Debt and Deficit Fluctuations in the Time-Consistent Setup*

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Abstract

This paper considers the implications of optimal taxation for the stochastic behavior of debt and deficit in the economy with discretionary government, focusing on Markov perfect equilibria. It concludes that in such time-consistent setup in case of market incompleteness the properties of the variables are very similar to those in the full commitment case. Moreover, debt shows more persistence than other variables and it increases in response to shocks that cause a higher deficit, which is in accordance with empirical evidence from U.S. data. This result, in contrast to the full commitment case, holds regardless whether the government pursues an optimal fiscal policy under complete markets, or under incomplete markets.

Keywords: optimal public debt; Markov-perfect equilibrium; time-consistent policy; market incompleteness.

JEL Classification Numbers: E61, E62, H21, H63.

1 Introduction

Recent economic literature devotes a lot of attention to the stochastic properties of optimal fiscal policy in the full commitment environment. However, the hypothesis of "once and forever committed" government can hardly ever be accepted. This paper proposes to look at the opposite extreme case: an economy in which the government reoptimizes its policy every period. While reality is most probably positioned somewhere in between the two extreme commitment assumptions, studying deeper this second case gives more insides to the existing evidence of the optimal fiscal policy characteristics.

In particular, there is a big discussion in the related studies about relative performance of complete versus incomplete markets assumption for the ability of the model to replicate

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empirical data. The incomplete markets structure in the framework of Lucas and Stokey (1983) model was considered to fit much better U.S. data, than the corresponding complete markets counterpart, assuming full commitment of the government to its fiscal policy plan (Marcet and Scott, 2009). This paper aims to compare performance of complete versus incomplete markets when the full commitment assumption of the government is relaxed. Is the complete markets structure still unsatisfactory in reproducing empirical evidence when the government is allowed to reoptimize its fiscal plan? Is the modelled world without commitment too different from the world with full commitment of the authorities?

To that end, I take the view that commitment of the government to its fiscal plan from the first period and forever is restricting and unrealistic. Instead, I propose to model a policy conduct as a continuous interplay between subsequent governments, which realize that their policies will be reoptimized by the successors, both because of changing economic conditions and because of optimality of such reoptimizing (as showed by Kydland and Prescott (1977)).

I do not study here the complete set of sustainable equilibria, assuming that reputational mechanisms are not operating (they may not be accessible the same way as commitment mechanisms). Therefore, I concentrate on the Markov perfect equilibria, which are consistent by construction, well-interpretable, and relatively easy to solve for.

For the case of full commitment there exists a numerous characterization of the cyclical properties of the main aggregate economic variables. See for example Chari, Christiano and Kehoe (1994) for the description of the stochastic properties of the business-cycle model; Aiyagari, Marcet, Sargent and Seppala (2002) for characterization of the stochastic behavior of fiscal variables in the model without capital when financial markets are incomplete; Scott (2007) for study of the properties of labor taxes in the stochastic world; Marcet and Scott (2009) for the comparison of stochastic properties of a variety of models when there are complete vs incomplete financial markets. The last three papers propose a firm support for the eligibility of the full commitment-incomplete markets setup to replicate correctly a set of the stylized facts from U.S. economy: the existence of unit root component in labor taxes and debt series; positive correlation of the debt and deficit series; higher persistence of debt series when compared to GDP series.

The no commitment outcomes are a-priori known to be Pareto inferior to the full commitment case. But how much different are the time-consistent allocations from those implied by Ramsey equilibrium, and how the responses to the exogenous shocks of the variables in the no commitment differ from those when full commitment is imposed?

Corresponding analysis for the no commitment case is not so rich. The related literature on time-consistent fiscal policy can be broadly divided in two parts. There is a body of papers which try to analyze the entire set of equilibria (reputation mechanisms are operating) when government policy is time consistent. For instance, Chari and Kehoe (1989, 1993), who first define and describe a notion of sustainable equilibrium, Phelan and Stacchetti (2001), who propose a generalized approach for the numerical characterization of the entire set of equilibria of time-consistent policy, Fernandez-Villaverde and Tsyvinski (2002), who *and only who* apply the approach of Phelan and Staccetti to analyze the *stochastic properties* of neoclassical growth model for the entire set of sustainable equilibria. Lucas and Stokey (1983) and Persson et al. (2006) propose different strategies to make the government policy time-consistent.

A second approach to study the time-consistent fiscal policy is employed in this paper and consists of characterizing only the Markov equilibria (when reputation mechanisms are not operating and payoffs depend only on the payoff-relevant states). Markov equilibria represent only a small subset of the whole set of time-consistent outcomes; however they have a tractable interpretation and are quite easy to analyze. The closely related studies that have discussed the properties of Markov perfect equilibria in line with this paper include:

Klein and Rios-Rull (2003) compare *stochastic properties* of optimal fiscal policy without commitment with those under full commitment in *the growth model*. In their model government cannot commit to capital income tax. They restrict the government to run balanced budget every period and conclude that the stochastic properties of the no commitment case contrast significantly with those of full commitment case (capital income tax is positive and high, labor income tax is highly volatile in the no commitment). Klein, Krusell and Rios-Rull (2007) give a theoretical and numerical characterization of the time-consistent policy in the growth model when the government has to choose also the public expenditures level. They hold, however, the balanced budget assumption, justifying it by the immediate default of the government in case of positive debt. Ortigueira and Pereira (2007) extended the model of Klein, Krusell and Rios-Rull with introduction of government bonds. They solved the *deterministic* growth model for Markov perfect equilibria and found two steady states, one with no distortionary taxation in which the government holds positive assets, and another with positive debt and income taxes. There is also a related body of research on the optimal monetary policy. For example, Martin (2006) studies a model without capital in which the government operates nominal debt and issue money. The author obtains positive debt levels in equilibrium, with debt and inflation positively correlated with bad exogenous government expenditure shocks.

In this paper I focus on the simplest model with production, consumption, and government policy consisting of the choice of taxes, government expenditures, and bonds (with bonds levels restricted to lye inside the 60% of GDP bound), and try to quantify the importance of the full commitment assumption for the model's predictions. In particular, I repeat the analysis of the stochastic properties of taxes, debt, deficit, output, and consumption for the model without capital, but with uncertainty, done before for the full commitment, assuming now that the governments cannot commit to their announced policy, and as a result, are involved in the dynamic game with their successors. I characterize Markov perfect equilibria and compare the properties of the model to the corresponding outcomes when government is assumed to fully commit to its plan. This paper applies the full commitment solution based on the one described in Aiyagari et al. (2002) and no commitment solution algorithm based on that proposed in Ortigueira (2006).

I find that under the incomplete markets structure the stochastic behavior of the variables in the no commitment, when studied as a Markov perfect equilibrium, and in the full commitment are very similar. This suggests that the conclusions about stochastic behavior of the variables in the models with artificially imposed full commitment assumption may approximately hold for those models where such assumption is relaxed.

I also find that, surprisingly, under the complete markets structure, the no commitment assumption results in the debt and deficit series, reacting so differently to stochastic shocks under full commitment, now being very similar with the stochastic behavior close to that observed in the incomplete markets environment. This result was obtained by numerically approximating the government debt function as dependent on the states: previous government debt position and current shock for incomplete markets, previous government debt position and future possible shocks for complete markets. The findings of this paper reinforces the complete markets assumption as a market structure plausible for the empirical success of the model, but instead casts doubts on the perfection of the full commitment assumption.

The rest of the paper is organized as follows: section two describes the main concepts of the model used; sections three defines the problem of the households and the government, with government reoptimizing each period, under complete and under incomplete markets. Section four briefly reviews the problem of the households and the government, with government fully committing to its policy plan. Section five compares the properties of the Markov perfect equilibria under both markets' structures. Section six compares the properties of the full and no commitment outcomes for the incomplete market structure. Section seven concludes. The appendix contains all proofs and descriptions of computational methods.

2 The Model

This section briefly describes the main properties of the economic model considered in the paper. The model represents a version of Lucas and Stokey (1983) economy, with the only difference that the public goods now deliver utility to the agents[1], and bonds traded between the government and households have one-period structure, and can be risk-free.

