Sharing a resource with concave benefits

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Abstract

A group of agents are collectively entitled to a perfectly divisible good or resource. They enjoy concave and satiable benefit functions from consuming it. They also value money (transfers). The resource is scarce in the sense that not everybody can consume its satiated consumption level. This paper characterizes the unique (resource and money) allocation that is efficient, incentive-compatible and equal-sharing individual rational. It then discusses its implementation and its link with other axioms.

Key words: Common resource, single peak, incentive compatible, no envy, fairness.
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1 Introduction

Imagine a community of peasants who produce irrigated crops. They share the same pool of water but have different needs: some might have more land or better irrigation technology than others, or grow different crops. How to share water among them? The answer is obvious if there is enough water to cover their needs: just assign to each peasant what she or he needs. A problem arises when water is scarce in the precise sense that not all needs can be fulfilled. Simple sharing rules like equal sharing might be inefficient because some users might have more than needed. Other sharing rules such as an efficient allocation of water might be perceived as unfair. Those who are assigned less water might ask for some sort of monetary compensation, whereas the others should somehow "pay" to consume more. This system might also be difficult to implement when productivity is private information since users would over-report their productivity to obtain more water.

This paper deals with the problem of sharing a common resource among a group of agents and designing monetary contributions/compensations in a context similar to the above example. The model is based on the following assumptions. First, the agents' benefits from consuming the resource are represented by concave and single-peak functions. Examples include irrigation production functions. Production increases as more input (water) is available until satiation. Marginal productivity is decreasing. It is positive before satiation and becomes negative above this point because over-consumption may cause flooding or increase sanitation costs.

Second, the agents can be ranked according to their marginal benefit from consuming the resource up to their peak: those who value the resource marginally more than others also have equal or higher needs/peak consumption. For instance, they might have more land or grow a crop that requires more water.

Third, the agents might be asked to pay a contribution or to receive a compensation. An allocation specifies how much of the resource each agent consumes and what he pays or receives. It should be feasible, in the sense that what is consumed does not exceed what is available. Individuals' transfers are not bounded (e.g. agents have deep pocket). Nevertheless, transfers must be budget balanced.

Our first concern is to share the resource efficiently in the Pareto sense. This requires us to equalize marginal benefits. Our second concern is to implement the efficient resource allocation with asymmetric information. Transfers must provide every agent with incentives to report benefits truthfully, no matter what the others do. The allocation should be dominant strategy incentive-compatible or strategy-proof. The last concern is to ensure that everybody accepts the allocation. It should guarantee everybody at least the benefit of consuming up to an equal share of the resource.[1]

This paper characterizes the unique allocation that is efficient, incentive-compatible and equal-sharing individual rational. It is called the Walrasian allocation from equal endowments. It can be implemented (i) by selling the resource at a fixed price and dividing the money collected equally, or (ii) by assigning property rights on equal shares in a competitive resource market.

In our framework, incentive-compatibility (or strategy-proofness) is equivalent to no envy (or envy-freeness), which is a central concept in equity theory (Foley, 1967, Varian, 1974, Baumol, 1986). An agent does not envy another agent if his payoff is higher with his own assigned consumption bundle (resource and money) than with somebody else's bundle. An allocation satisfies no envy if no agent envies the bundle assigned to another agent. The allocation proposed here is therefore the unique allocation that satisfies no envy, efficiency and equal-sharing individual rationality.

Our model is a two-good version (resource and money) of the fair division problem in exchange economies.[2] This literature has established the compatibility between Pareto efficiency and envy-freeness when preferences are continuous and convex (Varian, 1974, Baumol 1986). Under the assumption of non-satiated preferences, the equilibrium allocation of a Walrasian

[1] This guaranteed equal split condition is the oldest axiom in the fair division literature, often taken as the definition of fairness (Steinhaus, 1948, Moulin, 1991). Since the equal division of the resource seems a natural solution (albeit an inefficient one), an agent could object to an allocation by arguing that he would enjoy a higher benefit with equal sharing (the status-quo). Therefore, all agents should prefer their assigned bundles rather than consuming (up to) an equal share of the resource.

[2] Note that the deep-pocket assumption on side-payments implies that agents have unlimited financial endowments. Nevertheless, since the allocation satisfies individual rationality, utilities are positive. Therefore, the transfers paid are bounded by individuals' benefits from consuming the resource.

exchange economy with equal initial endowments is envy-free and Pareto efficient in addition to being obviously equal-sharing individual rational (Foley, 1967, Schmeidler and Vind, 1972, see Thomson and Varian, 1985, for a survey). This paper extends this result to specific satiated preferences. It establishes the uniqueness of such allocation with a large number (a continuum) of agents.

In exchange economies with a finite number of agents, strategy-proofness is not compatible with efficiency (Zhou, 1991), and therefore with a Walrasian allocation. This is so even in the limit as the economy grows (Barberà and Jackson, 1995).[3] Under our assumption of a continuum of agents with specific weakly increasing utility (under free disposal), strategy-proofness is compatible with efficiency and equal sharing individual rationality. Furthermore, the unique strategy-proof individual rational and efficient allocation can be implemented as a Walrasian equilibrium.

