Sharing a River among Satiable Agents[*]

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Abstract

We consider the problem of efficiently sharing water from a river among a group of satiable agents. Since each agent's benefit function exhibits a satiation point, the environment can be described as a cooperative game with externalities. We show that the downstream incremental distribution is the unique distribution which both is fair according to the "aspiration welfare" principle and satisfies the non-cooperative core lower bounds. On the other hand, the cooperative core may be empty. Furthermore, the downstream incremental distribution satisfies all core lower bounds for all connected coalitions if and only if each agent's individual rationality constraint is independent of the behavior of the other agents.

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1 Introduction

In many economic and political environments the characteristics of a prisoner's dilemma are present: the non-cooperative equilibrium is inefficient and enforcing the efficient outcome requires cooperation (Hardin, 1968; Ostrom, 1990). International agreements determine how to achieve cooperation among a group of countries and specify how to make monetary compensations to distribute the resulting benefits. Examples are the European Union, the GATT, and the Kyoto Protocol on greenhouse emissions.

In all these environments the benefit of a group of countries depends on *how* the other countries behave. Cooperation of the other countries exerts an externality on the value (or benefit) of a coalition (or group) of countries. In other words, these environments can be described as a cooperative game with externalities.[1]

A natural requirement of any agreement is that a subgroup of countries should not be better off by signing a separate agreement. An agreement belongs to the core if it is not blocked by a subgroup of countries. In the presence of externalities the stability of an agreement depends on how countries act after the deviation of a coalition. We might make two extreme assumptions about the behavior of countries outside the blocking coalition: either they continue to cooperate by signing their own agreement or they do not cooperate at all. The non-cooperative core requires that the agreement is not blocked by a subgroup assuming that the countries outside do not cooperate. This concept corresponds to Hart and Kurz (1983)'s notion of γ-stability of agreements whereby an agreement is disbanded once a coalition deviates. Similarly, the cooperative core imposes that the agreement is not blocked by a subgroup assuming that the countries outside do all cooperate. This concept corresponds to Hart and Kurz (1983)'s notion of δ-stability whereby countries continue to act together after the deviation of a coalition.

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[1]For cooperative games with externalities several recent papers offer extensions of the Shapley value of games without externalities (see Maskin (2003), de Clippel and Serrano (2005), Navarro (2007) and Macho-Stadler, Pérez-Castrillo, and Wettstein (2007)).

We consider international agreements for sharing water resources of a river.[2] A set of agents is located along the river and the river picks up volume along its course. Each agent extracts water from the river for consumption and/or production. Agents value water differently in the sense that some have higher needs and higher marginal utility /productivity than others. These heterogeneous valuations are represented by concave and single-peaked benefit functions, where the peak consumption corresponds to an agent's satiation point.

Kilgour and Dinar (2001) and Ambec and Sprumont (2002) considered the special case when each agent's benefit function is strictly increasing and satiation points do not exist. This assumption appears unnatural because in reality overconsumption may cause flooding or increase sanitation costs with higher water extraction costs. We show that under single-peaked benefit functions the environment can be described as a cooperative game with externalities. The values assigned to coalitions are the result of a non-cooperative game.

Because property rights over water are not well defined, there are two conflicting doctrines invoked by riparian countries in international river disputes: the theory of *absolute territorial sovereignty* (ATS) and the theory of *unlimited territorial integrity* (UTI) respectively (see Godona, 1985). Core lower bounds are inspired by ATS. Under UTI an agent (or group of agents) could freely use the full stream of water originating upstream from its location if the other agents are absent, thereby enjoying a benefit called "aspiration welfare". Since water is scarce, not everybody can enjoy its aspiration welfare. A welfare distribution that assigns to any agent or group of agents more than its aspiration welfare should be perceived as unfair. The aspiration welfare defines upper bounds on welfare for any coalition of agents.[3]

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[2]The importance of this problem has been empirically shown by Godana (1985) and Barrett (1994). The field of research on water allocation is increasingly important with diminishing water reserves (Young and Haveman, 1995; Carraro, Marchiori, and Sgobbi, 2005, Griffin, 2006).

[3]Notice that, in a recent paper, Ni and Wang (2007) apply the ATS and UTI principles to the problem of dividing the cost of cleaning a polluted river.

Under non-cooperative behavior there exist distributions satisfying the core lower bounds. Our first main result shows that the downstream incremental distribution is the unique distribution satisfying the non-cooperative core lower bounds and the aspiration upper bounds. The downstream incremental distribution is the incremental distribution corresponding to the natural order of the river. Our second main result shows that for more than three agents, there may not exist any distribution satisfying the cooperative core lower bounds. Therefore, the cooperative core lower bounds are above the non-cooperative core lower bounds. In general cooperation exerts a positive externality on the value of a coalition compared to its value under non-cooperative behavior.

Since the core is empty, we may allow only connected coalitions to block. Even if blocking is restricted to these coalitions, the core may still be empty. Our third main result gives a simple necessary and sufficient condition for the existence of a distribution satisfying the aspiration upper bounds and the core lower bounds for all connected coalitions independently of the other agents' behavior. The condition is that cooperation exerts no externality on the value of any agent. Since all core lower bounds are above the non-cooperative core lower bounds, it follows that the downstream incremental distribution is not blocked by any connected coalition independently of the other agents' behavior if and only if the individual rationality constraints are identical under all behaviors of the other agents.

The paper is organized as follows. In Section 2 we introduce the problem of sharing a river among satiable agents and we determine necessary and sufficient conditions for an efficient water consumption plan. In Section 3 we calculate the value of a coalition for each partition of the agents via the backwards induction algorithm applied to a dynamic game induced by the structure of the river and the partition of the agents. In Section 4 we focus on non-cooperative behavior and show that the downstream incremental distribution is the unique distribution satisfying the non-cooperative core lower bounds and the aspiration upper bounds. In Section 5 we turn to cooperative

behavior and show that for more than three agents there may not exist any distribution satisfying the cooperative core lower bounds. Furthermore, the downstream incremental distribution satisfies all core lower bounds for all connected coalitions if and only if the cooperation exerts no externality on the value of any agent. In Section 6 we conclude.

2   The Problem

Let $N = \{1, \ldots, n\}$ denote the set of agents (or countries). We identify agents with their locations along the river and number them from upstream to downstream: $i < j$ means that i is upstream from j. A coalition is a non-empty subset of N. Given two coalitions S and T, we write $S < T$ if $i < j$ for all $i \in S$ and all $j \in T$. Given a coalition S, we denote by $\min S$ and $\max S$, respectively, the smallest and largest members of S. Let $Pi = \{1, \ldots, i\}$ denote the set of predecessors of agent i and $P^0 i = Pi \backslash \{i\}$ denote the set of strict predecessors of agent i. Similarly, let $Fi = \{i, i + 1, \ldots, n\}$ denote the set of followers of agent i and let $F^0 i = Fi \backslash \{i\}$ denote the set of strict followers of i. A coalition S is connected if for all $i, j \in S$ and all $k \in N$, $i < k < j$ implies $k \in S$. Given a coalition S, let $\mathcal{C}(S)$ denote the set of connected components of S, i.e. $\mathcal{C}(S)$ is the coarsest partition of S such that any $T \in \mathcal{C}(S)$ is connected. We often omit set brackets for sets and write i instead of $\{i\}$ or $v(i, j)$ instead of $v(\{i, j\})$.

