such that $\eta \to 0$, one obtains a retirement age that goes to its lower bound: $\omega - 1$, and consequently $dz/d\omega \to 1$. Equivalently, the effect of longevity on aggregate wealth is, at a first glance, ambiguous since it both increases the average age of the consumer and the average age of the worker. However, since $dz/d\omega \in (0,1)$, the final effect is always positive. Remark that Chang (1991) and Kalemli-Ozcan and Weil (2004) have pointed out the importance of the assumption of certainty on individual life-span. In case of uncertain life-span, an increase in longevity may reduce the age of retirement.

The effect of an increase of the elasticity of inter-temporal substitution on retirement is negative. Indeed, at the monetary steady-state such that the interest rate is equal to zero, the elasticity does not influence the individual consumption growth rate. It then only modifies the arbitrage between consumption and leisure in favor of the second, which ultimately means less human wealth and consequently less consumption. The effect of the elasticity on aggregate wealth is then the opposite of the one, described above, of the retirement age on wealth.

4 Monetary cycles

In this section, the linearized dynamics in the neighborhood of the monetary steady-state given by Lemma 3 is studied. We successively analyze the exogenous and endogenous retirement cases and look for particular long-run fluctuations defined as follows:

Definition 3 A monetary cycle is a periodic solution of the inter-temporal equilibrium in the vicinity of the monetary steady-state.

4.1 Exogenous retirement

Assume that the retirement age is mandatory; replacing (2) in (7) yield the size of the active population: $P(t) = 1$ for all t. The inter-temporal equilibrium is hence characterized by the following functional differential equation:

$$(R(t))^{\sigma} = \int_{t-\omega}^{t} \frac{\int_{s}^{s+1} R(v)\,dv}{\int_{s}^{s+\omega} (R(v))^{(1-\sigma)}\,dv} ds. \tag{10}$$

The dynamics of $R(t)$ is indeed dependent on the entire set of realizations of R on the interval $[t-\omega, t+\omega]$. Past realizations, that yield delays in the dynamics, are generated by the vintage structure of human capital as in Boucekkine, de la Croix and Licandro (2002), while future realizations that yield advances in the dynamics, come from the assumption of perfect foresight. Remark that Eq. (10) is non linear for any $\sigma \neq 1$, meaning that revenue and substitution effects do not compensate each others.

Following Rustichini (1989), the proof of the existence of periodic solutions uses the Hopf bifurcation theorem. We consider the local dynamics around $R = 1$: it is the one of $x(t)$ defined such that $R(t) = 1 + \varepsilon x(t)$.

Property 1 The characteristic function of $x(t)$, denoted $H(\lambda)$, satisfies:

$$H(\lambda) = \int_{-\omega}^{0} \left(\int_{s}^{s+1} e^{\lambda v} dv \right) ds - \omega\sigma - \frac{(1-\sigma)}{\omega} \int_{-\omega}^{0} \left(\int_{s}^{s+\omega} e^{\lambda v} dv \right) ds. \quad (11)$$

Proof: Replace $R(t) = 1 + \varepsilon x(t)$ in Eq. (10) and do a Taylor expansion in the neighborhood of $\varepsilon = 0$. Finally, $H(\lambda)$ is obtained by the following change of variable: $x(t) = e^{\lambda t}$ and some algebra. $\square$

It can be shown that the characteristic function $H(\lambda)$ has an infinity of complex roots with negative real parts and an infinity of complex roots with positive real parts. This implies that the linearized dynamics is initially characterized by oscillations that eventually disappear. These fluctuations are consequently of few interest because it is the dynamics in the neighborhood of the steady-state which is studied. The following lemma hence focuses on permanent fluctuations yielded by the pure imaginary roots of $H(\lambda)$.

Lemma 4 *There exist (ω, σ) such that $H(\lambda)$ has pure imaginary roots which are Hopf bifurcation values.*

Proof: The proof proceeds in two steps: **1**) it supposes $\lambda = iq$ and proves there exists a least one $q > 0$ such that $H(iq) = 0$; it hence defines $(\omega_0, \sigma_0(\omega_0))$ the pair of parameters for which such a root exists. **2**) It uses σ as a bifurcation parameter and shows that there exists a neighborhood of σ_0 such that $d\,\mathrm{Re}(H(\lambda))/d\sigma$ is not equal to zero while $d\,\mathrm{Im}(H(\lambda))/d\sigma = 0$.

