

On the singular set of the parabolic obstacle problem

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Abstract

This paper is devoted to regularity results and geometric properties of the singular set of the parabolic obstacle problem with variable right hand side. Making use of a monotonicity formula for singular points, we prove the uniqueness of blow-up limits at singular points. These results apply to parabolic obstacle problem with variable coefficients.

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1. Introduction

Points in $\mathbb{R}^d \times \mathbb{R}$ are denoted (x, t) , where the space variable $x = (x_1, \dots, x_d)$ belongs to \mathbb{R}^d and the time variable, t , belongs to \mathbb{R} . To $x_0 \in \mathbb{R}^d$, $P_0 = (x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}$ and $R > 0$, we associate the Euclidian open ball $B_R(x_0) := \{x \in \mathbb{R}^d : |x - x_0|^2 < R^2\}$ and the open parabolic cylinder

$$Q_R(P_0) := B_R(x_0) \times \{t \in \mathbb{R} : |t - t_0| < R^2\}.$$

For $D \subset \mathbb{R}^d \times \mathbb{R}$ we denote $\mathcal{D}'(D)$ the set of distributions.

We consider a solution $u \in \mathcal{D}'(Q_R(P_0))$ of the following parabolic obstacle problem:

$$\begin{cases} \Delta u(x, t) - \frac{\partial u}{\partial t}(x, t) = [1 + f(x, t)] \mathbb{1}_{\{u=0\}}(x, t), \\ u(x, t) \geq 0 \end{cases} \quad \text{a.e. in } Q_R(P_0), \quad (1.1)$$

where $\mathbb{1}_{\{u>0\}}$ denotes the characteristic function of the set $\{u = 0\} := \{(x, t) \in \mathbb{R}^{d+1} : u(x, t) = 0\}$.

The set $\{u = 0\}$ and its boundary $\Gamma := Q_R(P_0) \cap \partial\{u = 0\}$ are respectively called the *coincidence set* and the *free boundary* of the parabolic obstacle problem (1.1).

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Up to a transformation the parabolic obstacle problem with variable coefficients reduces to this problem (see appendix). This model is the generalisation of the Stefan problem (case $f \equiv 0$) which describes the melting of an ice cube in a glass of water (see [16,19,11,13] and reference therein). This problem also appears in the valuation of American option in the Black-Scholes model with local volatility (see [2,12,15]).

Let $P_1 = (x_1, t_1) \in \Gamma$, we define $\sigma_{P_1} : \mathbb{R}_+ \mapsto \mathbb{R}_+$ a non-decreasing function such that $\sigma_{P_1}(0) = 0$, $\lim_{t \rightarrow 0} \sigma_{P_1}(t) = 0$ and $|f(P) - f(P_1)| \leq \sigma_{P_1} \left(\sqrt{|x - x_1|^2 + |t - t_1|} \right)$ for all $P = (x, t) \in Q_R(P_0)$. We assume that:

$$\begin{cases} [0, 1] \ni \alpha \mapsto \frac{1}{\alpha} \int_0^\alpha \frac{\sigma_{P_1}(\theta)}{\theta} d\theta \text{ is integrable for all } P_1 \in \Gamma, \\ f(P) \geq -\frac{1}{2} \text{ for all } P \in Q_r(P_1), \text{ for some } r > 0. \end{cases} \quad (1.2)$$

Notice that for all $P_1 \in \Gamma$, $f(P_1) = 0$ and there exists r such that $f(P) \geq -1/2$ for all $P \in Q_r(P_1)$.

Under Assumption (1.2), consider u solution of (1.1), $P_1 \in \Gamma$ and $(P_n)_{n \in \mathbb{N}} \in \Gamma^\mathbb{N}$ converging to P_1 . The blow-up sequence $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$ associated to $u \in \mathcal{D}'(Q_r(P_n))$ is the sequence of generic term

$$u_{P_n}^{\varepsilon_n}(x, t) = \frac{1}{\varepsilon_n^2} u \left(x_n + x \frac{\varepsilon_n}{\sqrt{1 + f(P_1)}}, t_n + t \frac{\varepsilon_n^2}{1 + f(P_1)} \right) \quad \forall (x, t) \in Q_{r/2\varepsilon_n}(0). \quad (1.3)$$

Proposition 1.1 (Classification of blow-up limits in \mathbb{R}^{d+1}) *Let $P_1 \in \Gamma$. Under Assumption (1.2), consider a solution u of (1.1). There exists a sub-sequence $(\varepsilon_{n_k})_{k \in \mathbb{N}}$ of $(\varepsilon_n)_{n \in \mathbb{N}}$ such that the blow-up sequence at the fixed point P_1 , $(u_{P_1}^{\varepsilon_{n_k}})_{k \in \mathbb{N}}$ converges to one of the following:*

- (i) $u_e^0(x, t) := \frac{1}{2} (\langle x, e \rangle)^2$, for a unit vector e , where $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^d ,
- (ii) $u_{m,A}^0$ the unique non-negative solution in the distributional sense of $\Delta u - \frac{\partial u}{\partial t} = \mathbb{1}_{\{u=0\}}$ which coincides with $mt + \frac{1}{2} X^T \cdot A \cdot X$, in $\mathbb{R}^d \times (-\infty, 0)$, for given $m \in [-1, 0]$ and A in the set \mathcal{M}_m of the $d \times d$ -matrix satisfying $\text{Tr}(A) = m + 1$.

The blow-up limit can depend on the choice of the sub-sequence. We define the singular set as the set of points such that there exists a blow-up limit of type (ii). We denote \mathcal{S} as the set of singular points. The set $\Gamma \setminus \mathcal{S}$ is the set of regular points. L. Caffarelli, A. Petrosyan and H. Shahgholian prove in [9] that the free boundary is a $\mathcal{C}_{x,t}^\infty$ -manifold locally around regular points for the constant coefficients case. We give in Proposition 3.3 an energy characterisation of regular and singular points.

This paper is devoted to the study of the *singular set* \mathcal{S} . For these points we have

Proposition 1.2 (Uniqueness of blow-up limits at singular points) *Let $P_1 \in \mathcal{S}$. Under Assumption (1.2), consider a solution u of (1.1). Let $(P_n)_{n \in \mathbb{N}} \in \mathcal{S}^\mathbb{N}$ converging to P_1 . There exists a unique $(m_{P_1}, A_{P_1}) \in [-1, 0] \times \mathcal{M}_m$ such that for any sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ converging to 0, the whole blow-up sequence $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$ locally uniformly converges to $u_{m_{P_1}, A_{P_1}}^0$ where $u_{m,P}^0$ is defined in Proposition 1.1.*

To a point $P_1 \in \mathcal{S}$ we can hence associate a unique $(m_{P_1}, A_{P_1}) \in [-1, 0] \times \mathcal{M}_m$.