Consider an economy inhabited by identical households. The representative household derives utility from private and public consumption and leisure, and receives income from labor and interest on its holdings of government bonds. There is a benevolent government which has to tax households to finance a stream of government expenditures, and which maximizes the households' welfare by optimally choosing the sequence of bond issues and public expenditures. The technology of the economy satisfies:

$$c_t + g_t = \theta_t(1 - x_t), \tag{1}$$

where c_t denotes consumption of the household, g_t - the amount of government expenditures, and x_t - the amount of leisure of the household at period t. The time endowment of the household is 1 for each period of time, θ_t represents stochastic shock to the labor productivity, which is assumed to follow a Markov process. The utility of the household is assumed to be separable in its three arguments. I consider the utility with the general form[2]

[1] Endogenizing public goods in the considered model allows for the existence of continuous policy functions in the solution of the government optimization problem in case when the government is discretionary. Otherwise the corner solutions with step policy functions may arise (see Krusell, Martin, and Rios-Rull 2006).

[2] By making the utility separable in all arguments I am assuming that in this economy public and private goods are not perfectly substitutable. The extreme case of CRRA utility, usually considered in the literature (separable, logarithmic in arguments) results in the qualitatively same conclusions as those reported in the subsequent sections.

$$u(c_t, x_t, g_t) = (1-\phi_g)\phi_c \frac{c_t^{1-\sigma_c}-1}{1-\sigma_c} + (1-\phi_g)(1-\phi_c)\frac{x_t^{1-\sigma_x}-1}{1-\sigma_x} + \phi_g \frac{g_t^{1-\sigma_g}-1}{1-\sigma_g}. \tag{2}$$

There is also a representative firm maximizes profits. Households and firms are competitive which implies that the wage is equal to θ_t. The tax on labor income is proportional and denoted by τ_t. Households, firms and government observe all shocks up to the current period. Government faces a budget constraint which restricts the government public expenditures and expenditures on debt services not to exceed tax revenues. Moreover, to preclude the accumulation of infinite debt or infinite assets by the government, the debt constraints are imposed (see Aiyagari, Marcet, Sargent and Seppala (2002) for discussion):

$$\underline{M} \leq b_t \leq \bar{M}, \tag{3}$$

with $\underline{M} = -\bar{M}$. These constraints should be rarely binding in equilibrium however, and will not matter much for the conclusions stressed by the paper.

In this paper I consider a government which lacks a commitment device to stick to the unique fiscal policy plan chosen "once and for all periods". Instead, the government is allowed to reoptimize its policy plan each time period. It commits, however, to previous debt repayments, so the possibility of default is precluded, say by huge productivity loses. I restrict attention to the Markov perfect equilibria of such a policy, which are time-consistent by construction. Furthermore, I assume that government is a Stackelberg leader in the sense that it has instantaneous leadership in choosing taxes and public expenditures before the households choose their consumption and leisure levels. Therefore the government takes into account possible response of households to its policy choices (see Ortigueira, 2006).

Below I define the problems of the household and the government, as well as a Markov perfect equilibrium, for cases when the financial markets are complete (the government is able to issue state-contingent bonds), and when the full insurance is not available (the government bonds are risk-free). Then the comparison of the outcomes of the both market structures will be made.

3 The Model Under No Commitment

3.1 Complete Markets

Although this section assumes that the government is able to issue state-contingent bonds to smooth allocations in response to any possible risk, the bonds are restricted to be only of one period maturity. This assures that the fiscal policy cannot be made time-consistent by the particular restructuring of the bonds as in Lucas and Stokey (1983). Therefore the time-consistent solution considered here does not coincide with corresponding full commitment solution.

The budget constraint of the government in this case is:

$$g_t + b_{t-1} \leq \tau_t \theta_t (1-x_t) + \int_\Omega p_t(\bar{\theta}) b_t(\bar{\theta}) dF(\bar{\theta}|\theta_t), \tag{4}$$

where $dF(\bar{\theta}|\theta_t)$ is the conditional distribution of the shocks next period, given today's realization θ, Ω is a complete space of possible realizations of θ.

The problem of the household

Given the government policy choice (the level of government expenditures and taxes), households choose their consumption, leisure and savings, conditional on the expected future policies of the government.

Therefore, the problem of the household that holds the amount b of government assets, that has to pay tax τ on its labor income, that enjoys the amount g of public goods, and that expects the future government to set debt and public goods expenditures according to the policies $G(b,\theta), B(b,\theta)$, given the current state of the economy θ, can be written as:

$$v(b,\theta;g,B,\tau) = \max_{c,x,b'} \; \{u(c,x,g) + \beta E\tilde{v}(b',\theta';G',B')\}, \tag{5}$$

$$s.t. \;\; c + \int_{\Omega} p'(.)b'(.)dF(\theta) = (1-\tau)\theta(1-x) + b,$$

$$c + g = \theta(1-x),$$

where $E\tilde{v}(b',\theta';G',B')$ is the expected continuation value as foreseen by the households.

Given the representative agent assumption, $b = B$. It turns out that when the government expenditures are given by $g = G(b,\theta)$, and government *contingent* debt policy is given by $b = B(b,\theta)$, the competitive equilibrium optimality conditions for (5) result in the following implementability constraint to be satisfied by the consumption function and savings function of the households:

$$cu_c + \beta \int_{\Omega} u_c(c')b'dF(\bar{\theta}|\theta) = (c+g)u_x + bu_c,$$

$$c = \Psi(b,\theta), b' = B(b,\theta').$$

The problem of the government

The government behaves as a Stackelberg leader in the economy. It repays previous period debt, sets the levels of labor taxes and public expenditures, and the issues of new *contingent* debt, taking into account the effect of these choices on the consumption function of households (via the competitive equilibrium implementability constraint), and the expected future policies of itself or its successors. I use the primal approach, substituting taxes away from the problem. Then, the problem of the government can be written as:

$$V(b,\theta) = \max_{c,g,b'}\{u(c, 1-(c+g)/\theta, g) + \beta E\tilde{V}(b',\theta')\}, \tag{6}$$

$$s.t. \;\; cu_c + \beta \int_{\Omega} u_c(c'(.))b'(.)dF(\bar{\theta}|\theta) = (c+g)u_x + bu_c,$$

where $E\tilde{V}(b',\theta')$ is next-period value as foreseen by the time-t government.

Proposition: The optimal government policy satisfies the following necessary optimality conditions:

- the generalized Euler equation:

$$\frac{u_x - u_g}{-u_x + (c+g)u_{xx}} \int_\Omega (u_{cc}(\Psi(B(b,\bar\theta),\bar\theta)\Psi_b(B(b,\bar\theta);\bar\theta)B(b,\bar\theta) + u_c(\Psi(B(b,\bar\theta),\bar\theta))dF(\bar\theta|\theta) \quad (7)$$
$$= \int_\Omega \frac{u_{x'} - u_{c'}}{u_{c'} + (c' - b')u_{cc'} - u_{x'} + (c' + g')u_{xx'}} u_c(\Psi(B(b,\bar\theta),\bar\theta))dF(\bar\theta|\theta),$$

- first order condition for the government maximization problem:

$$\frac{u_x - u_c}{u_c + (c-b)u_{cc} - u_x + (c+g)u_{xx}} = \frac{u_x - u_g}{-u_x + (c+g)u_{xx}}, \quad (8)$$

- the debt constraints:

$$\underline{M} \le b_t \le \bar M. \quad (9)$$

(see proof in appendix).

Definition: *A stationary Markov perfect equilibrium in a given complete markets economy consists of a value function V, policy functions $\Psi(b,\theta)$, $G(b,\theta)$ and a state-contingent debt function $B(b,\theta)$ such that:*

1. Given the policy functions $\Psi(b,\theta)$, $G(b,\theta)$ and $B(b,\theta)$, the value function V satisfies functional equation:

$$V(b,\theta) = u(\Psi(b,\theta), 1 - (\Psi(b,\theta) + G(b,\theta))/\theta, G(b,\theta)) + \beta \int_\Omega \tilde V(B(b,\bar\theta),\bar\theta)dF(\bar\theta|\theta), \quad (11)$$

and $V(b,\theta) = \tilde V(b,\theta)$.