With single-peak preferences but only one good (i.e. without transfers), the uniform allocation rule is the unique sharing rule that is efficient, strategy-proof and anonymous (Sprumont, 1991, Ching, 1992).[4] It assigns to each agent the maximum among its peak and a uniform quota. With unbounded side-payments and decreasing marginal utility on the shared good, the uniform rule is not efficient since it fails to equalize marginal benefits. As a consequence, there is room for Pareto improvements through resource re-allocation and transfers.[5]

The rest of the paper proceeds as follows. Section 2 introduces the model and defines the axioms. Section 3 characterizes the Walrasian allocation from equal endowments. Section 4 discusses the axioms. It also relates the characterized allocation to three other axioms: the peak upper bound, consistency, and strict no-envy.

[3] More precisely, Barberà and Jackson (1995) show that with a finite number of agents, two goods and strictly increasing utilities, any strategy-proof and individual rational allocation must be implemented with a fixed-trading mechanism in which agents set selling and buying prices before the utility functions of the agents are realized.

[4] Here, the combination of strategy-proofness and anonymity is equivalent to our definition of incentive-compatibility.

[5] Notice that in our framework, while being inefficient, the uniform rule is nevertheless incentive-compatible and equal-sharing individual rational.

2 Model and definitions

Users who share a quantity X of a private good or resource are labeled according to their "type" θ. It is common knowledge that θ is distributed according to a density f and a cumulative F in $[\underline{\theta}, \bar{\theta}] \equiv \Theta$, with $f(\theta) > 0$ for every $\theta \in \Theta$, within the population of users. Agent θ enjoys a benefit $b(x, \theta)$ for consuming a level x of the resource. $b(., \theta)$ is assumed positive, twice continuously differentiable, strictly concave in x and strictly increasing up to a maximum denoted $\hat{x}_\theta$ for every $\theta \in \Theta$. Agents can be ranked according to their productivity parameter θ in the sense that an agent with higher θ has a higher marginal productivity, formally $\frac{\partial^2 b}{\partial\theta\partial x}(x, \theta) > 0$ for every $x \leq \hat{x}_\theta$ and $\theta \in \Theta$. This assumption is often called the "single-crossing" property. It is standard in the mechanism design literature. It implies that $\hat{x}_\theta$ is non-decreasing in θ. It is assumed that $b(0, \theta) = 0$ and $\frac{\partial b}{\partial x}(0, \theta) \geq k$ for every $\theta \in \Theta$ with $k > 0$ being sufficiently high.[6] Figure 1 below represents an example of benefit functions for three types $\theta' < \theta'' < \theta'''$.

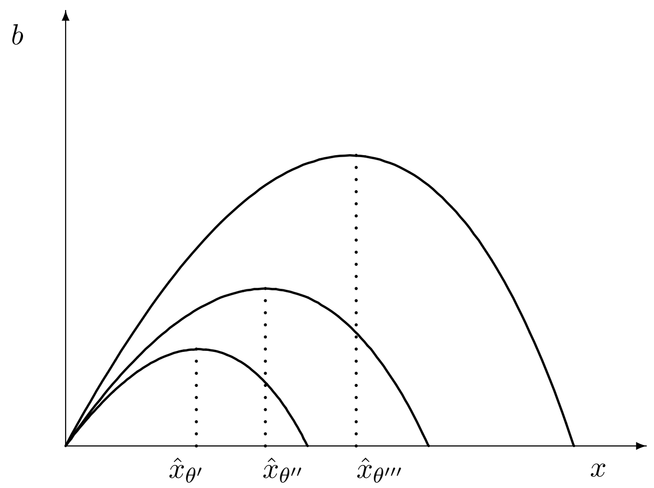

Figure 1: Benefit functions

Examples of such benefit functions include the quadratic functions $b(x, \theta) = \theta x - \frac{x^2}{2}$ (with $\underline{\theta}$ high enough) or the log function up to the peak consumption, i.e., $b(x, \theta) = \theta log(x + 1)$ for

[6] More precisely, k is assumed higher than the shadow cost of the resource constraint λ (defined later) so that it is efficient to serve everybody. We could have for instance $k = \infty$.

every $x \leq \hat{x}_\theta$. It could take the form $\theta b(x)$ with b concave, in which case the agents have the same peak consumption $\hat{x}_\theta$.