Definition 1.3 (The sets $\mathcal{S}(k)$) *For $k \in \{0, d\}$ we define $\mathcal{S}(k)$ as the set of singular points P such that $\dim \text{Ker } A_P = k$ and the smallest of the k non-zero eigenvalues is bounded from below by a positive constant δ fixed.*

To state our results on the regularity of $\mathcal{S}(k)$ we need to define the set $\mathcal{C}_{x,t}^{1/2}$, of holderian function $D \subset \mathbb{R}^{d+1} \rightarrow \mathbb{R}$ of exponent 1/2:

$$\mathcal{C}_{x,t}^{1/2}(D) := \{u \in C_{x,t}^0(D) : \sup_{\substack{(x,t),(y,s) \in D \\ (x,t) \neq (y,s)}} \frac{|u(x,t) - u(y,s)|}{\sqrt{|x-y| + |t-s|}} < \infty\}.$$

This leads to the definition of a $\mathcal{C}_{x,t}^{1/2}$ -manifold.

For the sets $\mathcal{S}(k)$, $k \in \{0, \dots, d\}$ we state:

Theorem 1.4 (Regularity of $\mathcal{S}(k)$) *Under Assumption (1.2), consider a solution u of (1.1).*

(i) *If $P_1 \in \mathcal{S}(d)$ then there exists $\tilde{\Gamma}$, a \mathcal{C}_x^2 -graph in space, such that*

$$\mathcal{S}(d) \cap Q_\rho(P_1) \subset \tilde{\Gamma},$$

for some $\rho = \rho(d, \sup_{Q_R(P_0)} |u|) > 0$ small enough.

(ii) *If $P_1 \in \mathcal{S}(k)$, for $k \in \{0, \dots, d-1\}$, then there exists Γ , a k -manifold of class $\mathcal{C}_{x,t}^{1/2}$, such that*

$$\mathcal{S}(k) \cap Q_\rho(P_1) \subset \Gamma,$$

for some $\rho = \rho(d, \sup_{Q_R(P_0)} |u|) > 0$ small enough.

As a consequence we prove

Corollary 1.5 (Regularity of $\cup_{k=0}^d \mathcal{S}(k)$) *Under Assumption (1.2), consider a solution u of (1.1).*

If $P_1 \in \cup_{k=0}^d \mathcal{S}(k)$ then there exists $\hat{\Gamma}$, a d -manifold of class $\mathcal{C}_{x,t}^{1/2}$ such that

$$\bigcup_{k=0}^d \mathcal{S}(k) \cap Q_\rho(P_1) \subset \hat{\Gamma},$$

for some $\rho = \rho(d, \sup_{Q_R(P_0)} |u|) > 0$ small enough.

The study of the singular sets in obstacle problems has been through many developments over the past twenty years. Especially for the elliptic obstacle problem. In this area a lot of questions have been conjectured by the pioneering work of D. G. Schaeffer [17]. L. Caffarelli developed in [5] a new theory to study obstacle problems by introducing *the blow-up method*. This theory has been largely simplified by *the monotonicity method* of G. Weiss (see [20]). A further step in the study of the singular set of the elliptic obstacle problem is [14] where R. Monneau takes the formula of G. Weiss further to obtain a *monotonicity formula for singular points*. In particular he proves the uniqueness of blow-up limits in singular points and gives sharp geometric results on the singular set. For the elliptic obstacle problem with no sign assumption on the solution, L. Caffarelli and H. Shahgholian prove in [10] regularity properties on the singular set making use of the monotonicity formula of [1] and [7].

For the parabolic obstacle problem the analysis is quite recent. G. Weiss introduced in [21] a monotonicity formula for the parabolic obstacle problem. In a recent paper L. Caffarelli, A. Petrosyan and H. Shahgholian make an in-depth analysis of the parabolic obstacle problem with no sign assumption on the solution with constant coefficients. However they do not study the singular set. For the parabolic obstacle problem with assumption (1.2), J. Dolbeault, R. Monneau and the author make a study of the singular set but in one-dimension, in [4].

In Section 2 we prove preliminary results. In Section 3 we prove a monotonicity formula of Weiss' type. This energy gives an energy criterion to characterise the regular and singular points of the free boundary. Proposition 1.1 is a consequence of this characterisation. In Section 4 we prove a monotonicity formula for singular points and prove the uniform convergence of the whole blow-up sequence to the blow-up limit at singular points. In Section 5 we prove the geometric and regularity results of Theorem 1.4 and Corollary 1.5.

Notation 1 For u smooth enough, u_t denotes $\frac{\partial u}{\partial t}$, D_i the derivative $\frac{\partial u}{\partial x_i}$, D_{ij} the derivative $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and H the heat operator $\Delta - \frac{\partial}{\partial t}$. We define the open parabolic lower half-cylinder $Q_R^-(P_0) := B_R(x_0) \times \{t \in (-\infty, 0) : |t - t_0| < R^2\}$.

2. Preliminaries

This section is quite classic but it has not been proved in this framework. For the constant coefficients case the reader can refer to [9]. For more detailed proofs the interested reader can refer to [4], where these kinds of proofs are demonstrated in one dimension. For $D \subset \mathbb{R}^d \times \mathbb{R}$, we define the Sobolev space

$$W_{x,t}^{2,1;\infty}(D) := \left\{ u \in L^q(D) : \left(\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial u}{\partial t} \right) \in (L^\infty(D))^3 \right\} .$$

Theorem 2.1 (A priori regularity estimates) *Assume (1.2) and consider a solution u of (1.1). For all $R' < R$, u is bounded in $W_{x,t}^{2,1;\infty}(Q_{R'}(P_0))$.*

This proof has been pointed out to us by H. Shahgholian. It has been done in one dimension in [3].

Lemma 2.2 *Let u be a solution of (1.1), assume (1.2) and consider a point $0 \in \Gamma$. Then there exists a constant C such that*

$$\sup_{Q_r(0)} |u| \leq C r^2$$

for any $r \in [0, 1]$.

Proof. This proof follows the first part of the proof of Lemma 4.2 in [9] which was adopted from [8]. We introduce

$$S_j(u) := \sup_{Q_{2^{-j}}(0)} |u| \quad \text{and} \quad N(u) := \{k \in \mathbb{N} : 2^{2k} S_{j+1}(u) \geq S_j(u)\} .$$

By a recursion argument we easily see that it is sufficient to prove that there exists $C_0 \geq 1$ such that

$$S_{j+1}(u) \leq C_0 M 2^{-2j} \quad \text{for } j \in N(u) \tag{2.1}$$

where $M := \sup_{Q_R(P_0)} |u|$ to complete the result with $C := 16 M C_0$.

Assume by contradiction that (2.1) is false and that there exists $k_j \in N(u)$ such that

$$S_{k_j+1}(u) \geq j 2^{-2k_j} . \tag{2.2}$$

We define

$$u_j(x, t) := \frac{1}{S_{k_j+1}(u)} u(2^{-k_j} x, 2^{-2k_j} t) \quad \text{in } Q_1 .$$

By (1.1) and (2.2), the functions u_j satisfy

$$\lim_{j \rightarrow \infty} \sup_{Q_1} |Hu_j| = 0 .$$

By definition of S_j and u_j

$$\sup_{Q_{1/2}} u_j = 1 . \tag{2.3}$$

Furthermore u_j is non-negative and $u_j(0, 0) = 0$. Up to the choice of a sub-sequence, $(u_j)_{j \in \mathbb{N}}$ converges to a function u^0 . The function u^0 is caloric and bounded in $Q_{1/2}$ and achieves its minimum in 0. By the strong maximum principle u^0 is constant which contradicts (2.3).