2. Given the policy functions $\Psi(b,\theta)$, $B(b,\theta)$, and *the value function V, $G(b,\theta)$ delivers an optimal choice for the government:*

$$G(b,\theta) \in \arg\max_g \{u(\Psi(b,\theta), 1 - (\Psi(b,\theta) + g)/\theta, g) + \beta \int_\Omega V(B(b,\bar\theta),\bar\theta)dF(\bar\theta|\theta)\}, \quad (12)$$

3. Given the policy functions $G(b,\theta)$, $B(b,\theta)$, and *the value function V, $\Psi(b,\theta)$ solves the first-order condition of household's maximization problem:*

$$\Psi(b,\theta)u_c + \beta \int_\Omega u_c(\Psi(B(b,\bar\theta),\bar\theta))B(b,\bar\theta)dF(\bar\theta|\theta) = (\Psi(b,\theta) + g)u_x + bu_c, \quad (13)$$

4. Given the policy functions $G(b,\theta)$, $\Psi(b,\theta)$, and *the value function V, $B(b,\theta)$ solves the generalized Euler equation of government's maximization problem:*

$$\lambda \int_\Omega (u_{cc}(\Psi(B(b,\bar\theta),\bar\theta)\Psi_b(B(b,\bar\theta);\bar\theta)B(b,\bar\theta) + u_c(\Psi(B(b,\bar\theta),\bar\theta))dF(\bar\theta|\theta) \;=\; \quad (14)$$
$$= \int_\Omega (\lambda(B(b,\bar\theta),\bar\theta)u_c(\Psi(B(b,\bar\theta),\bar\theta))dF(\bar\theta|\theta),$$

with λ defined as

$$\lambda = \frac{u_x - u_c}{u_c + (c-b)u_{cc} - u_x + (c+g)u_{xx}} = \frac{u_x - u_g}{-u + (c+g)u_{xx}}, \quad (15)$$

5. The government debt constraints are satisfied:

$$\underline{M} \leq b \leq \bar{M}. \tag{16}$$

The functions $c = \Psi(b, \theta)$, $b' = B(b, \theta')$, and $g = G(b, \theta)$ are assumed to be continuous and twice continuously differentiable.

3.2 Incomplete Markets

In the case of incomplete markets the set of assets available to the government is restricted to uncontingent bonds (again maturing in one period).

The budget constraint of the government in this case is:

$$g_t + b_{t-1} \leq \tau_t \theta_t (1 - x_t) + p_t b_t.$$

Now the debt function is not state contingent, and takes the form: $b' = B(b, \theta)$, where θ is a current period shock value (and not future value, to which the bonds are indexed in the complete markets case). In everything else the problems of households and government are similar to those when the full insurance is available.

Definition: *A stationary Markov perfect equilibrium in a given incomplete markets economy consists of a value function V, policy functions $\Psi(b, \theta)$, $G(b, \theta)$ and a debt function $B(b, \theta)$ such that:*

1. Given the policy functions $\Psi(b, \theta)$, $G(b, \theta)$ and $B(b, \theta)$, the value function V satisfies functional equation:

$$V(b, \theta) = u(\Psi(b, \theta), 1 - (\Psi(b, \theta) + G(b, \theta))/\theta, G(b, \theta)) + \beta E\tilde{V}(B(b, \theta), \theta'), \tag{17}$$

and $V(b, \theta) = \tilde{V}(b, \theta)$.

2. Given the policy functions $\Psi(b, \theta)$, $B(b, \theta)$, and *the value function V, $G(b, \theta)$ delivers an optimal choice for the government:*

$$G(b, \theta) \in \arg\max_g \{u(\Psi(b, \theta), 1 - (\Psi(b, \theta) + g)/\theta, g) + \beta EV(B(b, \theta), \theta')\}, \tag{18}$$

3. Given the policy functions $G(b, \theta)$, $B(b, \theta)$, and *the value function V, $\Psi(b, \theta)$ solves the first-order condition of household's maximization problem:*

$$\Psi(b, \theta) u_c + \beta E u_c(\Psi(B(b, \theta), \theta')) B(b, \theta) = (\Psi(b, \theta) + g) u_x + b u_c, \tag{19}$$

4. Given the policy functions $G(b, \theta)$, $\Psi(b, \theta)$, and *the value function V, $B(b, \theta)$ solves the generalized Euler equation of government's maximization problem:*

$$\begin{aligned} \lambda E(u_{cc}(\Psi(B(b, \theta), \theta') \Psi_b(B(b, \theta); \theta) B(b, \theta) + u_c(\Psi(B(b, \theta), \theta')) &= \\ = E(\lambda' u_c(\Psi(B(b, \theta), \theta')), \end{aligned} \tag{20}$$

with λ defined as

$$\lambda = \frac{u_x - u_c}{u_c + (c - b) u_{cc} - u_x + (c + g) u_{xx}} = \frac{u_x - u_g}{-u + (c + g) u_{xx}}, \tag{21}$$

5. The government debt constraints are satisfied:

$$\underline{M} \leq b \leq \bar{M}. \quad (22)$$

Again, the policy functions $c = \Psi(b, \theta)$, $b' = B(b, \theta)$, and $g = G(b, \theta)$ are assumed to be continuous and twice continuously differentiable.

3.3 Steady State Markov Perfect Equilibrium

This subsection explores the characteristics of the steady states of the Markov perfect equilibrium. To simplify the analysis, in this subsection all uncertainty is removed from the model. Than, the deterministic steady state for the model considered can be defined as:

A *steady-state deterministic Markov perfect equilibrium* is defined as a list of sequences of allocations $\{c_t, x_t\}$, fiscal variables $\{g_t, \tau_t, b_t\}$, and wages $\{w_t\}$ such that they are generated by a Markov perfect equilibrium, and its values do not change over time, with the aggregate shock θ level being constant over time, that is: $g_{t+1} = g_t$, $\tau_{t+1} = \tau_t$, $b_{t+1} = b_t$, $\theta_{t+1} = \theta_t = 1$, and thus $c_{t+1} = c_t$, $x_{t+1} = x_t$, $w_{t+1} = w_t$ for all t.

The generalized Euler equation of the government problem in such a deterministic setup takes a form:

$$\lambda(u_{cc}\Psi_b b + u_c) = \lambda' u_{c'}, \quad (23)$$

and dictates three possible steady states:

$$\lambda = 0, \ \ \Psi_b = 0, \ \ b = 0. \quad (24)$$

Debortoli and Nunes (2007) have studied the stationarity of the conditions (24), and concluded that only two of three possible steady states are stationary: first with government debt being zero in equilibrium ($b = 0$), and second with positive level of government assets and no distortionary taxation ($\lambda = 0$). They characterize the third steady state ($\Psi_b = 0$) as nonstationary, with the negative debt level, situated between the previous two stationary steady states. As this paper intends to study the economy with distortionary taxation, and imposes the debt limits (which are usually more stringent than those needed for accumulation of large asset positions by the government), I will consider only the condition $b = 0$ as the relevant one defining the deterministic steady state Markov perfect equilibrium for a given economy, and will further explore the stochastic properties of the economy, using $b = 0$ as the initial guess for the numerical computations.

4 The Model Under Full Commitment

For the sake of completeness of the exposition, this section proposes a brief overview of the main theoretical concepts for the case when the government commits to follow its announced in initial period fiscal plan (described in detail by Aiyagari, Marcet, Sargent and Seppala, 2002). I review here the incomplete markets case, noticing below that the complete markets solution can be extracted from the one described in this section in the way as Aiyagari et al. (2002) did.

Consider an economy like the one in section two, but assume that the government is obliged by some commitment mechanism to follow the fiscal plan it announced in period zero fiscal plan for all remaining infinite periods of the economy life.