The resource is scarce in the sense that needs are higher than what is available:

$$\int_\Theta \hat{x}_\theta dF(\theta) > X.$$

An allocation $\{x_\theta, t_\theta\}_{\theta \in \Theta}$ (hereafter denoted $\{x_\theta, t_\theta\}$) is a list of bundles (x_θ, t_θ) per type θ. It specifies θ's resource consumption $x_\theta \in \mathbb{R}_+$ and transfer $t_\theta \in \mathbb{R}$ (possibly negative) for every $\theta \in \Theta$. $\{x_\theta\}$ is called a resource allocation whereas $\{t_\theta\}$ is a transfer scheme. It must be *feasible* and *budget-balanced* in the sense defined below.
A resource allocation $\{x_\theta\}$ is *feasible* if it satisfies the following resource constraint:

$$\int_\Theta x_\theta dF(\theta) \leq X. \tag{1}$$

A transfer scheme $\{t_\theta\}$ is *budget-balanced* if it satisfies the following budget constraint:

$$\int_\Theta t_\theta dF(\theta) \leq 0. \tag{2}$$

The allocation $\{x_\theta, t_\theta\}$ yields to agent θ a utility or payoff $u(x_\theta, t_\theta, \theta) = b(x_\theta, \theta) + t_\theta$. Without loss of generality, we restrict attention to resource allocations that satisfy $x_\theta \leq \hat{x}_\theta$ for every $\theta \in \Theta$ because, since the resource is scarce, assigning to any agent θ more than its peak $\hat{x}_\theta$ is inefficient. I now define the basic axioms.

Definition 1 *The allocation $\{x_\theta, t_\theta\}$ is Incentive Compatible (IC) if and only if it satisfies the following incentive-compatibility constraints,*

$$b(x_\theta, \theta) + t_\theta \geq b(\min\{x_{\theta'}, \hat{x}_\theta\}, \theta) + t_{\theta'} \text{ for every } \theta' \in \Theta \qquad IC(\theta),$$

for every $\theta \in \Theta$.

An allocation is IC if every agent θ prefers his assigned bundle (x_θ, t_θ) to any other bundle. We assume free disposal: any agent is free not to consume all the resource. His consumption level is therefore the maximal value among his peak and his assigned resource consumption for any report $\theta' \in \Theta$. The above incentive compatibility constraints $IC(\theta)$ (with free disposal) are more stringent than the standard incentive-compatibility constraints (without free disposal).

Definition 2 *The allocation* $\{x_\theta, t_\theta\}$ *is Equal-Sharing Individual Rational (ESIR) if and only if it satisfies the following equal-sharing individual rationality constraints:*

$$b(x_\theta, \theta) + t_\theta \geq b(\min\{X, \hat{x}_\theta\}, \theta) \qquad ESIR(\theta),$$

for every $\theta \in \Theta$.

The condition $ESIR(\theta)$ guarantees that agent of type θ obtains at least its benefit with an equal division of the resource under free disposal. To be ESIR, the allocation should assign at least the benefit from consuming an equal share of the resource X or less to every agent. Since, with a continuum of agents of mass 1, X is both the total and the per capita amount of the resource, giving X to everybody is a feasible resource allocation.

3 Theorem

In this transferable utility set-up, efficiency (Pareto optimality) alone selects a unique feasible resource allocation $\{x^*_\theta\}$. It is the resource allocation that maximizes the sum of the benefit functions (e.g. total production) subject to the resource constraint. It solves,

$$\max_{\{x_\theta\}} \int_\Theta b(x_\theta, \theta) dF(\theta) \text{ subject to } \int_\Theta x_\theta dF(\theta) \leq X.$$

Denoting λ the Langragian multiplier associated with the feasibility constraint (1), the efficient allocation satisfies the following first order conditions,

$$\frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda \text{ for every } \theta \in \Theta \text{ with } \lambda > 0. \tag{3}$$

Since $\frac{\partial^2 b}{\partial\theta\partial x} > 0$ for every $x \leq \hat{x}_\theta$, then x^*_θ is increasing in θ.

The efficient resource allocation is unique. It equalizes marginal benefits to the shadow price of the resource. Lets define the transfer scheme $t^*_\theta = \lambda(X - x^*_\theta)$ for every $\theta \in \Theta$. It is what agent θ would pay or receive by trading $(X - x^*_\theta)$ at price λ. Call $\{x^*_\theta, t^*_\theta\}$ the *Walrasian allocation from equal endowments.*

Theorem *The Walrasian allocation from equal endowments* $\{x^*_\theta, t^*_\theta\}$ *is the only (feasible and budget-balanced) allocation that is efficient, incentive-compatible and equal-sharing individual* *rational.*

The formal proof of the Theorem is provided in the Appendix. The main argument is explained here using the following graphic illustration.