Proof of Theorem 2.1. Let $P_1 = (x_1, t_1) \in \Gamma \cap Q_{1/4}^-$ and

$$d := \sup\{r : Q_r^-(P_1) \cap \{u > 0\} \cap Q_1^-\}.$$

Introduce the function

$$u^d(x, t) := \frac{1}{d^2} u(dx + x_1, d^2t + t_1).$$

By Lemma 2.2, u^d is uniformly bounded in Q_1^- . By definition of d , u^d satisfies $\text{Hu}^d = 1$. The standard parabolic L^p estimates gives the result. \square

Lemma 2.3 (Non-degeneracy lemma) *Let $f \in L^\infty(\mathbb{R}^{d+1})$. Consider a solution u in the distributional sense of*

$$\begin{cases} \Delta u(x, t) - u_t(x, t) \geq [1 + f(x, t)] \mathbf{1}_{\{u > 0\}} > 0 \\ u(x, t) \geq 0 \end{cases} \quad (2.4)$$

Let $P_1 \in \overline{\{u > 0\}}$. If $r > 0$ is such that $Q_r^-(P_1) \subset Q_R(P_0)$ then

$$\sup_{Q_r^-(P_1)} u \geq \bar{C} r^2.$$

with $\bar{C} = \frac{1}{2d+1} (1 + \|f\|_{L^\infty(Q_R(P_0))})$.

This kind of lemma has been proved for the first time by L. Caffarelli for the elliptic obstacle problem in [5].

Proof. The proof lies on the maximum principle. Consider first $P' = (x', t') \in \{u > 0\} \cap Q_r^-(P_1)$. We set for all $(x, t) \in Q_r^-(P')$

$$w(x, t) := u(x, t) - u(P') - \bar{C} ((x - x')^2 + |t - t'|).$$

By the maximum principle, for any $\rho \leq r$ the maximum of the sub-caloric function w in $\overline{Q_\rho^-(P') \cap \{u > 0\}}$ is achieved in the parabolic boundary of $Q_\rho^-(P') \cap \{u > 0\}$. As w is negative in $\partial\{u = 0\} \cap \overline{Q_\rho^-(P')}$ and $w(P') = 0$, there exists $P_2 = (x_2, t_2)$ the parabolic boundary of $\partial^p Q_\rho^-(P') \cap \{u > 0\}$ such that

$$\sup_{\overline{Q_\rho^-(P') \cap \{u > 0\}}} w = w(P_2) = u(P_2) - u(P') - \bar{C} ((x_2 - x')^2 + |t_2 - t'|) \geq 0.$$

So for any $\rho \leq r$

$$\sup_{Q_\rho^-(P')} u \geq u(P_2) \geq u(P') + \bar{C} \rho^2. \quad (2.5)$$

By continuity of u we achieve the result when P' converges to P_1 . \square

This lemma is very useful and will be used several times throughout this paper. The proof of Section 5.3 in [9] applies to prove

Lemma 2.4 (Measure of Γ) *Let $f \in L^\infty(\mathbb{R}^{d+1})$. Consider a solution $u \in \mathcal{D}'(Q_R(P_0))$ of (2.4). The set $\partial\{u = 0\}$ is a closed set of zero $(d+1)$ -Lebesgue measure.*

We can now state the main result of Section 2.

Proposition 2.5 (Blow-up limit) *Under Assumption (1.2), consider a solution u of (1.1), $P_1 \in \Gamma$, $(P_n)_{n \in \mathbb{N}} \in \Gamma^{\mathbb{N}}$ converging to P_1 and a blow-up sequence $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$. There exists a sub-sequence and a function $u_{P_1}^0 \in W_{x,t}^{2,1;\infty}(\mathbb{R}^{d+1})$ such that the blow-up sequence $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$ uniformly converges in every*

compact $K \subset\subset \mathbb{R}^{d+1}$ to $u_{P_1}^0$. Furthermore $u_{P_1}^0$ is a solution of the following global parabolic obstacle problem with constant coefficients :

$$\begin{cases} \Delta u_{P_1}^0(x, t) - \frac{\partial u_{P_1}^0}{\partial t}(x, t) = \mathbb{1}_{\{u_{P_1}^0=0\}}(x, t), & \text{a.e. in } \mathbb{R}^{d+1}. \\ u_{P_1}^0(x, t) \geq 0 \end{cases} \quad (2.6)$$

Moreover, $0 \in \partial\{u_{P_1}^0 = 0\}$.

Proof. By Ascoli-Arzelà's theorem, there exists a sub-sequence $(\varepsilon_{n_k})_{n \in \mathbb{N}}$ and a non-negative function $u_{P_1}^0 \in \mathcal{D}'(\mathbb{R}^{d+1})$ such that $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$ uniformly converges to $u_{P_1}^0$ for any compact $K \subset \mathbb{R}^{d+1}$. By uniform convergence we can pass to the limit in the equation satisfied by $u_{P_n}^{\varepsilon_n}$ to obtain:

$$\frac{\partial^2 u_{P_1}^0}{\partial x^2} - \frac{\partial u_{P_1}^0}{\partial t} = 1 \quad \text{in } \{u_{P_1}^0 > 0\}.$$

By Lemma 2.4, $\partial\{u_{P_1}^0 > 0\}$ has zero $(d+1)$ -Lebesgue measure, so $u_{P_1}^0$ is a solution of (2.6).

By non-degeneracy lemma (Lemma 2.3)

$$\bar{C} r^2 \leq \sup_{Q_r^-(0)} u_{P_n}^{\varepsilon_n} \rightarrow \sup_{Q_r^-(0)} u_{P_1}^0 \quad \text{as } n \rightarrow \infty,$$

which proves that $0 \in \partial\{u_{P_1}^0 = 0\}$. □

3. Classification of blow-up limits

A crucial tool for our study is the monotonicity formula of Weiss' type. G. Weiss introduced this kind of tool in [20] to prove the scale-invariance of blow-up limits in the elliptic obstacle problem. This scale-invariance of blow-up limits is very interesting because of the following Liouville's type theorem for self-similar solutions of (2.6) (see Lemma 6.3 and Theorems I and 8.1 in [9]):

Proposition 3.1 (Liouville's type theorem for $t < 0$) *If $u^0 \in \mathcal{D}'(\mathbb{R}^{d+1})$ for any compact $K \subset \mathbb{R}^{d+1}$ is a non-zero self-similar solution of (2.6) i.e. solution of (2.6) under the constrain*

$$u^0(\lambda x, \lambda^2 t) = \lambda^2 u^0(x, t) \quad \forall \lambda > 0 \quad \forall (x, t) \in \mathbb{R}^d \times (-\infty, 0)$$

then, either there exists a unit vector e such that $u^0 = u_e^0$ or there exists $(m, A) \in [-1, 0] \times \mathcal{M}_m$ such that $u^0 = u_{m,A}^0$; where u_e^0 and $u_{m,A}^0$ are defined in Proposition 1.1.

Furthermore $u_t^0 \leq 0$ and $D_{\nu\nu} u^0 \geq 0$, for any spatial direction $\nu \in \mathbb{R}^d$.