In the decentralized equilibrium the households maximize their utilities choosing consumption, leisure and government bonds holdings subject to their budget constraint:

$$\max_{\{c_t, x_t, b_t^g\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - x_t, g_t), \tag{25}$$
$$s.t. \;\; c_t + p_t b_t = (1 - \tau_t)\theta_t(1 - x_t) + b_{t-1},$$

where b_t represents households' holdings of government bonds.

Definition: *A competitive equilibrium in the considered economy consists of stochastic processes for one-period risk-free bond prices, $\{p_t\}_{t=0}^{\infty}$, tax rates $\{\tau_t\}_{t=0}^{\infty}$, government expenditures $\{g_t\}_{t=0}^{\infty}$, allocations $\{c_t, x_t\}_{t=0}^{\infty}$, and bonds $\{b_t^g\}_{t=0}^{\infty}$, such that:*

1. given prices, taxes and government expenditures, the allocations $\{c_t, x_t\}_{t=0}^{\infty}$ maximize the households objective function $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, x_t, g_t)$ subject to budget constraint $c_t + p_t b_t = (1 - \tau_t)\theta_t(1 - x_t) + b_{t-1}$;

2. The government budget constraint is satisfied and natural debt limit of the government is not violated for each state of the world: $g_t + b_{t-1} = \tau_t(1 - x_t) + p_t b_t$, $\underline{M} \le b_t \le \bar{M}$;

3. Feasibility constraint is satisfied: $c_t + g_t = \theta_t(1 - x_t)$; bonds market clear: b_t is government debt held by the public..

Households debt limit is assumed to be less stringent that the government one, so that the households' problem always has interior solution. The government natural debt limit is difficult to find under the given assumptions of the problem, so in what follows I impose an ad hoc limit. However, the debt limit restriction should be rarely binding because of non-optimality of going too much in debt (it is optimal not to approach debt limit to avoid large cuts in consumption when the debt constraint is binding and the sequence of bad shocks takes place).

Proposition: The assumption of the market incompleteness leads to the following necessary conditions to be satisfied in the competitive equilibrium (see Aiyagari el al, 2002):

$$b_{-1} = \frac{1}{u_{c,0}} E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{x,t}(1 - x_t)), \tag{26}$$

$$\underline{M} \le b_{t-1} \le \bar{M}, \tag{27}$$

$$b_{t-1} = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} (c_{t+j} - \frac{u_{x,t+j}}{u_{c,t+j}} (1 - x_{t+j})). \tag{28}$$

Definition: *A Ramsey problem for a given incomplete markets economy consists of the choice of the government policy $\{g_t, \tau_t, b_t\}$ that maximizes the lifetime utility of the representative agent over competitive equilibria.*

The solution to Ramsey problem is obtained by maximizing (1) subject to (26)-(28). Note, that as stated, the problem (1)-(26)-(28) cannot be written in the recursive form, because the constraint (28) involves the forward-looking variables. To obtain recursive formulation Aiyagari et al.(2002) introduce an additional co-state variable Lagrange multiplier Ψ_t. The resulting equilibrium motion of the system is described by the equations (proof in the appendix):

$$\Psi_t = \frac{E_t u_{c,t+1}\Psi_{t+1} - E_t u_{c,t+1}(v_{1,t+1} - v_{2,t+1})}{E_t u_{c,t+1}}, \tag{29}$$

$$\frac{u_{c,t} - u_{g,t} + u_{cc,t}(\underline{M}v_{1,t} - \bar{M}v_{2,t} + (\Psi_t - \Psi_{t-1} - v_{1,t} + v_{2,t})b_{t-1})}{u_{cc,t}c_t + u_{c,t}} = \Psi_t,$$

$$\frac{u_{c,t} - \theta_t u_{g,t}}{u_{x,t} - u_{xx,t}(1 - x_t)} = \Psi_t,$$

$$b_t = \frac{u_{x,t}(1 - x_t) + b_{t-1}u_{c,t} - c_t u_{c,t}}{\beta E_t u_{c,t+1}},$$

where $v_{1,t+1}$ and $v_{2,t+1}$ are the Lagrange multipliers on the condition (27).

The resulting debt policy is a function of the state variables: $b_t = B(b_{t-1}, \Psi_{t-1}, \theta_t)$. Note, that this is different from the corresponding problem in the no commitment case, where the *co-state* Lagrange multiplier was not needed for the construction of equilibrium.

In the complete markets version of this economy government would be able to trade state-contingent debt. This would result in only the first equation (26) being necessary optimality condition for the solution (thus, $v_{1,t} = v_{2,t} = \Psi_t = 0, \forall t > 1, \Psi_1 = \Delta < 0$), and the debt function being defined as $b_t = B(\theta_t)$.

Marcet and Scott (2009) apply a test of the two market structures (complete vs incomplete, as described here), under full commitment, for their appropriateness in explanation of the stylized fiscal facts of the U.S. macroeconomic data: persistence of the government debt and co-movement of debt and primary deficit. The complete markets version of the model fails to reproduce both empirical facts. While incomplete markets generate both debt persistence (due to the presence of additional co-state variables from which one has a unit root characteristics - Lagrange multiplier), and its positive co-movement with deficit.

The following two sections discuss the ability of the no commitment model to reproduce the stylized fiscal facts, and compare the outcomes of the models under full and no commitment.

5 Comparison of Complete and Incomplete Markets Outcomes under No Commitment

The aim of this section is to compare the responses of debt and deficit to unexpected productivity shocks, both under complete and incomplete markets assumption, and to show that these responses are very similar, when the set of fiscal policies of the government is restricted to time-consistent Markov perfect equilibria.

First note, that the time-consistency results, even in the complete markets setup, in the presence of endogenous state variable: current debt position of the government. Citing Proposition 1 from Marcet and Scott (2009), *the portfolio of bonds issued at time t is independent of the realization of current period shocks. It does however depend on shocks predictable one period ahead and on the state variables,* thus, on the portfolio of bonds issued in period $t-1$. The issue of contingent bonds in period t by the discretionary government under complete markets may be represented as a time-invariant function:

$$b_t^{CM_{MPE}} = \int_\Omega B(b_{t-1}, \bar{\theta}) dF(\bar{\theta}|\theta_t) = \int_\Omega f^{CM_{MPE}}(b_{t-1}, \bar{\theta}) dF(\bar{\theta}|\theta_t). \tag{30}$$

Given the realization $\bar{\theta}_{t+1}$ of aggregate shock at date $t+1$, the debt to pay by the government at date $t+1$ is defined as

$$b_t^{CM_{MPE}}(\bar{\theta}_{t+1}) = f^{CM_{MPE}}(b_{t-1}, \theta_{t+1}). \tag{31}$$

In comparison, in the full commitment economy the government would face a unique implementability constraint for all time periods:

$$b_{-1} = \frac{1}{u_{c,0}} E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{x,t}(1 - x_t)). \tag{32}$$

Corresponding value of government debt to be repayed given the realization $\bar{\theta}_{t+1}$ of aggregate shock at date $t+1$, has a following general form:

$$b_t^{CM_{FC}}(\bar{\theta}_{t+1}) = f^{CM_{FC}}(\theta_{t+1}), \tag{33}$$

thus the resulted time-series are not persistent (they are determined only by the value of current aggregate shock), and contradict the behavior of U.S. data.

Relaxing the assumption of full commitment breaks the unique once-for-all periods implementability constraint (32) of the government problem, as now the government may reoptimize every period. This results in the appearance of a new state variable, previous government debt position, thus making the variable government debt much more persistent.

Under incomplete markets structure, the state variable b_{t-1} is still present in the debt function of the discretionary government. However, now the bonds issues are not contingent on future realizations of uncertainty. Instead, the issues of debt are defined by the realizations of current unexpected shocks. Given the realization $\bar{\theta}_{t+1}$ of aggregate shock at date $t+1$, the debt to be repayed by the government may be represented as a time-invariant function:

$$b_t^{IM_{MPE}}(\bar{\theta}_{t+1}) = f^{IM_{MPE}}(b_{t-1}, \theta_t). \tag{34}$$

Before turning to the numerical simulations of the solutions to the corresponding optimal policy functions (31),(33),(34), one can conjecture just looking on equations (31),(34) that the optimal debt policy in the Markov perfect equilibrium case should have similar stochastic behavior in the complete and incomplete markets, given that the stochastic shock θ_t follows some ergodic Markov process, with $\theta_t \in [\theta_{\min}; \theta_{\max}]$.