1) Replace $\lambda = iq$ in Eq. (11) to obtain $H(iq) = \mathrm{Re}(H(iq)) + i\,\mathrm{Im}(H(iq))$ with:

$$\begin{aligned} \mathrm{Re}(H(iq)) &= -\left(\frac{\cos(q) - 1 - \cos(q(\omega-1)) + \cos(q\omega)}{q^2}\right) \\ &\quad -\omega\sigma + \frac{2(1-\sigma)}{\omega}\left(\frac{\cos(q\omega)-1}{q^2}\right), \qquad (12) \\ \mathrm{Im}(H(iq)) &= -\left(\frac{\sin(q) + \sin(q(\omega-1)) - \sin(q\omega)}{q^2}\right). \qquad (13) \end{aligned}$$

The first step of the proof proceed as follows: (i) it shows there exist, for any σ, some $(\omega_0, q(\omega_0))$ such that $\mathrm{Im}(H(iq(\omega_0))) = 0$. (ii) it shows that $\mathrm{Re}(H(iq(\omega_0))) = 0$ is compatible with some $\sigma > 0$.

(i) Observe that Eq. (13) rewrites:

$$\mathrm{Im}(H(iq)) = -\frac{4\sin\left(\frac{q(\omega-1)}{2}\right)\sin\left(\frac{q}{2}\right)\sin\left(\frac{q\omega}{2}\right)}{q^2\omega}. \qquad (14)$$

Roots of $\mathrm{Im}(H(iq))$ are then $q = 2k\pi$, $q = 2k\pi/\omega$ and $q = 2k\pi/(\omega-1)$ for $k \in \mathbb{Z}$. Equation $\mathrm{Re}(H(iq)) = 0$ rewrites:

$$\sigma = \frac{-4\sin^2\left(\frac{q\omega}{2}\right) + 2\omega\left[\sin^2\left(\frac{q\omega}{2}\right) + \sin^2\left(\frac{q}{2}\right)\right] - \omega + \omega\cos(q(\omega-1))}{-4\sin^2\left(\frac{q\omega}{2}\right) + (\omega q)^2}. \qquad (15)$$

Consider now the roots $q = 2k\pi/(\omega-1)$ for $k \in \mathbb{Z}$, replace them in Eq. (15) and rearrange to obtain:

$$\sigma = \frac{(\omega-1)}{\frac{(q\omega)^2}{4\sin^2\left(\frac{q}{2}\right)} - 1}. \qquad (16)$$

With $\omega > 1$, conclude that the RHS of Eq. (16) is positive. Then for any $(\omega, 2k\pi/(\omega-1))$, $k \in \mathbb{Z}$, $\mathrm{Im}(H(iq)) = 0$ is satisfied and there exists a $\sigma > 0$ such that $\mathrm{Re}(H(iq)) = 0$. Notice that $\mathrm{Re}(H(iq)) \neq 0$ for $q = 2k\pi$.

2) Let $\lambda = p + iq$. Using Eq. (11), one has:

$$\frac{d\,\mathrm{Re}(H(\lambda))}{d\sigma} = -\omega + \frac{\cos(q\omega)}{\omega}\int_{-\omega}^{0}\left(\int_{s}^{s+\omega} e^{pv}dv\right)ds. \qquad (17)$$

Then, compute Eq. (17) for $p = 0$ and observe it is not zero because $q \neq 2k\pi/\omega$. $\square$

The following lemma gives an indication on the space of parameters that yield a cycle.

Lemma 5 *There exists a monetary cycle of period q only if $\sigma < q/\omega$.*

Proof: In what follows, we prove the lemma for $\omega < 2$, which is the realistic case for a retirement age which equals 1. The proof for $\omega \geq 2$ is similar and available upon request. We aim at showing that $\sigma - q/\omega < 0$ when σ is defined by Eq. (16) and $q = 2k\pi/(\omega - 1)$. Observe first that $\sigma - q/\omega < \phi(k, \omega)$ with

$$\phi(k, \omega) = \frac{(\omega - 1)}{\left(\frac{\omega k\pi}{(\omega-1)}\right)^2 - 1} - \frac{2k\pi}{\omega(\omega - 1)}. \tag{18}$$

Now, it is sufficient to show that $\phi(k, \omega) < 0$. Because $\partial\phi(k, \omega)/\partial k < 0$, we have to prove that $\phi(1, \omega) < 0$. Then, just observe that $\partial\phi(1, \omega)/\partial\omega > 0$ for $\omega > 1$ and that $\phi(1, 2) \simeq -3$. $\square$

Sufficiently strong revenue effects are then necessary to obtain monetary cycles. The magnitude of these effect depends on the periodicity of the price cycle or, equivalently, of the inflation rate cycle, relative to the individual life-span: to obtain a cycle with a lower periodicity, a stronger revenue effect is needed. Remark, nevertheless, that we obtain this result with a discount rate equal to zero while it is quite established in the literature that cycles are more likely to occur when individuals heavily discount the future.