In dimension 1, the proof of the first part of the theorem uses the self-similarity of the solutions to bring itself back to an ordinary differential equation. The reduction to the dimension 1 uses the monotonicity formula of Caffarelli ([6]) (see Lemma 6.3 in [9]). The second assertion is a direct consequence of Theorem I in [9]. Indeed, the case (iii) of Theorem I cannot happen because non-negative solutions are unique. So we cannot truncate the solution.

Let $Q_r(P_1) \subset Q_R(P_0)$. Consider a non-negative cut-off function $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that $\psi \equiv 1$ in $B_{1/(2\sqrt{1+f(P_1)})}$ and $\psi \equiv 0$ in $\mathbb{R}^d \setminus B_{1/(\sqrt{1+f(P_1)})}$. Define $\psi_r(x) := \psi(rx)$ and the function v_{P_1} (which depends on u , P_1 and r) for all $(x, t) \in \mathbb{R}^d \times (-r^2[1+f(P_1)], r^2[1+f(P_1)])$ by

$$v_{P_1}(x, t) := u \left(x_1 + \frac{x}{\sqrt{1+f(P_1)}}, t_1 + \frac{t}{1+f(P_1)} \right) \cdot \psi_r(x) \quad \mathbb{1}_{B_{1/(\sqrt{1+f(P_1)})}}(x). \quad (3.1)$$

For all $t \in (-r^2 [1 + f(P_1)], 0)$, define

$$\mathcal{E}_{u, P_1}(t, r) := \int_{\mathbb{R}^d} \left[\frac{1}{-t} \left(|\nabla v|^2 + 2v \right) - \frac{v^2}{t^2} \right] \mathcal{G}(x, t) dx - \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}^d} (\mathbb{H}v - 1) (\mathbb{L}v) \mathcal{G}(x, s) dx ds ,$$

with $\mathbb{H}v := \Delta v - v_t$, $\mathbb{L}v := -2v + x \cdot \nabla v + 2tv_t$ and $\mathcal{G}(x, t) := (2\pi(-t))^{-\frac{d}{2}} e^{(-|x|^2)/(-4t)}$.

Proposition 3.2 (Monotonicity formula for energy) *Under Assumption (1.2) consider a solution u of (1.1) and v_{P_1} defined in (3.1). The function $t \mapsto \mathcal{E}_{u, P_1}(t, r)$ is non-increasing, bounded in $W^{1, \infty}(-r^2 [1 + f(P_1)], 0)$ and such that for almost every $t \in (-r^2 [1 + f(P_1)], 0)$*

$$\frac{\partial}{\partial t} \mathcal{E}_{u, P_1}(t, r) = -\frac{1}{2(-t)^3} \int_{\mathbb{R}^d} |\mathbb{L}v_{P_1}(x, t)|^2 \mathcal{G}(x, t) dx .$$

Proof. The first part of the proof follows the idea of [9]. Assume first that $v_{P_1} =: v \in \mathcal{D}'(\mathbb{R}^d \times (-r^2 [1 + f(P_1)], 0))$. We begin to compute the time derivative of

$$\mathbf{e}(t; v) := \int_{\mathbb{R}^d} \left\{ \frac{1}{-t} \left(|\nabla v(x, t)|^2 + 2v(x, t) \right) - \frac{1}{t^2} v^2(x, t) \right\} \mathcal{G}(x, t) dx . \quad (3.2)$$

A change of variable gives $\mathbf{e}(\lambda^2 t; v) = \mathbf{e}(t; v^\lambda)$, for all t in $(-\lambda^{-2}, 0)$, where $v^\lambda(x, t) := \lambda^{-2} v(\lambda x, \lambda^2 t)$. Because $(\frac{\partial}{\partial \lambda} v^\lambda)|_{\lambda=1} = \mathbb{L}v$, we obtain at $\lambda = 1$

$$\frac{d\mathbf{e}}{dt}(t; v) = \frac{1}{2t} D_v \mathbf{e}(t; v) \cdot \mathbb{L}v ,$$

where $D_v \mathbf{e}$ is defined for all ϕ in $\mathcal{C}^\infty(\mathbb{R}^d \times (-r^2 [1 + f(P_1)], 0))$ by

$$D_v \mathbf{e}(t; v) \cdot \phi := \int_{\mathbb{R}^d} \left\{ \frac{1}{-t} \left(2 \nabla v(x, t) \cdot \nabla \phi(x, t) + 2 \phi(x, t) \right) - \frac{2}{t^2} v(x, t) \phi(x, t) \right\} \mathcal{G}(x, t) dx .$$

Integration by parts and a reordering of the terms give

$$\frac{d}{dt} \mathbf{e}(t; v) = \int_{\mathbb{R}^d} \left\{ \frac{1}{2t^3} |\mathbb{L}v(x, t)|^2 + \frac{1}{t^2} \mathbb{L}v(x, t) \left(\mathbb{H}v(x, t) - 1 \right) \right\} \mathcal{G}(x, t) dx .$$

This equality is still true for $v \in W_{x, t}^{2, 1; \infty}(\mathbb{R}^d \times (-r^2 [1 + f(P_1)], 0))$ with compact support in space by a density argument. Note that $t \mapsto \mathbf{e}(t; v)$ is bounded in $W_{\text{loc}}^{1, \infty}(-r^2 [1 + f(P_1)], 0)$ by $W_{x, t}^{2, 1; \infty}$ -regularity estimates on u .

The second part of the proof follows the idea of [4]. We have to control the function \mathbf{r} defined by

$$\mathbf{r}(t; v) := \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}^d} [(\mathbb{H}v - 1) \mathbb{L}v \mathcal{G}](x, s) dx ds . \quad (3.3)$$

We write $|\mathbf{r}(t; v)| \leq \mathbf{A}(t) + \mathbf{B}(t)$ with

$$\mathbf{A}(t) := \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}^d} |[\mathbb{H}v - (1 + f)] \mathbb{L}v \mathcal{G}|(x, s) dx ds , \quad (3.4)$$

$$\mathbf{B}(t) := \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}^d} |f \mathbb{L}v \mathcal{G}|(x, s) dx ds . \quad (3.5)$$

By definition of v , $[\mathbb{H}v - (1 + f)] \mathbb{L}v$ vanishes in $B_{1/2}(0) \cup [\mathbb{R}^{d+1} \setminus B_1(0)]$ and by $W_{x, t}^{2, 1; \infty}$ -regularity,

$$\mathbf{A}(t) \leq c \int_0^{|t|} \frac{ds}{s^2} \int_{B_1 \setminus B_{1/2}} \frac{e^{-|\rho|^2/4s}}{(2\pi s)^{d/2}} d\rho \leq \frac{c}{2(2\pi)^{d/2}} \int_0^{|t|} \frac{e^{-1/16s}}{s^{(d+4)/2}} ds ,$$

which gives a control of $\mathbf{A}(t)$ for any $t \in (-r^2 [1 + f(P_1)], 0)$.