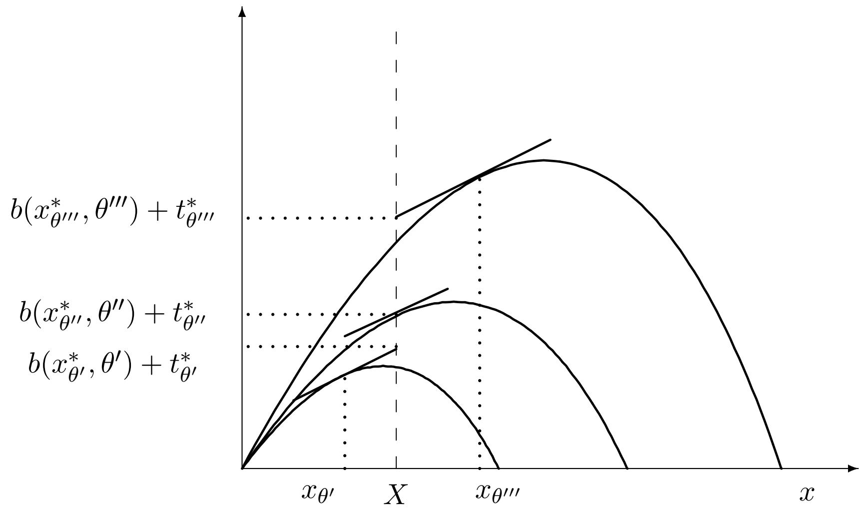

Figure 2: Payoffs with the Walrasian allocation from equal endowments

Figure 2 represents the benefit functions of three types of agent $\theta' < \theta'' < \theta'''$. They consume the efficient resource allocation which equalizes marginal benefits to the shadow cost of the resource constraint λ. The tangent of the benefit functions at the efficient resource allocation are all of slope λ. These tangents are represented in Figure 2 for the three types of agent. Agent θ' (resp. θ''') consumes less (resp. more) than the per capita resource X. Agent θ'' consumes exactly X the equal sharing level.

An agent θ's utility is $u(x^*_\theta, t^*_\theta, \theta) = b(x^*_\theta, \theta) + \lambda(X - x^*_\theta)$. It is evaluated in Figure 2 at the crossing point of the equal share vertical line X and the tangent to its benefit function at x^*_θ. Agent of type θ' (resp. θ'''), who consumes less (resp. more) than the per capita level X, receives a positive (resp. negative) transfer.

First, by definition, the resource allocation $\{x^*_\theta\}$ is efficient. Second, the Walrasian allocation from equal endowment is IC because the consumption level x^*_θ maximizes $b(x, \theta) + \lambda(X - x)$ with respect to x for every $\theta \in \Theta$. Therefore, each agent θ maximizes its utility by choosing its assigned bundle (x^*_θ, t^*_θ). Third, it is ESIR because, due to the concavity of b, the tangent of any agent's benefit function crosses the vertical X line above its benefit at X. Furthermore, when X is lower than the peak consumption $\hat{x}_\theta$, it crosses the vertical line above its benefit at $\hat{x}_\theta$.

Lastly, the proof that no other allocation satisfies all axioms proceeds as follow. It is first shown that first-best resource allocation $\{x^*_\theta\}$ can be implemented with incentive-compatible constraints only if transfers are such that $t_\theta = \gamma - \lambda x^*_\theta$ for every $\theta \in \Theta$, where $\gamma \in \mathbb{R}$ remains to be chosen. It must be lower than λX to satisfy the budget balance constraint (2) and higher than λX to be ESIR and efficient. Therefore, the budget and equal-sharing individual rationality constraints combined with efficiency impose $\gamma = \lambda X$.

The Walrasian allocation from equal endowments $\{x^*_\theta, t^*_\theta\}$ can be implemented by selling the resource at its shadow price λ and redistributing the money collected λX through a lump-sum subsidy $\sigma \equiv \lambda X$. Each agent consumes the amount that equalizes its marginal benefit to the price, i.e., $\frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$. Agent θ's payoff is $b(x^*_\theta, \theta) - \lambda x^*_\theta + \sigma$. Since $\sigma = \lambda X$, it simplifies to $b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) = b(x^*_\theta, \theta) + t^*_\theta$.

It can also be implemented by dividing the resource equally if agents can exchange the resource in a (Walrasian) competitive market. The equilibrium price p equals marginal benefits, i.e., $p = \frac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$. At this price, any arbitrary agent θ consumes x^*_θ. He sells or buys $X - x^*_\theta$ units of the resource.[7] His payoff is $b(x^*_\theta, \theta) + p(X - x^*_\theta) = b(x^*_\theta, \theta) + \frac{\partial b}{\partial x}(x^*_\theta, \theta)(X - x^*_\theta) = b(x^*_\theta, \theta) + t^*_\theta$.

Note that all agents must not be short of money to buy enough resource in the market. This is not a problem if the benefit is interpreted as a production function: producers trade future production (e.g. corn) with the resource. Otherwise a minimal endowment of money $p(X - x^*_\theta)$ for every agent $\theta \in \Theta$ is required to implement the proposed allocation.

[7] Recall that, with a continuum of agents of mass 1 as assumed here, X is both the total and average level of the resource.