The river picks up volume along its course. We denote by $e_i \geq 0$ the volume which the river picks up at agent i's location (or in country i). Each agent is endowed with a benefit function. Let $b_i : \mathbb{R}_+ \to \mathbb{R}$ denote agent i's benefit function. We assume that b_i is differentiable for all $x_i > 0$ and strictly concave. Furthermore, $b_i'(x_i)$ goes to infinity as x_i tends to 0 and there exists a satiation point $\hat{x}_i > 0$ such that $b_i'(\hat{x}_i) = 0$. In other words, $\hat{x}_i$ is agent i's optimal (water) consumption and if he consumes more than $\hat{x}_i$, then he will infer a loss (compared to consuming $\hat{x}_i$) from overconsumption.

A problem is a triple (N, e, b) where $e = (e_i)_{i \in N}$ and $b = (b_i)_{i \in N}$. Given a problem,

a consumption plan for N is a vector $x(N) \in \mathbb{R}^N_+$ such that for all $j \in N$

$$\sum_{i \in Pj} x_i(N) \leq \sum_{i \in Pj} e_i.$$

The above constraint says that the water e_i, which is picked up by the river at agent i's location, can only be consumed by i and the agents which are located downstream from i.[4]

Given a consumption plan $x(N)$ and an agent i, let

$$E_i(x(N)) = \sum_{j \in P^0 i} (e_j - x_j(N))$$

denote the amount of water which is passed to agent i from his strict predecessors $P^0 i$ in the consumption plan $x(N)$ (with the convention $E_1(x(N)) = 0$).[5]

We call $x^*(N)$ an optimal (or efficient) consumption plan if and only if it maximizes the sum of all agents' benefits. Note that here it is suboptimal for any agent to consume more than $\hat{x}_i$. Therefore, it may be suboptimal to use all the water $\sum_{i \in N} e_i$. Now analogously to Ambec and Sprumont (2002) there exists a unique optimal consumption plan $x^*(N)$ and that for $x^*(N)$ there exists a partition $\{N_k\}_{k=1,\dots,K}$ of N and a list $(\beta_k)_{k=1}^K$ of non-negative numbers such that[6]

$$N_k < N_{k'} \text{ and } \beta_k > \beta_{k'} \text{ whenever } k < k' \tag{1}$$

$$b_i'(x_i^*(N)) = \beta_k \text{ for every } i \in N_k \text{ and every } k = 1, \dots, K \tag{2}$$

$$x_i^*(N) \leq \hat{x}_i \text{ for all } i \in N \tag{3}$$

$$\sum_{i \in N_k} (x_i^*(N) - e_i) = 0 \text{ for every } k = 1, \dots, K - 1. \tag{4}$$

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[4]This makes our problem different from both the allocation of a private good with the possibility of sidepayments and queuing problems where the order of the agents is flexible and agents are compensated for the welfare maximizing queue (see among others Maniquet (2003) and Chun (2004)).

[5]Agent 1 does not receive any water from the other agents because agent 1 occupies the first location of the river.

[6]For a detailed description of the efficient allocation of water along a river, see Kilgour and Dinar (2001). Furthermore, recall that $S < T$ means $i < j$ for all $i \in S$ and all $j \in T$.

Thus, if $x_i^*(N) = \hat{x}_i$, then $i \in N_K$, i.e. the saturated agents belong to the last member N_K of the partition.

Money is available in unbounded quantities to perform sidepayments. Agent i's utility from consuming x_i units of water and the monetary transfer t_i is $u_i(x_i, t_i) = b_i(x_i) + t_i$. An allocation is a tuple $(x(N), t(N))$ where $x(N)$ is a consumption plan for N and $t(N) \in \mathbb{R}^N$ is a vector of monetary transfers such that $\sum_{i \in N} t_i(N) \leq 0$. A (welfare) distribution is any vector $z = (z_1 \ldots, z_n) \in \mathbb{R}^N$ which is the utility image of some allocation $(x(N), t(N))$ in the sense that $z_i = b_i(x_i(N)) + t_i(N)$ for all $i \in N$. We distribute the maximal welfare $\sum_{i \in N} b_i(x_i^*(N))$ among the agents.[7]

3 Externalities and Core Lower Bounds

Since each agent's benefit function is single-peaked, any agent never consumes more than his satiation point. If marginal benefits are higher for agents located more downstream, then it may be profitable for a coalition to pass some water from one component to another component even though some of the passed water is consumed by agents in between the two components. Therefore, the value of a coalition may be greater than the sum of the values of its connected components. However, it may be also profitable for the agents outside of S to pass some water from one component to the next one leaving some water for consumption for the agents in S. Hence, *the value of a coalition S will depend on both the components of S and the behavior of the agents outside of S.* In other words, the behavior of the agents outside of S exerts an externality on the value of coalition S. In what follows we will assume that the agents outside of S form a partition and each member of the partition is maximizing its surplus for any amount of water which is not used by the predecessors. Furthermore, by the structure of the river, any amount of unused water can only be transferred downstream and each member of the partition is maximizing its surplus

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[7]Note that any vector $z \in \mathbb{R}^N$ such that $\sum_{i \in N} z_i = \sum_{i \in N} b_i(x_i^*(N))$ is a distribution because it is the utility image of $(x^*(N), t^*(N))$ where $t_i^*(N) = z_i - b_i(x_i^*(N))$ for all $i \in N$.

at any of its connected components for any amount of water, which is not used by the predecessors of this connected component. Therefore, the outcome is a "subgame perfect Nash equilibrium of the dynamic game with perfect information given by the river".

Let $v(S, \mathcal{P})$ denote the value of S when the partition $\mathcal{P}$ of N forms where $S \in \mathcal{P}$. The calculation of $v(S, \mathcal{P})$ follows the simple backwards induction algorithm along the river. Here each coalition belonging to $\mathcal{P}$ is a player in the extensive form game with perfect information (given by the river). The nodes of play are given by the connected components of all coalitions in $\mathcal{P}$. Information is perfect because at any node of play the amount of unused water from the strict predecessors is observed (or equivalently the consumptions of the strict predecessors are observed). A subgame consists of an initial node of play and an amount of unused water which is passed to the initial node of play by its strict predecessors. In the subgame each node of play, which (weakly) follows the initial one, receives an amount of unused water from its strict predecessors (or equivalently observes the consumption plans chosen by the previous nodes) and chooses a feasible consumption plan given this amount of unused water. The backwards induction algorithm calculates for each subgame the feasible consumption plan of the initial node which maximizes the sum of their benefits plus the sum of the benefits of all components which belong to the same coalition and are further down the river. Here the reactions of the components further down the river are already given by the amount of water which the initial component passes to the following component. The outcome of the backwards induction algorithm is the consumption plan of the (sub)game starting with the first component of the river (agent 1 belongs to this component) and no amount of unused water is received by this first component. Then $v(S, \mathcal{P})$ is equal to the sum of the benefits all agents belonging to S receive in the outcome of the backwards algorithm.