Due to the $W_{x,t}^{2,1;\infty}$ -regularity of u and (1.2) of σ we have

$$\mathbf{B}(t) \leq \int_t^0 \frac{1}{s^2} \int_{\mathbb{R}^d} \sigma(\sqrt{x^2 + |s|}) (x^2 + |s|) \mathcal{G}(x, s) dx ds.$$

The change of variable $(x, s) \mapsto (y := \frac{x}{\sqrt{t}}, -s)$ and Fubini-Tonelli's theorem give

$$\mathbf{B}(t) \leq c \int_{\mathbb{R}^d} (y^2 + 1) e^{-y^2/4} \int_0^{-t} \frac{1}{s} \sigma(\sqrt{s} \sqrt{y^2 + 1}) ds dy.$$

A cylindric change of coordinates on y gives

$$\mathbf{B}(t) \leq c \int_0^{+\infty} (r^2 + 1) e^{-r^2/4} \int_0^{-t} \frac{1}{s} \sigma(\sqrt{s} \sqrt{r^2 + 1}) ds r^{d-1} dr$$

and the change of variable $(r, s) \mapsto (\beta := \sqrt{r^2 + 1}, \theta := \sqrt{s(r^2 + 1)})$ gives

$$\mathbf{B}(t) \leq c \int_1^{+\infty} (\beta^2 - 1)^{\frac{d-2}{2}} \beta^3 e^{-\beta^2/4} \left(\int_0^{\min(\beta \sqrt{|t|}, \sqrt{1+|t|})} \frac{\sigma(\theta)}{\theta} d\theta \right) d\beta.$$

Thus $\mathbf{B}(t)$ is bounded by Assumption (1.2). \square

Proposition 1.1 is a corollary of

Proposition 3.3 (Energy characterisation of the points of Γ) *Let $P_1 \in \Gamma$. Under Assumption(1.2), consider a solution u of (1.1) and $r > 0$ such that $Q_r(P_1) \subset Q_R(P_0)$. If $u_{P_1}^0$ is a blow-up limit associated to u at the fixed point P_1 , then*

$$\Lambda(P_1) := \lim_{\substack{t \rightarrow 0 \\ t < 0}} \mathcal{E}_{u, P_1}(t, r) = \mathcal{E}_{u_{P_1}^0, 0}(t, 0) \in \{2K, K\} \quad \forall (t, r) \in (-\infty, 0) \times \mathbb{R}^d,$$

where K is a positive constant which only depends on the dimension d . If $\Lambda(P_1) = K$, then $u_{P_1}^0 = u_e^0$, for a certain unit vector e . If $\Lambda(P_1) = 2K$, then $u_{P_1}^0 = u_{m,A}^0$ for some $(m, A) \in [-1, 0] \times \mathcal{M}_m$, where u_e^0 and $u_{m,A}^0$ are defined in Proposition 1.1.

Proof. By the monotonicity formula for the energy (Proposition 3.2), \mathcal{E} is non-increasing in time and bounded from below, so $\lim_{t \rightarrow 0} \mathcal{E}_{u, P_1}(t, r)$ is finite. A simple change of variable shows that $\mathcal{E}_{u_{P_1}^{\varepsilon_n}, 0}(t, \varepsilon_n r) = \mathcal{E}_{u, P_1}(\varepsilon_n^2 t, r)$. So $\lim_{t \rightarrow 0} \mathcal{E}_{u, P_1}(t, r) = \lim_{n \rightarrow \infty} \mathcal{E}_{u, P_1}(\varepsilon_n^2 t, r) = \lim_{n \rightarrow \infty} \mathcal{E}_{u_{P_1}^{\varepsilon_n}, 0}(t, \varepsilon_n r) = \mathcal{E}_{u_{P_1}^0, 0}(t, 0)$. Hence $\frac{\partial}{\partial t} \mathcal{E}_{u_{P_1}^0, 0}(t, 0) = 0$. And so $u_{P_1}^0$ is scale-invariant in $\{t < 0\}$.

By the classification of the scale-invariant solutions of (2.6) for $t < 0$ (Proposition 3.1) we identify $u_{P_1}^0$ in $\{t < 0\}$ as one of the functions u_e^0 and $u_{m,A}^0$. By the uniqueness of non-negative solutions of (2.6), $u_{P_1}^0$ is either u_e^0 or $u_{m,A}^0$ in $\mathbb{R}^d \times \mathbb{R}$.

A direct computation gives $\mathcal{E}(t; u_e^0) = K$ and $\mathcal{E}(t; u_{m,A}^0) = 2K$. \square

Proposition 3.3 allows the division of the free boundary into two sets, depending on the value of Λ : the points P of the free boundary such that $\Lambda(P) = K$ are the regular points and the points P of the free boundary such that $\Lambda(P) = 2K$ are the singular points.

Lemma 3.4 (Topological properties of \mathcal{R} and \mathcal{S}) *Under Assumption (1.2), \mathcal{S} is a closed set, and $\mathcal{R} = \Gamma \setminus \mathcal{S}$ is open in Γ .*

Proof. Let $(P_1, P_2) \in \Gamma^2$. By the energy characterisation of the points of Γ (Proposition 3.3), for all $\delta > 0$ there exists $t_0 = t_0(\delta)$ such that $|\mathcal{E}_{u_{P_1}^0, 0}(t_0, 0) - \Lambda(P_1)| < \delta/2$. By $W_{x,t}^{2,1;\infty}$ -regularity of u , for this t_0 there

exists a continuous function $\omega_{t_0} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\omega_{t_0}(0) = 0$ and $|\mathcal{E}_{u_{P_1}^0,0}(t_0,0) - \mathcal{E}_{u_{P_2}^0,0}(t_0,0)| < \omega_{t_0}(|P_1 - P_2|)$. We can choose P_2 close enough to P_1 such that $\omega_{t_0}(|P_1 - P_2|) < \delta/2$. With these choices we compute

$$\Lambda(P_2) - \Lambda(P_1) \leq \mathcal{E}_{u_{P_2}^0,0}(t_0,0) - \mathcal{E}_{u_{P_1}^0,0}(t_0,0) + \mathcal{E}_{u_{P_1}^0,0}(t_0,0) - \Lambda(P_1) < \omega_{t_0}(|P_1 - P_2|) + \delta/2 < \delta .$$

Hence the function $\Gamma \ni P \mapsto \Lambda(P)$ is upper semi-continuous. If $\Lambda(P_1) = K$, then $\Lambda(P_2) = K$ for P_2 in a neighbourhood of P_1 . This proves that \mathcal{R} is an open set in Γ and its complementary, \mathcal{S} is a closed set of Γ . \square

4. Study of the singular points of the free boundary

4.1. A monotonicity formula for singular points

R. Monneau developed in [14] a monotonicity formula to study the set of singular points in the elliptic obstacle problem. He used it to prove the uniqueness of blow-up limit in singular points. This tool has been extended to the parabolic obstacle problem in one-dimension in [4]. We write it in higher dimensions.

For v defined as in (3.1) and u^0 one of the functions of Proposition 1.1, we define, for $t \in (-r^2 [1 + f(P_1)], 0)$, the functional

$$\Phi_{u,P_1}^{u^0}(t; r) := \frac{1}{t^2} \int_{\mathbb{R}^d} |v - u^0|^2 \mathcal{G} \, dx - \int_t^0 \frac{2}{s^2} \int_{\mathbb{R}^d} (\text{H}v - 1) (v - u^0) \mathcal{G} \, dx \, ds + \int_t^0 \frac{2}{s} \mathbf{r}(s; v) \, ds .$$

where \mathbf{r} is defined in (3.3).