4 Discussion

4.1 More about the axioms

Significantly, the theorem holds without the free disposal assumption in the definitions of IC and ESIR. That is if agents are obliged to consume all their resource allocation. Clearly, since the IC and ESIR constraints are less stringent without free disposal, the Walrasian allocation from equal endowments $\{x^*_\theta, t^*_\theta\}$ does satisfy them. Furthermore, it is the only allocation to satisfy them because the binding IC and ESIR constraints are without free disposal. Indeed the binding IC constraints are "local", i.e., for types infinitively closed to θ which are assigned less than θ's satiated level $\hat{x}_\theta$ for every $\theta \in \Theta$. The only binding ESIR constraint is the one of agents of type θ'' in Figure 2, i.e., those who are assigned the per-capita level X, which is always lower than their satiated level $\hat{x}_{\theta''}$. Other (feasible and budget-balanced) efficient allocations violate one of these binding constraints.

The three axioms in the theorem are clearly independent since it is easy to find allocations that satisfy only two of them. First, as mentioned in the introduction, the uniform rule (with no monetary transfers) is IC and ESIR but not efficient. Second, ESIR and efficiency are achieved with an efficiency resource allocation $\{x^*_\theta\}$ and a transfer scheme that binds $ESIR(\theta)$ for every $\theta \in \Theta$, i.e., $t'_\theta = b(\min\{X, \hat{x}_\theta\}, \theta) - b(x^*_\theta, \theta)$ for every $\theta \in \Theta$. It is obviously budget balanced.[8] Yet the allocation $\{x^*_\theta, t'_\theta\}$ is not IC. It is not IC with free disposal for any agent θ whose satiated consumption level $\hat{x}_\theta$ is strictly lower than X because the agent prefers the bundle of *higher* types. He obtains his peak benefit $b(\hat{x}_\theta, \theta)$ with his assigned bundle (x^*_θ, t'_θ) whereas he can get as much of the resource and a strictly positive transfer with the bundle of any higher type agents. In other words, he envies those with higher θ because they obtain as much of the resource and more money. The allocation $\{x^*_\theta, t'_\theta\}$ is not IC even without free disposal for any agent θ with assigned consumption level $x^*_\theta > X$ because he prefers the bundle of *lower* types. He obtains $b(X, \theta)$ with his assigned bundle but can enjoy $b(x^*_{\theta'}, \theta) + b(X, \theta') - b(x^*_{\theta'}, \theta')$ with the bundle of an agent $\theta' < \theta$ which is strictly higher if $x^*_{\theta'} > X$ due to the concavity and single crossing assumptions.[9] Third, if the resource is sold

[8] We have $\int_\Theta t_\theta dF(\theta) = \int_\Theta b(\min\{X, \hat{x}_\theta\}, \theta) dF(\theta) - \int_\Theta b(x^*_\theta, \theta) dF(\theta) \leq 0$.

[9] If $X < x^*_{\theta'} < \hat{x}_{\theta'}$ and $\theta > \theta'$ then $b(x^*_{\theta'}, \theta) - b(x^*_{\theta'}, \theta') > b(X, \theta) - b(X, \theta')$.

at λ defined in (3) (or if resource extraction is taxed at a rate λ) but the money collected is not redistributed, then the allocation implemented $\{x^*_\theta, -\lambda x^*_\theta\}$ is efficient and IC but not ESIR. Yet this allocation satisfies another fairness principle called the peak upper bound.

4.2 The peak upper bound

This principle relies on the straightforward observation that, with the Walrasian allocation from equal endowments, some agents end up enjoying a payoff strictly higher than with their satiated (maximal) benefit $b(\hat{x}_\theta, \theta)$. This is particularly the case of agent θ' in Figure 2. He obtains more than he would get if the resource was abundant. This agent somehow exploits the scarcity of the resource to extract a share of the other agents' benefit in addition to his own maximal benefit. This might be perceived as unfair: the resource being scarce (in the sense that not everybody can enjoy its satiated benefit), by solidarity, it is fair that nobody ends up with more than their satiated benefit. This fairness axiom is called the peak upper bound (Moulin, 1991). It is formally defined as follows. An allocation $\{x_\theta, t_\theta\}$ satisfies the peak upper bound (PUB) if $b(x_\theta, \theta) + t_\theta \leq b(\hat{x}_\theta, \theta)$ for every $\theta \in \Theta$. Moulin (1991) shows that the PUB is compatible with efficiency and IC/no envy. In our framework, it is easy to show that the allocation $\{x^*_\theta, -\lambda x^*_\theta\}$ satisfies the PUB in addition to being efficient, IC and individual rational (in the sense that everybody obtains a non-negative payoff). This allocation yields a payoff $b(x^*_\theta, \theta) - \lambda x^*_\theta$ to any arbitrary agent $\theta \in \Theta$. It is IC because the payoffs obtained equal to the ones with the Walrasian allocation with equal endowment $\{x^*_\theta, t^*_\theta\}$ adjusted by a lump-sum "tax" λX, i.e., $b(x^*_\theta, \theta) - \lambda x^*_\theta = b(x^*_\theta, \theta) + t^*_\theta - \lambda X$ for every $\theta \in \Theta$. Since $\{x^*_\theta, t^*_\theta\}$ is IC, so is $\{x^*, -\lambda x^*_\theta\}$. Agent θ's payoff is located in Figure 3 below where the tangency of the benefit function at the efficient consumption level crosses the vertical axis.