4.2 Endogenous retirement

Assume now that the retirement age is endogenous. To define the intertemporal equilibrium, it is necessary to characterize the size of the active population and consequently to know how cohorts leave the labor market. It is then useful to define the "last in, last out" property.

Definition 4 For all $t \geq 0$, let s_0 be the greatest $s \in [t-\omega, t]$ such that $s_0 + z(s_0) = t$. Then, cohorts satisfy the "last in, last out" property if and only if for all $s < s_0$, $s + z(s) \geq t$.

According to definition 4, when there is "last in, last out", cohorts leave the labor market in the same order they have entered it; consequently, $P(t) = z(t - P(t))$. Otherwise, it may exist some date t_0 such that there is no cohort that leave the labor market and some date t_1 such that different cohorts leave it simultaneously. If such situations were to occur, the analytical characterization of the inter-temporal equilibrium would be drastically complicated. However, the following lemma eliminates such situations in the neighborhood of the steady-state.

Lemma 6 *In the neighborhood of the monetary steady-state, the "last in, last out" property holds.*

Proof: Observe first that optimal age of retirement, defined as the $z(.)$ that solve Eq. (5), is continuously differentiable with respect to s. Indeed, as $R(.) \in C^1(\mathbb{R}_+)$, then $z(s)$ is $C^1(\mathbb{R}_+)$. Consequently, there is "last in, last out" if and only if $1 + dz(s)/ds > 0$. In the neighborhood of the monetary steady-state this condition is satisfied because $dz(s)/ds = 0$. $\square$

It is now possible to characterize the inter-temporal equilibrium in the neighborhood of the monetary steady-state. It is the solution of the following system of non linear functional differential equations with state-dependent delays and advances:

$$\begin{cases} P(t)\,(R(t))^{\sigma} = \int_{t-\omega}^{t} (R(s+z(s)))^{\sigma}\,(\omega - z(s))^{\frac{\sigma}{\eta}}\,ds, \\ \int_{t}^{t+z(t)} R(v)\,dv = (R(t+z(t)))^{\sigma}\,(\omega - z(t))^{\frac{\sigma}{\eta}} \int_{t}^{t+\omega} (R(v))^{1-\sigma}\,dv, \\ P(t) = z(t - P(t)). \end{cases} \tag{19}$$

The first equation of system (19) is the equilibrium condition in the good market which has been modified by replacing the optimal condition on individual retirement (the second equation). Remark that the dynamics of R is governed by distributed delays and advances while the dynamics of z depends on the future realizations of R only. Finally, the third equation characterizes the size of the population when there is "last in, last out". The main difference between system (19) and Eq. (10) relies on the presence of state-dependent delays and advances.

Consider the local dynamics around the steady-state defined by Eq. (9). Following Cooke and Huang (1996), it is the one of $(x(t), y(t), h(t))$ defined such that:

$$\begin{cases} R(t) = 1 + \varepsilon x(t), \\ z(t) = z + \varepsilon y(t), \\ P(t) = z + \varepsilon h(t), \end{cases} \tag{20}$$

where z satisfies $\left(\frac{z}{\omega}\right)^{-\frac{1}{\sigma}} - (\omega - z)^{-\frac{1}{\eta}} = 0$. Remark that state-dependent delays and advances vanish in the linearized system and that it is consequently possible to apply Rustichini (1989).

Property 2 The characteristic function of the linearized system is denoted $Q(\lambda)$ and satisfies:

$$\begin{aligned} Q(\lambda) &= 2\omega + \frac{\sigma z \omega}{\eta(\omega - z)} + \frac{(\omega - z)\,\eta\omega}{\sigma z} - \left(1 + \frac{\sigma z}{(\omega - z)\,\eta\omega} \int_{-\omega}^{0} e^{\lambda(s+z)} ds\right) \\ &\quad * \left(\frac{(\omega - z)\,\eta\omega}{\sigma z} + \frac{\omega}{z\sigma} \int_{0}^{z} e^{\lambda(v-z)} dv - \frac{(1-\sigma)}{\sigma} \int_{0}^{\omega} e^{\lambda(v-z)} dv\right). \end{aligned} \tag{21}$$

Proof: Replace system (20) in system (19) and do a Taylor expansion in the neighborhood of $\varepsilon = 0$. This yields $h(t) = y(t-z)$ and a system of two equations. Write the Jacobian matrix $\mathbf{J}$ and then $Q(\lambda) = \det \mathbf{J}$. Some algebra yield Eq. (21). $\square$

As in the previous case, only pure imaginary roots are now considered.