Proposition 4.1 (Monotonicity formula for singular points) *Under Assumption (1.2), consider a solution u of (1.1), $P_1 \in \mathcal{S}$, $r > 0$ such that $Q_r(P_1) \subset Q_R(P_0)$ and u^0 one of the functions of Proposition 1.1. The function $t \mapsto \Phi_{u,P_1}^{u^0}(t; r)$ is non-increasing and bounded in $W^{1,1}(-r^2 [1 + f(P_1)], 0)$.*

Proof. By density, it is sufficient to prove the result for a smooth function v .

Let $w := v - u^0$ and $y := x \sqrt{-t}$. Using $\mathcal{G}(\sqrt{-t}y, t) \, dx = \mathcal{G}(y, 1) \, dy$, and $\text{L}u_{P_2}^0 = 0$ in $\{t < 0\}$ we have

$$\frac{d}{dt} \left[\frac{1}{t^2} \int_{\mathbb{R}^d} w^2(x, t) \mathcal{G}(x, t) \, dx \right] = \frac{1}{t^3} \int_{\mathbb{R}^d} \text{L}v(x, t) w(x, t) \mathcal{G}(x, t) \, dx . \quad (4.1)$$

On the one hand, by the monotonicity of \mathcal{E} (Proposition 3.2)

$$\begin{aligned} \mathcal{E}_{u,P_1}(t, r) - \mathcal{E}_{u^0,0}(t, r) &= \mathcal{E}_{u,P_1}(t, r) - \mathcal{E}_{u_{P_1}^0,0}(t, 0) + \mathcal{E}_{u_{P_1}^0,0}(t, 0) - \mathcal{E}_{u^0,0}(t, r) \\ &= \mathcal{E}_{u,P_1}(t, r) - \mathcal{E}_{u_{P_1}^0,0}(t, 0) \leq 0 . \end{aligned}$$

On the other hand

$$\mathcal{E}_{u,P_1}(t, r) - \mathcal{E}_{u^0,0}(t, r) = \mathbf{e}(t; v) - \mathbf{e}(t; u^0) - \mathbf{r}(t, v)$$

where \mathbf{e} is defined in (3.2) and \mathbf{r} in (3.3). By integration by part and reordering we obtain

$$\mathbf{e}(t; v_{P_1}) - \mathbf{e}(t; u_{P_2}^0) = \int_{\mathbb{R}} \left[\frac{1}{-t} [1 - \text{H}v_{P_1}(x, t)] + \frac{1}{2t^2} \text{L}v_{P_1}(x, t) \right] w(x, t) \mathcal{G}(x, t) \, dx . \quad (4.2)$$

Here we use $\text{H}u_{P_2}^0 = 1$ and $\text{L}u_{P_2}^0 = 0$ in $\{t < 0\}$.

Finally combining (4.1) and (4.2) and adding and subtracting f give

$$\frac{\partial}{\partial t} \Phi_{u, P_1}^{u^0}(t; r) = \frac{2}{t} [\mathcal{E}_{u, P_1}(t, r) - \mathcal{E}_{u^0, 0}(t, r)] \leq 0. \quad (4.3)$$

Remains to control

$$\mathbf{C}(t) = \int_t^0 \frac{2}{s^2} \int_{\mathbb{R}^d} (\mathbb{H}v_{P_1} - 1) w \mathcal{G} dx ds \quad \text{and} \quad \mathbf{D}(t) = \int_t^0 \frac{2}{s} \mathbf{r}(s; v_{P_1}) ds.$$

The term $\mathbf{C}(t)$ can be controlled in the same way as $\mathbf{B}(t)$ in the proof of Proposition 3.2 by replacing $|\mathbb{L}v| \leq C(|x|^2 + |t|)$ by $|w| \leq C(|x|^2 + |t|)$. The last term to control is $\mathbf{D}(t)$. With $\mathbf{B}(t)$ and $\mathbf{A}(t)$ defined in (3.4) we have

$$\begin{aligned} \mathbf{D}(t) &\leq \int_t^0 \frac{2}{s} [\mathbf{A}(s) + \mathbf{B}(s)] ds \\ &\leq c \int_t^0 2 \int_0^{|t|} \frac{e^{-\frac{1}{16s}}}{s^{\frac{n+6}{2}}} ds + c \int_t^0 \frac{1}{s} \int_1^{+\infty} (\beta^2 - 1)^{\frac{n-2}{2}} \beta^3 e^{-\beta^2/4} \left(\int_0^{\min(\beta \sqrt{|s|}, \sqrt{1+|s|})} \frac{\sigma(\theta)}{\theta} d\theta \right) d\beta. \end{aligned}$$

Which is bounded by Assumption (1.2). \square

4.2. Uniqueness of blow-up limit in singular points

By the monotonicity formula for singular points (Proposition 4.1), $t \mapsto \Phi_{u, P_1}^{u^0}(t; r)$ is non-increasing and bounded from below, so $\lim_{t \rightarrow 0} \Phi_{u, P_1}^{u^0}(t; r)$ is finite. A simple change of variable shows that $\Phi_{u_{\varepsilon_n}^0, 0}(t; \varepsilon_n r) = \Phi_{u, P_1}^{u^0}(\varepsilon_n^2 t; r)$. Let $u_{P_1}^0$ be one of the blow-up limits at P_1 . We have

$$\lim_{t \rightarrow 0} \Phi_{u, P_1}^{u^0}(t; r) = \lim_{n \rightarrow \infty} \Phi_{u, P_1}^{u^0}(\varepsilon_n^2 t; r) = \lim_{n \rightarrow \infty} \Phi_{u_{\varepsilon_n}^0, 0}(t; \varepsilon_n r) = \Phi_{u_{P_1}^0, 0}(t; 0).$$

We can apply this computation to two different limits of $(u_{P_1}^{\varepsilon_n})_{n \in \mathbb{N}}$ to prove that the blow-up limit is unique at a fixed point P_1 .

Hence $\lim_{t \rightarrow 0} \Phi_{u, P_1}^{u_{P_1}^0}(t; r) = 0$. By the monotonicity formula for singular points (Proposition 4.1), for all $\delta > 0$ there exists $t_0 = t_0(\delta)$ such that $|\Phi_{u, P_1}^{u_{P_1}^0}(t; r)| < \delta/2$. By *a priori* regularity of u , for this t_0 there exists a continuous function $\omega_{t_0} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\omega_{t_0}(0) = 0$ and $|\Phi_{u, P_1}^{u_{P_1}^0}(t; r) - \Phi_{u, P_2}^{u_{P_1}^0}(t; r)| < \omega_{t_0}(|P_1 - P_2|)$. So there exists $\eta = \eta(t_0(\delta))$ such that $|P_1 - P_2| < \eta$ implies $\omega_{t_0}(|P_1 - P_2|) < \delta/2$. With the above choice of δ , $t_0(\delta)$ and $\eta(t_0(\delta))$, take $N \in \mathbb{N}$ such that $\varepsilon_n^2 < t_0$ and $|P_1 - P_n| < \eta$. Let u^0 be one of the blow-up limits of $(u_{P_n}^{\varepsilon_n})_{n \in \mathbb{N}}$,

$$\begin{aligned} \Phi_{u^0, 0}^{u_{P_1}^0}(t; 0) &= \lim_{n \rightarrow \infty} \Phi_{u_{P_n}^{\varepsilon_n}, 0}^{u_{P_1}^0}(t; \varepsilon_n r) = \lim_{n \rightarrow \infty} \Phi_{u, P_n}^{u_{P_1}^0}(\varepsilon_n^2 t; r) \leq \lim_{n \rightarrow \infty} \Phi_{u, P_n}^{u_{P_1}^0}(t_0; r) \\ &\leq \Phi_{u, P_1}^{u_{P_1}^0}(t_0; r) + \omega_{t_0}(|P_1 - P_n|) < \delta. \end{aligned}$$