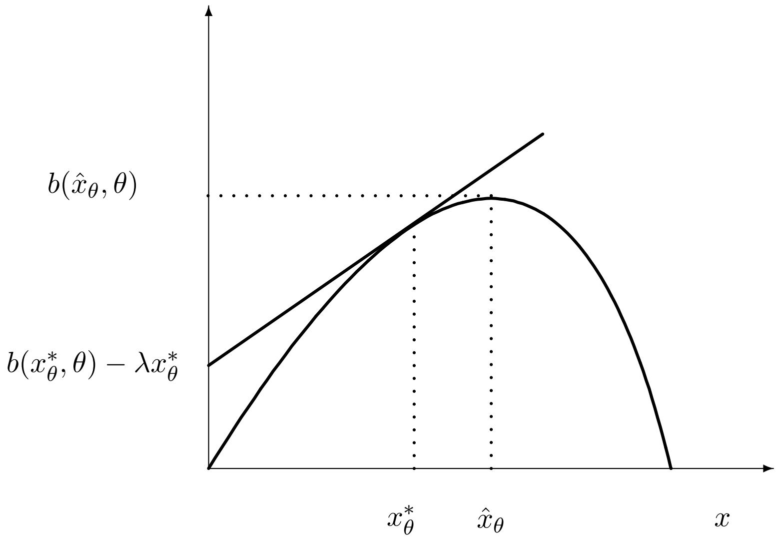

Figure 3: The peak upper bound

By concavity of the benefit function, it is located above 0. Therefore $\{x^*, -\lambda x^*_\theta\}$ is IR. Furthermore, since agent θ's utility is below the peak benefit $b(\hat{x}_\theta, \theta)$ so long as $\lambda > 0$ (i.e. the resource is scarce), the allocation is also PUB. Hence the PUB principle can be obtained by replacing ESIR with individual rationality.

4.3 Consistency

We now focus on another axiom called "consistency". An allocation is consistent if, in any economy, it assigns the same bundles to the "reduced" economy obtained by allowing to some subgroups of agents to leave the scene with their allotted bundles. The remaining agents share their own allotted bundles in this reduced economy. The allocation prescribed in the reduced economy among those remaining agents should not be different to the one prescribed to them in the whole the economy.

In atomless economies (e.g. with a continuum of agents like here) and satiable preferences, Thomson and Zhou (1993) show that the allocations which are supported by equal-budget Walrasian equilibria with "slack" are the only ones that are efficient, ESIR and consistent.[10] The slack comes from the fact that, since preferences might be satiated, some agents may select points in the interior of their budget set. This will never happen here because of the

[10] I wish to thank an anonymous referee for pointing out this paper.

scarcity of the resource and the form of the utility function (diminishing marginal benefit for the resource and no satiated preferences for money).[11]

To see if the Walrasian allocation from equal endowments is consistent, we shall rephrase the model as a fair division problem with two goods: the resource and money. Denote by T the amount of money being shared. So far we assumed that $T = 0$, but we could have any other positive or negative level (e.g. a surplus or a debt). Agents are collectively entitled to X and T. The Walrasian allocation from equal endowments prescribes $t^*_\theta = \lambda(X - x^*_\theta) + T$. Consider the economy reduced to any arbitrary measurable subset of agents $\Omega \in \Theta$. If the others leave with their assigned bundles, the agents in Ω are collectively entitled to $X_\Omega = \int_\Omega x^*_\theta dF(\theta)$ and $T_\Omega = \int_\Omega t^*_\theta dF$. Straightforward computation shows $T_\Omega = \lambda(X - X_\Omega) + T$. Remark that if the subgroup of agents is assigned with less of the resource in average, i.e. $X_\Omega < X$, they have more money in average, i.e. $T_\Omega > T$, and vice-versa.[12] Denote by $\{x^\Omega_\theta, t^\Omega_\theta\}_{\theta\in\Omega}$ the Walrasian allocation from equal endowments of the reduced economy. It prescribes equalizing marginal benefit among agents to the shadow value of the resource which is also λ because the agents in Ω share the same amount of resource in the reduced economy as in the whole economy. Therefore $x^\Omega = x^*_\theta$ for every $\theta \in \Omega$. Second, it assigns $t^\Omega_\theta = \lambda(X - x^*_\theta) + T_\Omega$ for every $\theta \in \Omega$. Since $T^\theta_\Omega = \lambda(X - X_\Omega) + T$, we have $t^\Omega_\theta = t^*_\theta$ for every $\theta \in \Omega$. Therefore, the Walrasian allocation from equal endowments is the same in any reduced economy. It is therefore consistent. According to Thomson and Zhou (1993), it is the only one to also be efficient and ESIR.