Formally, let $\cup_{T \in \mathcal{P}} \mathcal{C}(T) = \{T_1, \ldots, T_k\}$ be such that $T_1 < \cdots < T_k$. The backwards induction algorithm calculates for each component and each amount of unused water

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received by this component a feasible consumption plan which is optimal for this component and the components further down the river which belong to the same coalition:[8]

(k) For all $E' \geq 0$, let $x^*(T_k, E')$ be the optimal consumption plan for $(T_k, (e_{\min T_k} + E', e_{T_k \backslash \{\min T_k\}}), b_{T_k})$;

(k-1) For all $E' \geq 0$, let $x^*(T_{k-1}, E')$ be the optimal consumption plan for $(T_{k-1}, (e_{\min T_{k-1}} + E', e_{T_{k-1} \backslash \{\min T_{k-1}\}}), b_{T_{k-1}})$; note that T_{k-1} and T_k necessarily belong to different members of $\mathcal{P}$; after the choice of $x^*(T_{k-1}, E')$, the amount $E_k(x^*(T_{k-1}, E')) = E' + \sum_{i \in T_{k-1}}(e_i - x_i^*(T_{k-1}, E'))$ of unused water is passed from T_{k-1} to T_k and T_k chooses the consumption plan $x^*(T_k, E_k(x^*(T_{k-1}, E')))$.

$\vdots$

(l) Given E' and the volume the river picks up along the locations in T_l, $x(T_l, E')$ is a feasible consumption plan for T_l if $E' + \sum_{i \in T_l \cap Pj}(e_i - x_i(T_l, E')) \geq 0$ for all $j \in T_l$. By backwards induction, suppose that for all components $T_{l'}$ following T_l ($l' \in \{l+1, \dots, k\}$) and all amounts of water $E' \geq 0$ we have defined $x^*(T_{l'}, E')$. Given these choices, a fixed $E' \geq 0$ and a feasible consumption plan $x(T_l, E')$, let $E_{l+1}(x(T_l, E')) = E' + \sum_{i \in T_l}(e_i - x_i(T_l, E'))$ be the amount of water passed from T_l to T_{l+1}, let $E_{l+2}(x(T_l, E')) = E_{l+2}(x^*(T_{l+1}, E_{l+1}(x(T_l, E'))))$ be the amount of water passed from T_l and T_{l+1} to T_{l+2}, and in general, for $t \in \{1, \dots, k-l\}$, let $E_{l+t}(x(T_l, E')) = E_{l+t}(x^*(T_{l+t-1}, E_{l+t-1}(x(T_l, E'))))$ be the amount of water passed from $T_l, \dots, T_{l+t-1}$ to T_{l+t}.

Let $T \in \mathcal{P}$ be such that $T_l \subseteq T$. Then for all $E' \geq 0$, let $x^*(T_l, E')$ be the consumption plan for T_l which solves

$$\max_{x(T_l, E')} \sum_{i \in T_l} b_i(x_i(T_l, E')) + \sum_{l' \in \{l+1, \dots, k\}: T_{l'} \subseteq T} \sum_{i \in T_{l'}} b_i(x_i^*(T_{l'}, E_{l'}(x(T_l, E'))))$$

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[8]For any $S \subseteq N$, let $b_S = (b_i)_{i \in S}$ and $e_S = (e_i)_{i \in S}$.

where $x(T_l, E')$ is a feasible consumption plan for T_l given E'. In other words, $x(T_l, E')$ maximizes the surplus of T in the subgame starting at T_l given E' and how the other components react on any amount of water which arrives at each component following T_l.

$\vdots$

From the concavity of the b_i we obtain that each component's optimal consumption plan is unique. We denote the outcome of the backwards induction algorithm applied to $\mathcal{P}$ by $x^{\mathcal{P}}(N)$ where $x_{T_1}^{\mathcal{P}}(N) = x^*(T_1, 0)$ and for all $l \in \{2, \ldots, k\}$, $x_{T_l}^{\mathcal{P}}(N) = x^*(T_l, E_l(x^*(T_1, 0)))$. Then for $S \in \mathcal{P}$ we define

$$v(S, \mathcal{P}) = \sum_{i \in S} b_i(x_i^{\mathcal{P}}(N)).$$

We will also call $v(S, \mathcal{P})$ the core lower bound of S given that partition $\mathcal{P}$ of N forms.

Remark 1 The outcome of the backwards induction algorithm may not be unique because some coalitions may be indifferent between passing water and not passing any water. In the rare case of indifference at the outcome $x^{\mathcal{P}}(N)$, we assume that any coalition is passing water instead of not passing any water. Given $\mathcal{P}$, this assumption ensures that the value of any coalition $S \in \mathcal{P}$ is maximal among all outcomes of the backwards induction algorithm.

We focus on subgame perfect equilibrium since we think that these are the only kind of equilibria that are expected to emerge given the sequential nature of the decisions made by the agents.

Remark 2 The following is an important observation. Suppose that for some agent we have $e_i > \hat{x}_i$. Then for any partition $\mathcal{P}$ of N, in the outcome of the backwards induction algorithm agent i will never consume more than $\hat{x}_i$, i.e. $x_i^{\mathcal{P}}(N) \leq \hat{x}_i$ and i will always dispose $e_i - \hat{x}_i$ (independently of whether i is a singleton in $\mathcal{P}$ or not). Now define $e_i' = \hat{x}_i$ and $e_{i+1}' = e_{i+1} + (e_i - \hat{x}_i)$ and let $e' = (e_{N \setminus \{i, i+1\}}, e_i', e_{i+1}')$.[9] It

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[9] If $i = n$, then we set $e' = (e_{N \setminus \{n\}}, e_n')$, i.e. the amount $e_n - \hat{x}_n$ is not consumed by any agent.

is immediate that for both problems (N, e, b) and (N, e', b), $x^{\mathcal{P}}(N)$ is the outcome of the backwards induction algorithm applied to $\mathcal{P}$. Furthermore, we also obtain from (1)-(4) that $x^*(N)$ is an optimal consumption plan for the problem (N, e, b) if and only if $x^*(N)$ is an optimal consumption plan for the problem (N, e', b). Thus, from now on we may suppose without loss of generality that in the problem (N, e, b) we have $e_i \leq \hat{x}_i$ for all $i \in N$.

The two extremes of behavior of the agents outside of S are the following: either they do not cooperate at all or they all cooperate.

Non-Cooperative Core Lower Bounds: For all coalitions S, let $\underline{v}(S) = v(S, \{S\} \cup \{\{i\}|i \in N \backslash S\})$.