5.1 Numerical Simulations

To compare the stochastic behavior of the fiscal variables for the different market structures and commitment assumptions, the corresponding models were solved numerically. This subsection summarizes the choice of the models' parameters and proposes the discussion of the numerical solution for the Markov perfect equilibrium complete and incomplete market cases. In the next section the comparison of the numerical solutions for the full and no commitment is made.

Simulation Parameters

The parameters of utility function were chosen so, that labor account for approximately 2/3 of the time endowment in equilibrium, consumption to about 65% of output, and government expenditures - about 35% of output, in particular: $\phi_c = 0.20$, $\phi_g = 0.2$, $\sigma_c = 2$, $\sigma_x = 3$, $\sigma_g = 0.95$; $\beta = 0.96$.

The technology (labor productivity) shock is assumed to follow AR(1) process:

$$\log\theta_t = \rho\log\theta_{t-1} + \sigma\varepsilon_t.$$

with $\rho = 0.91$, $\varepsilon_t\tilde{}N(0,\sigma)$, where ρ was calibrated from the U.S. economic data of 1950-2007 (by assuming $Y_t = A_t\theta_t L_t$, $A_t = e^{0.02t}$, and using OECD data on labor force for 1950-2007). The value of deviation σ, implied by the considered US data is $\sigma = 0.0106$. Debt limits imposed are: $\underline{M} = -0.2; \bar{M} = 0.2$, with 0.2 corresponding to approximately 60% of economy's output.

Numerical Algorithm

The numerical solution[3] for the no commitment case (Markov perfect equilibrium) relies on the approximation of unknown policy functions ($c = \Psi(b,\theta)$, and $g = G(b,\theta)$ for the incomplete markets case, $c = \Psi(b,\theta)$, $g = G(b,\theta)$, and $b' = B(b,\theta')$ for the complete markets case) by Chebyshev polynomials, and then solving the system of equations that determines equilibrium by the collocation method on the two-dimensional grid of state variables (b,θ). The expectations are approximated by Gauss-Hermite quadrature on 10 points for possible values of θ_{t+1} given θ_t. The initial guess of the parameters of the unknown policy functions is obtained from the deterministic case (when the variance of the shock $\sigma^2 = 0$). Then gradual increase in the number of points of the grid and variance of the shock is applied to achieve the needed precision of the solution for the chosen set of parameters. The numerical algorithm is described more in detail in appendix.

The aim of this section is to compare the outcomes of the model for complete and incomplete market when the government is discretionary. To relate this study to the previous literature, the case of complete markets under full commitment is also presented. As was already noted before, complete markets perform bad in reproducing stylized facts about stochastic behavior of fiscal variables (see Marcet and Scott, 2009). I solve the complete markets full commitment case by considering the equation (32) for each time period, and approximating the expectation appearing in the right hand side of the equation.

[3]The MATLAB codes for the solutions of all models considered in the paper are available from the author upon request.

Below, the results of the numerical simulations are discussed.

Simulation results

Figure 1 below represents the responses of the debt, deficit, and output series to the positive productivity shock in the cases of Markov perfect equilibrium under complete markets, under incomplete markets, and, for comparison, in the case of full commitment under the complete markets.

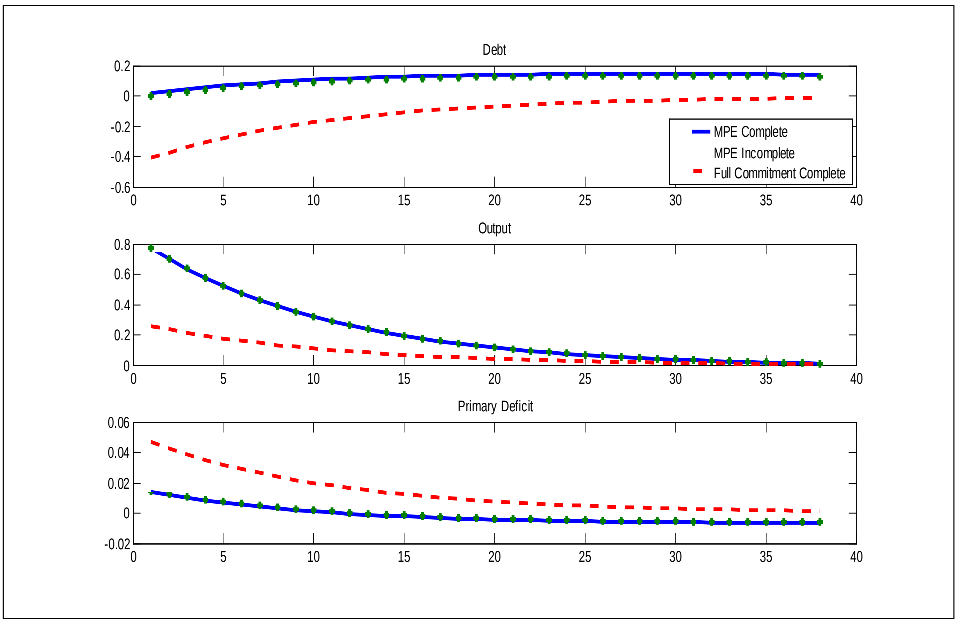

Figure 1. Impulse-responses to a st.dev. positive innovation to productivity, Markov perfect equilibrium complete markets ——, incomplete markets · · ·, and Full commitment complete markets - - - .

For the full commitment complete markets structure, as was already noted by the previous literature, the debt decreases for the shocks which cause deficit to increase (when public expenditures increase). For the Markov perfect equilibria, both complete and incomplete markets series show persistent response of the debt series, and in the same direction as that of deficit. Notice also, that the deviations of the debt and deficit series are much larger for the full commitment setup. The Markov perfect equilibrium series, independently of the market structure (complete or incomplete) react to the shock by almost the same magnitude. The similarity of responses is due to the fact that both series are the function of state variable - previous debt position, and due to the persistence of the aggregate shock (so that θ_t and θ_{t+1} can not be too different).

Figure 2 proposes simulations of the same variables in the stochastic steady state. Again, the full commitment debt is not persistent, and deviates significantly with aggregate productivity shocks. Whereas the no commitment series are persistent and have lower deviation, thus, result in less than optimal smoothness of optimal consumption of the agents.

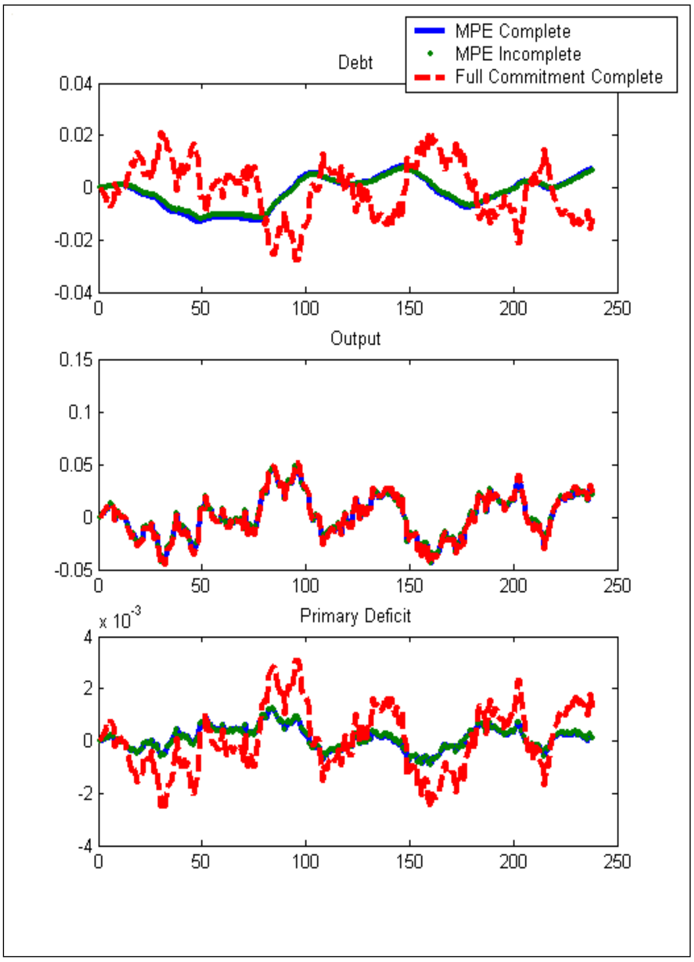

Figure 2. Simulations of Markov perfect equilibrium complete markets ——, incomplete markets · · ·, and Full commitment complete markets - - - .