Lemma 7 *$Q(\lambda)$ has no pure imaginary roots.*

Proof: The proof shows that $|Q(iq)| > 0$. Using Eq. (21), one has:

$$\begin{aligned} |Q(iq)| &= \left| 2\omega + \frac{\sigma z \omega}{\eta(\omega - z)} + \frac{(\omega - z)\eta\omega}{\sigma z} - \left| \left(1 + \frac{\sigma z}{(\omega - z)\eta\omega} \int_{-\omega}^{0} e^{iq(s+z)} ds\right) \right.\right. \\ &\quad \left.\left. * \left(\frac{(\omega - z)\eta\omega}{\sigma z} + \frac{\omega}{z\sigma}\int_0^z e^{iq(v-z)}dv - \frac{(1-\sigma)}{\sigma}\int_0^{\omega} e^{iq(v-z)}dv\right)\right|\right| . \end{aligned} \quad (22)$$

Consequently, $|Q(iq)| \geq |\phi(\sigma)|$ where:

$$\begin{aligned} \phi(\sigma) &= 2\omega + \frac{\sigma z \omega}{\eta(\omega - z)} + \frac{(\omega - z)\eta\omega}{\sigma z} \\ &\quad - \frac{\omega}{\sigma}\left(1 + \frac{\sigma z}{(\omega - z)\eta}\right)\left(\frac{(\omega - z)\eta}{z} + (1+\varepsilon) + |1-\sigma|\right). \end{aligned} \quad (23)$$

with $\varepsilon > 0$. Showing that $\phi(\sigma) < 0$ for all $\sigma > 0$ is then sufficient to conclude. One indeed has: $\phi'(\sigma) > 0$ while $\phi(1) < 0$ and $\lim_{\sigma \to +\infty} \phi(\sigma) < 0$. $\square$

The main result of this section is presented in the following proposition:

Proposition 2 *When the retirement age is optimally chosen by the individuals, the occurrence of monetary cycles is ruled out.*

Proof: Given lemmas 4 and 7, the proof is immediate. $\square$

Let us now turn to the general comment of this section's results. In the exogenous retirement case, we have shown that increasing the frequency of trade within generations, is not sufficient to smooth the strong revenue effects that may yield cycles in OLG models. Conversely, introducing a simple leisure choice modelled as a retirement decision has been proved to be crucial for the occurrence of monetary cycles. The intuition is the following: for mandatory retirement age, an anticipation of high prices, which corresponds to low interest rate, increases the individual savings when the elasticity of inter-temporal substitution is small. This increases the aggregate wealth and increases the prices. But if individuals are allowed to choose their retirement age, an anticipation of high prices is an incitation to retire later because high prices implies high nominal wages. This consequently lower the incitation to save and the final effect on aggregate wealth is then lower than in the mandatory retirement case.

Proposition 2 should extend to more general economies. First, allowing for endogenous entree in the labor market, with a schooling decision as in de la Croix and Licandro (1999), would produce the same result: cycles would be more likely with a mandatory age for the end of education than with an optimally chosen one. In production economies, some non linearities are added, and then endogenous cycles are possible even with endogenous labour supply. This was notably shown by Whitesell (1986) and Matsuyama (2005). However, Cazzavillan and Pintus (2004) and Nourry and Venditti (2006) prove, in a two-period framework, that the occurrence of cycles is reduced if individuals are supposed to consume during their youth.

5 Conclusion

In this article, we analyze the existence of long-run fluctuations in OLG economies. We show that when individuals choose their retirement age, some periodic solutions of the inter-temporal equilibrium dynamics vanish. It should be now interesting to study the existence and uniqueness of the inter-temporal equilibrium. For linear MFDE, the existence problem was studied by d'Albis and Augeraud-Véron (2004) while the indeterminacy issue was analyzed by Demichelis and Polemarchakis (2007). However, the problem is increased with state-dependent delays because of the characterization of the initial conditions.

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