Hence $u^0 = u_{P_1}^0$ in $\{t < 0\}$. By the uniqueness of non-negative solutions of (2.6), $u^0 = u_{P_1}^0$ in \mathbb{R}^{d+1} . \square

To any $P \in \Gamma$, we can therefore associate a unique $(m_P, A_P) \in [-1, 0] \times \mathcal{M}_m$ such that the blow-up limit of a solution at this point is u_{m_P, A_P}^0 where $u_{m, A}^0$ is defined in Proposition 1.1.

5. Geometric properties of \mathcal{S}

In Section 5.1 we deduce some regularity properties on the set $\mathcal{S}(d)$ and in Section 5.2 we study the sets $\mathcal{S}(k)$, $k \in \{0, \dots, d-1\}$.

5.1. Proof of Theorem 1.4 (i): the set $\mathcal{S}(d)$

Lemma 5.1 (Regularity property of $\mathcal{S}(d)$) *Under Assumption (1.2), consider a solution u of (1.1) and $P_1 \in \mathcal{S}(d)$.*

$$\sup_{\substack{(x,t),(y,s) \in \mathcal{S}(d) \cap Q_\rho(P_1) \\ (y,s) \neq (x,t)}}} \frac{|s-t|}{|y-x|^2} = 0,$$

for some $\rho = \rho(d, \sup_{Q_R(P_0)} |u|) > 0$.

Proof. Assume by contradiction there are $(P_n = (x_n, t_n))_{n \in \mathbb{N}}$ and $(P'_n = (x'_n, t'_n))_{n \in \mathbb{N}}$ two sequences of points of $\mathcal{S}(d)$ converging to P_1 such that

$$\lim_{n \rightarrow \infty} \frac{|t'_n - t_n|}{|x'_n - x_n|^2} =: \delta > 0. \quad (5.1)$$

With no restriction we can assume that $t'_n \leq t_n$. Introduce $\varepsilon_n := \sqrt{|x_n - x'_n|^2 + |t_n - t'_n|}$. There exists a sub-sequence $(n_k)_{k \in \mathbb{N}}$ and a vector $\mu = (x_\mu, t_\mu)$ such that $\mu_n := \left(\frac{x_n - x'_n}{\varepsilon_n}, \frac{t_n - t'_n}{\varepsilon_n^2} \right)$ converges to $\mu = (x_\mu, t_\mu)$ in the boundary of $B_1^p(0) := \{(x, t) \in \mathbb{R}^{d+1} : |x|^2 + |t| < 1\}$. By Proposition 1.2, there exists a function $u_{P_1}^0$ such that $\lim_{n \rightarrow \infty} u_{P_n}^{\varepsilon_n} = u_{P_1}^0$. By non-degeneracy lemma (Lemma 2.3) μ belongs to $\partial\{u_{P_1}^0 = 0\}$. By Proposition 3.1, $u_{P_1}^0$ is decreasing in time, so $0 = u_{P_1}^0(0) = u_{P_1}^0(0, t_\mu) = 0$. By the convexity of $u_{P_1}^0$ (see Proposition 3.1), $u_{P_1}^0 = 0$ in $[0, x_\mu] \times \{t_\mu\}$. By (5.1), $x_\mu \neq 0$. This is a contradiction because x_μ does not belong to $\{x = 0\} = \text{Ker } A_{P_1}$. \square

Proof of Theorem 1.4 (i). Let $g : \mathcal{S}(d) \ni P = (x, t) \mapsto x \in \mathbb{R}^{d+1}$. By Lemma 5.1, there exists $\rho > 0$ such that $g|_{\mathcal{S}(d) \cap Q_\rho(P_1)}$ is one-to-one. We introduce the closed set $\mathcal{S}^*(d) := g|_{\mathcal{S}(d) \cap Q_\rho(P_1)}(\mathcal{S}(d))$ and define $(g|_{\mathcal{S}(d) \cap Q_\rho(P_1)})^{-1} : \mathcal{S}^*(d) \rightarrow \mathbb{R}^{d+1}$ which associates to the projection the unique point of $\mathcal{S}(d)$. Thanks to Lemma 5.1, $(g|_{\mathcal{S}(d) \cap Q_\rho(P_1)})^{-1}$ is a $\mathcal{C}_x^2(\mathcal{S}^*(d))$ function in space. By Whitney's extension theorem (see [22]) there exists a $\mathcal{C}_x^2(\mathbb{R}^d)$ function which extends $(g|_{\mathcal{S}(d) \cap Q_\rho(P_1)})^{-1}$ in \mathbb{R}^{d+1} . The graph of this function contains $\mathcal{S}(d) \cap Q_\rho(P_1)$, for ρ small enough. \square

5.2. Proof of Theorem 1.4 (ii): the sets $\mathcal{S}(k)$

Lemma 5.2 *Under Assumption (1.2), consider a solution u of (1.1). For $k \in \{0, \dots, d-1\}$, consider a point $P_1 \in \mathcal{S}(k)$. There exists $\nu \in (\text{Ker } A_{P_1})^\perp \subset \mathbb{R}^d$ such that*

$$\sup_{\substack{P, P' \in \mathcal{S}(k) \cap Q_r(P_1) \\ P' \neq P}} \frac{|\langle P, (\nu, 0) \rangle - \langle P', (\nu, 0) \rangle|}{|\text{Proj}_{(\nu, 0)}(P) - \text{Proj}_{(\nu, 0)}(P')|^{1/2}} < \infty,$$

for $r = r(d, \sup_{Q_R(P_0)} |u|) > 0$ small enough, where $\text{Proj}_{(\nu, 0)}$ is the projection in $\nu^\perp \times \mathbb{R}$ defined by

$$\begin{aligned} \text{Proj}_{(\nu,0)} : \mathcal{S}(k) &\rightarrow \nu^\perp \times \mathbb{R} \\ P &\mapsto P - \langle P, (\nu, 0) \rangle (\nu, 0) . \end{aligned}$$

and $|\cdot|$ denotes the Euclidian distance in \mathbb{R}^{d+1} .