Taking into account the results of the numerical solutions of the models considered in this section, and the conclusions of previous related studies (Marcet and Scott, 2009), I suggest that the complete markets on their own may not be so bad in reproduction of the stylized empirical facts. Instead, what makes the model to departure from the reality is the full commitment assumption.

6 Comparison of Full Commitment and No Commitment Outcomes under Incomplete Markets

This section checks whether the full commitment assumption is important for the model's outcomes in the incomplete markets world. Indeed, the previous section concluded that once full commitment is relaxed, the stochastic behavior of simulated debt and deficit series approach that observed in the U.S. data, regardless of the market completeness assumption.

Consider now the incomplete markets full commitment model, generally approved for its good replication of empirical facts, and remove the assumption of the commitment of the government.

Before turning to the numerical comparisons, consider again the optimal debt policy functions of the government as time-invariant functions of the states.

In the Markov perfect equilibrium with incomplete markets, as was already stated in the previous section, the debt function of the government, given the realization $\bar{\theta}_{t+1}$ of aggregate shock at date $t+1$, may be represented as a time-invariant function:

$$b_t^{IM_{MPE}}(\bar{\theta}_{t+1}) = f^{IM_{MPE}}(b_{t-1}, \theta_t). \tag{35}$$

In the Markov equilibrium the interest rate is determined as a reaction function to the government choice of the current bonds issues, so that there is no future period states in the optimality conditions.

It is different from the full commitment, where interest rate is determined by future consumption level. In the full commitment with incomplete markets, the debt policy is a function of three states: $b_{t-1}, \Psi_{t-1}, \theta_t$, where Ψ_{t-1} is a Martingale-type Lagrange multiplier. Thus, given the realization $\bar{\theta}_{t+1}$ of aggregate shock at date $t+1$, the debt function of the government in the full commitment may be represented as a time-invariant function:

$$b_t^{IM_{FC}}(\bar{\theta}_{t+1}) = f^{IM_{FC}}(b_{t-1}, \Psi_{t-1}, \theta_t). \tag{36}$$

As will be seen in the next subsection, the presence of additional co-state Ψ_{t-1} does not affect significantly the direction and persistence of the reaction of debt series to the aggregate shocks. The magnitude of the reaction of deficit, output, public and private consumption and leisure is neither affected significantly by Ψ_{t-1}.

However, the presence of Ψ_{t-1} was emphasized by the literature (see Scott, 2007) to be crucial for the success of the incomplete markets models in replication of the empirical behavior of labor taxes. While in general, the degree of persistence of the labor taxes depends on the parameters of the utility function, Ψ_{t-1} being martingale-type component of the time-invariant labor tax function, makes optimal taxes to approach random walk behavior. In the no commitment case Ψ_{t-1} does not enter optimal tax policy function, so for certain set of parameters the resulting optimal taxes are not persistent at all. To understand better these arguments consider the numerical simulations of both models.

Numerical Algorithm

The numerical solution for the full commitment incomplete markets case was obtained by approximation of expectations appearing in the necessary optimality conditions of the government problem by Parameterized expectation algorithm[4]. To compare the outcomes of the two commitment assumptions, I solve the full commitment version of the model numerically, and simulate the two economies with different commitment assumptions 1000 times for 250 periods, taking the last 50 observations from each simulation.

Below, the results of the numerical simulations are discussed.

[4]The alternative method, which delivers qualitatively same results, and uses the same approach as the one, employed for the no commitment model solution, is to approximate unknown functions by Chebyshev polynomials, with two states (b, θ) and co-state Ψ, dependent on the previous two exogenous states.

Simulation results

Table 1 below proposes summary statistics for the obtained outcomes in the full and no commitment incomplete markets economy.

Table 1. Summary statistics for simulations of the full commitment and no commitment incomplete markets economies

Variable	Full Commitment			MPE		
	mean	std	autocorr	mean	std	autocorr
$\sigma = 2, 3, 0.95$						
output	0.3212	0.0110	0.7985	0.3201	0.0108	0.7994
consumption	0.2481	0.0067	0.7991	0.2473	0.0066	0.8002
leisure	0.6796	0.0031	0.8063	0.6808	0.0032	0.8015
gov.exp.	0.0731	0.0043	0.8764	0.0728	0.0042	0.8213
debt	-0.0723	0.0193	0.9376	-0.0013	0.0072	0.9413
primary def.	0.0031	0.0016	0.8219	0.0001	0.0007	0.8149
deficit	-0.0678	0.0224	0.6672	-0.0012	0.0080	0.6906
labor tax	0.2177	0.0041	0.8905	0.2270	0.0037	0.8058
tech. shock	1.0033	0.0439	0.7997	1.0033	0.0439	0.7997

First thing to note is that the characteristics of the variables in the both commitment cases are relatively similar. Full commitment is characterized by the higher output and consumptions levels, which is expected due to its Pareto dominance with respect to the no commitment case. Labor taxes are higher in the no commitment.

However, the taxes in the full commitment case have higher persistence (0.89 vs 0.8 in the no commitment). The increase in the persistence of taxes when moving from the no to full commitment is the result of influence of the martingale-type Lagrange multiplier.

Figure 3 demonstrates the impulse-responses to the one standard deviation increase in the labor productivity[5]. The variables exhibit similar responses except the taxes which are *much less persistent* in the no commitment economy.

[5]In the simulated economies debt levels react positively to the positive innovations that increase output. This seems to be at odds with empirical evidence, as debt is commonly characterized as a counter cyclical variable. However, the model considered here assume an important role for the government expenditures in the utility of the agents. Therefore any increase in output (for the parameters studied here), will lead to the increase in the government spendings that overweights rise in tax revenues due to increased productivity, so that government deficit, and therefore debt will increase. For different set of elasticities and weights in utility function ($\phi_c = 0.30$, $\phi_g = 0.13$, $\sigma_c = 1$, $\sigma_x = 1$, $\sigma_g = 1$) the reaction of debt and deficit to the positive productivity shock is negative, as one would expect.

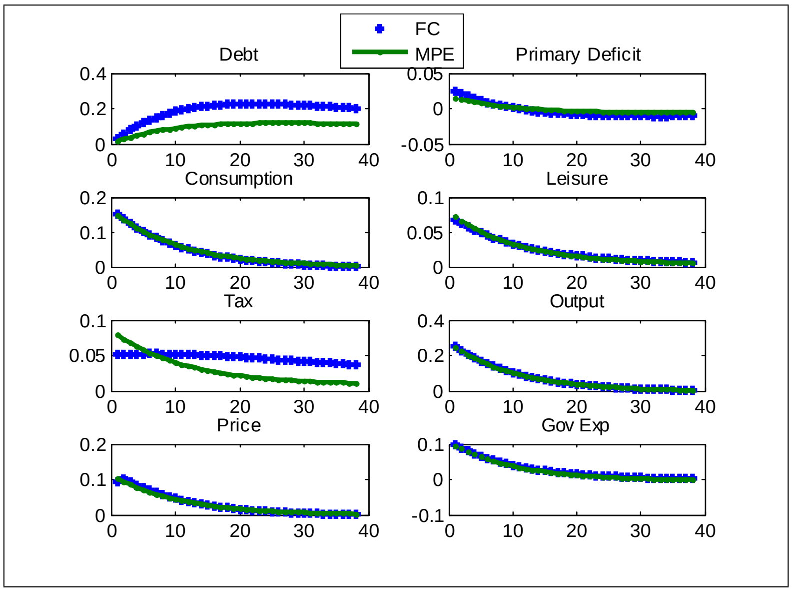

Figure 4. Impulse-responses to a st.dev. positive innovation to productivity, Markov perfect equilibrium incomplete markets ——, and Full commitment incomplete markets +++ .