4.4 Strict no envy

Lastly, we mention that the Walrasian allocation from equal endowments satisfies a stronger variant of the no envy concept which is strict no envy, as defined in Thomson and Zhou (1993). An allocation is strictly envy-free if no agent prefers the average holding of any group of agents

[11] If the resource was abundant (in the sense that everybody could consume their peak consumption) then it would have no value (zero price). As long as it has a positive price, agents equalize marginal benefit to price, which implies a consumption lower than $\hat{x}_\theta$ for every $\theta \in \Theta$.

[12] For instance, a subgroup composed by agents of type lower (resp. higher) than θ'' such that $x^*_{\theta''} = X$ in Figure 2 have less (resp. more) water per capita to share but more (resp. less) money.

that does not include him to his own.[13] Here, with a continuum of agents in $[\underline{\theta}, \bar{\theta}] = \Theta$, the average consumption level X_Ω of any arbitrary subgroup $\Omega \subset \Theta$ is in the range of the efficient consumption levels $[x^*_{\underline{\theta}}, x^*_{\bar{\theta}}]$. Therefore there exists an agent type $\tilde{\theta}$ that is assigned the average consumption level $X_\Omega = x^*_{\tilde{\theta}}$. Straightforward computation shows that the average amount of money in Ω is $\lambda(X - x^*_{\tilde{\theta}})$ which is also $\tilde{\theta}$'s transfer.[14] Since any agent θ does not envy $\tilde{\theta}$, he does not envy the average bundle of the subgroup Ω. Since the argument applies to any subgroup $\Omega \subset \Theta$, the Walrasian allocation from equal endowments is strictly envy-free.

[13]Note that in the atomless model the exclusion of the agent from the group is irrelevant.

[14]If the amount of money to be shared is $T \neq 0$, then the average money in Ω is $\lambda(X - x^*_{\tilde{\theta}}) + T = t^*_{\tilde{\theta}}$.

Appendix

<u>Part 1</u>: *The allocation* $\{x^*_\theta, t^*_\theta\}$ *is efficient, IC and ESIR.*

First, the resource allocation $\{x^*_\theta\}$ is efficient by definition. Second, since $\int_\Theta t_\theta dF(\theta) = \lambda X - \lambda \int_\Theta x^*_\theta dF(\theta) = 0$, where the last equality is due to the binding resource constraint, the transfer scheme is budget balanced. Third, I show that the allocation $\{x^*_\theta, t^*_\theta\}$ is ESIR. The constraint $ESIR(\theta)$ writes

$$b(x^*_\theta, \theta) - \lambda x^*_\theta + \lambda X \geq b(\min\{X, \hat{x}_\theta\}, \theta). \tag{4}$$

The proof relies on the concavity of the benefit function. Consider all $\theta \in \Theta$ such that $x^*_\theta \geq X$. Since b is concave,

$$b(x^*_\theta, \theta) - b(X, \theta) \geq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(x^*_\theta - X).$$

Substitute $\frac{\partial b}{\partial x}(x^*_\theta, \theta)$ with λ to obtain:

$$b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(X, \theta),$$

which is (4) when $X \leq x^*_\theta \leq \hat{x}_\theta$. Consider now all agents with $x^*_\theta \leq X \leq \hat{x}_\theta$. Since b is concave,

$$b(X, \theta) - b(x^*_\theta, \theta) \leq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(X - x^*_\theta).$$

Replace $\frac{\partial b}{\partial x}(x^*_\theta, \theta)$ by λ to obtain:

$$b(X, \theta) \leq b(x^*_\theta, \theta) + \lambda(X - x^*_\theta),$$

which is (4) for $X \leq \hat{x}_\theta$. Finally, for agents θ such that $x^*_\theta \leq X$ but $X \geq \hat{x}_\theta$, the equal sharing benefit is $b(\hat{x}_\theta, \theta)$, i.e. the peak benefit. Again, b concave and $x^*_\theta \leq \hat{x}_\theta$ imply:

$$b(\hat{x}_\theta, \theta) - b(x^*_\theta, \theta) \leq \frac{\partial b}{\partial x}(x^*_\theta, \theta)(\hat{x}_\theta - x^*_\theta),$$

for every $\theta \in \Theta$, which is equivalent to,

$$b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(\hat{x}_\theta, \theta) + \lambda(X - \hat{x}_\theta). \tag{5}$$

For $\hat{x}_\theta \leq X$, the above condition implies:

$$b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(\hat{x}_\theta, \theta),$$

which is (4) for $X \geq \hat{x}_\theta$.