Cooperative Core Lower Bounds: For all coalitions S, let $\overline{v}(S) = v(S, \{S, N \backslash S\})$.

We say that *cooperation exerts no externality on a coalition S* if for any partition $\mathcal{P}$ of N such that $S \in \mathcal{P}$,

$$\underline{v}(S) = v(S, \mathcal{P}).$$

Then the value of a coalition is independent of the interactions of the other agents. We say that *cooperation exerts a positive externality on a coalition S* if for any partition $\mathcal{P}$ of N such that $S \in \mathcal{P}$,

$$\underline{v}(S) \leq v(S, \mathcal{P}).$$

Then cooperation does not decrease the value of a coalition compared to the value under non-cooperative behavior.

The following proposition contains some basic relations among the core lower bounds of a coalition for different behaviors of its complement. First, cooperation exerts a positive externality on a coalition. Second, the following super-additivity property is true: for any partition of N, if two coalitions belonging to the partition

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merge, then their joint payoff does not fall compared to the payoff when they are separate.

Proposition 1 *Let $\mathcal{P}$ be partition of N and $S \in \mathcal{P}$.*

(i) $\underline{v}(S) \leq v(S, \mathcal{P})$.

(ii) $\underline{v}(S) \leq \overline{v}(S)$.

(iii) For any two disjoint coalitions $S, T \in \mathcal{P}$, $v(S, \mathcal{P}) + v(T, \mathcal{P}) \leq v(S \cup T, \mathcal{P}')$ where $\mathcal{P}' = (\mathcal{P} \backslash \{S, T\}) \cup \{S \cup T\}$.

The proof of Proposition 1 is obvious and is left to the reader (or see Ambec and Ehlers, 2006).

It is immediate from our definition that the value of a coalition consisting of an agent and his predecessors is independent of how the other agents behave, i.e. for all $i \in N$ and all $Pi \in \mathcal{P}$ we have

$$\underline{v}(Pi) = v(Pi, \mathcal{P}) = \overline{v}(Pi).$$

Thus, cooperation exerts no externality on the coalition Pi. Even though the value of a coalition may depend on how the other agents behave, the structure of the river induces a unique natural incremental distribution, namely the downstream incremental distribution z^*: for all $i \in N$, let

$$z_i^* = \underline{v}(Pi) - \underline{v}(P^0 i).$$

4 Non-Cooperative Core Lower Bounds and Aspiration Upper Bounds

The aspiration upper bounds are implied by the UTI doctrine. Contrary to the core lower bounds, these upper bounds do not depend on how the agents outside of a

coalition behave. The aspiration welfare of a coalition S is the highest welfare it could achieve in the absence of the agents in $N \backslash S$ while the volumes, which the river picks up at the locations in $N \backslash S$, exist. It is obtained by choosing a consumption plan $y(S) \in \mathbb{R}_+^S$ maximizing $\sum_{i \in S} b_i(y_i(S))$ subject to the constraints

$$\sum_{i \in Pj \cap S} y_i(S) \leq \sum_{i \in Pj} e_i \text{ for all } j \in S.$$

Since all benefit functions are strictly concave, the maximization problem has a unique solution, which we denote by $y^*(S)$. Then the aspiration welfare of S is

$$w(S) = \sum_{i \in S} b_i(y_i^*(S)).$$

A distribution z satisfies the aspiration upper bounds if $\sum_{i \in S} z_i \leq w(S)$ for all coalitions S. In the Lockean tradition, coalition S has a legitimate right to the welfare level $w(S)$ but not to more. Unfortunately the aspirations of two complementary coalitions S and $N \backslash S$ are incompatible: $w(S) + w(N \backslash S) > \underline{v}(N)$. It is even the case that for any partition $\mathcal{P}$ of N such that $S \in \mathcal{P}$ we have $\sum_{T \in \mathcal{P}} w(T) > \underline{v}(N)$, i.e. the aspiration of S is never compatible with the aspiration(s) of $N \backslash S$ independently of how $N \backslash S$ cooperates. Therefore, if $\sum_{i \in S} z_i > w(S)$, then $\sum_{i \in N \backslash S} z_i < \sum_{T \in \mathcal{P}: T \neq S} w(T)$. This means that S benefits from the existence of $N \backslash S$ while $N \backslash S$ suffers from the existence of S. If none of the agents bears any responsibility for the existence of the others, no coalition is ought to enjoy more than its aspiration upper bound.

Remark 3 Both the ATS and the UTI doctrines are also inspired by Moulin's (1990) group externalities depending on how we define property rights over water. In the absence of the other agents and the water entering the river at their locations, any agent i enjoys $\underline{v}(i)$. Since $\sum_{i \in N} \underline{v}(i) \leq \underline{v}(N)$, then our problem has positive group externalities and any agent i should receive at least $\underline{v}(i)$. This inspires the ATS doctrine for individuals and groups. In the absence of the other agents and the presence of the water entering the river at their locations, any agent enjoys i enjoys $w(i)$. Since $\sum_{i \in N} w(i) \geq \underline{v}(N)$, then our problem has negative group externalities

and any agent i should receive at most $w(i)$. This inspires the UTI doctrine for individuals and groups.

Remark 4 There is an obvious relation between the non-cooperative core lower bounds and the aspiration upper bounds: $\underline{v}(Pi) = w(Pi)$ for all $i \in N$. Now the following is easy to show: if a distribution satisfies the non-cooperative lower bounds and the aspiration upper bounds, then it must be the downstream incremental distribution.[10]

The main challenge of our paper is to find distributions which satisfy core lower bounds. These bounds depend on the behavior of the agents outside of a coalition. In the case of non-cooperative behavior, there are distributions satisfying the core lower bounds (or belonging to the γ-core). It turns out that in the presence of optimal water consumptions and non-cooperative behavior, the downstream incremental distribution is the only compromise between the ATS and the UTI doctrines.

Theorem 1 *The downstream incremental distribution is the unique distribution satisfying the non-cooperative core lower bounds and the aspiration upper bounds.*

Proof. By Remark 4, if a distribution z satisfies the non-cooperative core lower bounds and the aspiration upper bounds, then we must have $z = z^*$.

Next we show that z^* satisfies the non-cooperative lower bounds. Let S be connected and $\mathcal{P} = \{S\} \cup \{\{i\}|i \in N\backslash S\}$. Because behavior is non-cooperative, we have for all $i \in P^0 \min S$, $x_i^{\mathcal{P}}(N) = e_i$. Thus, $E_{\min S}(x^{\mathcal{P}}(N)) = 0$. Since S is connected, $\{S, P^0 \min S\}$ is a partition of $P \max S$. Hence, by $E_{\min S}(x^{\mathcal{P}}(N)) = 0$,

$$\underline{v}(P \max S) \geq \underline{v}(S) + \underline{v}(P^0 \min S).$$

[10]The proof is identical to Ambec and Sprumont (2002): Let z be a distribution satisfying the non-cooperative lower bounds and the aspiration upper bounds. Since $\underline{v}(1) = w(1)$, we have $z_1 = \underline{v}(1) = z_1^*$. Let $z_i = z_i^*$ for all $i < j \leq n$. Since $\underline{v}(Pj) = w(Pj)$, we have $\sum_{i \in Pj} z_i = \underline{v}(Pj)$. Thus, by $\sum_{i \in P^0j} z_i = \sum_{i \in P^0j} z_i^* = \underline{v}(P^0j)$, we obtain $z_j = \underline{v}(Pj) - \sum_{i \in P^0j} z_i = \underline{v}(Pj) - \underline{v}(P^0j) = z_j^*$, the desired conclusion.