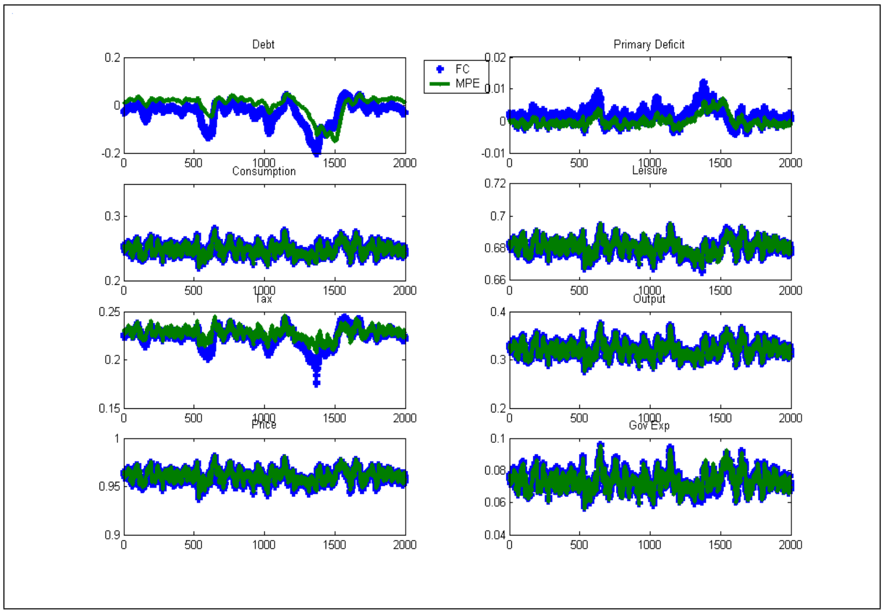

Figure 4. Simulations of the economies in full commitment and in Markov perfect equilibrium, incomplete markets.

As one can observe from the figure 3, the debt and deficit series demonstrate similar persistence properties in both models. However, the unit-root properties of the labor taxes disappear when the full commitment assumption is relaxed. This may be signal for the two conjectures: 1) full commitment to the government fiscal policy is a better description of the reality than the no commitment to the fiscal policy; or 2) incomplete markets structure is not so good in replicating the reality as one would suppose, as it is not robust to the ease of full commitment assumption. In this paper I will not try to justify any of the two conjectures, in the aim, however, to raise further discussion in the subsequent research.

7 Conclusions

The notion of time-consistency has been proven to be relevant for the description of the fiscal policy (see Kydland and Prescott, 1977), but it still does not enjoy many attention of the researchers. Partially because time-consistent problems are more difficult to investigate; partially because there is a few evidence of the consistency of the fiscal policy, as it is unclear how to test the degree of commitment.

In this paper I considered two principal questions about the properties of stochastic behavior of fiscal variables in the full commitment and no commitment real economies with aggregate uncertainty: 1) are these properties significantly different for two different assumptions about the ability of the government to commit to its fiscal plan; 2) is the complete markets structure still unable to replicate a set of stylized facts from U.S. economy when the full commitment assumption is relaxed.

I found that allowing the government to reoptimize its fiscal plan results in almost no change in the stochastic behavior of the fiscal variables, when compared with those from the full commitment case. That suggests that the conclusions about stochastic behavior of the variables in the models with artificially imposed full commitment assumption may approximately hold for those models where such assumption is relaxed (and equilibria are restricted to the Markov perfect ones). However, the absence of commitment almost completely eliminates the unit-root properties of the labor taxes, stressed in the incomplete markets with full commitment literature, and puts into doubt the performance of the incomplete markets models in replicating U.S. evidence of the behavior of labor taxes.

In a search for the answer for second question this paper relies on the evidence from the previous literature where the tests were proposed to discriminate between complete vs incomplete markets structure by looking at the reactions of debt and deficit to stochastic shocks. For these tests the assumption of full commitment was imposed. This paper finds that when such assumption is relaxed, there is no more distinguishing difference between the stochastic properties of debt and deficit in the complete and incomplete markets.

This study also suggests that the martingale component resulted from the availability of the full commitment and market incompleteness is only necessary for the correct replication of the empirical properties of taxes, and it is not decisive for the replication of debt and deficit series when the full commitment assumption is relaxed.

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Appendix

Derivation of the Generalized Euler Equation

The problem of the government in recursive form, given the optimality conditions of the competitive equilibrium and substituting leisure is:

$$V(b,\theta) = \max_{b',g} \left\{ u(\Psi(b,\theta), 1 - \frac{\Psi(b,\theta) + G(b,\theta)}{\theta}, G(b,\theta)) + \beta E V(b',\theta') \right\}, \qquad (MPE)$$

$$s.t.\ \ F = \Psi(b,\theta)u_c + \beta E u_c(\Psi(b',\theta'))b' - (\Psi(b,\theta) + G(b,\theta))u_x - b u_c = 0.$$

First order condition with respect to b':

$$(u_G - u_x/\theta)G_{b'} + \beta E V_{b'} = 0,$$

where

$$G_{b'} = -\frac{dF/db'}{dF/dG},$$

Envelope condition:

$$V_b = (u_\Psi - u_x/\theta)\Psi_b,$$

$$\Psi_b = -\frac{dF/db}{dF/d\Psi}.$$

Combining first order condition and envelope condition:

$$(u_G - u_x/\theta)G_{b'} + \beta E(u_{\Psi'} - u_{x'}/\theta')\Psi_b' = 0,$$
$$-(u_G - u_x/\theta)\frac{dF/db'}{dF/dG} - \beta E(u_{\Psi'} - u_{x'}/\theta')\frac{dF/db}{dF/d\Psi} = 0,$$
$$(u_g - u_x/\theta)\frac{\beta E(u_{cc}\Psi_{b'}b' + u_c)}{-u_x + u_{xx}(\Psi + G)/\theta} = \beta E(u_{c'} - u_{x'}/\theta')\frac{u_{c'}}{u_{cc}(c-b) - u_x + u_{xx}(\Psi+G)/\theta},$$

Description of Ramsey Problem in The Full Commitment Incomplete Markets Economy

The government maximize the utility of representative agent subject to the implementability constraints:

$$\max_{\{c_t,x_t,b_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t, x_t, \theta_t(1-x_t) - c_t)$$

$$b_{-1} = \frac{1}{u_{c,0}} E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t}c_t - u_{x,t}(1-x_t)),$$

$$\underline{M} \le b_{t-1} \le \bar{M},$$

$$b_{t-1} = E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}}(c_{t+j} - \frac{u_{x,t+j}}{u_{c,t+j}}(1 - x_{t+j})).$$

with corresponding Lagrangian given by:

$$\begin{aligned} L(.) = \max_{\{c_t,x_t,b_t\}_{t=0}^{\infty}} & E_0 \sum_{t=0}^{\infty} \beta^t \{u(c_t, x_t, \theta_t(1-x_t) - c_t) + \\ & +(-\gamma_t - v_{1,t} + v_{2,t})(u_{c,t}c_t - u_{x,t}(1-x_t)) + \\ & +u_{c,t}(v_{1,t}\underline{M} - v_{2,t}\bar{M} + \gamma_t b_{t-1})\}, \end{aligned}$$

introduce new co-state variable to make the problem recursive: $\Psi_t = \Psi_{t-1} + v_{1,t} - v_{2,t} + \gamma_t$.