Proof. Consider $(P_n = (x_n, t_n))_{n \in \mathbb{N}}$ and $(P'_n = (x'_n, t'_n))_{n \in \mathbb{N}}$ two sequences of points of $\mathcal{S}(k)$ converging to P_1 . Assume

$$\frac{|\langle P_n - P'_n, (\nu, 0) \rangle|}{|\text{Proj}_{(\nu,0)}(P_n - P'_n)|^{1/2}} = \infty . \quad (5.2)$$

Introduce $\varepsilon_n := \sqrt{|x_n - x'_n|^2 + |t_n - t'_n|^2}$. There exists a sub-sequence $(n_k)_{k \in \mathbb{N}}$ and a vector $\mu = (x_\mu, t_\mu)$ such that $\mu_n := \left(\frac{x_n - x'_n}{\varepsilon_n}, \frac{t_n - t'_n}{\varepsilon_n^2} \right)$ converges to $\mu = (x_\mu, t_\mu)$ in the boundary of $B_1^p(0) := \{(x, t) \in \mathbb{R}^{d+1} : |x|^2 + |t| < 1\}$. By Proposition 1.2, there exists a function $u_{P_1}^0$ such that $\lim_{n \rightarrow \infty} u_{P_n}^{\varepsilon_n} = u_{P_1}^0$. But $\mu \in B_1^p(0)$ implies $|x_n - x'_n|^2 + |t_n - t'_n|^2 \leq |x_n - x'_n|^2 + |t_n - t'_n| = 1$. Hence (5.2) implies that μ belongs to $\text{Vect}(\nu, 0)$. By convexity of $u_{P_1}^0$ (see Proposition 3.1) $u_{P_1}^0(\cdot, 0) \equiv 0$ in $\text{Vect}(\nu)$. This is a contradiction with ν in $[\text{Ker } A_{P_1}]^\perp$. \square

A further step toward the proof of Theorem 1.4 (ii) is

Lemma 5.3 ($\mathcal{C}_{x,t}^{1/2}$ -regularity of $\mathcal{S}(k)$) *Under Assumption (1.2), consider a solution u of (1.1). For $k \in \{0, \dots, d-1\}$, consider $P_1 \in \mathcal{S}(k)$ and $\nu_i \in \text{Ker}(A_{P_1})^\perp$. There exists Γ_i , a d -dimensional manifold of class $\mathcal{C}_{x,t}^{1/2}$, such that*

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma_i$$

for some $r = r(d, \sup_{Q_R(P_0)} |u|) > 0$.

Proof. By Lemma 5.2, there exists $\rho > 0$ such that the restriction of $\text{Proj}_{(\nu_i,0)}$ in $Q_\rho(P_1)$, denoted $\text{Proj}_{(\nu_i,0)|_{\mathcal{S}(k) \cap Q_\rho(P_1)}}$ is one-to-one. We introduce the closed set $\mathcal{S}^*(k) := \text{Proj}_{(\nu_i,0)|_{\mathcal{S}(k) \cap Q_\rho(P_1)}}(\mathcal{S}(k))$ and define $\text{Proj}_{(\nu_i,0)|_{\mathcal{S}(k) \cap Q_\rho(P_1)}}^{-1} : \mathcal{S}^*(k) \rightarrow \mathbb{R}$ which associates to the projection the unique point of $\mathcal{S}(k)$. Thanks to Lemma 5.2, $\text{Proj}_{(\nu_i,0)|_{\mathcal{S}(k) \cap Q_\rho(P_1)}}^{-1}$ is a $\mathcal{C}_{x,t}^{1/2}(\mathcal{S}^*(k))$ function. By Whitney's extension theorem generalised to holderian functions by Stein (Theorem 3 p.174, [18]) there exists a $\mathcal{C}_{x,t}^{1/2}(\nu_i^\perp \times \mathbb{R})$ function which extends $\text{Proj}_{(\nu_i,0)|_{\mathcal{S}(k) \cap Q_\rho(P_1)}}^{-1}$ in $\nu_i^\perp \times \mathbb{R}$. The d -manifold of this function is denoted Γ_i and contains $\mathcal{S}(k) \cap Q_\rho(P_1)$ for ρ small enough. \square

Proof of Theorem 1.4 (ii). Let $P_1 \in \mathcal{S}(k)$, for $k < d$. By Lemma 5.3, for the k independent $(\nu_i)_{\{1, \dots, k\}}$ in $\text{Ker}(A_m)^\perp$ there exists a $\mathcal{C}_{x,t}^{1/2}$ -manifold, Γ_i such that $\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma_i$. Hence

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma := \bigcap_{i=1}^{d-k} \Gamma_i \quad \text{and} \quad \dim \Gamma = d - k .$$

\square

Proof of Corollary 1.5. Points of $\mathcal{S}(d)$ are isolated. Let $P_1 \in \mathcal{S} \setminus \mathcal{S}(d) = \cup_{i=0}^{d-1} \mathcal{S}(k)$. There exists ν in $\text{Ker } A_{P_1}^\perp$. By Lemma 5.3, there exists Γ_i , a d -dimensional manifold of class $\mathcal{C}_{x,t}^{1/2}$, such that

$$\mathcal{S}(k) \cap Q_r(P_1) \subset \Gamma_i$$

for some $r = r(d, \sup_{Q_R(P_0)} |u|) > 0$. We achieve the result with $\hat{\Gamma} := \Gamma_i$. \square

Appendix: Transformation

Let D be a domain of D^{d+1} . Let a_{ij}, b_i, c and g be continuous function of space and time in D . Consider a solution v of the following parabolic obstacle problem with variable coefficients

$$\begin{cases} v \in W_{x,t}^{2,1;\infty}(D) \\ a_{ij}(y, s) \frac{\partial^2 v}{\partial y_i \partial y_j} + b_i(y, s) \frac{\partial v}{\partial y_i} + c(y, s) v - \frac{\partial v}{\partial s} = g(y, s) \mathbb{1}_{\{y>0\}}(y, s) & \text{a.e. } (y, s) \in D \\ v(y, s) \geq 0. \end{cases} \quad (5.3)$$

The reduction of a general parabolic operator to the heat operator is done by a classical transformation which goes as follows. Let $P_1 = (y_1, s_1) \in \partial\{v = 0\}$ and take $r > 0$ such that $Q_r(P_1) \subset D$. For all $P \in Q_r(P_1) \cap \{v > 0\}$, Equation (5.3) can be rewritten as

$$\begin{aligned} a_{ij}(P_1) \frac{\partial^2 v}{\partial y_i \partial y_j}(P) - \frac{\partial v}{\partial s}(P) \\ = g(P_1) + (g(P) - g(P_1)) - (a_{ij}(P) - a_{ij}(P_1)) \frac{\partial^2 v}{\partial y_i \partial y_j}(P) - b_i(P) \frac{\partial v}{\partial y_i}(P) - c(P) v(P). \end{aligned}$$

Consider the affine change of variables

$$(y, s) \mapsto \left(x := \sqrt{\frac{f(P_1)}{a_{ij}(P_1)}} y, \quad t := f(P_1) s \right).$$

The function $u(x, t) := v(y, s)$ is a solution of (1.1) with

$$1 + f(x, t) := \frac{1}{g(P_1)} \left((g(P) - g(P_1)) - (a_{ij}(P) - a_{ij}(P_1)) \frac{\partial^2 v}{\partial y_i \partial y_j}(P) - b_i(P) \frac{\partial v}{\partial y_i}(P) - c(P) v(P) \right).$$

By construction, $P_1 \in \partial\{u = 0\}$ and $f(P_1) = 0$. Note that if a_{ij}, b_i, c and g are $\mathcal{C}_{x,t}$ function then (1.2) is satisfied.

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