Thus, for any connected S,

$$\sum_{i \in S} z_i^* = \underline{v}(P \max S) - \underline{v}(P^0 \min S) \geq \underline{v}(S). \tag{5}$$

Before we proceed, we note the following: for all $i \in N$ we have $\underline{v}(P^0 i) + b_i(\hat{x}_i) \geq \underline{v}(Pi)$. Thus, for all $i \in N$,

$$b_i(\hat{x}_i) \geq \underline{v}(Pi) - \underline{v}(P^0 i) = z_i^*. \tag{6}$$

Let S be an arbitrary coalition and let $\mathcal{P} = \{S\} \cup \{\{i\} | i \in N \backslash S\}$. Since $e_i \leq \hat{x}_i$ for all $i \in N$, we have $\sum_{i \in P^0 \min S}(x_i^{\mathcal{P}}(N) - e_i) = 0$. Hence, $E_{\min S}(x^{\mathcal{P}}(N)) = 0$. Let $\mathcal{C}(S) = \{S_1, \dots, S_L\}$ where $S_1 < S_2 < \cdots < S_L$. Choose the minimal $l \in \{1, \dots, L\}$ such that $E_{\max S_l+1}(x^{\mathcal{P}}(N)) = 0$ and set $T_1 = \cup_{t=1}^l S_t$. Then, by $e_i \leq \hat{x}_i$ for all $i \in N$, again we have $E_{\min S_{l+1}}(x^{\mathcal{P}}(N)) = 0$. Now choose the $l' > l$ minimal such that $E_{\max S_{l'}+1}(x^{\mathcal{P}}(N)) = 0$ and set $T_2 = \cup_{t=l+1}^{l'} S_t$. Continuing this way we find a partition $\mathcal{T} = \{T_1, T_2, \dots, T_M\}$ of S. By construction, $T_1 < T_2 < \cdots < T_M$ and

$$\underline{v}(S) = \sum_{T \in \mathcal{T}} \underline{v}(T). \tag{7}$$

For each $T \in \mathcal{T}$, let $\bar{T} = P \max T \backslash P^0 \min T$. Then by definition of T, we have for all $i \in \bar{T} \backslash T$, $E_i(x^{\mathcal{P}}(N)) > 0$ and therefore, $x_i^{\mathcal{P}}(N) = \hat{x}_i$ for all $i \in \bar{T} \backslash T$. Now we have

$$
\begin{aligned}
\sum_{T \in \mathcal{T}} \sum_{i \in \bar{T}} z_i^* &\geq \sum_{T \in \mathcal{T}} \underline{v}(\bar{T}) \\
&\geq \sum_{T \in \mathcal{T}} \left(\underline{v}(T) + \sum_{i \in \bar{T} \backslash T} b_i(\hat{x}_i) \right) \\
&= \underline{v}(S) + \sum_{T \in \mathcal{T}} \sum_{i \in \bar{T} \backslash T} b_i(\hat{x}_i),
\end{aligned}
$$

where the first equality follows from (5) and the fact that each $\bar{T}$ is connected, the second inequality follows from the fact that $x_i^{\mathcal{P}}(N) = \hat{x}_i$ for all $i \in \bar{T} \backslash T$ and the consumption plan $(x_i^{\mathcal{P}}(N))_{i \in \bar{T}}$ is feasible for $\bar{T}$, and the equality follows from (7). Therefore, we have

$$\sum_{i \in S} z_i^* = \sum_{T \in \mathcal{T}} \sum_{i \in T} z_i^* \geq \underline{v}(S) + \sum_{T \in \mathcal{T}} \sum_{i \in \bar{T} \backslash T} (b_i(\hat{x}_i) - z_i^*).$$

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From (6) we know that $b_i(\hat{x}_i) - z_i^* \geq 0$ for all $i \in N$. Hence, $\sum_{i \in S} z_i^* \geq \underline{v}(S)$ and z^* satisfies the non-cooperative core lower bounds.

The proof that z^* satisfies the aspiration upper bounds uses a straightforward modification of the corresponding lemma in Ambec and Sprumont (2002).

Lemma 1 If $S \subseteq T \subseteq N$ and $T < i$, then $w(S \cup i) - w(S) \geq w(T \cup i) - w(T)$.

Then for any coalition S we obtain

$$\sum_{i \in S} z_i^* = \sum_{i \in S} (w(Pi) - w(P^0 i)) \leq \sum_{i \in S} (w(Pi \cap S) - w(P^0 i \cap S)) = w(S),$$

where the inequality follows from Lemma 1 and the last equality follows from the fact that all terms cancel out except $w(P \max S \cap S) = w(S)$ and $-w(P^0 \min S \cap S) = w(\emptyset) = 0$. $\qquad\square$

Remark 5 It can be easily checked that Theorem 1 and its proof remain true if agents are allowed to have benefit functions which either have a satiation point or are strictly increasing (as in Ambec and Sprumont (2002)). Therefore, Theorem 1 generalizes the theorem of Ambec and Sprumont (2002). In the presence of satiation points the main difference and (non-trivial) difficulty is to show that the downstream incremental distribution satisfies the non-cooperative core lower bounds. In Ambec and Sprumont (2002) this was straightforward because with strictly increasing benefit functions it is never optimal for a coalition to pass water from one component to another and cooperation exerts no externality on any coalition. Therefore, their game is consecutive (Greenberg and Weber, 1986) meaning that the value of a coalition equals the sum of the values of its connected components. Then for showing that a distribution satisfies the core lower bounds, it is sufficient to show that the distribution satisfies the core lower bounds for connected coalitions.

Remark 6 The grand coalition N needs not necessarily to form in order to implement the efficient allocation and downstream incremental distribution. Instead of having a global agreement, local agreements among coalitions belonging to the "efficient" partition $\{N_k\}_{k=1,\ldots,K}$ (defined by conditions (1) to (4)) equivalently implement

both the optimal consumption plan $x^*(N)$ and the downstream incremental distribution z^*. It is straightforward that the consumption plan $(x_i^*(N))_{i \in N_k}$ of each subset N_k coincides with the optimal consumption plan for the subriver sharing problem (N_k, b_{N_k}, e_{N_k}), i.e. the portion N_k of the river N. Moreover, for any N_k it is easy to show from (1) to (4) that $(z_i^*)_{i \in N_k}$ is the downstream incremental distribution of the subriver sharing problem (N_k, b_{N_k}, e_{N_k}).[11]

5   (Cooperative) Core Lower Bounds

The downstream incremental distribution satisfies the non-cooperative core lower bounds. We investigate whether there exist distributions satisfying the core lower bounds when agents cooperate, i.e. once a coalition S forms the complement $N \backslash S$ can also from coalitions. First, we focus on the other extreme of non-cooperative behavior, namely on cooperative behavior.