Then:

$$\begin{aligned} L(.) = \max_{\{c_t,x_t,b_t\}_{t=0}^{\infty}} & E_0 \sum_{t=0}^{\infty} \beta^t (u(c_t, x_t, \theta_t(1-x_t) - c_t) - \\ & -\Psi_t(u_{c,t}c_t - u_{x,t}(1-x_t)) + u_{c,t}(v_{1,t}\underline{M} - v_{2,t}\bar{M} + \gamma_t b_{t-1})), \\ & \Psi_{-1} = 0, b_0 \cdot given. \end{aligned}$$

Description of Solution Algorithms

Full Commitment, Incomplete Markets

The motion of the system is described by the equations:

$$
\begin{aligned}
[b_t] &: \quad \Psi_t = \frac{E_t u_{c,t+1}\Psi_{t+1} - E_t u_{c,t+1}(v_{1,t+1} - v_{2,t+1})}{E_t u_{c,t+1}}, \\
[c_t] &: \quad \frac{u_{c,t} - u_{g,t} + u_{cc,t}(\underline{M} v_{1,t} - \bar{M} v_{2,t} + (\Psi_t - \Psi_{t-1} - v_{1,t} + v_{2,t})b_{t-1})}{u_{cc,t}c_t + u_{c,t}} = \Psi_t, \\
[x_t] &: \quad \frac{u_{c,t} - \theta_t u_{g,t}}{u_{x,t} - u_{xx,t}(1 - x_t)} = \Psi_t, \\
[IC] &: \quad b_t = \frac{u_{x,t}(1 - x_t) + b_{t-1}u_{c,t} - c_t u_{c,t}}{\beta E_t u_{c,t+1}}.
\end{aligned}
$$

To make the solution possible one needs to approximate two unknown expectations, which are the functions of the state variables (see Marcet, Marimon (1989)). The functional form used for approximation is exponential polynomials:

$$
\begin{aligned}
E_t(\mu_{t+1}u_{c,t+1}) &= \exp(\Psi_{t-1}(\beta^1), b_{t-1}(\beta^1), \theta_t; \beta^1), \\
E_t(u_{c,t+1}) &= \exp(\Psi_{t-1}(\beta^2), b_{t-1}(\beta^2), \theta_t; \beta^2).
\end{aligned}
$$

The parameterized expectations algorithm described in Den Haan and Marcet (1990) gives easily a numerical solution to this problem.

No Commitment, Complete Markets

The motion of the system is described by the equations:

$$
\frac{u_{x,t} - u_{c,t}}{u_{c,t} + (c_t - b_{t-1})u_{cc,t} - u_{x,t} + (c_t + g_t)u_{xx,t}} E_t(u_{cc,t+1}(\Psi(b_t))\Psi_{b,t}(b_t)b_t + u_{c,t+1}(\Psi(b_t))) =
$$

$$
= E_t \frac{u_{x,t+1} - u_{c,t+1}}{u_{c,t+1} + (c_{t+1} - b_t)u_{cc,t+1} - u_{x,t+1} + (c_{t+1} + g_{t+1})u_{xx,t+1}} u_{c,t+1},
$$

$$
\frac{u_{c,t} - u_{g,t}}{u_{c,t} + (c_t - b_{t-1})u_{cc,t}} = \frac{u_{x,t} - \theta_t u_{g,t}}{-u_{x,t} + (c_t + g_t)u_{xx,t}},
$$

$$
c_t u_{c,t} + \beta E_t(u_{c,t+1}(\Psi(b_t)))b_t = (1 - x_t)u_{x,t} + b_{t-1}u_{c,t}.
$$

To solve the system, one needs to approximate two unknown functions of the state variables: consumption and government expenditures, and a state-contingent bond function. I use collocation method described in Judd (1992). The unknown functions are approximated by two-dimensional Chebyshev polynomials:

$$
c_t = \Psi^1(b_{t-1}, \theta_t), \;\; g_t = \Psi^2(b_{t-1}, \theta_t), \;\; b_t = \Psi^3(b_{t-1}, \theta_{t+1}),
$$

where
$$
\Psi(b_{t-1}, \theta_t) = \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} T_i(b_{t-1}) T_i(\theta_t),
$$

$$
T_0(b_t) = 1, T_1(b_t) = b_t, ..T_j(b_t) = 2b_t T_{j-1}(b_t) - T_{j-2}(b_t), \quad j = 2, 3, ..,
$$

on the grid $[b_{t-1};\theta_t]:[[\underline{M};\bar{M}];[-3\sigma_\theta;3\sigma_\theta]]$, where $\sigma_\theta=\frac{\sigma}{\sqrt{1-\rho^2}}$, with polynomials evaluated on the grid rescaled to zeros of Chebyshev polynomials. Expectations appearing in the Euler equation and in the budget constraint are approximated by Gauss-Hermite quadrature. The $N\times N$ points of the grid result in the system of $N\times N\times 3$ equations which can be solved for $N\times 2$ unknown coefficients and N values of b_t.

No Commitment, Incomplete Markets

The motion of the system is described by the equations:

$$\frac{u_{x,t}-u_{c,t}}{u_{c,t}+(c_t-b_{t-1})u_{cc,t}-u_{x,t}+(c_t+g_t)u_{xx,t}}E_t(u_{cc,t+1}(\Psi(b_t))\Psi_{b,t}(b_t)b_t+u_{c,t+1}(\Psi(b_t)))\;=$$

$$=E_t\frac{u_{x,t+1}-u_{c,t+1}}{u_{c,t+1}+(c_{t+1}-b_t)u_{cc,t+1}-u_{x,t+1}+(c_{t+1}+g_{t+1})u_{xx,t+1}}u_{c,t+1},$$

$$\frac{u_{c,t}-u_{g,t}}{u_{c,t}+(c_t-b_{t-1})u_{cc,t}}=\frac{u_{x,t}-\theta_t u_{g,t}}{(-u_{x,t}+(c_t+g_t)u_{xx,t}},$$

$$c_t u_{c,t}+\beta E_t(u_{c,t+1}(\Psi(b_t)))b_t=(1-x_t)u_{x,t}+b_{t-1}u_{c,t}.$$

To solve the system, one needs to approximate two (in case of exogenous government expenditures one) unknown functions of the state variables: consumption and government expenditures. I use collocation method described in Judd (1992). The unknown functions are approximated by two-dimensional Chebyshev polynomials:

$$c_t=\Psi^1(b_{t-1},\theta_t),\qquad g_t=\Psi^2(b_{t-1},\theta_t),$$

where

$$\Psi(b_{t-1},\theta_t)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}T_i(b_{t-1})T_i(\theta_t),$$

$$T_0(b_t)=1,T_1(b_t)=b_t,..T_j(b_t)=2b_tT_{j-1}(b_t)-T_{j-2}(b_t),\quad j=2,3,..,$$

on the grid $[b_{t-1};\theta_t]:[[\underline{M};\bar{M}];[-3\sigma_\theta;3\sigma_\theta]]$, where $\sigma_\theta=\frac{\sigma}{\sqrt{1-\rho^2}}$, with polynomials evaluated on the grid rescaled to zeros of Chebyshev polynomials. Expectations appearing in the Euler equation and in the budget constraint are approximated by Gauss-Hermite quadrature. The $N\times N$ points of the grid result in the system of $N\times N\times 3$ equations which can be solved for $N\times 2$ unknown coefficients and N values of b_t.

The stability of the Chebyshev coefficients with increase of the order of polynomial and their gradual decrease within the polynomial with increase of the order of state variables served as an informal accuracy check.

All initial guesses were obtained from the no uncertainty balanced budget case, with subsequent solution involving introduction of shocks, gradual increase of the debt limits and of orders of approximating functions.

Accuracy Check

The examination of Euler equation errors by method of Judd (1992) for the solution of the no commitment case:

Table 2. Accuracy check

Elasticities (grid for $b, \theta : 100x50$)	$log_{10}\|\|E\|\|_1$	$log_{10}\|\|E\|\|\infty$
$\sigma = 2, 3, 0.95$	-7.02	-6.29

Errors implied: on maximum 1 dollar lost per 10000000 of consumption in this period relatively to the next period (Judd, 1992).