Fourth, I turn to the IC condition. Pick any agent θ. If he selects the bundle $(x_{\hat{\theta}}, t_{\hat{\theta}})$, he obtains $u(x^*_\theta, t^*_\theta, \theta) = b(x_{\hat{\theta}}, \theta) + \lambda(X - x_{\hat{\theta}})$. Since $\dfrac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$ is the first-order condition of the maximization program $\max_x b(x,\theta) - \lambda x + \lambda X$, $b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(x^*_{\theta'}, \theta) + \lambda(X - x^*_{\theta'})$ for every $\theta' \in \Theta$. Replace $\lambda(X - x^*_\theta)$ and $\lambda(X - x^*_{\theta'})$ by, respectively, t^*_θ and $t^*_{\theta'}$ to obtain

$$b(x^*_\theta, \theta) + t^*_\theta \geq b(x_{\theta'}, \theta) + t_{\theta'} \tag{6}$$

for every $\theta' \in \Theta$, which is the incentive-compatibility condition *without* free disposal. Condition (5) implies $b(x^*_\theta, \theta) + \lambda(X - x^*_\theta) \geq b(\hat{x}_\theta, \theta) + \lambda(X - x_{\theta'})$ for every $x_{\theta'} \geq \hat{x}_\theta$. Since $t_\theta = \lambda(X - x_\theta)$ and $t^*_{\theta'} = \lambda(X - x_{\theta'})$, it leads to

$$b(x^*_\theta, \theta) + t^*_\theta \geq b(\hat{x}_\theta, \theta) + t_{\theta'},$$

for every θ' such that $x_{\theta'} \geq \hat{x}_\theta$. The last conditions and inequality (6) for every $\theta' \in \Theta$ imply $b(x^*_\theta, \theta) + t^*_\theta \geq b(\min\{x^*_{\theta'}, \hat{x}_{\theta'}\}, \theta) + t_{\theta'}$ which is $IC(\theta)$. Since the argument applies for every $\theta \in \Theta$, the allocation $\{x^*_\theta, t^*_\theta\}$ is IC.

Part 2: Any efficiency, IC and ESIR allocation $\{x_\theta, t_\theta\}$ is such that $x_\theta = x^*_\theta$ and $t_\theta = \lambda(X - x^*_\theta) = t^*_\theta$ for every $\theta \in \Theta$.

Suppose that $\{x_\theta, t_\theta\}$ is an efficient, IC and ESIR (budget-balanced and feasible) allocation. We already know that efficiency and feasibility implies $x_\theta = x^*_\theta$ for every $\theta \in \Theta$.

Since $\{x^*_\theta, t_\theta\}$ satisfies $IC(\theta)$ for an arbitrary $\theta \in \Theta$, then $\theta \in \arg\max_{\tilde{\theta} \in \Theta} b(x^*_{\tilde{\theta}}, \theta) + t_{\tilde{\theta}}$. The first-order condition of the above maximization program yields,

$$\frac{\partial b}{\partial x_\theta}(x^*_\theta, \theta)\frac{dx^*_\theta}{d\theta} + \frac{dt_\theta}{d\theta} = 0.$$

Substitute $\dfrac{\partial b}{\partial x}(x^*_\theta, \theta) = \lambda$ to obtain

$$\frac{dt_\theta}{d\theta} = -\lambda\frac{dx^*_\theta}{d\theta}.$$

The above equality implies that $\exists\gamma$ such that $t_\theta = -\lambda x^*_\theta + \gamma$. Hence, we have shown that $IC(\theta)$ implies $t_\theta = \gamma - \lambda x^*_\theta$ for every $\theta \in \Theta$ where $\gamma \in \mathbb{R}$ remains to be specified.

The budget constraint becomes:

$$\gamma - \lambda \int_\Theta x^*_\theta dF(\theta) \leq 0.$$

Due to the feasibility constraint, it implies $\gamma \leq \lambda X$. Now by ESIR

$$b(x_\theta^*, \theta) - \lambda x_\theta^* + \gamma \geq b(\min\{X, \hat{x}_\theta\}, \theta).$$

Since $\underline{\theta} < \bar{\theta}$ (i.e. there is some heterogeneity among agents), the efficiency first-order condition (3) and the feasibility constraint imply $x_{\underline{\theta}}^* < X < x_{\bar{\theta}}^*$. Moreover, the assumptions on the benefit functions (including the single crossing property) imply that x_θ^* is continuously increasing in $[x_{\underline{\theta}}^*, x_{\bar{\theta}}^*]$. Therefore there exists an agent type θ'' who is assigned the average level of the resource $x_{\theta''}^* = X < \hat{x}_{\theta''}$ (see Figure 2). For this agent, the above ESIR condition becomes

$$b(X, \theta'') - \lambda X + \gamma \geq b(X, \theta''),$$

which leads to $\gamma \geq \lambda X$. Hence, combining the budget constraint, which implies $\gamma \leq \lambda X$, with ESIR and efficiency, which imply $\gamma \geq \lambda X$, shows that $\gamma = \lambda X$. Therefore $t_\theta = \lambda(X - x_\theta^*) = t_\theta^*$ for every $\theta \in \Theta$.

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