Note that for two agents we have $\underline{v} = \overline{v}$ and the non-cooperative core lower bounds and the cooperative core lower bounds are identical. Although the downstream incremental distribution may violate the cooperative core lower bounds for three agents, the cooperative core is always non-empty for three agents (Ambec and Ehlers, 2006). Unfortunately, for more than three agents there may not exist any distribution satisfying the cooperative core lower bounds.

Theorem 2 *When there are more than three agents all distributions may violate the cooperative core lower bounds.*

The following example establishes Theorem 2.

Example 1 *(The cooperative core may be empty)* Let $N = \{1, 2, 3, 4\}$ and the benefit functions b be such that $b_1(x) = 50x - \frac{x^2}{2}$ for all $x \in [20, 100]$, $b_2(x) = b_3(x) =$

---

[11]It follows from (1) to (4) that $b'_{\min N_2}(e_{\min N_2}) \leq \beta_2$ and (where $P^0 \min N_2 = N_1$) $\underline{v}(P \min N_2) = \underline{v}(N_1) + \underline{v}(\min N_2)$. Therefore, $z^*_{\min N2} = \underline{v}(\min N_2)$ which is identical with agent $\min N_2$'s value in the subriver sharing problem (N_2, b_{N_2}, e_{N_2}). The same argument applies to any N_k.

$100x - 10x^2$ for all $x \in [3, 10]$ and $b_4(x) = 2b_1(x)$. The river inflows are $e_1 = 33$, $e_2 = e_3 = 4$, $e_4 = 37$.

We show that $\bar{v}(2) = b_2(\hat{x}_2)$, $\bar{v}(3) = b_3(\hat{x}_3)$, and $\bar{v}(1, 2, 3, 4) < \underline{v}(1) + \bar{v}(2) + \bar{v}(3) + \underline{v}(4)$. The last condition implies that no distribution satisfies each agent i's cooperative core lower bound $\bar{v}(i)$ (note that $\underline{v}(1) = \bar{v}(1)$ and $\underline{v}(4) = \bar{v}(4)$).

First, the optimal consumption plan $x^*(N)$ solves the maximization program defined by $\bar{v}(1, 2, 3, 4)$. Because $b_4 = 2b_1$, $x^*(N)$ equalizes agents' marginal benefits, i.e., $50 - x_1^*(N) = 100 - 20x_2^*(N) = 100 - 20x_3^*(N) = 100 - 2x_4^*(N)$, and satisfies the global resource constraint $x_1^*(N) + x_2^*(N) + x_3^*(N) + x_4^*(N) = e_1 + e_2 + e_3 + e_4 = 78$. The solution is $(30, 4, 4, 40)$. Therefore $\bar{v}(1, 2, 3, 4) = b_1(30) + b_2(4) + b_3(4) + b_4(40) = 3930$.

Second, we show that coalition $\{1, 3, 4\}$ chooses to pass three units of water from 1 to $\{3, 4\}$. Therefore, 2 consumes $\hat{x}_2 = 5$ units of water and $\bar{v}(2) = b_2(\hat{x}_2)$. Doing so, coalition $\{1, 3, 4\}$ loses $\hat{x}_2 - e_2 = 1$ unit of water (which is consumed by 2) and 1, 3, and 4, respectively, can consume 30, 4, and 39 units of water. The welfare achieved is $b_1(30) + b_3(4) + b_4(39) = 3690$. If no water is passed from 1 to $\{3, 4\}$, then 1 consumes $e_1 = 33$ and 3 and 4 share optimally 41 units of water by consuming respectively $\frac{41}{11}$ and $\frac{410}{11}$. The welfare is then $b_1(33) + b_3(\frac{41}{11}) + b_4(\frac{410}{11}) \approx 3677.32 < 3690$.

Third, we show that the coalition $\{1, 2, 4\}$ chooses to pass three units of water from $\{1, 2\}$ to 4. Therefore, 3 consumes $\hat{x}_3 = 5$ units of water and $\bar{v}(3) = b_3(\hat{x}_3)$. Doing so, coalition $\{1, 2, 4\}$ loses $\hat{x}_3 - e_3 = 1$ unit of water (which is consumed by 3) and 1 and 2, respectively, can consume 30 and 4 units whereas 4 consumes $e_4 + 2 = 39$ units. The welfare achieved is $b_1(30) + b_2(4) + b_4(39) = 3690$. If no water is passed from $\{1, 2\}$ to 4, then 1 and 2 share optimally $e_1 + e_2 = 37$ units of water by consuming respectively (approximatively) 32.9 and 4.1 and 4 consumes $e_4 = 37$. The welfare is then $b_1(32.9) + b_3(4.1) + b_4(37) \approx 3676.7 < 3690$.

Finally, $\underline{v}(1) + \bar{v}(2) + \bar{v}(3) + \underline{v}(4) = b_1(e_1) + b_2(\hat{x}_2) + b_3(\hat{x}_3) + b_4(e_4) = b_1(33) + b_2(5) + b_3(5) + b_4(37) = 3936.5 > 3930 = \bar{v}(1, 2, 3, 4)$. Hence, all distributions violate the cooperative core lower bounds.

Given Theorem 2, next we investigate when there exist distributions satisfying the core lower bounds. When the expectations of any coalition are pessimistic, the downstream incremental distribution satisfies the core lower bounds. When the expectations of any coalition are optimistic, there may not be any distribution satisfying the core lower bounds. This is even true if blocking is restricted to connected coalitions (see Example 1). Allowing blocking only for connected coalitions is natural for a river because coordination is easier for neighboring countries. Empirically, Murgai et al. (2002) show that water exchanges among farmers located on irrigation canals in Pakistan is negatively correlated with geographical distance. In the literature on line-graph games such as the river game, most papers assume that only consecutive players can communicate with each other (Greenberg and Weber, 1986; Brink, Laan and Vasil'ev, 2007). Demange (2004) also considers only blocking by consecutive or connected coalitions. The following result gives a simple necessary and sufficient condition for the existence of a compromise between the UTI doctrine and the ATS doctrine for all connected coalitions under optimistic expectations.

We will say that a distribution z satisfies for any connected coalition S all core lower bounds if $\sum_{i \in S} z_i \geq v(S, \mathcal{P})$ for all partitions $\mathcal{P}$ of N such that $S \in \mathcal{P}$. Then the core lower bound of S is always satisfied independently of how the agents outside of S are organized.

Theorem 3 *The following are equivalent:*

(i) *There exists a distribution satisfying the aspiration upper bounds and for any connected coalition all core lower bounds.*

(ii) *The downstream incremental distribution satisfies for any connected coalition all core lower bounds.*

(iii) *Cooperation exerts no externality on the value of any agent, i.e. $\underline{v}(i) = v(i, \mathcal{P})$ for all $i \in N$ and all partitions $\mathcal{P}$ of N such that $\{i\} \in \mathcal{P}$.*

Proof. (i)⇒(ii): Let z be a distribution satisfying the aspiration upper bounds and for any connected coalition all core lower bounds. By Proposition 1, all core lower bounds are greater than or equal to the non-cooperative core lower bounds. Hence, z is a distribution satisfying the aspiration upper bounds and for any connected coalition the non-cooperative core lower bounds. Then, by Remark 4, we have $z = z^*$ and the downstream incremental distribution satisfies for any connected coalition all core lower bounds.

(ii)⇒(i): By Theorem 1, z^* satisfies the aspiration upper bounds. Hence, z^* is a distribution satisfying the aspiration upper bounds and for any connected coalition all core lower bounds.

(ii)⇒(iii): Let z^* satisfy all core lower bounds for all connected coalitions. Let $i \in N$ and $\mathcal{P}$ be a partition of N such that $\{i\} \in \mathcal{P}$. Since $\{i\}$ is connected, we have $\underline{v}(Pi) - \underline{v}(P^0i) = z_i^* \geq v(i, \mathcal{P})$. Hence, by $v(i, \mathcal{P}) = b_i(x_i^{\mathcal{P}}(N))$,

$$\underline{v}(Pi) \geq \underline{v}(P^0i) + b_i(x_i^{\mathcal{P}}(N)). \tag{8}$$

On the other hand, by $\sum_{j \in P^0i} x_j^{\{P^0i, N \setminus P^0i\}}(N) = \sum_{j \in P^0i} e_j \geq \sum_{j \in P^0i} x_j^{\{Pi, N \setminus Pi\}}(N)$, $x_{P^0i}^{\{Pi, N \setminus Pi\}}(N)$ is a consumption plan for P^0i. Therefore, $\underline{v}(P^0i) \geq \sum_{j \in P^0i} b_j(x_j^{\{Pi, N \setminus Pi\}}(N))$ and

$$\underline{v}(P^0i) + b_i(x_i^{\{Pi, N \setminus Pi\}}(N)) \geq \underline{v}(Pi). \tag{9}$$

Hence, from (8) and (9) we obtain $b_i(x_i^{\{Pi, N \setminus Pi\}}(N)) \geq b_i(x_i^{\mathcal{P}}(N))$. Since agent i's consumption is always smaller than or equal to $\hat{x}_i$ and b_i is strictly increasing between 0 and $\hat{x}_i$, the previous inequality is equivalent to

$$x_i^{\{Pi, N \setminus Pi\}}(N) \geq x_i^{\mathcal{P}}(N) \tag{10}$$

By $\{i\} \in \mathcal{P}$, we have $x_i^{\mathcal{P}}(N) \in \{e_i, \hat{x}_i\}$. If $x_i^{\mathcal{P}}(N) = e_i$, then $v(i, \mathcal{P}) = b_i(e_i) = \underline{v}(i)$, the desired conclusion. If $x_i^{\mathcal{P}}(N) \neq e_i$, then $x_i^{\mathcal{P}}(N) = \hat{x}_i$. Hence, by (10), $x_i^{\{Pi, N \setminus Pi\}}(N) = \hat{x}_i$, and by $\hat{x}_i \geq e_i$, $\hat{x}_i > e_i$. But then, by $x_i^{\{Pi, N \setminus Pi\}}(N) = \hat{x}_i > e_i$, we have

$$\sum_{j \in P^0i} x_j^{\{Pi, N \setminus Pi\}}(N) < \sum_{j \in P^0i} e_j.$$

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Therefore,

$$\sum_{j \in P^0 i} b_j(x_j^{\{Pi, N \setminus Pi\}}(N)) < \underline{v}(P^0 i). \tag{11}$$

Hence,

$$\underline{v}(Pi) = \sum_{j \in P^0 i} b_j(x_j^{\{Pi, N \setminus Pi\}}(N)) + b_i(x_i^{\{Pi, N \setminus Pi\}}(N)) < \underline{v}(P^0 i) + b_i(\hat{x}_i),$$

where the inequality follows from (11) and $x_i^{\{Pi, N \setminus Pi\}}(N) = \hat{x}_i$. Now, by $x_i^{\mathcal{P}}(N) = \hat{x}_i$, this inequality contradicts (8). Thus, we have to have $x_i^{\mathcal{P}}(N) = e_i$ and $v(i, \mathcal{P}) = \underline{v}(i)$ for all $i \in N$ and all $\mathcal{P}$ such that $\{i\} \in \mathcal{P}$.

(iii)$\Rightarrow$(ii): Let S be a connected coalition and $\mathcal{P}$ be a partition such that $S \in \mathcal{P}$. We show $\underline{v}(S) = v(S, \mathcal{P})$. Since S is connected, we have either $\underline{v}(S) = v(S, \mathcal{P})$ or $\underline{v}(S) < v(S, \mathcal{P}) = \sum_{i \in S} b_i(\hat{x}_i)$. Suppose that

$$\underline{v}(S) < v(S, \mathcal{P}) = \sum_{i \in S} b_i(\hat{x}_i). \tag{12}$$

Then there exists $i \in S$ such that $e_i < \hat{x}_i$. Let $\mathcal{P}' = (\mathcal{P} \setminus S) \cup \{\{j\} | j \in S\}$. By (12), $x_j^{\mathcal{P}}(N) = \hat{x}_j$ for all $j \in S$. Then $x^{\mathcal{P}}(N)$ is also the outcome of the backwards induction algorithm when agents cooperate according to $\mathcal{P}'$. Hence, $x_j^{\mathcal{P}'}(N) = \hat{x}_j$ for all $j \in S$ and $v(i, \mathcal{P}') = b_i(\hat{x}_i)$. Since $e_i < \hat{x}_i$ and $\underline{v}(i) = b_i(e_i)$, we obtain $\underline{v}(i) < v(i, \mathcal{P}')$, which contradicts (ii). Hence, (12) was wrong and we have $\underline{v}(S) = v(S, \mathcal{P})$.

By Theorem 1, z^* satisfies the non-cooperative core lower bounds. Hence, by $\underline{v}(S) = v(S, \mathcal{P})$ for all connected coalitions S and all partitions $\mathcal{P}$ such that $S \in \mathcal{P}$, z^* satisfies for any connected coalition all core lower bounds, the desired conclusion.$\square$

By Theorem 3, the downstream incremental distribution satisfies all core lower bounds for all connected coalitions if and only if the individual rationality constraints are identical under all behaviors of the other agents. Condition (iii) of Theorem 3 is trivially satisfied in Ambec and Sprumont (2002) because in their paper no coalition is passing water from one of its connected components to another one independently of the behavior of the other agents. Basically, it requires that no agent can "free

ride" on the other agents' cooperative behavior. Such a condition can be estimated in real-world river sharing problems.

Remark 7 The equivalence in Theorem 3 does not remain true under cooperative behavior, i.e. if we require that the cooperative core lower bounds are satisfied for any connected coalition. Below we provide an example showing that $\underline{v}(i) = \overline{v}(i)$ for all $i \in N$ but the downstream incremental distribution violates a cooperative core lower bound for a connected coalition.

By Proposition 1, cooperation exerts a positive externality on a coalition (compared to non-cooperative behavior). Then one may wonder whether starting from any partition "more" cooperation of the other agents always exerts a positive externality on the value of a coalition. Here "more" cooperation means that from a partition we obtain a coarser partition by merging some coalitions. If this were true, then the cooperative core lower bound of a coalition is maximal among all core lower bounds for all behaviors of the other agents. The following example shows that merging some coalitions may exhibit a negative externality on the value of a coalition (compared to the value of the coalition before the merger).[12]

Example 2 (*For a coalition the cooperative core lower bound may not be maximal among all core lower bounds*) Let $N = \{1, 2, 3, 4\}$ and the benefit functions b be such that $b_1(x) = 50x - \frac{x^2}{2}$ for all $x \in [20, 100]$, $b_2(x) = b_3(x) = 100x - 10x^2$ for all $x \in [3, 10]$ and $b_4(x) = 2b_1(x)$. The river inflows are $e_1 = 33$, $e_2 = 4$, $e_3 = \hat{x}_3 = 5$, $e_4 = 35$.

We show the following: $\underline{v}(i) = \overline{v}(i)$ for all $i \in N$ and $\overline{v}(2) < v(2, \{\{1, 4\}, \{2\}, \{3\}\}) = b_2(\hat{x}_2)$. Therefore, if coalitions $\{1, 4\}$ and $\{3\}$ merge, then $\{2\}$ obtains strictly less than $v(2, \{\{1, 4\}, \{2\}, \{3\}\})$. The welfare achieved by a coalition might decrease with a coarser partition of its complement.

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[12]This is in contrast to industrial environments where collusive agreements or cartels reduce market competition or R&D agreements with spillovers.

First, we show that $\{1,4\}$ passes some water from 1 to 4 and 2 consumes his peak $\hat{x}_2$ under the partition $\{\{1,4\},\{2\},\{3\}\}$, i.e. $v(2,\{\{1,4\},\{2\},\{3\}\}) = b_2(\hat{x}_2)$. Without passing water, the welfare achieved by $\{1,4\}$ is $b_1(e_1) + b_4(e_4) = b_1(33) + b_4(35) = \frac{6761}{2} = 3380.5$. By passing some water, the coalition looses 1 unit of water (consumed by 2 because $\hat{x}_2 - e_2 = 1$), and agents 1 and 4 share optimally 33+35-1=67. They equalize marginal benefits $50 - x_1 = 100 - 2x_4$ and satisfy the resource constraint $x_1 + x_4 = 67$. Thus, 1 and 4, respectively, consume 28 and 39. Their welfare is $b_1(28) + b_4(39) = 3387 > 3380.5$.

Second, we show that if 3 joins the coalition $\{1,4\}$, then coalition $\{1,3,4\}$ chooses not to pass any water from 1 to $\{3,4\}$. Doing so 3 and 4 share optimally $e_3 + e_4$ and they consume respectively $\frac{40}{11}$ and $\frac{400}{11}$. Then the welfare achieved by $\{1,3,4\}$ is $b_1(33) + b_3(\frac{40}{11}) + b_4(\frac{400}{11}) = \frac{80321}{22} > 3650$. If some water is passed from 1 to $\{3,4\}$, then $e_1 + e_3 + e_4 - 1 = 72$ units of water are shared optimally between the members of $\{1,3,4\}$. Agents 1, 3 and 4, respectively, consume $\frac{890}{31}$, $\frac{122}{31}$ and $\frac{1220}{31}$. The welfare is then $b_1(\frac{890}{31}) + b_3(\frac{122}{31}) + b_4(\frac{1220}{31}) = \frac{113110}{31} < 3649$. Therefore, $\{1,3,4\}$ chooses not to pass any water from 1 to $\{3,4\}$ and $\overline{v}(2) = b_2(e_2) = \underline{v}(2)$.

Since $e_3 = \hat{x}_3$, we have $\underline{v}(i) = \overline{v}(i)$ for all $i \in N$. Furthermore, by $e_2 < \hat{x}_2$, $\overline{v}(2) < v(2,\{\{1,4\},\{2\},\{3\}\})$.

Finally, we show that the downstream incremental distribution violates the cooperative core lower bounds although we have $\underline{v}(i) = \overline{v}(i)$ for all $i \in N$. Since $e_2 < \hat{x}_2$ and $e_3 = \hat{x}_3$, we obtain

$$\overline{v}(2,3) = v(2,\{\{1,4\},\{2\},\{3\}\}) + v(3,\{\{1,4\},\{2\},\{3\}\}) = b_2(\hat{x}_2) + b_3(\hat{x}_3) > \underline{v}(2,3).$$

Hence, by Theorem 3, z^* violates the cooperative core lower bounds.

6 Conclusion

Theorems 1 and 2 are consistent with the literature on international agreements for pollution reduction. This literature disagrees on the stability of a global agreement

(the "grand coalition") because of opposite assumptions about the behavior of the nonmembers of an agreement. On the one hand, Chander and Tulkens (1997) show that the non-cooperative core is non-empty, thereby leading to a "grand coalition" agreement. On the other hand, Carraro and Siniscalco (1993) assume that coalitions still cooperate when an individual country deviates. They conclude that any global agreement is not stable because at least one individual country blocks it and the core is empty. In our river sharing problem, the non-cooperative core is non-empty since it includes the downstream incremental distribution while the cooperative core might be empty.

An important work related to ours is Demange (2004). She considers hierarchies and shows that the "hierarchical outcome" satisfies the core lower bounds for all connected coalitions for all super-additive cooperative games. If we insist that the core lower bounds are satisfied for some non-connected coalitions, then there exists a large class of super-additive games where the "hierarchical outcome" violates the core lower bounds. If the hierarchy is a river, then the hierarchical outcome corresponds to the upstream incremental distribution u^*. This is the analogue of the downstream incremental distribution defined by $u_i^* = \underline{v}(Fi) - \underline{v}(F^0i)$ (here again it does not matter how the agents in P^0i behave). Both her and our work have in common that the cooperative game is super-additive and that an incremental distribution corresponding to the structure of the river (or the hierarchy) is proposed as a solution to the game under consideration. The important differences between Demange (2004) and our work are that here externalities do exist whereas in hers they do not and that the downstream incremental distribution satisfies the non-cooperative core lower bounds for all coalitions (connected or non-connected).

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